



Estimating the order of vanishing at infinity of Drinfeld quasi-modular forms

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Federico Pellarin

July 26, 2009

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1 Introduction, results.

Foreword about the classical theory. For w, l non-negative integers, let $\widetilde{M}_w^{\leq l}$ be the \mathbb{C} -vector space generated by the classical *quasi-modular forms* (for $\mathbf{SL}_2(\mathbb{Z})$) which have weight w and depth $\leq l$ (definition of Kaneko and Zagier in [20]).

We recall that in [4] it was noticed, by a simple resultant argument, for $f \neq 0$ a classical quasi-modular form of weight w and depth $\leq l$, that

$$\nu_\infty(f) \leq \frac{(l+1)(w-l)}{6}. \quad (1)$$

In [19], Kaneko and Koike highlight the following hypothesis; for all w, l such that $\widetilde{M}_w^{\leq l} \neq \{0\}$ the image of the function

$$\nu_\infty : \widetilde{M}_w^{\leq l} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0},$$

which associates to every form its *order of vanishing at infinity* ⁽¹⁾, is equal to the interval $[0, \dots, \dim_{\mathbb{C}} \widetilde{M}_w^{\leq l} - 1]$. Writing $\nu_\infty^{\max}(w, l) = \max\{\nu_\infty(f) : f \in \widetilde{M}_w^{\leq l} \setminus \{0\}\}$, we obviously get from this hypothesis,

$$\lim_{w \rightarrow \infty} \frac{\dim_{\mathbb{C}} \widetilde{M}_w^{\leq l}}{\nu_\infty^{\max}(w, l)} = 1, \quad l \geq 0 \text{ fixed}. \quad (2)$$

Unexpected difficulties occur when one tries to formulate and prove analogues of (1) and (2) in the framework of *Drinfeld quasi-modular forms*. The main theorem of the paper is a slightly weaker analog of inequality (1); finding a reasonable substitute of (2) in the drinfeldian framework remains an open problem.

Drinfeldian theory. Let $q = p^e$ be an integer power of a prime number p with $e > 0$, let \mathbb{F}_q be the field with q elements. Let us write $A = \mathbb{F}_q[\theta]$ and $K = \mathbb{F}_q(\theta)$, with θ an indeterminate over \mathbb{F}_q , and define an absolute value $|\cdot|$ on K by $|a| = q^{\deg_\theta a}$, a being in K , so that $|\theta| = q$. Let $K_\infty := \mathbb{F}_q((1/\theta))$ be the completion of K for this absolute value, let $K_\infty^{\text{alg.}}$ be an algebraic closure of K_∞ , let C be the completion of $K_\infty^{\text{alg.}}$ for the unique extension of $|\cdot|$ to $K_\infty^{\text{alg.}}$, and let $K^{\text{alg.}}$ be the algebraic closure of K in C .

¹Terminology explained, for example, in [4].

Following Gekeler in [13], we denote by Ω the rigid analytic space $C \setminus K$ and write Γ for $\mathbf{GL}_2(A)$, group that acts on Ω by homographies. In this setting we have three functions $E, g, h : \Omega \rightarrow C$, holomorphic in the sense of [11, Definition 2.2.1], such that, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \Omega$:

$$\begin{aligned} g(\gamma(z)) &= (cz + d)^{q-1} g(z), \\ h(\gamma(z)) &= (cz + d)^{q+1} \det(\gamma)^{-1} h(z), \\ E(\gamma(z)) &= (cz + d)^2 \det(\gamma)^{-1} \left(E(z) - \frac{c}{\tilde{\pi}(cz + d)} \right) \end{aligned} \quad (3)$$

where $\gamma(z) = (az + b)/(cz + d)$ and

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in K_{\infty}((-\theta)^{\frac{1}{q-1}}) \setminus K_{\infty},$$

a choice of a $(q-1)$ -th root having been made once and for all ⁽²⁾.

The functional equations above tell that g, h are *Drinfeld modular forms*, of weights $q-1$, $q+1$ and types $0, 1$ respectively. More precisely, the function g is proportional to a variant of Eisenstein series (constructed by Goss in [15]), while h is proportional to a variant of a Poincaré series (constructed by Gekeler in [13]).

The function E is not a Drinfeld modular form. In [13], Gekeler calls it “False Eisenstein series” of weight 2 and type 1; it is often considered as a reasonable substitute of the normalised (complex) Eisenstein series E_2 of weight 2, although it “vanishes at infinity” (see later).

The C -algebra $\widetilde{M} := C[E, g, h]$ has dimension 3. Weights and types of E, g, h associated to the functional equations (3) determine a graduation of \widetilde{M} by the group $\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$. A *degree* is a couple of integers $(w, m) \in \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$. By convention, we identify the class m with its unique representative in the interval $[0, q-1[$. A polynomial $f \in \widetilde{M} \setminus \{0\}$ is a *Drinfeld quasi-modular form* of *weight* w and *type* m if it is homogeneous (or isobaric) of degree (w, m) .

The algebra \widetilde{M} is also filtered by the *depths*. The depth $l(f)$ of a non-zero polynomial $f \in \widetilde{M}$ is by definition its degree $\deg_E f$. By convention, the zero polynomial of \widetilde{M} is a quasi-modular form of weight w , type m and depth l for all w, m, l . In all the following, we denote by $\widetilde{M}_{w,m}^{\leq l}$ the finite-dimensional C -vector space of Drinfeld quasi-modular forms of weight w , type m and depth $\leq l$; we recall that if $w \not\equiv 2m \pmod{q-1}$, then $\widetilde{M}_{w,m}^{\leq l} = (0)$. Obviously, if $f \neq 0$ is quasi-modular, then $w(f) \geq 2l(f)$. The dimension of $\widetilde{M}_{w,m}^{\leq l}$ can be easily computed with

²See [27, Section 2.1], where the notation $\overline{\pi}$ is adopted; there is an analogy with the number $2\pi i$.

the formulas of [14, p. 33]; one deduces that for l, m fixed,

$$\lim_{w \rightarrow \infty} \frac{\dim_C \widetilde{M}_{w,m}^{\leq l}}{w-l} = \frac{l+1}{q^2-1}, \quad (4)$$

the limit being taken over the w 's such that $w \equiv 2m \pmod{q-1}$, because if $w \not\equiv 2m$, then $\widetilde{M}_{w,m}^{\leq l} = (0)$.

Let $e_{\text{Car}} : C \rightarrow C$ be the Carlitz exponential function (see below, (21)), and let us write $u : \Omega \rightarrow C$ for the “parameter at infinity” of Ω , that is, the function $u(z) = 1/e_{\text{Car}}(\pi z)$.

The C -algebra \widetilde{M} embeds in $C[[u]]$ (cf. [13]). If $f \in \widetilde{M}$, then, for $u = u(z)$ with $|u|$ small enough, we have a converging u -expansion

$$f(z) = \sum_{i=0}^{\infty} c_i u^i, \quad c_i \in C. \quad (5)$$

Let us write, with f as in (5), $\nu_{\infty}(f) := \inf\{i \text{ such that } c_i \neq 0\}$ with the convention $\inf \emptyset = \infty$. If $f \neq 0$, then $\nu_{\infty}(f) < \infty$.

The main motivation of this paper is the following:

Question *Let w, m, l be such that $\widetilde{M}_{w,m}^{\leq l} \neq (0)$. How big can $\nu_{\infty}(f)$ be, with $f \in \widetilde{M}_{w,m}^{\leq l}$ non-zero?*

If $\widetilde{M}_{w,m}^{\leq l} \neq (0)$, let us write

$$\nu_{\infty}^{\max}(w, m, l) = \max\{\nu_{\infty}(f) : f \in \widetilde{M}_{w,m}^{\leq l} \setminus \{0\}\}$$

and let us first look at the case $l = 0$. We write $M_{w,m} = \widetilde{M}_{w,m}^{\leq 0}$. Let $M = \oplus_{w,m} M_{w,m} = C[g, h]$ be the graded C -algebra of Drinfeld modular forms, of dimension 2. It is easy to show (the brackets $[\cdot]$ denote the lower integer part) that

$$\left\lfloor \frac{\nu_{\infty}^{\max}(w, m, 0)}{q-1} \right\rfloor = \dim_C M_{w,m} - 1, \quad (6)$$

so that, for all $f \in M_{w,m} \setminus \{0\}$,

$$\nu_{\infty}(f) \leq \frac{w}{q+1}, \quad (7)$$

which also is the best possible bound linear in w ⁽³⁾.

³Riemann-Roch's Theorem over the rigid analytic curve compactification of $\Gamma \backslash \Omega$ also implies (7), see [13, (5.14)].

Different phenomena arise in the vector spaces $\widetilde{M}_{w,m}^{\leq l}$ for $l > 0$. In this case, it is unclear how to extend (6).

An example and a conjecture. We try to better illustrate this difficulty, proper to the non-zero characteristic theory, by considering the case of depth ≤ 1 . In [4] we have constructed a family of Drinfeld quasi-modular forms $(x_k)_{k \geq 0}$ with $x_k \in \widetilde{M}_{q^{k+1},1}^{\leq 1} \setminus M$ *extremal* in the sense that it attains, in the indicated vector space, the biggest possible order of vanishing at infinity; we have also proved that $\nu_\infty(x_k) = q^k$ for all k . From this construction, one can in fact furnish a normalised extremal quasi-modular form $f_{w,m,1}$ in every non-trivial vector space $\widetilde{M}_{w,m}^{\leq 1}$ (4). If for example $q = 5, m = 1$, the sequence $(f_{4n+2,1,1})_{n \geq 0}$ is:

$$-x_0, \underbrace{-\frac{x_1}{[1]}, -g \frac{x_1}{[1]}, \dots, -g^4 \frac{x_1}{[1]} - \frac{x_2}{[1][2]}, -g \frac{x_2}{[1][2]}, \dots, -g^{24} \frac{x_2}{[1][2]}}_{5 \text{ terms}}, \dots,$$

where $[i] := \theta^{q^i} - \theta$ (if w is not of the form $4n + 2$, the space $\widetilde{M}_{w,1}^{\leq 1}$ is trivial). For general q and m it can be proved, by using the forms x_k 's, that the sequences $(f_{(q-1)n+2m,m,1})_{n \geq 0}$ involve quasi-modular forms which are monomials $\lambda g^a h^b x_k$ ($\lambda \in C^\times, a, b, k \geq 0$). An accurate study of these forms (that we skip here), implies that for all q, m :

$$0 < \liminf_{w \rightarrow \infty} \frac{\dim_C \widetilde{M}_{w,m}^{\leq 1}}{\nu_\infty^{\max}(w, m, 1)} < \limsup_{w \rightarrow \infty} \frac{\dim_C \widetilde{M}_{w,m}^{\leq 1}}{\nu_\infty^{\max}(w, m, 1)} < \infty,$$

the limits being taken in sequences with w such that $\dim_C \widetilde{M}_{w,m}^{\leq 1} \neq 0$; so that there is no close analog of (2). The smallest infimum limit precisely occurs in the sequence of spaces $\widetilde{M}_{q^{k+1},1}^{\leq 1}$, for $k \geq 0$. Induction on $k \geq 0$ starting with the equality $\dim_C \widetilde{M}_{2,1}^{\leq 1} = 1$ yields the computation of the dimensions:

$$\dim_C \widetilde{M}_{q^{k+1}+1,1}^{\leq 1} = q \dim_C \widetilde{M}_{q^k+1,1}^{\leq 1} + r_k,$$

where $r_{2s} = -1$ and $r_{2s+1} = -2$ if $q = 2$, and $r_{2s} = -q + 2$ and $r_{2s+1} = -2q + 3$ if $q > 2$. Hence,

$$\lim_{k \rightarrow \infty} \frac{\dim_C \widetilde{M}_{q^{k+1},1}^{\leq 1}}{\nu_\infty^{\max}(q^k + 1, 1, 1)} = \frac{2}{q^2 - 1}.$$

Combining with (4) we find that for all w big enough with $\widetilde{M}_{w,m}^{\leq 1} \neq (0)$ and $f \in \widetilde{M}_{w,m}^{\leq 1} \setminus \{0\}$, $\nu_\infty(f) \leq w - 1$ (that is, $\leq l(w - l)$ with $l = 1$).

⁴A formal series $\sum_{i \geq i_0} c_i u^i \in L[[u]]$ (L being a field) with $c_{i_0} \neq 0$ is *normalised* if its leading coefficient c_{i_0} is one.

These arguments have been extended with the help of several numerical computations to higher depths and seem to justify the following (cf. [4]):

Conjecture *Let q and $l > 0$ be fixed. For all m , for all w big enough such that $\widetilde{M}_{w,m}^{\leq l} \neq (0)$, and for all $f \in \widetilde{M}_{w,m}^{\leq l} \setminus \{0\}$,*

$$\nu_{\infty}(f) \leq l(w - l). \quad (8)$$

This upper bound cannot be improved, as the choice $f = E^n$ indicates (indeed, we have $\nu_{\infty}(E) = 1$).

In this paper we demonstrate a partial but quite satisfactory result in the direction of the inequality (8), by proving the following slightly weaker analog of inequality (1):

Theorem *Let w, l, m be integers, with $0 \leq m < q - 1$, $l > 0$, let $f \in \widetilde{M}_{w,m}^{\leq l}$ be a non-zero Drinfeld quasi-modular form. Then,*

$$\nu_{\infty}(f) \leq 6q^4(4q + 5)^3(q^2 + 1)l^2w(12 + \log_q w), \quad (9)$$

where \log_q denotes the logarithm in base q . If moreover

$$w \geq 4l(2q(q + 2)(3 + 2q)l + 3(q^2 + 1))^{3/2}, \quad (10)$$

Then we have the more precise estimate

$$\nu_{\infty}(f) \leq 16q^3(3 + 2q)^2lw. \quad (11)$$

Condition (10) makes it clear that our Theorem does not overlap with [4, Theorem 1.4] (joint work with Bosser), a rather precise estimate which, however, holds for Drinfeld quasi-modular forms of depth $\leq q^2$ only.

1.1 Difficulties encountered by using hyperdifferential structures

A sketch of proof of inequality (1) is given in the introduction of [4]. If $f \in \mathbb{C}[E_2, E_4, E_6]$ is a non-zero (classical) quasi-modular form ⁽⁵⁾, Ramanujan's differential system implies that $(d/dz)f$ is again a non-zero quasi-modular form. The

⁵The E_{2i} 's denote normalisations of classical Eisenstein series of weights $2i$.

bound follows easily remarking that if f and $(d/dz)f$ are coprime, then the resultant

$$\mathbf{Res}_{E_2}(f, (d/dz)f) \quad (12)$$

of the polynomials $f, (d/dz)f \in \mathbb{C}[E_2, E_4, E_6]$ with respect to E_2 is a non-zero modular form whose weight is controlled by elementary considerations and whose order of vanishing at infinity is controlled by the well known suitable analog of (7).

If f and $(d/dz)f$ are not coprime, one combines the resultant argument with a variant of the *separation property* of Brownawell and Masser [5, Lemma, p. 212] which is Nesterenko's Lemma 5.2 of [24, Chapter 10].

Inequality (1) is in fact a very simple *multiplicity estimate*. Multiplicity estimates are important tools in transcendence and algebraic independence techniques. A much deeper result was obtained by Nesterenko [24, Chapter 10, Theorem 1.3], and was the key tool in his theorem on the algebraic independence of values of normalised Eisenstein series and the function $e^{2\pi iz}$. The theory of multiplicity estimates in differential polynomial algebras gave general results when the base field is algebraically closed of zero characteristic; see for example [5, 6, 23].

Analogues in positive characteristic of differential algebras are called *iterative differential*, or *hyperdifferential* algebras (cf. [22]); very little is known about multiplicity estimates in this framework. In a joint paper with Bosser, we proved [3, Theorem 2] that the algebra \widetilde{M} is hyperdifferential, endowed with a *hyperdifferential structure* D , that is, a collection of linear operators $D = (D_n)_{n \geq 0}$ satisfying certain properties. However, for a given non-constant polynomial $f \in \widetilde{M}$, the sequence of polynomials $(D_n f)_{n \geq 0}$ behaves rather erratically (despite the algorithms we developed in [4, Section 4.2]) and it is a serious computational problem to characterise, with enough generality, the set of forms f that divide $D_1 f, \dots, D_n f$ for a given $n \geq 0$. Hence, there is not a suitable separation property analog.

At least, we constructed in [4] families of *extremal* Drinfeld quasi-modular forms which allowed, by using a resultant argument, partial multiplicity estimates in the direction of (8).

1.2 Methods of proof with Frobenius structures

The main idea behind the proof of Theorem is to work with certain *deformations* of Drinfeld quasi-modular forms, that we called *almost A -quasi-modular forms* ⁽⁶⁾, introduced in this paper after a variation on Anderson's idea of t -motives, also called A -motives. Hence, in Section 2 we review and develop tools which have essentially been introduced by Anderson in [1], concerning *rigid analytic trivialisations* of A -motives associated to rank 2 Drinfeld A -modules. We use the

⁶The reader might find this terminology rather heavy. It has been chosen because in forthcoming works, we will also need to deal with *A -quasi-modular forms* and *A -modular forms*.

exposition in [27] as a basis to build the necessary background to proceed further: linear τ -difference equations over all.

While spaces of Drinfeld quasi-modular forms embed in $C[[u]]$ and are spanned by forms with coefficients in $\mathbb{F}_q[\theta]$, the spaces of almost A -quasi-modular forms we are interested in, embed in $C[[t, u]]$, with t a new indeterminate, are spanned by forms with coefficients in $\mathbb{F}_q[t, \theta]$ ⁽⁷⁾, and the algebra they generate is endowed with the extension $\tau : C[[t, u]] \rightarrow C[[t, u]]$ of Frobenius map $\tau : C \rightarrow C$, $\tau c = c^q$, defined by:

$$\tau \sum_{m,n \geq 0} c_{m,n} t^m u^n := \sum_{m,n \geq 0} c_{m,n}^q t^m u^{qn}$$

the $c_{m,n}$'s being in C .

Replacing t by θ essentially pushes almost A -quasi-modular forms down to Drinfeld quasi-modular forms (a special care is required to check convergence of our series). We will lift the quasi-modular form E to an almost A -quasi-modular form $\mathbf{E} \in \mathbb{F}_q[t, \theta][[u]]$ of which we will study the main properties, in Section 3. This section contains the most important tools of our paper, and its crucial features will be separately reviewed in its introduction. With the help of the results proved there, we will construct, in Section 4, a four dimensional sub-algebra \mathbb{M}^\dagger of almost A -quasi-modular forms which is at once:

- Graded by the group $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ (a *degree* will be a triple (μ, ν, m) , with (μ, ν) the *weight*, m the *type*).
- Stable under the action of τ , in a way which is compatible with the graduation.
- Endowed with a set of generators contained in $\mathbb{F}_q[t, \theta][[u]]$, whose coefficients c_n of their u -expansions have the property that the degrees in t grow “slowly” as n increases, unlike the degrees in θ .

The Theorem will be proved in Section 5 by using the properties above, avoiding resultants such as (12), in two steps. The first step is made by a multiplicity estimate in \mathbb{M}^\dagger itself (Proposition 19), with the use of resultants like

$$\mathbf{Res}_{\mathbf{E}}(\mathbf{f}, \tau \mathbf{f}), \tag{13}$$

with $\mathbf{f} \in \mathbb{M}^\dagger$, which essentially lands in $C[[t]][g, h]$, after rescaling by a well controlled A -quasi-modular form. The required variant of the separation property is easy to obtain, but the grading by $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ is unavoidable at this stage.

⁷These forms do not seem to have a counterpart in the classical framework yet, but could perhaps be related to *bimodular forms* (the author learned about the latter forms in a seminar of Zagier at “Collège de France” on a joint work by himself and Stienstra).

The second step will use *transcendental techniques*. With a variant of Siegel's lemma we construct a collection of non-trivial auxiliary forms $\mathbf{f}_{\mu,\nu,m} \in \mathbb{M}^\dagger$ of weight (μ, ν) and type m with certain technical conditions on $\mu, \nu \in \mathbb{Z}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. These forms vanish with high order at infinity, but we can bound from above this order by Proposition 19. Let f be a Drinfeld quasi-modular form. The proof of our Theorem ends with the study of a second resultant which lands in Drinfeld modular forms:

$$\mathbf{Res}_E(f, \varepsilon(\tau^k \mathbf{f}_{\mu,\nu,m})),$$

making good choice of the parameters μ, ν, m . Here, we also need to choose k not too big; this choice will be made possible by the crucial property highlighted above, that the u -expansions of the generators of \mathbb{M}^\dagger have their coefficients whose degrees in t grow slowly as the index increases.

After having discovered the benefits of the graduation by $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ of \mathbb{M}^\dagger , we have, for a while, searched for similar structures above classical quasi-modular forms. We believe that this theme should be investigated in the sequel. The reader might be interested in Remark 15, where we collect the results we know in this domain.

The paper ends with Section 6, where we describe a link to extremal quasi-modular forms as defined in [4]. In forthcoming works, we will develop the hyperdifferential viewpoint of almost A -quasi-modular forms and the appropriate extension of the theory of Hecke operators and Hecke eigenforms.

2 Anderson's functions

In this section we recall some tools introduced in [13, Section 2], [1, 2] and described in [27, Section 2 and Section 4.2].

As *A-lattice of rank $r > 0$* we mean a free sub- A -module of C of rank r , discrete in the sense that in every compact subset of C only finitely many elements of it lie. Let $\Lambda \subset C$ be an A -lattice of rank r and let us consider, for $\zeta \in C$, the function defined by the product:

$$e_\Lambda(\zeta) := \zeta \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{\zeta}{\lambda}\right), \quad (14)$$

which converges for all $\zeta \in C$. For $\lambda \in C^\times$, we have by the product expansion (14) defining e_Λ :

$$e_{\lambda\Lambda}(\zeta) = \lambda e_\Lambda(\lambda^{-1}\zeta). \quad (15)$$

There exist elements $1 = \alpha_0(\Lambda), \alpha_1(\Lambda), \alpha_2(\Lambda), \dots \in C$, depending on Λ only, such that:

$$e_\Lambda(\zeta) = \sum_{n \geq 0} \alpha_n(\Lambda) \zeta^{q^n}, \quad (16)$$

the series having infinite radius of convergence (cf. [13, 16]).

The construction of the exponential function by (14) is the main tool to prove that the category whose objects are A -lattices of rank r and morphisms are inclusions, is dually equivalent to the category whose objects are Drinfeld A -modules of rank r and morphisms are isogenies (see [13, Section (2.6)] or [27, Section 2]; references that contain a definition of Drinfeld A -module). For Λ as above, there is a Drinfeld A -module ϕ_Λ such that

$$\phi_\Lambda(a)e_\Lambda(\zeta) = e_\Lambda(a\zeta) \quad (17)$$

(for all $\zeta \in C$ and $a \in A$), which is uniquely determined by its value $\phi_\Lambda(\theta) \in \mathbf{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a)$ in θ . This value is a polynomial of degree r in τ , that we recall, is the Frobenius endomorphism $\tau : c \mapsto c^q$.

On the other side, any Drinfeld A -module ϕ of rank r , a lattice Λ_ϕ of rank r can be associated, so that the functors $\Lambda \mapsto \phi_\Lambda$ and $\phi \mapsto \Lambda_\phi$ are inverse of each other up to isomorphisms.

Let t be a new indeterminate. With Λ an A -lattice of rank $r > 0$ and e_Λ as in (14), let us consider $\omega \in \Lambda \setminus \{0\}$ and introduce, following Anderson in [1], the formal series:

$$s_{\Lambda, \omega}(t) := \sum_{i=0}^{\infty} e_\Lambda \left(\frac{\omega}{\theta^{i+1}} \right) t^i.$$

For a positive real number r , we denote by $\mathbb{T}_{<r}$ the sub- C -algebra of $C[[t]]$ whose elements are formal series $\sum_{i \geq 0} c_i t^i$ that converge for any $t \in C$ with $|t| < r$, and $\mathbb{T}_{\leq r}$ for the C -sub-algebra of formal series $\sum_{i \geq 0} c_i t^i$ that converge for $|t| \leq r$. We denote by $\mathbb{T}_{>0}$ the sub- C -algebra of $C[[t]]$ whose series converge in some non-empty open disk containing 0, and all the series of $\mathbb{T}_{\leq q}$ converge at $t = \theta$. We also denote by \mathbb{T}_∞ the sub- C -algebra of series that converge everywhere in C .

If $r_1 > r_2 > 0$, we have

$$\mathbb{T}_{>0} \supset \mathbb{T}_{<r_2} \supset \mathbb{T}_{\leq r_2} \supset \mathbb{T}_{<r_1} \supset \mathbb{T}_{\leq r_1} \supset \mathbb{T}_\infty.$$

The *Tate algebra* $\mathbb{T}_{\leq 1}$ will be also denoted by \mathbb{T} .

It is easy to verify that, with Λ and $\omega \in \Lambda$ as above, $s_{\Lambda, \omega} \in \mathbb{T}_{<q} \subset \mathbb{T}$. More precisely, if $\Lambda \subset K_\infty^{\text{alg.}}$, it can be proved that $s_{\Lambda, \omega}(t) \in K_\infty^{\text{alg.}}[[t]]$.

We extend the operator τ from C to $C[[t]]$ as follows:

$$f = \sum_{n \geq 0} c_n t^n \mapsto \tau f := \sum_{n \geq 0} c_n^q t^n.$$

We will also write $f^{(k)}$ for $\tau^k f$, $k \in \mathbb{Z}$ (the operator τ^{-1} is well defined). One checks that τ sends $\mathbb{T}_{<r}$ in $\mathbb{T}_{<r^q}$ and $\mathbb{T}_{\leq r}$ in $\mathbb{T}_{\leq r^q}$. The extension τ so constructed is an \mathbb{F}_q -automorphism of \mathbb{T} .

We write $\overline{A} = \mathbb{F}_q[t]$, $\overline{K} = \mathbb{F}_q(t)$. If $a = a(\theta) \in A$ we also write $\overline{a} = a(t) \in \overline{A}$. If Λ is an A -lattice of rank r and if ϕ_Λ is the Drinfeld A -module of rank r in (17), then, for all $a_1, a_2 \in A$ and $\omega_1, \omega_2 \in \Lambda$,

$$\phi_\Lambda(a_1)s_{\Lambda, \omega_1} + \phi_\Lambda(a_2)s_{\Lambda, \omega_2} = s_{\Lambda, a_1\omega_1 + a_2\omega_2} = \overline{a}_1 s_{\Lambda, \omega_1} + \overline{a}_2 s_{\Lambda, \omega_2}. \quad (18)$$

These identities, which hold in \mathbb{T} , are proved in [27, Section 4.2.2].

On another side, from (15) it immediately follows that, for $\lambda \in C^\times$,

$$s_{\lambda\Lambda, \lambda\omega}(t) = \lambda s_{\Lambda, \omega}(t). \quad (19)$$

We also have the series expansion (cf. [27, Section 4.2.2])

$$s_{\Lambda, \omega}(t) = \sum_{n=0}^{\infty} \frac{\alpha_n(\Lambda)\omega^{q^n}}{\theta^{q^n} - t}, \quad (20)$$

uniformly convergent in every compact subset of $C \setminus \{\theta, \theta^q, \dots\}$, and $s_{\Lambda, \omega}(t) - \omega/(\theta - t)$ extends to a rigid holomorphic function for $|t| < q^q$. We will then often say that $s_{\Lambda, \omega}$ has a *simple pole of residue* $-\omega$ in $t = \theta$. Notice that other poles occur at $t = \theta^q, \theta^{q^2}, \dots$, but we do not need to focus on them in this paper.

Example: rank one case. For $\Lambda = \tilde{\pi}A$ (rank 1), the exponential function is:

$$e_{\text{Car}}(\zeta) = \sum_{n \geq 0} \frac{\zeta^{q^n}}{d_n}, \quad (21)$$

where $d_0 := 1$ and $d_i := [i][i-1]^q \dots [1]^{q^{i-1}}$, recalling that $[i] = \theta^{q^i} - \theta$ if $i > 0$. The relations (17) become, for all $a \in A$,

$$\phi_{\text{Car}}(a)e_{\text{Car}}(\zeta) = e_{\text{Car}}(a\zeta),$$

where ϕ_{Car} is Carlitz's module defined by

$$\phi_{\text{Car}}(\theta) = \theta\tau^0 + \tau \in \mathbf{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a)$$

(see Section 4 of [13]).

We will write $s_{\text{Car}} = s_{\tilde{\pi}A, \tilde{\pi}}$. The function s_{Car} has a simple pole in θ with residue $-\tilde{\pi}$.

By (18) (cf. [27, Section 4.2.5]), the following τ -difference equation holds:

$$s_{\text{Car}}^{(1)}(t) = (t - \theta)s_{\text{Car}}. \quad (22)$$

After [11, Theorem 2.2.9], \mathbb{T} is a principal ideal domain. This property can be used to verify that the subfield of constants $\mathbb{L}^\tau := \{l \in \mathbb{L}, \tau l = l\}$, where \mathbb{L} is the fraction field of \mathbb{T} , is equal to $\overline{K} := \mathbb{F}_q(t)$ (see also [25, Lemma 3.3.2]). We deduce, just as in the proof of [25, Lemma 3.3.5], that the τ -difference equation $f^{(1)} = (t - \theta)f$ has, as a complete set of solutions in \mathbb{L} , the $\mathbb{F}_q(t)$ -vector space $\mathbb{F}_q(t)s_{\text{Car}}$. In fact, for all $a = a(\theta) \in A$, we have $s_{\tilde{\pi}A, a\tilde{\pi}} = \overline{a}s_{\text{Car}}$.

2.1 Drinfeld modules of rank 2

We recall some tools described in [27, Section 4.2.5] (see also [8, 25]). Let z be in Ω , and consider the A -lattice $\Lambda = \Lambda_z = A + zA$ of rank 2, with associated exponential function $e_z = e_\Lambda$. Let us consider the Drinfeld module ϕ_z defined by

$$\phi_z : \theta \mapsto \phi_z(\theta) = \theta\tau^0 + \tilde{g}(z)\tau^1 + \tilde{\Delta}(z)\tau^2, \quad (23)$$

where $\tilde{g}(z) = \tilde{\pi}^{q-1}g(z)$, $\tilde{\Delta}(z) = \tilde{\pi}^{q^2-1}\Delta(z)$, with $\Delta = -h^{q-1}$. Then,

$$\phi_z(a)e_z(\zeta) = e_z(a\zeta) \quad (24)$$

for all $a \in A$ and $\zeta \in C$ ([13, Section 5], [27, Section 4.2.5], see also [25]).

We can write, for $\zeta \in C$,

$$e_z(\zeta) = \sum_{i=0}^{\infty} \alpha_i(z)\zeta^{q^i}, \quad (25)$$

for functions $\alpha_i : \Omega \rightarrow C$ with $\alpha_0 = 1$. By (24) we deduce, with the initial values $\alpha_0 = 1, \alpha_{-1} = 0, \alpha_{-2} = 0$, the recursive relations

$$\alpha_i = \frac{1}{[i]}(\tilde{g}\alpha_{i-1}^q + \tilde{\Delta}\alpha_{i-2}^{q^2}), \quad i > 0. \quad (26)$$

This implies that the function $\alpha_i(z)$ is a modular form of weight $q^i - 1$ and type 0 for all $i \geq 0$.

In all the following, we shall write:

$$\mathbf{s}_1(z, t) = s_{\Lambda_z, z}(t), \quad \mathbf{s}_2(z, t) = s_{\Lambda_z, 1}(t).$$

These are functions $\Omega \times B_q \rightarrow C$, where B_q is the set $\{z \in C, |z| < q\}$.

In fact the definition of the functions $s_{\Lambda, \omega}$ tells that $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{Hol}(\Omega)[[t]]$, where $\mathbf{Hol}(\Omega)$ denotes the C -algebra of rigid holomorphic functions $\Omega \rightarrow C$. After (25) and (20) we see that, for any couple $(z, t) \in \Omega \times B_q$, the following convergent series expansions hold:

$$\begin{aligned} \mathbf{s}_1(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)z^{q^i}}{\theta^{q^i} - t} \\ \mathbf{s}_2(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}. \end{aligned}$$

Our notations stress the dependence on two variables $z \in \Omega, t \in B_q$. However, we can also fix $z \in \Omega$ and study the functions $\mathbf{s}_1(z, \cdot), \mathbf{s}_2(z, \cdot)$, or look at the functions $\mathbf{s}_1(\cdot, t), \mathbf{s}_2(\cdot, t) : \Omega \rightarrow \mathbb{T}_{< q}$ with formal series as values. In the next section, we provide the necessary analysis of the functions $\mathbf{s}_1(z, \cdot), \mathbf{s}_2(z, \cdot)$. Hence, we fix now $z \in \Omega$.

2.1.1 Studying functions of the variable t , with z fixed.

At θ , the functions $\mathbf{s}_i(z, \cdot)$ have simple poles. Their respective residues are, according to Section 2, $-z$ for the function $\mathbf{s}_1(z, \cdot)$ and -1 for $\mathbf{s}_2(z, \cdot)$. Moreover, we have $\mathbf{s}_1^{(1)}(z, \theta) = \eta_1$ and $\mathbf{s}_2^{(1)}(z, \theta) = \eta_2$, where η_1, η_2 are the quasi-periods of Λ_z (see [27, Section 4.2.4] and [12, Section 7]).

Let us consider the matrix function:

$$\widehat{\Psi}(z, t) := \begin{pmatrix} \mathbf{s}_1(z, t) & \mathbf{s}_2(z, t) \\ \mathbf{s}_1^{(1)}(z, t) & \mathbf{s}_2^{(1)}(z, t) \end{pmatrix}.$$

By [27, Section 4.2.3] (see in particular equation (15)), we have:

$$\widehat{\Psi}(z, t)^{(1)} = \widetilde{\Theta}(z) \cdot \widehat{\Psi}(z, t), \quad \text{where } \widetilde{\Theta}(z) = \begin{pmatrix} 0 & 1 \\ \frac{t-\theta}{\widetilde{\Delta}(z)} & -\frac{\widetilde{g}(z)}{\widetilde{\Delta}(z)} \end{pmatrix}, \quad (27)$$

yielding the following τ -difference linear equation of order 2:

$$\mathbf{s}_2^{(2)} = \frac{t-\theta}{\widetilde{\Delta}} \mathbf{s}_2 - \frac{\widetilde{g}}{\widetilde{\Delta}} \mathbf{s}_2^{(1)}. \quad (28)$$

Remark 1 By [1], there is a fully faithful contravariant functor from the category of Drinfeld A -modules over $K^{\text{alg.}}$ to the category of *Anderson's A -motives* over $K^{\text{alg.}}$. Part of this correspondence is sketched in [27, Section 4.2.2], where the definition of A -motive is given and discussed (see also [8]); it is based precisely on Anderson's functions $s_{\Lambda, \omega}$. In the language introduced by Anderson, $\widehat{\Psi}$ is a *rigid analytic trivialisation* of the A -motive associated to the Drinfeld module $\phi = \phi_{\Lambda}$.

In the next section, we study the functions $\mathbf{s}_1, \mathbf{s}_2$ as functions $\Omega \rightarrow \mathbb{T}_{>0}$.

2.1.2 Studying functions $\Omega \rightarrow \mathbb{T}_{<q}$.

We observe, by the definitions of $\mathbf{s}_1, \mathbf{s}_2$, and by the fact, remarked in (26), that α_i is modular of weight $q^i - 1$ and type 0 for all i , and by (18), that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$:

$$\begin{aligned} \mathbf{s}_2(\gamma(z), t) &= \sum_{i=0}^{\infty} (cz + d)^{q^i - 1} \frac{\alpha_i(z)}{\theta^{q^i} - t} \\ &= (cz + d)^{-1} s_{\Lambda_z, cz+d}(z) \\ &= (cz + d)^{-1} (\overline{c} \mathbf{s}_1(z, t) + \overline{d} \mathbf{s}_2(z, t)). \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{s}_1(\gamma(z), t) &= \sum_{i=0}^{\infty} (cz + d)^{q^i - 1} \frac{\alpha_i(z)(\gamma(z))^{q^i}}{\theta^{q^i} - t} \\
&= (cz + d)^{-1} s_{\Lambda_z, az+b}(z) \\
&= (cz + d)^{-1} (\bar{a}\mathbf{s}_1(z, t) + \bar{b}\mathbf{s}_2(z, t)).
\end{aligned}$$

Let us write

$$\Sigma(z, t) := \begin{pmatrix} \mathbf{s}_1(z, t) \\ \mathbf{s}_2(z, t) \end{pmatrix}.$$

We have proved:

Lemma 2 *For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and for all $z \in \Omega$, we have the following identity of series in $\mathbb{T}_{< q}$:*

$$\Sigma(\gamma(z), t) = (cz + d)^{-1} \bar{\gamma} \cdot \Sigma(z, t), \quad (29)$$

where $\bar{\gamma}$ is the matrix $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in \mathbf{GL}_2(\bar{A})$.

3 The function \mathbf{E}

The function of the title is defined, for $z \in \Omega$ and $t \in B_q$, by:

$$\mathbf{E}(z, t) = (t - \theta)^{-1} \tilde{\pi}^q h(z) s_{\text{Car}}^{-1}(t) \mathbf{s}_2^{(1)}(z, t).$$

This section is entirely devoted to the description of its main properties. Three Propositions will be proved here.

In Proposition 3 we show that \mathbf{E} satisfies a linear τ -difference equation of order 2 with coefficients isobaric in $C[[t]][g, h]$; a special case of a general phenomenon to be compared with a result of Stiller in [28].

In Proposition 4, we analyse the functional equations relating the values of \mathbf{E} at z and $\gamma(z)$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$; they involve the factors of automorphy:

$$J_\gamma(z) = cz + d, \quad \mathbf{J}_\gamma(z) = \bar{c} \frac{\mathbf{s}_1(z, t)}{\mathbf{s}_2(z, t)} + \bar{d},$$

with values convergent in $C[[t]]$.

Proposition 5 describes the third important feature of the function \mathbf{E} ; the existence of a u -expansion in $\mathbb{F}_q[t, \theta][[u]]$. For Drinfeld quasi-modular forms, the

degree in θ of the n -th coefficient of the u -expansion grows pretty rapidly with n in contrast of the classical framework. The function \mathbf{E} does not make exception to this principle. However, the degree in t of the n -th coefficient grows slowly, and this property is used in the proof of the multiplicity estimate. Another important property studied in this section is that $\mathbf{E}(z, \theta)$ is equal to Gekeler's function E .

3.1 linear τ -difference equations

Proposition 3 *For all $z \in \Omega$, the function $\mathbf{E}(z, \cdot)$ can be developed as a series of $\mathbb{T}_{\leq q}$. Moreover, The following linear τ -difference equation holds in $\mathbb{T}_{\leq q}$, for all $z \in \Omega$:*

$$\mathbf{E}^{(2)} = \frac{1}{t - \theta^{q^2}}(\Delta \mathbf{E} + g^q \mathbf{E}^{(1)}). \quad (30)$$

Proof. After having chosen a $(q-1)$ -th root of $-\theta$, let us write, following Anderson, Brownawell, and Papanikolas in [2, Section 3.1.2],

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{n=1}^{\infty} \left(1 - \frac{t}{\theta^{q^n}}\right) \in (\mathbb{T}_{\infty} \cap K_{\infty}((-\theta)^{\frac{1}{q-1}})[[t]]) \setminus K_{\infty}(t)^{\text{alg}}.$$

It is plain that

$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

Thanks to the remark on the \overline{K} -vector space structure of the set of solutions of (22) and after the computation of the constant of proportionality, we get

$$s_{\text{Car}}(t) = \frac{1}{\Omega^{(-1)}(t)}. \quad (31)$$

At once, we obtain that the function s_{Car} has no zeros in the domain $C \setminus \{\theta, \theta^q, \dots\}$ from which it follows that $((t - \theta)s_{\text{Car}})^{-1} \in \mathbb{T}_{\leq q}$. Moreover, for all $z \in \Omega$, we have $\mathbf{s}_2 \in \mathbb{T}_{< q}$ so that $\mathbf{s}_2^{(1)} \in \mathbb{T}_{< q^2} \subset \mathbb{T}_{\leq q}$. Multiplying the factors that define the function \mathbf{E} , we then get, for all $z \in \Omega$, that $\mathbf{E}(z, \cdot) \in \mathbb{T}_{\leq q}$, which gives the first part of the proposition.

In order to prove the second part of the proposition, we remark, from (27), that

$$\mathbf{s}_2^{(3)} = \frac{t - \theta^q}{\widetilde{\Delta}^q} \mathbf{s}_2^{(1)} - \frac{\widetilde{g}^q}{\widetilde{\Delta}^q} \mathbf{s}_2^{(2)}.$$

By the definition of \mathbf{E} and the relation (22) we find the relation:

$$\mathbf{E}^{(k)} = (t - \theta^{q^k})^{-1} (t - \theta^{q^{k-1}})^{-1} \dots (t - \theta)^{-1} \widetilde{\pi}^{q^{k+1}} h^{q^k} s_{\text{Car}}^{-1} \mathbf{s}_2^{(k+1)}, \quad (32)$$

for $k \geq 0$. Substituting the above expression for $\mathbf{s}_2^{(3)}$ in it, we get what we expected.

□

3.2 Factors of automorphy, modularity

In the next proposition, the function \mathbf{E} is considered as a function $\Omega \rightarrow C[[t]]$. In order to state the proposition, we first need a preliminary discussion.

If $\omega \notin \theta\Lambda$, $e_\Lambda(\omega/\theta) \neq 0$ and $s_{\Lambda,\omega}(t) \in \mathbb{T}_{>0}^\times$ (group of units of $\mathbb{T}_{>0}$), so that $\mathbf{s}_2(z, \cdot)^{-1} \in \mathbb{T}_{>0}^\times$ (the radius of convergence, depending on z , seems very difficult to compute). Hence, we have a well defined map

$$\begin{aligned} \mathbf{z} &: \Omega \rightarrow \mathbb{T}_{>0}^\times \\ \mathbf{z} &\mapsto \frac{\mathbf{s}_1(z, t)}{\mathbf{s}_2(z, t)}, \end{aligned}$$

and we can consider the map

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mapsto \mathbf{J}_\gamma := \bar{c}\mathbf{z} + \bar{d} \in \mathbb{T}_{>0}.$$

Since c, d are coprime, we have $cz + d \notin \theta\Lambda_z$ implying that $\bar{c}\mathbf{s}_1 + \bar{d}\mathbf{s}_2 = s_{\Lambda_z, cz+d} \in \mathbb{T}_{>0}^\times$. Therefore, for all $\gamma \in \Gamma$ and $z \in \Omega$, $\mathbf{J}_\gamma \in \mathbb{T}_{>0}^\times$.

Moreover, by (29) we have, for all $\gamma \in \Gamma$ and $z \in \Omega$,

$$\mathbf{z}(\gamma(z)) = \bar{\gamma}(\mathbf{z}(z)) \in C((t)), \quad (33)$$

so that, for $\gamma, \delta \in \Gamma$ and $z \in \Omega$,

$$\mathbf{J}_{\gamma\delta}(z) = \mathbf{J}_\gamma(\delta(z))\mathbf{J}_\delta(z).$$

the map $\mathbf{J} : \Gamma \times \Omega \rightarrow \mathbb{T}_{>0}^\times$ is our “new factor of automorphy”, to be considered together with the more familiar factor of automorphy

$$J_\gamma(z) := cz + d.$$

Let us also write, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$:

$$\begin{aligned} L_\gamma(z) &= \frac{c}{cz + d}, \\ \mathbf{L}_\gamma(z) &= \frac{\bar{c}}{\bar{c}\mathbf{s}_1 + \bar{d}\mathbf{s}_2} \in \mathbb{T}_{>0}. \end{aligned}$$

The fact that $\mathbf{L}_\gamma(z)$ belongs to $\mathbb{T}_{>0}$ follows from the discussion that led us to the conclusion that $\mathbf{J}_\gamma(z) \in \mathbb{T}_{>0}^\times$.

We further define the sequence of functions $(g_k^*)_{k \geq 0}$ by:

$$g_{-1}^* = 0, \quad g_0^* = 1, \quad g_1^* = g, \quad g_k^* = (t - \theta^{q^{k-1}})g_{k-2}^* \Delta^{q^{k-2}} + g_{k-1}^* g^{q^{k-1}}, \quad k \geq 2,$$

so that for all $k \geq 0$, we have the identity $g_k^*(z, \theta) = g_k(z)$, the function introduced in [13, Equation (6.8)].

We have:

Proposition 4 *For all $z \in \Omega$, $\gamma \in \Gamma$ and $k \geq 0$ the following identity of formal series of $\mathbb{T}_{>0}$ holds:*

$$\begin{aligned} \mathbf{E}^{(k)}(\gamma(z), t) &= \det(\gamma)^{-1} J_\gamma(z)^{q^k} \mathbf{J}_\gamma(z) \times \\ &\quad \left(\mathbf{E}^{(k)}(z, t) + \frac{g_k^*}{\tilde{\pi}(t - \theta)(t - \theta^q) \cdots (t - \theta^{q^k})} \mathbf{L}_\gamma(z) \right). \end{aligned} \quad (34)$$

Proof. Let $f(z, t)$ be the function $\det(\widehat{\Psi}(z, t))h(z)\tilde{\pi}^{1+q}$, for $z \in \Omega$ and $t \in B_q$. We have:

$$f^{(1)}(z, t) = -(t - \theta)\tilde{\Delta}(z)^{-1} \det(\widehat{\Psi}(z, t))h(z)^q \tilde{\pi}^{q+q^2} = (t - \theta)f(z, t).$$

For fixed $z \in \Omega$, $\mathbf{s}_i^{(k)}(z, \cdot) \in \mathbb{T}_{< q^{k+1}} \subset \mathbb{T}$ for all $k \geq 0$. Hence, $f(z, \cdot) \in \mathbb{T}$ for all $z \in \Omega$. By arguments used in the proof of Proposition 3, $f(z, t)$ is equal to $\lambda(z, t)s_{\text{Car}}(t)$, for some $\lambda(z, t) \in \overline{A}$; the matter is now to compute λ , which does not depend on $z \in \Omega$ as follows easily by fixing $t = t_0 \in B_q$ transcendental over \mathbb{F}_q and observing that $f(z, t_0)$ is holomorphic over Ω with values in a discrete set.

Now, for z fixed as $t \rightarrow \theta$,

$$\lim_{t \rightarrow \theta} \widehat{\Psi}(z, t) - \begin{pmatrix} -\frac{z}{t-\theta} & -\frac{1}{t-\theta} \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

η_1, η_2 being the quasi-periods (periods of second kind) of the lattice $A\omega_1 + A\omega_2$ [12, Section 7, Equations (7.1)], with generators $\omega_1 = z, \omega_2 = 1$, where the asterisks denote continuous functions of the variable z . Hence, we have $\lim_{t \rightarrow \theta} (t - \theta) \det(\widehat{\Psi}(z, t)) = -z\eta_2 + \eta_1$. By [12, Theorem 6.2], $-z\eta_2 + \eta_1 = \tilde{\pi}^{-q}h(z)^{-1}$ (notice that the function denoted by h in [12] is equal to *minus* h , the function of our paper and of [13]). At once,

$$\lim_{t \rightarrow \theta} (t - \theta) \det(\widehat{\Psi}(z, t)) = \lambda(\theta) \tilde{\pi}^{-q-1} h(z)^{-1} \lim_{t \rightarrow \theta} (t - \theta) s_{\text{Car}}(t) = -\lambda(\theta) \tilde{\pi}^{-q} h(z)^{-1},$$

which implies that $\lambda = \lambda(\theta) = -1$ (θ is transcendental over \mathbb{F}_q). We get:

$$\det(\widehat{\Psi}) = -\tilde{\pi}^{-1-q} h(z)^{-1} s_{\text{Car}}(t). \quad (35)$$

From (35) we deduce that

$$\mathbf{s}_1^{(1)} = \frac{\mathbf{s}_1}{\mathbf{s}_2} (\mathbf{s}_2^{(1)} + \tilde{\pi}^{-1-q} h^{-1} s_{\text{Car}}). \quad (36)$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Applying τ on both left and right hand sides of

$$\mathbf{s}_2(\gamma(z), t) = J_\gamma^{-1} \mathbf{J}_\gamma \mathbf{s}_2(z, t) = J_\gamma^{-1} (\bar{c} \mathbf{s}_1 + \bar{d} \mathbf{s}_2), \quad (37)$$

consequence of Lemma 2, we see that

$$\mathbf{s}_2^{(1)}(\gamma(z), t) = J_\gamma^{-q}(\bar{c}\mathbf{s}_1^{(1)} + \bar{d}\mathbf{s}_2^{(1)}).$$

Eliminating $\mathbf{s}_1^{(1)}$ from this identity and (36), we get:

$$\mathbf{s}_2^{(1)}(\gamma(z), t) = J_\gamma^{-q} \left(\mathbf{s}_2^{(1)}(z, t) \left(\bar{c} \frac{\mathbf{s}_1(z, t)}{\mathbf{s}_2(z, t)} + \bar{d} \right) + \tilde{\pi}^{-1-q} \frac{\bar{c}s_{\text{Car}}(t)}{h(z)\mathbf{s}_2(z, t)} \right),$$

that is,

$$\mathbf{s}_2^{(1)}(\gamma(z), t) = J_\gamma^{-q} \mathbf{J}_\gamma \left(\mathbf{s}_2^{(1)}(z, t) + \frac{\tilde{\pi}^{-1-q}s_{\text{Car}}(t)}{h(z)} \mathbf{L}_\gamma \right). \quad (38)$$

This already implies, by the definition of \mathbf{E} and the modularity of h :

$$\mathbf{E}(\gamma(z), t) = \det(\gamma)^{-1} J_\gamma \mathbf{J}_\gamma \left(\mathbf{E}(z, t) + \frac{1}{\tilde{\pi}(t - \theta)} \mathbf{L}_\gamma \right).$$

Now, the joint application of (37), (38) and (27) implies, for all $k \geq 0$ and $\gamma \in \Gamma$:

$$\mathbf{s}_2^{(k)}(\gamma(z), t) = J_\gamma^{-q^k} \mathbf{J}_\gamma \left(\mathbf{s}_2(z, t) + \frac{s_{\text{Car}}(t)g_{k-1}^*}{h^{q^{k-1}}\tilde{\pi}^{q^{k+1}}} \mathbf{L}_\gamma \right). \quad (39)$$

By the definition of \mathbf{E} and the functional equation (22), we end the proof of the proposition. \square

3.3 u -expansions

Proposition 5 *We have*

$$\mathbf{E}(z, t) = u \sum_{n \geq 0} c_n(t) u^{(q-1)n} \in u\mathbb{F}_q[\theta, t][[u^{q-1}]],$$

where the formal series on the right-hand side converges for all t, u with $|t| \leq q$ and $|u|$ small. The first terms (with respect to the ordering of the degrees in u) of the u -expansion of \mathbf{E} are:

$$\mathbf{E} = u(1 + u^{(q-1)^2} - (t - \theta)u^{(q-1)q} + \dots). \quad (40)$$

Moreover, for all $n > 0$, we have the following inequality for the degree in t of $c_n(t)$:

$$\deg_t c_n \leq \log_q n,$$

where \log_q denotes the logarithm in base q and where we have adopted the convention $\deg_t 0 = -\infty$.

Proof. We begin with the proof of the existence and the convergence of the u -expansion. We recall the modular forms α_i , defined by (26). There exist elements $c_{i,m} \in C$ such that

$$\alpha_i(z) = \sum_{m \geq 0} c_{i,m} u^m, \quad i \geq 0,$$

with convergence for $z \in \Omega$ such that $|u|$ is small enough. We need to provide upper bounds for the $|c_{i,m}|$'s, with explicit dependence on i, m .

Let us write $\tilde{g} = \sum_{i \geq 0} \gamma_i u^i$ and $\tilde{\Delta} = \sum_{i \geq 0} \delta_i u^i$ with $\gamma_i, \delta_i \in C$. Looking at [13, Definition (5.7), (iii)], there exists $B \geq q$ such that, for all $i \geq 0$, $\max\{|\gamma_i|, |\delta_i|\} \leq B^i$. We know that $\alpha_0 = 1$ and that $|c_{1,m}| \leq q^{-q} B^m$. Now, the recursive relations (26) imply, for $i > 1$, $m \geq 0$ and j, k non-negative integers:

$$c_{i,m} = \frac{1}{[i]} \left(\sum_{j+qk=m} \gamma_j c_{i-1,k}^q + \sum_{j+q^2k=m} \delta_j c_{i-2,k}^{q^2} \right).$$

After induction and the equality $[i] = q^{q^i}$ ($i > 0$), we deduce from these identities,

$$|c_{i,m}| \leq q^{-q^i} B^m, \quad (i, m \geq 0). \quad (41)$$

For $z \in \Omega$ such that $|u| < B^{-1}$ with B as above, and for $|t| < q$:

$$\begin{aligned} s_2(z, t) &= \frac{1}{\theta - t} + \sum_{i \geq 1} \frac{\alpha_i(z)}{\theta^{q^i} - t} \\ &= \frac{1}{\theta - t} + \sum_{i \geq 1} \sum_{m \geq 0} c_{i,m} u^m \frac{1}{\theta^{q^i} - t} \\ &= \frac{1}{\theta - t} + \sum_{m \geq 0} u^m \sum_{i \geq 1} \frac{c_{i,m}}{\theta^{q^i} - t} \\ &= \frac{1}{\theta - t} + \sum_{m \geq 1} \kappa_m(t) u^m, \end{aligned}$$

where, in the second equality we have u -expanded α_i , in the third equality, expanded $1/(\theta^{q^i} - t)$, and where for $m \geq 1$,

$$\kappa_m(t) = \sum_{j \geq 0} t^j \sum_{i \geq 0} c_{i,m} \theta^{-q^i(1+j)} \in \mathbb{T}_{\leq q}.$$

We see, by using (41), that for all $t \in C$ such that $|t| \leq q$, $|\kappa(t)| \leq B^m q^{-1}$. Since

$$s_2^{(1)}(z, t) = \frac{1}{\theta^q - t} + \sum_{m \geq 1} \kappa_m^{(1)}(t) u^{qm},$$

we obtain, from the definition of \mathbf{E} , the convergence of the u -expansion of h , the identity (31) and the convergence of the product defining Ω , that \mathbf{E} has a u -expansion

$$\mathbf{E}(z, t) = \sum_{i \geq 0} \lambda_i(t) u^i,$$

with $\lambda_i \in \mathbb{T}_{\leq q}$, converging for all t, u such that $|t| \leq q$ and $|u|$ small enough.

If $a = d = 1, c = 0$ and $b \in A$ in (29) we find that $\mathbf{s}_2(z + b, t) = \mathbf{s}_2(z, t)$. Taking further $\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{F}_q)$, one sees that in fact $\mathbf{E} \in u\mathbb{T}_{\leq q}[[u^{q-1}]]$.

We need now to prove that this series is defined over $\mathbb{F}_q[t, \theta]$. From (30) we get:

$$\mathbf{E} = \Delta^{-1}((t - \theta^{q^2})\mathbf{E}^{(2)} - g^q \mathbf{E}^{(1)}). \quad (42)$$

Let us write $v = u^{q-1}$. We have the series expansions (cf. [13, Section 10]):

$$\begin{aligned} g &= 1 - [1]v + \cdots = \sum_{n=0}^{\infty} a_n v^n \in A[[v]], \\ \Delta &= -v(1 - v^{q-1} + \cdots) \in vA[[v]], \\ \Delta^{-1} &= -v^{-1}(1 + v^{q-1} + \cdots) = -v^{-1} + \sum_{n=0}^{\infty} b_n v^n \in v^{-1}A[[v]], \end{aligned}$$

so that $a_0 = 1, a_1 = -[1], a_2 = \cdots = a_q = 0$ and $b_0 = b_1 = \cdots = b_{q-3} = 0, b_{q-2} = -1$ if $q > 2$, while for $q = 2$, the u -expansion of Δ^{-1} begins with $u^{-1}(1 + u + u^5 + \cdots)$. The integrality of the coefficients of the u -expansions of Δ^{-1} comes from the fact that the leading coefficient of the u -expansion of Δ lies in \mathbb{F}_q^\times .

We know, after the definition of \mathbf{E} , that

$$\mathbf{E} = u \sum_{i \geq 0} c_i(t) v^i,$$

for elements $c_i(t) \in \mathbb{T}_{\leq q}$ that we want to determine. But, for $k \in \mathbb{Z}$,

$$\mathbf{E}^{(k)} = u^{q^k} \sum_{i \geq 0} c_i^{(k)}(t) v^{q^k i},$$

and (42) becomes:

$$\begin{aligned} u \sum_{i \geq 0} c_i(t) v^i &= (-v^{-1} + \sum_{n=0}^{\infty} b_n v^n) \times \\ &\quad ((t - \theta^{q^2}) u^{q^2} \sum_{i \geq 0} c_i^{(2)}(t) v^{q^2 i} - \\ &\quad \sum_{n=0}^{\infty} a_n^q v^{nq} u^q \sum_{i \geq 0} c_i^{(1)}(t) v^{qi}). \end{aligned} \quad (43)$$

Therefore, for $r \geq 0$:

$$c_r = (t - \theta^{q^2}) \left(- \sum_{q^2\alpha+q=r} c_\alpha^{(2)} + \sum_{\alpha+q^2\beta+q+1=r} b_\alpha c_\beta^{(2)} \right) + \sum_{q\alpha+q\beta=r} a_\alpha^q c_\beta^{(1)} - \sum_{1+\alpha+q\beta+q\gamma=r} b_\alpha a_\beta c_\gamma^{(1)}.$$

If $r = 0$, this gives $c_0^{(1)} = c_0$, that is, $c_0 \in \mathbb{F}_q[t]$ because $c_0 \in \mathbb{T}_{\leq q} \subset \mathbb{T}$ and $\mathbb{T}^\tau = \mathbb{F}_q[t]$, information contained in a previous remark. But this is not enough to determine this coefficient. We recall that $(t - \theta)\mathbf{s}_2(z, t) \rightarrow -1$ and $(t - \theta)\mathbf{s}_1(z, t) \rightarrow -z$ as $t \rightarrow \theta$. Hence, $\lim_{t \rightarrow \theta} \frac{\mathbf{s}_1}{\mathbf{s}_2} = z$. This implies that $\lim_{t \rightarrow \theta} \mathbf{J}_\gamma(z) = J_\gamma(z)$. In a similar way we see that

$$\lim_{t \rightarrow \theta} (t - \theta)^{-1} \mathbf{L}_\gamma(z) = -L_\gamma(z). \quad (44)$$

Let $\phi(z)$ be the limit $\lim_{t \rightarrow \theta} \mathbf{E}(z, t)$. We obtain, by using (34) with $k = 0$ and the limits above, that

$$\phi(\gamma(z)) = \det(\gamma)^{-1} (cz + d)^2 \left(\phi(z) - \tilde{\pi}^{-1} \frac{c}{cz + d} \right),$$

but this is the collection of functional equations of the Drinfeld quasi-modular form E (3), whose u -expansion begins with the term u . Hence, we get $\phi = E$, $c_0(\theta) = 1$ and $c_0 = 1$ because θ is transcendental over \mathbb{F}_q . If $r > 0$, we see by induction that c_r is a polynomial with coefficients in $\mathbb{F}_q[t]$ evaluated in the coefficients a_i, b_j which are in A , as remarked above, and this implies that $c_r(t) \in \mathbb{F}_q[t, \theta]$ for $r \geq 0$.

Equation (43) easily delivers, after some calculations, the first coefficients of the u -expansion of \mathbf{E} given in (40), which agree, substituting t by θ , with the u -expansion of E that we know already after [13, Corollary (10.5)]:

$$E = u(1 + v^{(q-1)} + \dots).$$

To end the proof of the proposition, it remains to prove the growth estimate for the degrees in t of the coefficients c_i . We know from (40) that $c_0 = 1$, $c_n = 0$ for $n = 1, \dots, q - 2$ (this corresponds to empty data if $q = 2$), $c_{q-1} = 1$ and $c_q = \theta - t$. Therefore, $\deg_t c_n \leq 0$ for $n = 0, \dots, q - 2$ and $\deg_t c_q = 1$, which agrees with the estimate. By the recursive computation of c_n above, the identities $\deg_t r = \deg_t r^{(k)}$ for all $k \in \mathbb{Z}$ and $r \in C[t]$, and $\deg_t a_i = \deg_t b_j = 0$ for all i, j ,

we see by induction on $n \geq q$ that

$$\begin{aligned}
\deg_t c_n &\leq \max\{1 + \deg_t c_{\frac{n-q}{q^2}}^{(2)}, 1 + \max_{\alpha+q^2\beta+q+1=n} \deg_t c_\beta^{(2)}, \\
&\quad \max_{q\alpha+q\beta=n} \deg_t c_\beta^{(1)}, \max_{\alpha+q\beta+q\gamma=n} \deg_t c_\gamma^{(1)}\} \\
&\leq \max\{1 + \deg_t c_{\frac{n-q}{q^2}}, 1 + \max_{1+\alpha+q^2\beta+q+1=n} \deg_t c_\beta, \\
&\quad \max_{q\alpha+q\beta=n} \deg_t c_\beta, \max_{\alpha+q\beta+q\gamma=n} \deg_t c_\gamma\} \\
&\leq \max\{1 + \log_q \max\{1, n/q^2\}, \log_q \max\{1, n/q\}\} \\
&\leq \log_q \max\{1, n\},
\end{aligned}$$

where by definition, if $s \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$ (the induction hypothesis is applied in the next to last inequality). \square

Remark 6 Let us introduce the function

$$\mu = \tilde{\pi}^{1-q} s_2^{(1)} / s_2 \in C[[t, u^{q-1}]].$$

By (27), μ satisfies:

$$\mu^{(1)} = \frac{(t - \theta)}{\Delta} \mu^{-1} - \frac{g}{\Delta}.$$

Hence, $\mu = (t - \theta)\Delta^{-1}(\mu^{(1)} + g/\Delta)^{-1}$. Although not needed in this paper, we point out that this functional equation gives the following continued fraction development, which turns out to be convergent for the u -adic topology:

$$\begin{aligned}
\mu &= \frac{(t - \theta)}{\Delta(t - \theta^q)} \in \mathbb{F}_q[t, \theta][[v]]. \\
&\quad g + \frac{\Delta^q(t - \theta^{q^2})}{g^q + \frac{\Delta^{q^2}(t - \theta^{q^3})}{g^{q^2} + \dots}}
\end{aligned} \tag{45}$$

This property should be compared with the connection of Atkin's polynomials with certain continued fraction developments in [20, Section 4, 5], or the continued fraction developments described after [18, Theorem 2].

4 Bi-weighted automorphic functions

In this section we prove that almost A -quasi-modular forms generate a $\mathbb{T}_{\leq q}$ -algebra $\widetilde{\mathcal{M}} \subset C[[t, u]]$ which is, thanks to the two kinds of factor of automorphy described

below, graded by the group $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$, and contains the four dimensional algebra $\mathbb{M}^\dagger := \mathbb{T}_{\leq q}[g, h, \mathbf{E}, \mathbf{F}]$, where $\mathbf{F} = \tau \mathbf{E}$.

Proposition 3 implies that τ acts on \mathbb{M}^\dagger : If $\mathbf{f} \in \mathbb{M}^\dagger$ is homogeneous of degree (μ, ν, m) then $\tau \mathbf{f}$ also is homogeneous of degree $(q\mu, \nu, m)$.

If $\mathbf{f} \in \mathbb{M}^\dagger$ is homogeneous of degree (μ, ν, m) , then the function

$$\begin{aligned} \Omega &\rightarrow C \\ \varepsilon(\mathbf{f}) : z &\mapsto \mathbf{f}(z)|_{t=\theta} \end{aligned}$$

is a well defined Drinfeld quasi-modular form of weight $\mu + \nu$, type m and depth $\leq \nu$.

4.1 Preliminaries on graduations and filtrations

Let us consider three matrices:

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \mathcal{C} = \mathcal{A} \cdot \mathcal{B} = \begin{pmatrix} * & * \\ x & y \end{pmatrix} \in \Gamma. \quad (46)$$

Lemma 7 *We have:*

$$\begin{aligned} L_{\mathcal{A}}(\mathcal{B}(z)) &= \det(\mathcal{B})^{-1} J_{\mathcal{B}}(z)^2 (L_{\mathcal{C}}(z) - L_{\mathcal{B}}(z)), \\ \mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)) &= \det(\mathcal{B})^{-1} J_{\mathcal{B}}(z) \mathbf{J}_{\mathcal{B}}(z) (\mathbf{L}_{\mathcal{C}}(z) - \mathbf{L}_{\mathcal{B}}(z)). \end{aligned}$$

Proof. We begin by proving the first formula, observing that $c = \det(\mathcal{B})^{-1}(x\delta - y\gamma)$:

$$\begin{aligned} J_{\mathcal{B}}(z)^2 (L_{\mathcal{C}}(z) - L_{\mathcal{B}}(z)) &= \\ &= (\gamma z + \delta)^2 \left(\frac{x}{xz + y} - \frac{\gamma}{\gamma z + \delta} \right) \\ &= \det(\mathcal{B}) \frac{c(\delta + \gamma z)}{(c\alpha + d\gamma)z + (c\beta + d\gamma)} \\ &= \det(\mathcal{B}) \frac{c}{\frac{(c\alpha + d\gamma)z + (c\beta + d\gamma)}{\delta + \gamma z}} \\ &= \det(\mathcal{B}) \frac{c}{c \frac{\alpha z + \beta}{\gamma z + \delta} + d} \\ &= \det(\mathcal{B}) L_{\mathcal{A}}(\mathcal{B}(z)). \end{aligned}$$

As for the second formula, we set

$$\tilde{L}_{\mathcal{A}} = \frac{\bar{c}}{\bar{c}z + \bar{d}}, \quad z := \frac{s_1}{s_2}.$$

By using (33) and the obvious identity $\det(\mathcal{B}) = \det(\overline{\mathcal{B}})$, we compute in a similar way:

$$\begin{aligned}
\mathbf{J}_{\mathcal{B}}(z)^2(\tilde{L}_{\mathcal{C}}(z) - \tilde{L}_{\mathcal{B}}(z)) &= \\
&= (\overline{\gamma}z + \overline{\delta})^2 \left(\frac{\overline{x}}{\overline{x}z + \overline{y}} - \frac{\overline{\gamma}}{\overline{\gamma}z + \overline{\delta}} \right) \\
&= \det(\overline{\mathcal{B}}) \frac{\overline{c}}{\overline{c} \frac{\overline{\alpha}z + \overline{\beta}}{\overline{\gamma}z + \overline{\delta}} + \overline{d}} \\
&= \det(\mathcal{B}) \tilde{L}_{\mathcal{A}}(\mathcal{B}(z)).
\end{aligned}$$

Hence,

$$\tilde{L}_{\mathcal{A}}(\mathcal{B}(z)) = \det(\mathcal{B})^{-1} \mathbf{J}_{\mathcal{B}}(z)^2 (\tilde{L}_{\mathcal{C}}(z) - \tilde{L}_{\mathcal{B}}(z)).$$

But

$$\tilde{L}_{\mathcal{A}}(z) = \mathbf{s}_2(z) \mathbf{L}_{\mathcal{A}}(z),$$

so that

$$\begin{aligned}
\tilde{L}_{\mathcal{A}}(\mathcal{B}(z)) &= \mathbf{s}_2(\mathcal{B}(z)) \mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)) \\
&= (\overline{\gamma} \mathbf{s}_1(z) + \overline{\delta} \mathbf{s}_2(z)) \mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)) \\
&= \mathbf{s}_2(z) \mathbf{J}_{\mathcal{B}}(z)^{-1} \mathbf{J}_{\mathcal{B}}(z) \mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)),
\end{aligned}$$

where $\mathbf{s}_1, \mathbf{s}_2$ are considered as functions $\Omega \rightarrow \mathbb{T}_{>0}$, from which we deduce the expected identity. \square

4.2 Almost A -quasi-modular forms.

We recall that for all $z \in \Omega$ and $\gamma \in \Gamma$, $J_{\gamma}, \mathbf{J}_{\gamma}, L_{\gamma}, \mathbf{L}_{\gamma} \in \mathbb{T}_{>0}$.

Definition 8 (Almost A -quasi-modular forms) Let $r > 0$ be a positive number. Let us consider a function and a finite collection of functions

$$\mathbf{f}, \quad (f_{i,j})_{i,j} : \Omega \rightarrow \mathbb{T}_{<r}.$$

Let us assume that

$$\mathbf{f}(z) = \sum_{s=0}^{\infty} \phi_s(z) t^s, \quad f_{i,j}(z) = \sum_{s=0}^{\infty} \phi_{i,j,s}(z) t^s \quad \text{for all } i, j \quad (47)$$

with $\phi_s; \phi_{i,j,s} : \Omega \rightarrow \mathbb{C}$ holomorphic for all s .

Let us assume that there exists a positive number B and, for all i, j, s , series $\psi_s, \psi_{i,j,s} \in \mathbb{T}_{<r}$, such that for $z \in \Omega$ with $|u| < B$ and for all $|t| < r$,

$$\mathbf{f}(z) = \sum_{s \geq 0} \psi_s(t) u^s, \quad f_{i,j}(z) = \sum_{s \geq 0} \psi_{i,j,s}(t) u^s. \quad (48)$$

We say that \mathbf{f} is an *almost A -quasi-modular form* of weight $(\mu, \nu) \in \mathbb{Z}^2$, type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, and depth $\leq l$ if the following identity of formal series in $\mathbb{T}_{>0}$ holds for all $z \in \Omega$ and $\gamma \in \Gamma$:

$$\mathbf{f}(\gamma(z)) = \det(\gamma)^{-m} J_\gamma^\mu \mathbf{J}_\gamma^\nu \sum_{j+k \leq l} f_{j,k}(z) L_\gamma^j \mathbf{L}_\gamma^k. \quad (49)$$

The supremum of the r 's such that $\mathbf{f}, f_{i,j}, \psi_m, \psi_{i,j,m} \in \mathbb{T}_{<r}$ for all z, i, j, m is called the *radius* of the almost A -quasi-modular form \mathbf{f} , denoted by $\rho(\mathbf{f})$. We will say that $\mu = \mu(\mathbf{f}), \nu = \nu(\mathbf{f}), m = m(\mathbf{f})$ are respectively the *first weight*, the *second weight* and the *type* of \mathbf{f} .

4.2.1 First remarks.

It is obvious that in (49), $\mathbf{f} = f_{0,0}$ (use $\gamma = \text{identity matrix}$).

If $\lambda \in \mathbb{T}_{>0}$, then it trivially is an almost A -quasi-modular form of weight $(0, 0)$, type 0, depth ≤ 0 . The radius $\rho(\lambda)$ is then just the radius of convergence of the series.

Examples of almost A -quasi-modular forms are Drinfeld quasi-modular forms. Any quasi-modular form of weight w , type m , depth $\leq l$ is an almost A -quasi-modular form of weight $(w, 0)$, type m , depth $\leq l$. In all these cases, the radius is infinite. More generally, the $\mathbb{T}_{>0}$ -algebra $\mathbb{T}_{>0}[g, h]$ is graded by the couples $(w, m) \in \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ of weights and types, and the isobaric elements are all almost A -quasi-modular forms.

The function \mathbf{s}_2 is, by Lemma 2, an almost A -quasi-modular form of weight $(-1, 1)$, depth 0, type 0. The radius is q , by the results of Section 2.1.

If \mathbf{f} is a A -quasi-modular form of weight (μ, ν) , type m , depth $\leq l$ and radius $> q$, then $\varepsilon(\mathbf{f}) := \mathbf{f}|_{t=\theta}$ is a well defined holomorphic function $\Omega \rightarrow C$. This is the case, for example, if $\mathbf{f} = \mathbf{E}^{(k)}$, for all $k \geq 0$, by Proposition 4, and we have that $\mu(\mathbf{E}^{(k)}) = q^k, \nu(\mathbf{E}^{(k)}) = 1, m(\mathbf{E}^{(k)}) = 1$, and the depths are always ≤ 1 . It is clear, from (44) and $\lim_{t \rightarrow \theta} \mathbf{J}_\gamma = J_\gamma$, that

$$\varepsilon(\mathbf{f}) \in \widetilde{M}_{\mu+\nu, m}^{\leq l}. \quad (50)$$

The computation of $\varepsilon(\mathbf{E}^{(k)})$ is made in Proposition 28. In particular, $\varepsilon(\mathbf{E}) = E$. We denote by $\varepsilon_{\mu, \nu, m}$ or again ε the map which sends an almost A -quasi-modular form \mathbf{f} of specified weight, type, with radius $> q$ to the Drinfeld quasi-modular form $\varepsilon(\mathbf{f})$. This map is C -linear and such that, if $\lambda \in \mathbb{T}_{\leq q}$,

$$\varepsilon(\lambda \mathbf{f}) = \varepsilon(\lambda) \varepsilon(\mathbf{f}).$$

The function \mathbf{s}_2 is not well defined at $t = \theta$. However, the function $\mathbf{f}(z) := (t - \theta) \mathbf{s}_2$, which is almost A -quasi-modular of same weight, type and depth as \mathbf{s}_2 ,

has radius q^2 . Therefore, $\varepsilon(\mathbf{f})$ is well defined, and is the constant function -1 by the results of Subsection 2.1.1. The function $\mathbf{s}_2^{(1)}$ is, on the other side, an almost A -quasi-modular form of weight $(-q, 1)$, type 0 and radius $> q$.

4.2.2 Grading by the weights, filtering by the depths.

For $\mu, \nu \in \mathbb{Z}, m \in \mathbb{Z}/(q-1)\mathbb{Z}, l \in \mathbb{Z}_{\geq 0}$, we denote by $\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$ the $\mathbb{T}_{>0}$ -module of almost A -quasi-modular forms of weight (μ, ν) , type m and depth $\leq l$. We have

$$\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l} \widetilde{\mathcal{M}}_{\mu', \nu', m'}^{\leq l'} \subset \widetilde{\mathcal{M}}_{\mu+\mu', \nu+\nu', m+m'}^{\leq l+l'}.$$

We also denote by $\widetilde{\mathcal{M}}$ the $\mathbb{T}_{>0}$ -algebra generated by all the almost A -quasi-modular forms. We prove below that this algebra is graded by the group $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$, filtered by the depths (Proposition 13), and contains five algebraically independent functions (Proposition 14).

Let \mathcal{K} be any field extension of $\mathbb{F}_q(t, \theta)$. The key result of this section is the following elementary lemma.

Lemma 9 *The subset $\Gamma = \{(d, \bar{d}), d \in A\} \subset \mathbb{A}^2(\mathcal{K})$ is Zariski dense.*

Proof. Let us assume by contradiction that the lemma is false and let $\bar{\Gamma}$ be the Zariski closure of Γ . Then, we can write

$$\bar{\Gamma} = \bigcup_{i \in \mathcal{I}} \Gamma_i \cup \bigcup_{j \in \mathcal{J}} \widetilde{\Gamma}_j,$$

where the Γ_i 's are irreducible closed subsets of $\mathbb{A}^2(\mathcal{K})$ of dimension 1, the $\widetilde{\Gamma}_j$'s are isolated points of $\mathbb{A}^2(\mathcal{K})$, and \mathcal{I}, \mathcal{J} are finite sets.

From $\Gamma = \Gamma + (d, \bar{d})$ for all $d \in A$ we deduce $\bar{\Gamma} = \bar{\Gamma} + (d, \bar{d})$. The translations of $\mathbb{A}^2(\mathcal{K})$ by points such as (d, \bar{d}) being bijective, they induce permutations of the sets $\{\Gamma_i\}$ and $\{\widetilde{\Gamma}_j\}$, from which we easily deduce that $\mathcal{J} = \emptyset$. Therefore, the ideal of polynomials $R \in \mathcal{K}[X, Y]$ such that $R(\Gamma) \subset \{0\}$ is principal, generated by a non-zero polynomial P .

Now, if $b \in A$, $m_b(\bar{\Gamma}) \subset \bar{\Gamma}$, where $m_b(x, y) := (bx, \bar{b}y)$. Hence, $P(m_b(X, Y)) \in (P)$ and there exists $\kappa_b \in \mathcal{K}^\times$ such that

$$P(bX, \bar{b}Y) = \kappa_b P(X, Y).$$

Let us write:

$$P(X, Y) = \sum_{\alpha, \beta} c_{\alpha, \beta} X^\alpha Y^\beta,$$

and choose $b \notin \mathbb{F}_q$. If $c_{\alpha,\beta} \neq 0$, then $\kappa_b = b^{-\alpha}\bar{b}^{-\beta}$. If P is not a monomial, we have, for $(\alpha, \beta) \neq (\alpha', \beta')$, $c_{\alpha,\beta}, c_{\alpha',\beta'} \neq 0$, so that $b^{-\alpha}\bar{b}^{-\beta} = b^{-\alpha'}\bar{b}^{-\beta'}$, yielding a contradiction, because $b \notin \mathbb{F}_q$.

If P is a monomial, however, it cannot vanish at $(1, 1) \in \Gamma$; contradiction. \square

Lemma 10 *Let us suppose that for elements $\psi_{\alpha,\beta} \in C((t))$ and for a certain element $z \in \Omega$,*

$$\sum_{\alpha,\beta} \psi_{\alpha,\beta} J_\gamma^\alpha J_\gamma^\beta = 0, \quad (51)$$

in $C((t))$, for all $\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} \in \Gamma$ with determinant 1, the sum being finite. Then, $\psi_{\alpha,\beta} = 0$ for all α, β .

Proof. Let us suppose by contradiction the existence of a non-trivial relation (51). We have, with the hypothesis on γ , $J_\gamma = z + d$, $J_\gamma = \mathbf{z} + \bar{d} \in C((t))$, so that the relation of the lemma implies the existence of a relation:

$$\sum_{\alpha,\beta} \ell_{\alpha,\beta} d^\alpha \bar{d}^\beta = 0, \quad d \in A,$$

with $\ell_{\alpha,\beta} \in \mathcal{K} = C((t))$ not all zero, and all, but finitely many, vanishing. Lemma 9 yields a contradiction. \square

Another useful lemma is the following. The proof is again a simple application of Lemma 9 and will be left to the reader.

Lemma 11 *If the finite collection of functions $f_{i,j} : \Omega \rightarrow \mathbb{T}_{>0}$ is such that for all $z \in \Omega$ and for all $\gamma \in \Gamma$,*

$$\sum_{i,j} f_{i,j}(z) L_\gamma^i L_\gamma^j = 0,$$

then the functions $f_{i,j}$ are all identically zero.

Lemma 12 *Let \mathbf{f} be an almost A -quasi-modular form of type m with $0 \leq m < q - 1$. Then, with $v = u^{q-1}$,*

$$\mathbf{f}(z) = u^m \sum_{i \geq 0} c_i(t) v^i.$$

Proof. It follows the same ideas of the remark on p. 23 of [14]. Let us consider $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma$ with $\lambda \in \mathbb{F}_q^\times$. We have $\gamma(z) = \lambda z$, $\det(\gamma) = \lambda$, $J_\gamma = \mathbf{J}_\gamma = 1$, $L_\gamma = \mathbf{L}_\gamma = 0$, so that $\mathbf{f}(\lambda z) = \lambda^{-m} \mathbf{f}(z)$, for all $z \in \Omega$. Now, if $\mathbf{f} = \sum_i c_i(t) u^i$, since e_{Car} is \mathbb{F}_q -linear, we get $u(\lambda z) = \lambda^{-1} u(z)$ and if $c_i \neq 0$, then $i \equiv m \pmod{q-1}$. \square

Proposition 13 *The $\mathbb{T}_{>0}$ -algebra generated by the almost A -quasi-modular forms is graded by weights and types, hence by the group $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$, and filtered by the depths:*

$$\widetilde{\mathcal{M}} = \bigoplus_{(\mu, \nu, m) \in \mathbb{Z}^2 \times \frac{\mathbb{Z}}{(q-1)\mathbb{Z}}} \bigcup_{l=0}^{\infty} \widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l}.$$

Proof. We begin by proving the property concerning the grading by the group $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$. Let us consider distinct triples $(\mu_i, \nu_i, m_i) \in \mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$, $i = 1, \dots, s$, non-negative integers l_1, \dots, l_s and non-zero elements $f_i \in \widetilde{\mathcal{M}}_{\mu_i, \nu_i, m_i}^{\leq l_i}$. Then, we claim that $\sum_{i=1}^s f_i \neq 0$. To see this, we assume by contradiction that for some forms f_i as in the proposition, we have the identity in $\mathbb{T}_{>0}$:

$$\sum_{i=1}^s f_i = 0. \quad (52)$$

Recalling Definition 8 (identity (49)), we have, for all $i = 1, \dots, s$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in \Omega$:

$$f_i(\gamma(z), t) = \det(\gamma)^{-m_i} J_\gamma^{\mu_i} \mathbf{J}_\gamma^{\nu_i} \sum_{j+k \leq l} f_{i,j,k}(z, t) L_\gamma^j \mathbf{L}_\gamma^k,$$

for certain functions $f_{i,j,k} : \Omega \rightarrow \mathbb{T}_{>0}$.

Let us suppose first that γ is of the form $\begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ with $ad - b = 1$. We recall that $\mathbf{s}_2^{-1} \in \mathbb{T}_{>0}^*$ for all z . Therefore, for all $z \in \Omega$, (52) becomes the identity of formal series in $\mathbb{T}_{>0}$:

$$\sum_{i=1}^s \sum_{j+k \leq l_i} f_{i,j,k} \mathbf{s}_2^{-k} (z+d)^{\mu_i-j} (z+\bar{d})^{\nu_i-k} = 0. \quad (53)$$

By Lemma 10, (53) is equivalent to the relations:

$$\sum_{i,j,k} \phi_{i,j,k} = 0, \quad \text{for all } (\alpha, \beta) \in \mathbb{Z}^2 \quad (54)$$

where $\phi_{i,j,k} := f_{i,j,k} \mathbf{s}_2^{-k}$ and the sum runs over the triples (i, j, k) with $i \in \{1, \dots, s\}$ and j, k such that $\mu_i - j = \alpha$ and $\nu_i - k = \beta$, with obvious vanishing conventions on some of the $\phi_{i,j,k}$'s.

Let μ be the maximum value of the μ_i 's, and let us look at the relations (54) for $\alpha = \mu$. Since for all $\mu_i < \mu$ we get $\alpha = \mu > \mu_i - j$ for all $j \geq 0$, for such a choice of α we get:

$$\sum_{i,k} \phi_{i,0,k} = 0, \quad \text{for all } \beta \in \mathbb{Z}, \quad (55)$$

where the sum is over the couples (i, j) with i such that $\mu_i = \mu$ and $\nu_i - k = \beta$. Now, let \mathcal{E} be the set of indices i such that $\mu_i = \mu$ and write ν for the maximum of the ν_i with $i \in \mathcal{E}$. If j is such that $\mu_j = \mu$, and if $\nu \neq \nu_j$, then for all $k \geq 0$, $\nu > \nu_j - k$, so that for $\beta = \nu$, (55) becomes

$$\sum_i \phi_{i,0,0} = 0,$$

where the sum runs this time over the i 's such that $(\mu_i, \nu_i) = (\mu, \nu)$. But $\phi_{i,0,0} = f_{i,0,0} = f_i$ for $i = 1, \dots, s$. Since the types of the f_i 's with same weights are distinct by hypothesis, Lemma 12 implies that for all i such that $(\mu_i, \nu_i) = (\mu, \nu)$, $f_i = 0$. This contradicts our initial assumptions and proves our initial claim. Combining with Lemma 11, we end the proof of the proposition. \square

Proposition 14 *The functions*

$$E, g, h, \mathbf{s}_2, \mathbf{s}_2^{(1)} : \Omega \rightarrow \mathbb{T}_{>0}$$

are algebraically independent over $\mathbb{L}_{>0}$, the fraction field of $\mathbb{T}_{>0}$.

Proof. Assume by contradiction that the proposition is false. Since $E, g, h, \mathbf{s}_2, \mathbf{s}_2^{(1)} \in \widetilde{\mathcal{M}}$ are almost A -quasi-modular forms, by Proposition 13, there exist $(\mu, \nu), m \in \mathbb{Z}$, and a non-trivial relation (where the sum is finite):

$$\sum_{i,j \geq 0} P_{i,j} E^i \mathbf{s}_2^{(1)j} = 0,$$

with $P_{i,j} \in \mathbb{T}_{>0}[g, h, \mathbf{s}_2] \cap \widetilde{\mathcal{M}}_{\mu-2i+qj, \nu-j, m-i}^{\leq l}$ (for some $l \geq 0$). By Proposition 13, any vector space of almost A -quasi-modular forms of given weight and depth is filtered by the depths. Comparing with the functional equations (38) and [3, Functional equation (11)], and applying Lemma 11, we see that all the forms $P_{i,j}$ vanish. There are three integers α, m, n and a non trivial polynomial relation P among g, h, \mathbf{s}_2 , with coefficients in $\mathbb{T}_{>0}$:

$$\sum_{s=0}^n Q_s \mathbf{s}_2^s = 0,$$

where $Q_s \in \mathbb{T}_{>0}[g, h] \cap \widetilde{\mathcal{M}}_{\alpha+s, 0, m}^{\leq l}$ ($s = 0, \dots, n$), and for some s , Q_s is non-zero. Since $\nu(Q_s) = 0$ for all s such that $Q_s \neq 0$ and $\nu(\mathbf{s}_2) = 1$, The polynomial P , evaluated at the functions $E, g, h, \mathbf{s}_2, \mathbf{s}_2^{(1)}$ is equal to $Q\mathbf{s}_2^s$ for $Q \in \mathbb{T}_{>0}[g, h] \setminus \{0\}$ and $s \in \mathbb{Z}$ quantity that cannot vanish because g, h are algebraically independent over $\mathbb{T}_{\leq q}$: contradiction. \square

Remark 15 (classical framework) Linear q -difference equations seem to be legitimate analogues in zero characteristic of linear τ -difference equations. In this direction, Di Vizio has recently studied in [9] the notion of G_q -function, q -Gevrey functions etc. It is natural to investigate the analogues (if any) of the functions \mathbf{E} and allied, in her theory. Is there a \mathbb{Z}^2 -graded algebra $\mathbb{M}_{\mathbb{Q}}$ of formal series which are “ q -deformations” of classical quasi-modular forms? We include a rather heuristic discussion about this question.

Let \mathcal{H} be the complex upper-half plane, let Λ be the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ of \mathbb{C} , with the basis $\omega_1 = z \in \mathcal{H}, \omega_2 = 1$. Let us denote by $\eta_1(z), \eta_2(z)$ the quasi-periods $\eta(\omega_1)$ and $\eta(\omega_2)$ respectively, with $\eta : \Lambda \rightarrow \mathbb{C}$ the quasi-period map associated to the Weierstraß ζ -function for Λ . It is well known that $\eta_2(z) = \pi^2 E_2(z)/3$, and that $\eta_1(z)$ is related to $\eta_2(z)$ by Legendre’s formula

$$z\eta_2(z) - \eta_1(z) = 2\pi i. \quad (56)$$

Let $\iota : \mathcal{H} \rightarrow \mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ be the map defined by $\iota(z) = \eta_1(z)/\eta_2(z)$. From the relation (56) and the basic properties of the quasi-period map, we obtain the following identities, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$:

$$\eta_2(\gamma(z)) = (cz + d)^2 \left(\eta_2(z) - 2\pi i \frac{c}{cz + d} \right) \quad (57)$$

$$= (cz + d)(c\iota(z) + d)\eta_2(z) \quad (58)$$

$$= (c\iota(z) + d)^2 \left(\eta_2(z) + 2\pi i \frac{c}{c\iota(z) + d} \right), \quad (59)$$

where the second and the third functional equation hold for z which is not a zero of E_2 ⁽⁸⁾. We have, for all $\gamma \in \mathbf{SL}_2(\mathbb{Z})$, $\iota(\gamma(z)) = \gamma(\iota(z))$. Therefore, we can consider the map $\mathbf{SL}_2(\mathbb{Z}) \times \mathcal{H} \rightarrow \mathbb{P}_1(\mathbb{C})$ as something like a factor of automorphy. In this case, there is no graduation by \mathbb{Z}^2 but at least, the map η_2 can be elusively considered as of “bi-weight $(2, 0), (1, 1)$ and $(0, 2)$ ” ⁽⁹⁾.

⁸It is known that E_2 , hence η_2 , vanishes on \mathcal{H} infinitely many times in the vertical strip $0 < \Re(z) < 1$; see Heins paper [17]. It is in fact not too difficult to extend Heins result to prove that every non-empty vertical strip $a < \Re(z) < b$, $a, b \in \mathbb{R}$, contains a zero of E_2 . The set of zeroes deserves other surprises : read [10] for further details.

⁹Notice also that ι sends a certain domain $\mathcal{D}_{\infty} \subset \mathcal{H}$ contained in the half-plane $\Im(z) >$

5 Estimating the multiplicity

We prove our Theorem in this section.

5.1 Preliminaries

Let us denote by \mathbb{M}^\dagger the $\mathbb{T}_{\leq q}$ -algebra $\mathbb{T}_{\leq q}[g, h, \mathbf{E}, \mathbf{F}]$, where $\mathbf{F} := \mathbf{E}^{(1)}$; its dimension is 4, according to Proposition 14 and Proposition 4. By Proposition 13, this algebra is graded by the group $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$:

$$\mathbb{M}^\dagger = \bigoplus_{(\mu, \nu), m} \mathbb{M}_{\mu, \nu, m}^\dagger,$$

where $\mathbb{M}_{\mu, \nu, m}^\dagger = \widetilde{\mathcal{M}}_{\mu, \nu, m} \cap \mathbb{M}^\dagger$.

The operator τ acts on \mathbb{M}^\dagger by Proposition 3. More precisely, we have the homomorphism of $\mathbb{F}_q[t]$ -modules

$$\tau : \mathbb{M}_{\mu, \nu, m}^\dagger \rightarrow \mathbb{M}_{q\mu, \nu, m}^\dagger.$$

Let us write $\mathbf{h} = \widetilde{\pi} h s_{\text{Car}}^{-1} \mathbf{s}_2$.

Lemma 16 *The formula $\mathbf{h} = (g\mathbf{E} - (t - \theta^q)\mathbf{F})$ holds, so that $\mathbf{h} \in \mathbb{M}_{q,1,1}^\dagger$ and $\mathbb{M}^\dagger = \mathbb{T}_{\leq q}[g, h, \mathbf{E}, \mathbf{h}]$.*

Proof. From the definition of \mathbf{E} , (22) and (28), we find:

$$\begin{aligned} g\mathbf{E} - (t - \theta^q)\mathbf{F} &= \\ &= (t - \theta)^{-1} \widetilde{\pi}^q g h s_{\text{Car}}^{-1} \mathbf{s}_2^{(1)} - \widetilde{\pi}^{q^2} h^q (s_{\text{Car}}^{(1)})^{-1} \mathbf{s}_2^{(2)} \\ &= (t - \theta)^{-1} \widetilde{\pi}^q g h s_{\text{Car}}^{-1} \mathbf{s}_2^{(1)} - (t - \theta)^{-1} \widetilde{\pi}^{q^2} h^q s_{\text{Car}}^{-1} \left(\frac{t - \theta}{\widetilde{\Delta}} \mathbf{s}_2 - \frac{\widetilde{g}}{\widetilde{\Delta}} \mathbf{s}_2^{(1)} \right) \\ &= (t - \theta)^{-1} \widetilde{\pi}^q g h s_{\text{Car}}^{-1} \mathbf{s}_2^{(1)} - \widetilde{\pi} \Delta^{-1} h^q s_{\text{Car}}^{-1} \mathbf{s}_2 + (t - \theta)^{-1} \widetilde{\pi}^q g h^q \Delta^{-1} s_{\text{Car}}^{-1} \mathbf{s}_2^{(1)} \\ &= \widetilde{\pi} h s_{\text{Car}}^{-1} \mathbf{s}_2. \end{aligned}$$

This makes it clear that \mathbf{h} belongs to $\mathbb{M}_{q,1,1}^\dagger$ and that $\mathbb{M}^\dagger = \mathbb{T}_{\leq q}[g, h, \mathbf{E}, \mathbf{h}]$. \square

We have the following result.

1.91 biholomorphically on \mathcal{H} . The boundary of \mathcal{D}_∞ is a real curve of \mathcal{H} which is very close, homotopically equivalent, although not equal to a horocycle with center at ∞ . We can define a function $\phi : \mathcal{H} \rightarrow \mathbb{C}$ by $\phi(\xi) := \eta_2(\iota^{-1}(\xi))$, and (59) tells that ϕ has certain automorphic properties, and that lattices with \mathbb{Q} -linearly dependent quasi-periods play the role of cusps in this setting.

Proposition 17 For all $(\mu, \nu), m$, the map

$$\varepsilon : \mathbb{M}_{\mu, \nu, m}^\dagger \rightarrow \widetilde{M}_{\mu+\nu, m}^{\leq \nu}$$

is well defined and the inverse image of 0 is the $\mathbb{T}_{\leq q}$ -module $(t - \theta)\mathbb{M}_{\mu, \nu, m}^\dagger$.

Proof. Let \mathbf{f} be an element of $\mathbb{M}_{\mu, \nu, m}^\dagger$. Then, by Lemma 16,

$$\mathbf{f} = \sum_{i=0}^{\nu} \phi_i \mathbf{h}^{\nu-i} \mathbf{E}^i,$$

where $\phi_i \in M_{\mu-\nu q+i(q-1), m-\nu} \otimes_C \mathbb{T}_{\leq q}$. Since $\lim_{t \rightarrow \theta} s_{\text{Car}}^{-1} \mathbf{s}_2 = \widetilde{\pi}^{-1}$, we have $\varepsilon(\mathbf{h}) = h$. Moreover, by Proposition 28, $\varepsilon(\mathbf{E}) = E$, and

$$\varepsilon(\mathbf{f}) = \sum_{i=0}^{\nu} \varepsilon(\phi_i) h^{\nu-i} E^i$$

and $\varepsilon(\mathbf{f}) = 0$ if and only if $\varepsilon(\phi_i) = 0$ for all i . But for all i , ϕ_i is a polynomial in g, h with coefficients in $\mathbb{T}_{\leq q}$. If $\varepsilon(\phi_i) = 0$, then ϕ_i is a linear combination $\sum_{a,b} c_{a,b} g^a h^b$ with $c_{a,b} \in \mathbb{T}_{\leq q}$ such that $c_{a,b}(\theta) = 0$. Since $\mathbb{T}_{\leq q} \subset \mathbb{T}$, it is a principal ideal domain and the last condition is equivalent to $\phi_i \in (t - \theta)(M \otimes_C \mathbb{T}_{\leq q})$. Hence, $\varepsilon(\mathbf{f}) = 0$ if and only if, for all i , $\phi_i \in (t - \theta)(M \otimes_C \mathbb{T}_{\leq q})$. \square

5.2 Multiplicity estimate in \mathbb{M}^\dagger

By Proposition 5, $\mathbf{E} = u + \cdots \in u\mathbb{F}_q[t, \theta][[u^{q-1}]]$. If $f = \sum_{n \geq n_0} c_n(t) u^n$ with $c_n \in C((t))$, then

$$f^{(k)} = \sum_{n \geq n_0} c_n^{(k)}(t) u^{q^k n}, \quad k \in \mathbb{Z}. \quad (60)$$

Hence,

$$\mathbf{E}^{(k)} = u^{q^k} + \cdots \in u^{q^k} \mathbb{F}_q[t, \theta][[u^{(q-1)q^k}]], \quad k \geq 0,$$

and there is an injective map $\mathbb{M}^\dagger \rightarrow \mathbb{T}_{\leq q}[[u]]$. Let $f = \sum_{n \geq n_0} c_n(t) u^n$ be in $\mathbb{T}_{\leq q}[[u]]$, with $c_{n_0} \neq 0$. We write $\nu_\infty(f) := n_0$. We also set $\nu_\infty(0) := \infty$.

We recall that $\nu_\infty(g) = 0, \nu_\infty(h) = \nu_\infty(\mathbf{E}) = 1$ and $\nu_\infty(\mathbf{F}) = q$. Since $\nu_\infty(\mathbf{s}_2) = 0$, we also get $\nu_\infty(\mathbf{h}) = 1$. We begin with a rather elementary estimate, for \mathbf{f} of weight $(\mu, 0)$.

Lemma 18 If $\mathbf{f} \in \mathbb{M}_{\mu, 0, m}^\dagger$ is non-zero, then $\nu_\infty(\mathbf{f}) \leq \frac{\mu}{q+1}$.

Proof. A weight inspection shows that $\mathbb{M}_{\mu,0,m}^\dagger = \mathbb{T}_{\leq q}[g, h]_{\mu,m}$. We can write $\mathbf{f} = \sum_{a,b} d_{a,b} g^a h^b = \sum_{n \geq n_0} c_n u^n$, with $d_{a,b}, c_n \in \mathbb{T}_{\leq q}$ for all a, b, n , and $c_{n_0} \neq 0$. Let $t_0 \in C$ be such that $|t_0| \leq q$ and $c_{n_0}(t_0) \neq 0$. Then, setting $f_0 = \sum_{a,b} d_{a,b}(t_0) g^a h^b$, we see that $f_0 \in M_{\mu,m} \setminus \{0\}$ and $\nu_\infty(f_0) = \nu_\infty(\mathbf{f})$. In virtue of (7), $\nu_\infty(f_0) \leq \frac{\mu}{q+1}$. \square

In the next proposition, we study the case of \mathbf{f} of weight (μ, ν) with $\nu > 0$.

Proposition 19 *Let \mathbf{f} be a non-zero element of $\mathbb{M}_{\mu,\nu,m}^\dagger$ with $\nu \neq 0$. Then,*

$$\nu_\infty(\mathbf{f}) \leq \mu\nu.$$

It is not difficult to show that the statement of this proposition cannot be improved.

Before proving the proposition, we first need a lemma.

Lemma 20 *Let $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger$, $\mathbf{f}' \in \mathbb{M}_{\mu',\nu',m'}^\dagger$. Consider \mathbf{f}, \mathbf{f}' as polynomials in $\mathbb{T}_{\leq q}[g, h, \mathbf{E}, \mathbf{h}]$, which is licit by Lemma 16. Let l be the degree of \mathbf{f} in \mathbf{E} and l' be the degree of \mathbf{f}' in \mathbf{E} . Then,*

$$\phi := \mathbf{Res}_{\mathbf{E}}(\mathbf{f}, \mathbf{f}') = \mathbf{h}^{\nu l' + \nu' l - ll'} \phi_0,$$

where $\phi_0 \in M_{w^*, m^*} \otimes_C \mathbb{T}_{\leq q}$, with

$$w^* = \mu l' + \mu' l - ll' - q(\nu l' + \nu' l - ll'), \quad m^* := ml' + m' l - (\nu l' + \nu' l).$$

Proof. With an application of a natural variant of [26, Lemme 6.1] ⁽¹⁰⁾ we see that

$$\phi \in \mathbb{M}_{\mu l' + \mu' l - ll', \nu l' + \nu' l - ll', ml' + m' l - ll'}^\dagger.$$

But we have $\phi \in \mathbb{T}_{\leq q}[g, h, \mathbf{h}]$ (isobaric for the graduation by $\mathbb{Z}^2 \times \mathbb{Z}/((q-1)\mathbb{Z})$). Since $\nu(g) = \nu(h) = 0$ and $\nu(\mathbf{h}) = 1$, we have $\phi_0 := \phi / \mathbf{h}^{\nu l' + \nu' l - ll'} \in M \otimes \mathbb{T}_{\leq q}$. The computation of the weight and type of ϕ_0 is obvious, knowing that $\mu(\mathbf{h}) = q$. \square

Proof of Proposition 19. Since $\mathbb{T}_{\leq q} \subset \mathbb{T}$ and \mathbb{T} is a unique factorisation domain (cf. [11, Theorem 2.2.9]), $\mathbb{T}_{\leq q}$ itself is a unique factorisation domain and the ring $\mathbb{M}^\dagger = \mathbb{T}_{\leq q}[g, h, \mathbf{E}, \mathbf{h}]$ is a unique factorisation domain.

Let \mathbf{f} be in $\mathbb{M}_{\mu,\nu,m}^\dagger$, with $\nu > 0$. Assume first that \mathbf{f} , as a polynomial in $g, h, \mathbf{E}, \mathbf{h}$, is an irreducible polynomial with coefficients in $\mathbb{T}_{\leq q}$. Since the result is

¹⁰The first formula after the statement of the above cited lemma, mistakenly typed, must be replaced with

$$p(R) = p(F) \deg_{X_0}(G) + p(G) \deg_{X_0}(F) - p(X_0) \deg_{X_0}(F) \deg_{X_0}(G).$$

clear for \mathbf{f} proportional to \mathbf{h} , we can further exclude this possibility. There are two cases left.

Case (i). We suppose that \mathbf{f} divides $\mathbf{f}^{(1)} \in \mathbb{M}_{q\mu, \nu, m}^\dagger$ as a polynomial in $g, h, \mathbf{E}, \mathbf{h}$ with coefficients in $\mathbb{T}_{\leq q}$. For weight reasons, $\mathbf{f}^{(1)} = a\mathbf{f}$ with $a \in M_{\mu(q-1), 0} \otimes_C \mathbb{T}_{\leq q}$ and $a \neq 0$. We also have $\nu_\infty(\mathbf{f}^{(1)}) = q\nu_\infty(\mathbf{f})$ by (60), so that, by Lemma 18, $(q-1)\nu_\infty(\mathbf{f}) = \nu_\infty(a) \leq (q-1)(q+1)^{-1}\mu$. Hence, in this case, we get the stronger inequality ⁽¹¹⁾

$$\nu_\infty(\mathbf{f}) \leq \frac{\mu}{q+1}.$$

Case (ii). In this case, \mathbf{f} and $\mathbf{f}^{(1)}$ are coprime. Since \mathbf{f} is irreducible, $l = \nu > 0$, $\mathbf{f}, \mathbf{f}^{(1)}$ depend on \mathbf{E} , and their resultant ϕ with respect to \mathbf{E} is non-zero. We apply Lemma 20 with $\mathbf{f}' = \mathbf{f}^{(1)}$, finding

$$\phi = \mathbf{h}^{\nu^2} \phi_0,$$

with $\phi_0 \in M_{(q+1)\nu(\mu-\nu), m^*} \otimes_C \mathbb{T}_{\leq q}$, for a certain m^* to be computed with Lemma 20. By Lemma 18 again, $\nu_\infty(\phi_0) \leq \nu(\mu-\nu)$. Since $\nu_\infty(\mathbf{h}) = 1$, $\nu_\infty(\phi) \leq \nu(\mu-\nu) + \nu^2 = \mu\nu$. Now, the number $\nu_\infty(\phi)$ is an upper bound for $\nu_\infty(\mathbf{f})$ by Bézout identity for the resultant.

We have proved the proposition if $\mathbf{f} \in \mathbb{M}_{\mu, \nu, m}^\dagger$ is irreducible. If \mathbf{f} is not irreducible, we can write $\mathbf{f} = \prod_{i=0}^r \mathbf{f}_i$ with $\mathbf{f}_0 \in \mathbb{M}_{\mu_0, 0, m_0}^\dagger$, $\mathbf{f}_i \in \mathbb{M}_{\mu_i, \nu_i, m_i}^\dagger$ irreducible for all $i > 0$ with $\nu_i > 0$, and $\sum_i \mu_i = \mu$, $\sum_i \nu_i = \nu$, $\sum_i m_i \equiv m \pmod{q-1}$. Since $\nu_\infty(\mathbf{f}) = \sum_i \nu_\infty(\mathbf{f}_i)$, we get, applying Lemma 18,

$$\nu_\infty(\mathbf{f}) \leq \frac{\mu_0}{q+1} + \sum_{i>0} \mu_i \nu_i \leq \mu\nu.$$

□

5.3 Reduced forms

Let \mathbf{f} be in \mathbb{M}^\dagger . Since $\varepsilon(\mathbf{f}) \in \widetilde{M} \subset C[[u]]$, it is legitimate to compare the quantities $\nu_\infty(\mathbf{f})$ and $\nu_\infty(\varepsilon(\mathbf{f}))$. We have the inequality:

$$\nu_\infty(\mathbf{f}) \leq \nu_\infty(\varepsilon(\mathbf{f})), \quad (61)$$

but the equality is not guaranteed in general, because the leading term of the u -expansion of \mathbf{f} can vanish at $t = \theta$.

¹¹It can be proved that \mathbf{f} is in this case, a modular form, but we do not need this information here.

Definition 21 A function \mathbf{f} in \mathbb{M}^\dagger is *reduced* if $\nu_\infty(\mathbf{f}) = \nu_\infty(\varepsilon(\mathbf{f}))$, that is, if the leading coefficient of the u -expansion of \mathbf{f} does not vanish at $t = \theta$.

If $f \in \widetilde{M}_{w,m}^{\leq l}$ is a Drinfeld quasi-modular form which is not a modular form, and if $f = \varepsilon(\mathbf{f})$ with $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger$ reduced, and $w = \mu + \nu, l = \nu$, then (8) holds.

The next lemma provides a tool to construct reduced almost A -quasi-modular forms, useful in the sequel.

Lemma 22 Let $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger$ such that $\mathbf{f} = \sum_{n \geq n_0} b_n u^n$, with $b_n \in \mathbb{F}_q[t, \theta]$ for all n and $b_{n_0} \neq 0$. Then, for all $k > \log_q(\deg_t b_{n_0})$, the function $\mathbf{f}^{(k)}$ is reduced.

Proof. We have $b_{n_0}^{(k)}(\theta) = b_{n_0}(\theta^{q^{-k}})^{q^k} = 0$ if and only if $t - \theta^{1/q^k}$ divides the polynomial $b_{n_0}(t)$ in $K^{\text{alg}}[t]$. This polynomial having coefficients in K , $b_{n_0}^{(k)}(\theta) = 0$ if and only if the irreducible polynomial $t^{q^k} - \theta$ divides $b_{n_0}(t)$. However, this is impossible if $k > \log_q(\deg_t b_{n_0})$. \square

5.4 Construction of the auxiliary forms.

We recall the u -expansion of \mathbf{E} whose existence is proved in Proposition 5:

$$\mathbf{E} = u \sum_{i \geq 0} c_i(t) v^i,$$

where $c_0 = 1$, $c_i \in \mathbb{F}_q[t, \theta]$ for all $i > 0$ and $v = u^{q-1}$.

Proposition 23 The following properties hold.

- (i) Let $\alpha, \beta, \gamma, \delta$ be non-negative integers and let us write $\mathbf{f} = g^\alpha h^\beta \mathbf{E}^\gamma \mathbf{F}^\delta \in \mathbb{M}_{\mu,\nu,m}^\dagger$, with $\mu = \alpha(q-1) + \beta(q+1) + \gamma + q\delta$, $\nu = \gamma + \delta$ and $\beta + \gamma + \delta \equiv m \pmod{q-1}$, $m \in \{0, \dots, q-2\}$. Let us write

$$\mathbf{f} = u^m \sum_{n \geq 0} a_n(t) v^n$$

with $a_n \in \mathbb{F}_q[t, \theta]$ (this is possible after Proposition 5 and the integrality of the coefficients of the u -expansions of g, h). Then, for all $n \geq 0$,

$$\deg_t a_n(t) \leq \nu \log_q \max\{1, n\}.$$

(ii) Let λ be a positive real number. Let $\mathbf{f}_1, \dots, \mathbf{f}_\sigma$ be a basis of monic monomials in $g, h, \mathbf{E}, \mathbf{F}$ of the $\mathbb{T}_{\leq q}$ -module $\mathbb{M}_{\mu, \nu, m}^\dagger$. Let x_1, \dots, x_σ be polynomials of $\mathbb{F}_q[t, \theta]$ with $\max_{0 \leq i \leq \sigma} \deg_t x_i \leq \lambda$. Then, writing

$$\mathbf{f} = \sum_{i=1}^{\sigma} x_i \mathbf{f}_i = u^m \sum_{n \geq 0} b_n(t) v^n$$

with $b_n \in \mathbb{F}_q[t, \theta]$, we have, for all $n \geq 0$:

$$\deg_t b_n \leq \lambda + \nu \log_q \max\{1, n\}.$$

Proof. Since by definition $\mathbf{F} = \mathbf{E}^{(1)}$, we have

$$\mathbf{F} = u^q \sum_{n \geq 0} c_n^{(1)} v^{qn} = u \sum_{r \geq 0} d_r v^r,$$

where $d_r = 0$ if $q \nmid r - 1$ and $d_r = c_{(r-1)/q}^{(1)}$ otherwise. Now, the operator τ leaves the degree in t invariant. Therefore, $\deg_t d_r \leq \log_q \max\{1, r/q\} \leq \log_q \max\{1, r\}$.

Let us consider the u -expansions:

$$\begin{aligned} g &= \sum_{n \geq 0} a_n v^n, & \mathbf{E} &= u \sum_{n \geq 0} c_n v^n, \\ h &= u \sum_{n \geq 0} b_n v^n, & \mathbf{F} &= u \sum_{n \geq 0} d_n v^n. \end{aligned}$$

We can write:

$$\mathbf{f} = u^{m'} \sum_{n \geq 0} \kappa_n v^n,$$

where, for all n , $\kappa_n = \sum \prod_x a_{i_x} \prod_y b_{j_y} \prod_s c_{k_s} \prod_z d_{r_z}$, the sum being over the vectors of $\mathbb{Z}_{\geq 0}^{\alpha+\beta+\gamma+\delta}$ of the form

$$(i_1, \dots, i_\alpha, j_1, \dots, j_\beta, k_1, \dots, k_\gamma, r_1, \dots, r_\delta)$$

whose sum of entries is n , and the four products running over $x = 0, \dots, \alpha$, $y = 0, \dots, \beta$, $s = 0, \dots, \gamma$ and $z = 0, \dots, \delta$. Since the coefficients of the u -expansions of g, h do not depend on t and $\gamma + \delta = \nu$, $\deg_t \kappa_n \leq \nu \log_q \max\{1, n\}$.

If $m' = m + k(q - 1)$ with $k \geq 0$ integer, and $0 \leq m < q - 1$. We can write

$$\mathbf{f} = u^{m'} \sum_{n \geq 0} c'_n v^n = u^m \sum_{n \geq 0} c_n v^n,$$

where $c_n = c'_{n-k}$, with the assumption that $c'_{n-k} = 0$ if the index is negative. The inequalities $\deg_t c'_n \leq \nu \log_q \max\{1, n\}$ for $n \geq 0$ imply that $\deg_t c_n$ is submitted to the same bound, proving the first part of the proposition. The second part is a direct application of the first. \square

5.4.1

Lemma 24 *We have, for all m and $\mu, \nu \in \mathbb{Z}$ such that $\mu \geq (q+1)\nu \geq 0$,*

$$\sigma(\mu, \nu) - \nu - 1 \leq \dim_{\mathbb{T}_{\leq q}} \mathbb{M}_{\mu, \nu, m}^\dagger \leq \sigma(\mu, \nu) + \nu + 1,$$

where

$$\sigma(\mu, \nu) = \frac{(\nu + 1) \left(\mu - \frac{\nu(q+1)}{2} \right)}{q^2 - 1}.$$

Proof. By [14, p. 33], we know that

$$\delta(k, m) := \dim_C M_{k, m} = \left\lfloor \frac{k}{q^2 - 1} \right\rfloor + \dim_C M_{k^*, m},$$

where k^* is the remainder of the euclidean division of k by $q^2 - 1$. In the same reference, it is also proved that $\dim_C M_{k^*, m} = 0$ unless $k^* \geq m(q+1)$, case where $\dim_C M_{k^*, m} = 1$, so that, in all cases, $0 \leq \dim_C M_{k^*, m} \leq 1$.

A basis of $\mathbb{M}_{\mu, \nu, m}^\dagger$ is given by:

$$(\mathbf{b}_k)_{k=1, \dots, \dim \mathbb{M}_{\mu, \nu, m}^\dagger} = (\phi_{i, s} \mathbf{h}^s \mathbf{E}^{\nu-s})_{s=0, \dots, \nu, i=1, \dots, \sigma(s)}, \quad (62)$$

with, for all s , $(\phi_{i, s})_{i=1, \dots, \sigma(s)}$ a basis of $M_{\mu-s(q-1)-\nu, m-\nu}$ (hence $\sigma(s) = \delta(\mu - s(q-1) - \nu, m - \nu)$). We have (taking into account the hypothesis on μ which implies $\mu - s(q-1) - \nu \geq 0$ for all $0 \leq s \leq \nu$):

$$\dim \mathbb{M}_{\mu, \nu, m}^\dagger = \sum_{s=0}^{\nu} \left\lfloor \frac{\mu - s(q-1) - \nu}{q^2 - 1} \right\rfloor + \dim_C M_{(\mu-\nu-s(q-1))^*, m-\nu}.$$

But

$$\sum_{s=0}^{\nu} \frac{\mu - s(q-1) - \nu}{q^2 - 1} = \sigma(\mu, \nu),$$

from which we deduce the lemma easily. \square

5.4.2

We now prove the following:

Proposition 25 *Let $\mu, \nu \in \mathbb{Z}_{\geq 0}$ be such that $\mu \geq (q+1)\nu + 2(q^2 - 1)$ and $\nu \geq 1$, let m be an integer in $\{0, \dots, q-2\}$. There exists an integer $r > 0$ such that*

$$r \leq 4q\mu\nu \log_q(\mu + \nu + q^2 - 1) + \nu \quad (63)$$

and, in $\widetilde{M}_{r, m}^{\leq \nu}$, a quasi-modular form $f_{\mu, \nu, m}$ such that

$$\frac{1}{q(q+1)} \mu \nu^2 \log_q(\mu + \nu + q^2 - 1) \leq \nu_\infty(f_{\mu, \nu, m}) \leq 4q\mu\nu^2 \log_q(\mu + \nu + q^2 - 1). \quad (64)$$

A proof of the following variant of Siegel's Lemma can be found, for example, in [21, Lemma 1] (see also [7]).

Lemma 26 *Let U, V be positive integers, with $U < V$. Consider a system (65) of U equations with V unknowns:*

$$\sum_{i=1}^V a_{i,j} x_i = 0, \quad (1 \leq j \leq U) \quad (65)$$

where the coefficients $a_{i,j}$ are elements of $K[t]$. Let d be a non-negative integer such that $\deg_t a_{i,j} \leq d$ for each (i, j) . Then, (65) has a non-zero solution $(x_i)_{1 \leq i \leq V} \in (K[t])^V$ with $\deg_t x_i \leq Ud/(V - U)$ for each $i = 1, \dots, V$.

Proof of Proposition 25. We apply Lemma 26 with the parameters $V = \dim \mathbb{M}_{\mu, \nu, m}^\dagger$, $U = \lfloor V/2 \rfloor$, assuming that $\mu \geq (q+1)\nu \geq 0$ and $\nu \geq 1$. If $\mathbf{f} = \mathbf{b}_i$ as in (62), Writing

$$\mathbf{b}_i = u^m \sum_{j \geq 0} a_{i,j} v^j, \quad a_{i,j} \in A[t] \quad (66)$$

with $0 \leq m < q - 1$, Proposition 23 says that for all i and for all $j \geq 0$,

$$\deg_t a_{i,j} \leq \nu \log_q \max\{1, j\}. \quad (67)$$

Lemma 26 yields polynomials $x_1, \dots, x_V \in K[t]$, not all zero, such that if we write

$$\mathbf{f} = \sum_i x_i \mathbf{b}_i = u^m \sum_{n \geq n_0} b_n v^n, \quad 0 \leq m < q - 1 \quad (68)$$

with $b_n \in K[t]$ for all n and $b_{n_0} \neq 0$, we have the following properties. The first property is the last inequality below:

$$\begin{aligned} m + (q-1)n_0 = \nu_\infty(\mathbf{f}) &\geq m + (q-1)U \\ &\geq (q-1)(\sigma(\mu, \nu) - \nu - 1)/2 - 1 \\ &\geq \frac{(\nu+1)(\mu - \frac{\nu(q+1)}{2} - q^2 + 1)}{2(q+1)} - 1 \\ &\geq \frac{1}{4(q+1)}(\nu+1)\mu - 1, \end{aligned} \quad (69)$$

where we have applied Lemma 24 and used the hypothesis that $\mu \geq (q+1)\nu + 2(q^2 - 1)$. The second property is that

$$\deg_t b_n \leq 2\nu(\log_q(\mu + \nu + q^2 - 1) + \log_q \max\{1, n\}), \quad n \geq 0, \quad (70)$$

which follows from the following inequalities, with $d = \nu \log_q \max\{1, U - 1\}$

$$\begin{aligned}
\deg_t x_i &\leq Ud/(V - U) \\
&\leq \nu \log_q \max\{1, U - 1\} \\
&\leq \nu \log_q (\sigma(\mu, \nu) + \nu - 1) \\
&\leq \nu (\log_q(\nu + 1) + \log_q(\mu + q^2 - 1) - \log_q(q^2 - 1)) \\
&\leq 2\nu \log_q(\mu + \nu + q^2 - 1),
\end{aligned}$$

and the second part of Proposition 23.

By Proposition 19, we have $m + (q - 1)n_0 = \nu_\infty(\mathbf{f}) \leq \mu\nu$ so that $n_0 \leq \frac{\mu\nu}{q-1}$, where n_0 is defined in (68). Hence, by (70),

$$\deg_t b_{n_0} \leq 4\nu \log_q(\mu + \nu + q^2 - 1). \quad (71)$$

Lemma 22 implies that for every integer k such that

$$k \geq \log_q(4\nu) + \log_q \log_q(\mu + \nu + q^2 - 1), \quad (72)$$

the function $f_k := \varepsilon(\mathbf{f}^{(k)})$ satisfies $\nu_\infty(f_k) = \nu_\infty(\mathbf{f}^{(k)})$. Let k be satisfying (72). We have, by (50), (69), Proposition 19 and (60):

1. $f_k \in \widetilde{M}_{\mu q^k + \nu, m}^{\leq \nu}$,
2. $(4(q + 1))^{-1}(\nu + 1)\mu q^k \leq \nu_\infty(f_k) \leq \mu\nu q^k$.

Let us define the function

$$\kappa(\mu, \nu) := \lfloor \log_q(4\nu) + \log_q \log_q(\mu + \nu + q^2 - 1) \rfloor + 1$$

and write: $f_{\mu, \nu, m} := f_{\kappa(\mu, \nu)}$. This function satisfies the properties announced in the proposition. \square

5.5 Proof of the main Theorem

Let f be a Drinfeld quasi-modular form of weight w and depth l and assume first that, as a polynomial in E, g, h with coefficients in C , it is an irreducible polynomial. We can also assume, by (7) and, [4, Theorem 1.4], that $l > q$.

We begin with the proof of the second, more precise assertion of our Theorem. Let α be the function of a real variable defined, for $\mu \geq 0$, by $\alpha(\mu) = \mu l \log_q(\mu + Cl + q^2 - 1)$; we have $\alpha(\mu + 1) \leq 2\alpha(\mu)$. Since (the dash ' is the derivative) $\alpha'(\mu) \geq l \log_q(Cl + q^2 - 1) > 1$ for all $l \geq q$ and $\mu \geq 0$, for all $w \geq 0$ integer, there exist $\mu \in \mathbb{Z}_{\geq 0}$ such that

$$\alpha(\mu) \leq w < \alpha(\mu + 1), \quad (73)$$

and we choose one of them, for example the biggest one. Let us suppose that (10) holds and, at once, set

$$\nu = Wl,$$

with

$$W = q(2 + 4(q + 1)) = 2q(3 + 2q).$$

We define $\beta(l)$ to be the right hand side of (10), as a function of $l \geq q$. Condition (10) implies

$$\mu \geq \frac{\beta(l)}{2l \log_q(\mu + Wl + q^2 - 1)}.$$

Since $\log_q(x) \leq 2x^{1/2}$ for all $x \geq 1$ and $q \geq 2$, we get

$$(\mu + Wl + q^2 - 1)^{3/2} \geq \frac{\beta(l)}{4l},$$

that is,

$$\mu \geq \left(\frac{\beta(l)}{4l} \right)^{2/3} - Wl - q^2 + 1.$$

But replacing $\beta(l)$ by its value yields $\mu \geq (q+1)\nu + 2(q^2 - 1)$, which is the condition needed to apply Proposition 25.

Let us write $\mathcal{L} := \log_q(\mu + \nu + q^2 - 1)$. By Proposition 25, there exists a form $f_{\mu, \nu, m} \in \widetilde{M}_{r, m}^{\leq \nu}$ such that $l(f_{\mu, \nu, m}) = \nu$ and

$$\begin{aligned} 0 &< w(f_{\mu, \nu, m}) &\leq 4(q+1)\mu\nu\mathcal{L} \\ (q(q+1))^{-1}\mu\nu^2\mathcal{L} &\leq \nu_\infty(f_{\mu, \nu, m}) &\leq 4q\mu\nu^2\mathcal{L} \end{aligned} \quad (74)$$

We have two cases.

Case (i). If $f|f_{\mu, \nu, m}$, then

$$\nu_\infty(f) \leq \nu_\infty(f_{\mu, \nu, m}) \leq 4q\mu\nu^2\mathcal{L}. \quad (75)$$

Case (ii). If $f \nmid f_{\mu, \nu, m}$, then $\rho := \mathbf{Res}_E(f, f_{\mu, \nu, m})$ is a non-zero modular form, whose weight and type m^* can be computed with the help of [4, Lemma 2.5] (we do not need an explicit computation of m^*):

$$\begin{aligned} w(\rho) &= w\nu + w(f_{\mu, \nu, m})l - 2l\nu \\ &\leq w\nu + 4l(q+1)\mu\nu\mathcal{L} - 2l\nu \\ &\leq \nu(w + 4(q+1)\mu l\mathcal{L}) \\ &< \nu(\alpha(\mu + 1) + 4(q+1)\mu l\mathcal{L}) \\ &< \nu(2\alpha(\mu) + 4(q+1)\mu l\mathcal{L}) \\ &< (2 + 4(q+1))\nu\mu l\mathcal{L}. \end{aligned} \quad (76)$$

Let us suppose that $\nu_\infty(f) > (q(q+1))^{-1}\mu\nu^2\mathcal{L}$. Then, by Bézout identity for the resultant and (74), $\nu_\infty(\rho) \geq (q(q+1))^{-1}\mu\nu^2\mathcal{L}$. At the same time, by (7), $\nu_\infty(\rho) \leq \frac{w(\rho)}{q+1}$, yielding the inequality $W < q(2 + 4(q+1))$ which is contradictory with the definition of W .

Therefore, the case (ii) signifies that $\nu_\infty(f) \leq 4q\mu\nu^2\mathcal{L}$. Ultimately, we have shown that, in both cases (i), (ii),

$$\begin{aligned}\nu_\infty(f) &\leq 4q\mu\nu^2\mathcal{L} \\ &\leq 4q\mu W^2 l^2 \mathcal{L} \\ &\leq 4qW^2 lw,\end{aligned}$$

which is the estimate (11).

We now prove the weaker, but unconditional inequality (9). Let f be in $\widetilde{M}_{w,m}^{\leq l}$ be non-zero and irreducible as a polynomial in E, g, h . Let us set this time

$$\mu = BWw, \quad \nu = Wl,$$

with

$$B = \frac{3}{2}(q^2 + 1), \quad W = q(4(q+1) + 1).$$

Then, since $w \geq 2l$,

$$\begin{aligned}\mu &\geq 3(q^2 + 1)Wl \\ &\geq (q+1)Wl + 2(q^2 + 1) \\ &\geq (q+1)\nu + 2(q^2 + 1),\end{aligned}$$

we can apply Proposition 25 again. As before, there exists a form $f_{\mu,\nu,m} \in \widetilde{M}_{r,m}^{\leq \nu}$ such that the inequalities (74) hold. Again, we can distinguish two cases, according with f , if divides or not $f_{\mu,\nu,m}$. If f divides $f_{\mu,\nu,m}$, we get (75).

Let us assume that f and $f_{\mu,\nu,m}$ are coprime and form their non-vanishing resultant $\rho := \mathbf{Res}_E(f, f_{\mu,\nu,m})$, whose weight satisfies (76).

If $\nu_\infty(f) > (q(q+1))^{-1}\mu\nu^2\mathcal{L}$, by Bézout identity for the resultant and (74), $\nu_\infty(\rho) \geq (q(q+1))^{-1}\mu\nu^2\mathcal{L}$. At the same time, by (7), we find the inequality $W < q(4(q+1) + 1)$ which is contradictory with the definition of W .

Hence, in all cases,

$$\begin{aligned}\nu_\infty(f) &\leq 4q\mu\nu^2\mathcal{L} \\ &\leq 4q\mu W^2 l^2 \mathcal{L} \\ &\leq 4qBW^3 l^2 w \log_q(\mu + \nu + q^2 - 1) \\ &\leq 4qBW^3 l^2 w (\log_q + \log_q(BW + W/2 + q^2 - 1)),\end{aligned}$$

which yields the estimate (9), hence completing the proof of our Theorem. \square

Remark 27 The dependence on l in condition (10) can be relaxed, adding conditions on q . For all $\epsilon > 0$ there exists a constant $c > 0$ such that for all $q > c$, assuming that $w \gg_\epsilon l^{2+\epsilon}$, then, the inequality (11) holds. We do not pursue the proof of this fact here.

6 Link with extremal quasi-modular forms

As a conclusion of this paper, we would like to describe some links between the present work and [4]. In [4], we have introduced the sequence of Drinfeld quasi-modular forms $(x_k)_{k \geq 0}$ with $x_k \in \widetilde{M}_{q^{k+1},1}^{\leq 1} \setminus M$, defined by $x_0 = -E$, $x_1 = -Eg - h$ and by the recursion formula

$$x_k = x_{k-1}g^{q^{k-1}} - [k-1]x_{k-2}\Delta^{q^{k-2}}, \quad k \geq 2,$$

where we recall that $\Delta = -h^{q-1}$. In [4, Theorem 1.2], we have showed that for all $k \geq 0$, x_k is *extremal*, in the sense that $\nu_\infty(x_k)$ is the biggest possible value for $\nu_\infty(f)$, if $f \in \widetilde{M}_{q^{k+1},1}^{\leq 1} \setminus \{0\}$. We also computed the order of vanishing: $\nu_\infty(x_k) = q^k$ for all k .

Proposition 28 *For $k \geq 0$, we have*

$$\mathbf{E}^{(k)}(z, \theta) = (-1)^{k+1} \frac{x_k(z)}{[1][2] \cdots [k]}, \quad (77)$$

where the empty product equals 1.

Proof. By (34) and by the limits of $\mathbf{J}_\gamma, \mathbf{L}_\gamma$ that we have computed earlier, for $k \geq 0$, the function $\phi_k(z) := \mathbf{E}^{(k)}(z, \theta)$, is a well defined Drinfeld quasi-modular form in the space $\widetilde{M}_{q^{k+1},1}^{\leq 1}$. By (40),

$$\mathbf{E}^{(k)} = u^{q^k} + \cdots$$

Hence, ϕ_k is non-vanishing, and by [4, Theorem 1.2] normalised, extremal, therefore proportional to x_k for all k . By [4, Proposition 2.3],

$$x_k = (-1)^{k+1} L_k u^{q^k} + \cdots,$$

where $L_k = [k][k-1] \cdots [1]$ if $k > 0$ and $L_0 = 1$. This proves the proposition. \square

Combining with Proposition 5, we also obtain the following corollary, which gives a proof of the property claimed in [4, Remark 2.4].

Corollary 29 Define, for all $k \geq 0$, $E_k := \phi_k$. Then, E_k is normalised in $A[[u^{q^k}]]$.

Another interesting connection with [4] occurs with the sequence $(\xi_k)_{k \geq 0}$ introduced in [4, Identity (8)]. Let us define:

$$\mathbf{G} := \det \begin{pmatrix} \mathbf{E}^{(1)} & \mathbf{E} \\ \mathbf{E}^q & (\mathbf{E}^{(-1)})^q \end{pmatrix}.$$

It is straightforward to see that $\mathbf{G} \in \mathbb{F}_q[t, \theta][[u]]$. From (40) and a computation we deduce

$$\mathbf{G} = (t - t^q)u^{q^2+1}(1 - v^{q^2-1} + [1]v^{q^2} + \dots).$$

For all $k \geq 1$, $\mathbf{G}^{(k)} \in \mathbb{F}_q[t, \theta][g, h, \mathbf{E}, \mathbf{E}^{(1)}]$ is reduced and $\nu_\infty(\mathbf{G}^{(k)}) = (q^2 + 1)q^k$. By [4, Theorem 1.3], for all $k \geq 1$ and $q \geq 3$, $\varepsilon(\mathbf{G}^{(k)})$ is proportional to ξ_k . This property seems to hold also for $k = 0$ by numerical inspection for some small values of q , but we do not know if \mathbf{G} itself belongs to \mathbb{M}^\dagger . It is plausible that it is at least an almost A -quasi-modular form.

A result of Stiller [28] asserts that, if b is a non-constant meromorphic modular function for $\mathbf{SL}_2(\mathbb{Z})$ which generates the field of modular functions and f is a meromorphic modular form of weight k for $\mathbf{SL}_2(\mathbb{Z})$, then f , as a function of b , satisfies a linear differential equation with rational functions of b as coefficients.

Analogously, It is not too difficult to show that every non-zero element $\mathbf{f} \in \mathbb{F}_q[t, \theta][g, h, \mathbf{E}, \mathbf{E}^{(1)}]$ satisfies a non-trivial linear τ -difference equation of order $\leq \nu(\mathbf{f})$, with coefficients in $C(t)[g, h]$ (for example, (30) for \mathbf{E}). We were unable to explicitly determine such an equation for \mathbf{G} , leaving open the problem to prove or disprove that $\mathbf{G}/(t - t^q)$, normalised, is itself in $\mathbb{F}_q[t, \theta][[u]]$.

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