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HAMILTON CYCLE DECOMPOSITION OF THE BUTTERFLY NETWORK*

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ABSTRACT

In this paper, we prove that the wrapped Butterfly graph $WB\mathcal{F}(d, n)$ of degree d and dimension n is decomposable into Hamilton cycles. This answers a conjecture of Barth and Raspaud who solved the case $d = 2$.

Keywords: Butterfly graph, graph theory, Hamilton decomposition, Hamilton cycle, Hamilton circuit, perfect matching.

1. Introduction and notations

The construction of one, and if possible many edge-disjoint Hamilton cycles in a network can provide advantage for algorithms that make use of a ring structure. As example, the existence of many edge-disjoint Hamilton cycles allows the message traffic to be evenly distributed across the network. Furthermore, a partition of the edges into Hamilton cycles can be used in various distributed algorithms (termination, garbage collector, ...). So, many authors have considered the problem of finding how many edge-disjoint Hamilton cycles can be found in a given network. The most significant results have been obtained for the class of Cayley graphs on abelian groups, and for (underlying) line digraphs. Here we solve this problem for the *Butterfly networks*. These networks have been proposed as suitable topologies for parallel computers, due to their interesting structure (see [11,13]) because they are, when properly defined, both Cayley digraphs (on a non-abelian group) and iterated line digraphs.

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1.1. Definitions

First, we have to warn the reader that under the name *Butterfly* and with the same notation, different networks are described in the literature. Indeed, while some authors consider the *Butterfly networks* to be multistage networks used to route permutations, others consider them to be point-to-point networks. In what follows, we will study the point-to-point version, and use Leighton’s terminology [11], namely, *wrapped Butterfly*. Also, when we use the terms edge-disjoint or arc-disjoint, it obviously means *pairwise* edge-disjoint or arc-disjoint.

In this article, we will use the following definitions and notation, where \mathbb{Z}_q denotes the set of integers modulo q - For definitions not given here see [13].

Definition 1 *The wrapped Butterfly digraph of degree d and dimension n , denoted $\vec{\mathcal{WBF}}(d, n)$, has as vertices the ordered pairs (x, l) where x is an element of \mathbb{Z}_d^n , that is, a word $x_{n-1}x_{n-2} \cdots x_1x_0$ where the letters belong to \mathbb{Z}_d , and $l \in \mathbb{Z}_n$ (l is called the level). For any l , a vertex $(x_{n-1}x_{n-2} \cdots x_l \cdots x_1x_0, l)$ is joined by an arc to the d vertices $(x_{n-1} \cdots x_{l+1} x_l + \alpha x_{l-1} \cdots x_0, l + 1)$ where α is any element of \mathbb{Z}_d . Each one of these arcs is said to have the **slope** α .*

$\vec{\mathcal{WBF}}(d, n)$ is a d -regular digraph with nd^n vertices; its diameter is $2n - 1$. This network is sometimes considered as undirected, but its structure being indeed directed, we will always consider the digraph.

For convenience, we repeat the level 0 when drawing the wrapped Butterfly digraph. Hence, the reader has to remember that the two occurrences of level 0 have to be identified. Figure (1) displays $\vec{\mathcal{WBF}}(3, 2)$ with the arcs directed from left to right. Note that $\vec{\mathcal{WBF}}(d, n)$ is often represented (for example in [11,13]) in an opposite way to our drawing as the authors denote the nodes $(x_0x_1 \cdots x_{n-1}, l)$.

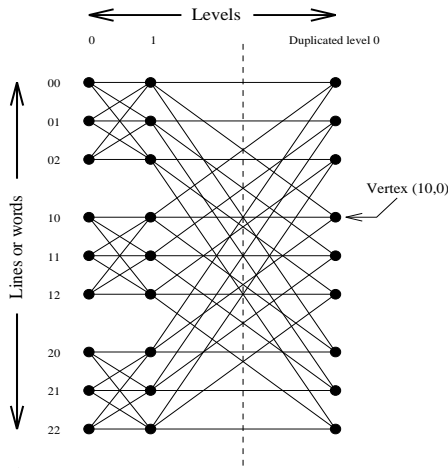


Figure 1: The digraph $\vec{\mathcal{WBF}}(3, 2)$, the arcs being directed from left to right.

Now, we define two other useful digraphs.

- \mathcal{K}_d^+ denotes the complete symmetric digraph with a loop on each vertex,
- $\vec{\mathcal{K}}_{d,d}$ denotes the complete bipartite digraph where each set of the bipartition has size d and with all the arcs directed from one part of the bipartition, called *left part*, to the other, called *right part*.

Note that $\mathcal{WB}\mathcal{F}(d, 1)$ is nothing else than \mathcal{K}_d^+ .

In digraphs, the concept of dipaths and circuits (directed cycles) is well-known. Here, we need to use more general concepts valid for digraphs of paths and cycles (which are also called oriented elementary paths and oriented elementary cycles).

Definition 2 A *path* of a digraph is a sequence $\mu = (v_0, e_0, v_1, e_1, \dots, v_k, e_k, v_{k+1})$ where the v_i 's are vertices and the e_i 's are arcs such that the end vertices of e_i are v_i and v_{i+1} and where the sequence μ does not meet twice the same vertex except maybe v_0 and v_{k+1} .

Definition 3 A path such that $v_{k+1} = v_0$ in the sequence μ is called a **cycle**.

Note that the arc e_i can be either directed from v_i to v_{i+1} or from v_{i+1} to v_i . If all the arcs of the path (resp. cycle) are directed from v_i to v_{i+1} we have a dipath (resp. circuit also called dicycle).

Definition 4 A vertex v_i of a cycle is said to be **of type +** (resp. **of type -**) for the cycle, if v_i is the terminal vertex of e_{i-1} (resp. e_i) and the initial vertex of e_i (resp. e_{i-1}).

Note that the type is not necessarily defined for all the vertices of a cycle. In a circuit, all vertices are of type +.

Definition 5 A vertex v is said to be **crossed by a cycle**, or a **cycle crosses the vertex** v , if v is of type + or of type - for the cycle. When a vertex v is crossed by a cycle, we will define its **sign function** ϵ by $\epsilon(v) = +1$ (resp. $\epsilon(v) = -1$) if v is of type + (resp. of type -).

Remark 1 We can also define the predecessor $p(v)$ and the successor $s(v)$ of the vertex v in the order induced by the cycle. Then, the vertex v is of type + (or has sign $\epsilon(v) = +1$) if $(p(v), v)$ and $(v, s(v))$ are both arcs of the digraph, and is of type - (or has sign $\epsilon(v) = -1$) if both $(s(v), v)$ and $(v, p(v))$ are arcs of the digraph.

Definition 6 A **Hamilton cycle** (resp. **circuit**) of a digraph is a cycle (resp. circuit) which contains every vertex exactly once.

Definition 7 We say that a digraph is **decomposable into Hamilton cycles** (resp. **circuits**) if its arcs can be partitioned into Hamilton cycles (resp. circuits).

Definition 8 A Hamilton cycle of $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ is said to be **l -crossing** if the cycle crosses all the vertices of level l and furthermore $\sum_{v=(x,l), x \in \mathbb{Z}_d^n} \epsilon(v) \equiv 0 \pmod{d}$.

Figure (3) shows examples of 1-crossing Hamilton cycles in $\mathcal{WB}\vec{\mathcal{F}}(3, 2)$ and $\mathcal{WB}\vec{\mathcal{F}}(3, 3)$. Note that a Hamilton circuit is l -crossing for all l .

1.2. Results

Various results have been obtained on the existence of Hamilton cycles in classical networks (see for example the surveys [2,9]). For example, it is well-known that any Cayley graph on an abelian group is Hamiltonian. Furthermore, it has been conjectured by Alspach [1] that:

Conjecture 1 (Alspach) *Every connected Cayley graph on an abelian group has a Hamilton decomposition.*

This conjecture has been verified for all connected 4-regular graphs on abelian groups in [8]. This includes in particular the toroidal meshes (grids). It is also known that $\mathcal{H}(2d)$, the hypercube of dimension $2d$, is decomposable into d Hamilton cycles (see [2,3]).

Concerning line digraphs, it has been shown in [10] that d -regular line digraphs always admit $\lfloor \frac{d}{2} \rfloor$ Hamilton circuits. In the case of de Bruijn and Kautz digraphs which are the simplest line digraphs, partial results have been obtained successively in [12] and [5] respectively, and near optimal results have been obtained for undirected de Bruijn and Kautz graphs [4].

The wrapped Butterfly digraph is actually a Cayley graph (on a non-abelian group) and a line digraph. So, the decomposition into Hamilton cycles (*resp. circuits*) of this digraph has received some attention. It is well-known that $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ has one Hamilton circuit (see [11, page 465] for a proof in the case $d = 2$ or [15]). In [6], Barth and Raspaud proved that the underlying multigraph associated with $\mathcal{WB}\vec{\mathcal{F}}(2, n)$ contains two arc-disjoint Hamilton cycles answering a conjecture of Rowley and Sotteau (private communication). In our terminology, their result can be stated as:

Theorem 1 (Barth, Raspaud) $\mathcal{WB}\mathcal{F}(2, n)$ is decomposable into 2 Hamilton cycles.

They conjectured that this result can be generalized for any degree:

Conjecture 2 (Barth, Raspaud) For $n \geq 2$, $\mathcal{WB}\mathcal{F}(d, n)$ is decomposable into d Hamilton cycles.

In this paper, we prove the conjecture (2). To do so, we use some techniques introduced in [7] where we studied the decomposition of $\mathcal{WB}\mathcal{F}(d, n)$ into Hamilton circuits. In fact, we prove that $\mathcal{WB}\mathcal{F}(d, n)$ is decomposable into d l -crossing Hamilton cycles. Indeed, the l -crossing property, combined with the recursive structure of $\mathcal{WB}\mathcal{F}(d, n)$, enables us to prove that the number of l -crossing arc-disjoint Hamilton cycles that $\mathcal{WB}\mathcal{F}(d, n)$ contains can only increase when n increases. Then, we prove mainly that $\mathcal{WB}\mathcal{F}(d, 2)$ contains d arc-disjoint l -crossing Hamilton cycles, by constructing two arc-disjoint l -crossing Hamilton cycles using only arcs of slopes 0 and 1 and $d - 2$ arc-disjoint Hamilton circuits using arcs of other slopes. The results are summarized in the following theorem:

Theorem 2 For $n \geq 2$,

- for $d \notin \{3, 4, 6\}$, $\mathcal{WB}\mathcal{F}(d, n)$ is decomposable into $d - 2$ Hamilton circuits and 2 Hamilton cycles,
- for $d \in \{4, 6\}$, $\mathcal{WB}\mathcal{F}(d, n)$ is decomposable into d Hamilton circuits,
- $\mathcal{WB}\mathcal{F}(3, n)$ is decomposable into 1 Hamilton circuit and 2 Hamilton cycles.

2. The general construction

We give below some additional definitions and properties enabling us to establish lemma (2) which is a strengthened version of the inductive lemma of [7]. This lemma is then applied in section 3 to construct inductively the decomposition.

2.1. Cyclic-potent families of permutations

In this paper, M will always denote a permutation of \mathbb{Z}_d which associates the element a with the element $M(a)$. To such a permutation, one can associate a perfect matching (denoted also M) of $\vec{\mathcal{K}}_{d,d}$ containing all the arcs $(a, M(a))$.

Let $x \in \mathbb{Z}_d^n$, M_x will denote a permutation; the label x will be useful in the proof of lemma (2), as we will associate M_x with a perfect matching of $\vec{\mathcal{K}}_{d,d}(x)$ where $\vec{\mathcal{K}}_{d,d}(x)$ denote the bipartite subgraph of $\mathcal{WB}\mathcal{F}(d, n + 1)$ with left part the vertices $(\star x, n)$ and right part the vertices $(\star x, 0)$. M_x contains the arcs joining (ax, n) to $(M_x(a)x, 0)$. In [7], M_x is said to be a permutation realizable in $\vec{\mathcal{K}}_{d,d}(x)$.

Definition 9 Let S be a set of slopes (that is a subset of \mathbb{Z}_d). Then, a **permutation** M of \mathbb{Z}_d **uses the slopes in S** if, for any $a \in \mathbb{Z}_d$, $M(a) \in \{a + s, s \in S\}$. A **family of d^n permutations** $\mathcal{M} = \{M_x, x \in \mathbb{Z}_d^n\}$ of \mathbb{Z}_d **uses the slopes in S** if, for any permutation M_x of the family, M_x uses the slopes in S .

Definition 10 A set of p permutations M_j , with $1 \leq j \leq p$, is said to be **compatible** if, $\forall a, M_j(a) \neq M_{j'}(a)$ for $j \neq j'$.

In other words the perfect matchings associated with the M_j are arc-disjoint.

Definition 11 For $1 \leq j \leq p$, let $\mathcal{M}_j = \{M_{x,j} \mid x \in \mathbb{Z}_d^n\}$ be p families, each consisting of d^n permutations. The **families \mathcal{M}_j are said to be compatible** if, for each x in \mathbb{Z}_d^n , the p permutations $M_{x,j}$ are compatible, i.e. $\forall a, M_{x,j}(a) \neq M_{x,j'}(a)$ for $j \neq j'$.

The **composition $M \cdot M'$ of two permutations** M and M' is the permutation which associates the element a with the element $M(M'(a))$.

Definition 12 A permutation M is **cyclic** if, for some x , all the elements $M^i(x)$ are distinct for $0 \leq i < d^n$.

Remark 2 Note that if M is cyclic, then for every x , the elements $M^i(x)$ are all distinct. In fact, to verify that M is cyclic, it suffices to verify that for a given x , $M^i(x) \neq x$, for $1 \leq i < d^n$. Indeed, if there exists j and k , with $j > k$, such that $M^j(x) = M^k(x)$, then $M^{j-k}(x) = x$.

For example, the permutation M which associates a with the element $a + \delta$ is clearly cyclic if and only if δ is prime with d , as $M^i(a) = a + \delta i$.

Definition 13 A family $\mathcal{M} = \{M_x, x \in \mathbb{Z}_d^n\}$ of d^n permutations of \mathbb{Z}_d satisfies the **cyclic-potent property** if, for any order of composition of the M_x and any set of sign $\{\epsilon_x \mid x \in \mathbb{Z}_d^n, \epsilon_x \in \{-1, 1\}\}$ such that $\sum_x \epsilon_x \equiv 0 \pmod{d}$, the permutation $\prod_x M_x^{\epsilon_x}$ is cyclic.

Definition 14 A family of d^n permutations $\mathcal{M} = \{M_x, x \in \mathbb{Z}_d^n\}$ is of **type (i, j)** if for $x \neq 0$, $M_x(a) = a + i$; and for $x = 0$, $M_0(a) = a + j$.

Lemma 1 A family of permutations of type (i, j) , $\mathcal{M} = \{M_x, x \in \mathbb{Z}_d^n\}$ is cyclic-potent if and only if $j - i$ is relatively prime to d .

Proof. As the permutations of the family commute, the permutation $\prod_x M_x^{\epsilon_x}$ of definition (13) can be simply expressed as $a \rightarrow a + \delta$. So, this permutation will

be cyclic if and only if δ is prime with d . Here $\delta = (\sum_{x \neq 0} \epsilon_x)i + \epsilon_0j$. As $\sum_x \epsilon_x = 0$, we have $\delta = (\sum_x \epsilon_x)i + \epsilon_0(j - i) = \epsilon_0(j - i)$. So, δ is clearly prime with d if and only if $j - i$ is prime with d . \square

We will represent a set of p families of permutations of type (i, j) : $\{(i_0, j_0), (i_1, j_1), \dots, (i_{p-1}, j_{p-1})\}$ by the array:

Families							
i_0	i_1	i_2	i_3	\dots	i_{p-2}	i_{p-1}	
j_0	j_1	j_2	j_3	\dots	j_{p-2}	j_{p-1}	

In section 3, we will need some very simple cyclic-potent families of permutations that we give as examples.

Families 1 There exist d compatible cyclic-potent families of permutations:

Families 1									
0	1	2	3	4	5	\dots	$d-2$	$d-1$	
1	2	3	4	5	6	\dots	$d-1$	0	

These families are cyclic-potent as, applying lemma (1), $1 - 0 = 2 - 1 = \dots = d - 1 - (d - 2) = 0 - (d - 1) = 1$ which is prime with d . These families use all the slopes.

Families 2 There exist 2 compatible cyclic-potent families using slopes $\{0, 1\}$:

Families 2	
0	1
1	0

According to lemma (1) they are two compatible cyclic-potent families and they use the slopes $\{0, 1\}$.

Families 3 When $d \neq 3$, there exist $d - 2$ compatible cyclic-potent families of permutations using the slopes $\{2, \dots, d - 1\}$. One possible solution is given below:

- when d is odd and $d \neq 3$, the following families can be used:

Families 3 (d odd and $d \neq 3$)							
2	3	4	5	\dots	$d-3$	$d-2$	$d-1$
4	5	6	7	\dots	$d-1$	2	3

- when d is even, we use the following families:

Families 3 (d even)						
2	3	4	5	\dots	$d-2$	$d-1$
3	2	5	4	\dots	$d-1$	$d-2$

These families are cyclic-potent as, applying lemma (1), we get:

- for d odd, $4-2 = 5-3 = \dots = d-1-(d-3) = 2$ and $2-(d-2) = 3-(d-1) = 4$, as 2 and 4 are prime with d ;
- for d even, $3-2 = 5-4 = \dots = (d-1)-(d-2) = 1$ and $2-3 = 4-5 = \dots = (d-2)-(d-1) = -1$, which are prime with d .

In both cases, the slopes used are in $\{2, \dots, d-1\}$.

2.2. Inductive construction

Lemma 2 *If $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ admits p arc-disjoint l -crossing Hamilton cycles and if there exist p compatible cyclic-potent families each of d^n permutations, then $\mathcal{WB}\vec{\mathcal{F}}(d, n+1)$ admits p arc-disjoint l -crossing Hamilton cycles.*

Proof. Let H be an l -crossing Hamilton cycle of $\mathcal{WB}\vec{\mathcal{F}}(d, n)$. As all the levels are equivalent, we can suppose without loss of generality and for simplicity in the notations that $l = 0$. Let $\mathcal{M} = \{M_x, x \in \mathbb{Z}_d^n\}$ be a cyclic-potent family of d^n permutations. The vertices of $\mathcal{WB}\vec{\mathcal{F}}(d, n+1)$ can be labeled (ax, l) with $a \in \mathbb{Z}_d$, $x \in \mathbb{Z}_d^n$ and $l \in \mathbb{Z}_{n+1}$. Now, we associate H and \mathcal{M} with a partial digraph H' in $\mathcal{WB}\vec{\mathcal{F}}(d, n+1)$ as follows (for an example of such a construction see figure (3)):

- for $0 \leq l \leq n-1$ and for each a , if the arc $(x, l)(x', l+1)$ belongs to H , we put in H' the arc $(ax, l)(ax', l+1)$ where the indices are taken modulo $n+1$, which means that to the arc $(x, n-1)(x', 0)$ of H is associated the arc $(ax, n-1)(ax', n)$ in H' ;
- between levels n and 0 of $\mathcal{WB}\vec{\mathcal{F}}(d, n+1)$ we put the arcs joining (ax, n) to $(M_x(a)x, 0)$.

With such a definition, each vertex of $\mathcal{WB}\vec{\mathcal{F}}(d, n+1)$ is incident to two arcs of H' . Hence, we can define for each vertex a predecessor and a successor on H' that enables us to prove that we can order H' in a cycle.

For $1 \leq l \leq n-1$, let (x', l') (*resp.* (x'', l'')) be the predecessor (*resp.* successor) of (x, l) in H , then the predecessor (*resp.* successor) of (ax, l) in H' will be (ax', l') (*resp.* (ax'', l'')).

For $l = 0$ and n , as H is a 0-crossing Hamilton cycle, vertices $(x, 0)$ are either of type $+$ or $-$ on H .

- When $(x, 0)$ is of type $+$, its predecessor (*resp.* successor) in the cycle H is $(x', n-1)$ (*resp.* $(x'', 1)$). Then, in H' the predecessor (*resp.* successor) of (ax, n) will be $(ax', n-1)$ (*resp.* $(M_x(a)x, 0)$); the predecessor (*resp.* successor) of $(ax, 0)$ will be $(M_x^{-1}(a)x, n)$ (*resp.* $(ax'', 1)$).

- When $(x, 0)$ is of type $-$, its predecessor (*resp.* successor) in H is $(x', 1)$ (*resp.* $(x'', n - 1)$). Then, in H' the predecessor (*resp.* successor) of $(ax, 0)$ will be $(ax', 1)$ (*resp.* $(M_x^{-1}(a)x, n)$); the predecessor (*resp.* successor) of (ax, n) will be $(M_x(a)x, 0)$ (*resp.* $(ax'', n - 1)$) in H' .

Therefore, when $(x, 0)$ is of type $+$ (*resp.* $-$), (ax, n) and $(ax, 0)$ are vertices of type $+$ (*resp.* $-$) in H' . Hence, all the vertices of levels 0 and n are crossed by H' ; furthermore, the sum of the signs of the vertices of H' of levels 0 or n will be d times the sum of the signs of the vertices of H of level 0, that is, by hypothesis, 0. Hence, H' is 0-crossing (and also n -crossing).

Now, we have to prove that H' is effectively a Hamilton cycle. For this it suffices to prove that if we start at some vertex $(ax, 0)$ and follow H' , we meet successively all the vertices of level 0 and n before coming back to $(ax, 0)$. Indeed, suppose that (y, l) was on the portion of cycle H between $(x_1, 0)$ and $(x_2, 0)$. Then, (ay, l) will be on the portion of H' between (ax_1, α) and (ax_2, β) , where $\alpha = 0$ (*resp.* $\alpha = n$) if $(x_1, 0)$ is of type $+$ (*resp.* $-$), and $\beta = 0$ (*resp.* $\beta = n$) if $(x_2, 0)$ is of type $-$ (*resp.* $+$). These cases are described on figure (2).

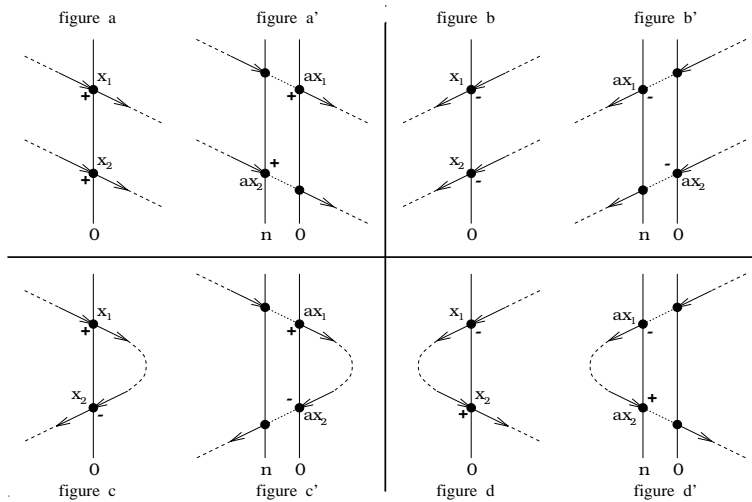


Figure 2: This figure shows the four possible cases when we perform the inductive construction of $\vec{WBF}(d, n + 1)$ from $\vec{WBF}(d, n)$. In figure a and a' (*resp.* b and b') the vertices x_1 and x_2 are of type $+$ (*resp.* $-$). Figure c and c' (*resp.* d and d') displays the case where the vertex x_1 is of type $+$ (*resp.* $-$) and the vertex x_2 is of type $-$ (*resp.* $+$).

Now, let $(x_0, 0), (x_1, 0), \dots, (x_{d^n} = x_0, 0)$ be the sequence of vertices of H at level 0 in the order we meet them on H . Starting from $(a_0x_0, 0)$ we will meet successively $(a_1x_1, 0), (a_2x_2, 0), \dots, (a_{d^n}x_{d^n} = a_{d^n}x_0, 0)$ on H' . Following such a path, we can meet either x_i of type $+$ by going from level n to level 0, in which case we will apply

the permutation M_{x_i} to some a , or x_i of type $-$ by going from level 0 to n , in which case we will apply $M_{x_i}^{-1}$ to a . So $a_{d^n} = \Pi M_{x_i}^{\epsilon_{x_i}}(a)$ where the product is taken in an order depending on x_0 . As all the x_i differ, we can meet again $(a_0 x_0, 0)$ only at some $a_{q d^n} x_0$, but M being cyclic-potent, the values $a_{d^n}, a_{2d^n}, \dots, a_{q d^n}, \dots, a_{(d)d^n}$ are all distinct. So, we meet again $(a_0 x_0, 0)$ only after having encountered the d^{n+1} vertices of level 0.

Now, note that we can perform this construction with p arc-disjoint 0-crossing cycles and p compatible cyclic-potent families. From construction, the p 0-crossing cycles that we will obtain, will be arc-disjoint. \square

Remark 3 When the 0-crossing Hamilton cycles used in the lemma above are circuits of $\vec{\mathcal{WBF}}(d, n)$, all the vertices are of type $+$, and the construction leads to circuits of $\vec{\mathcal{WBF}}(d, n+1)$, giving another proof of the inductive lemma of [7].

3. Decomposition of $\vec{\mathcal{WBF}}(d, n)$

We will use a decomposition of $\vec{\mathcal{WBF}}(d, n)$ into two partial digraphs.

Definition 15 *The Butterfly digraph $\vec{\mathcal{WBF}}(d, n)$ is the sum of two partial digraphs $\vec{\mathcal{WBF}}_{0,1}(d, n)$ and $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, n)$ defined as follows:*

- $\vec{\mathcal{WBF}}_{0,1}(d, n)$ contains the arcs which slopes belong to $\{0, 1\}$,
- $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, n)$ contains the arcs which slopes belong to $\{2, \dots, d-1\}$.

3.1. Decomposition of $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, n)$

The proof is by induction on n . We start the induction for $n = 1$.

Lemma 3 *When $d \notin \{4, 6\}$, $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, 1)$ is decomposable into $d-2$ Hamilton circuits.*

Proof. As $\vec{\mathcal{WBF}}(d, 1) = \mathcal{K}_d^+$, $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, 1)$ is obtained from \mathcal{K}_d^+ by removing the loops and the arcs of slope 1. Following Tillson [14], we know that \mathcal{K}_d^+ without the loops contains $d-1$ arc-disjoint Hamilton circuits when $d \neq 4, 6$. So, using Tillson's decomposition, we can label the vertices of \mathcal{K}_d^+ such that one of the circuits uses all the arcs of slope 1. By removing it, we get $d-2$ arc-disjoint Hamilton circuits in $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, 1)$. \square

Proposition 1 *For $d \notin \{3, 4, 6\}$, $\vec{\mathcal{WBF}}_{2,\dots,d-1}(d, n)$ is decomposable into $d-2$ Hamilton circuits.*

Proof. As $d \notin \{4, 6\}$, the proposition is proved for $n = 1$ by lemma (3). Then, as $d \neq 3$, the $d-2$ compatible cyclic-potent families (3) use the slopes

$\{2, \dots, d-1\}$ and satisfy the hypothesis of lemma (2). Hence, we can apply that lemma inductively, in order to construct $d-2$ arc-disjoint Hamilton circuits (see remark (3)) in $\vec{\mathcal{WBF}}_{2, \dots, d-1}(d, n)$. \square

3.2. Decomposition of $\vec{\mathcal{WBF}}_{0,1}(d, n)$

Lemma 4 $\vec{\mathcal{WBF}}_{0,1}(d, 2)$ is decomposable into 2 1-crossing Hamilton cycles.

Proof. For this proof, the vertices of $\vec{\mathcal{WBF}}_{0,1}(d, 2)$ will be denoted by the ordered pairs (xy, l) with $x \in \mathbb{Z}_d$, $y \in \mathbb{Z}_d$ and $l \in \mathbb{Z}_2$. We will show that we can build two arc-disjoint 1-crossing Hamilton cycles in $\vec{\mathcal{WBF}}_{0,1}(d, 2)$ by using two sets of arcs of $\vec{\mathcal{WBF}}_{0,1}(d, 2)$ defined by the next two rules:

1. Arcs of H_0 :

$$\begin{cases} \text{if } x \neq y, & (x(y-1), 0) \xrightarrow{\pm 1} (xy, 1) \xrightarrow{\pm 0} (xy, 0), & (1) \\ \text{if } x = y, & (xx, 0) \xrightarrow{\pm 0} (xx, 1) \xrightarrow{\pm 1} ((x+1)x, 0). & (2) \end{cases}$$

2. Arcs of H_1 :

$$\begin{cases} \text{if } x \neq y, & (xy, 0) \xrightarrow{\pm 0} (xy, 1) \xrightarrow{\pm 1} ((x+1)y, 0), & (1) \\ \text{if } x = y, & (x(x-1), 0) \xrightarrow{\pm 1} (xx, 1) \xrightarrow{\pm 0} (xx, 0). & (2) \end{cases}$$

It is easy to verify that H_0 and H_1 are arc-disjoint. With the arcs (1) of H_0 , we can define for each $x \in \mathbb{Z}_d$ a dipath P_x as follows:

$$P_x \begin{cases} \rightarrow & (xx, 0) & \rightarrow & (x(x+1), 1) & \rightarrow & (x(x+1), 0) & \rightarrow \\ \rightarrow & (x(x+2), 1) & \rightarrow & \dots & \rightarrow & (x(x+d-2), 1) & \rightarrow \\ \rightarrow & (x(x+d-2), 0) & \rightarrow & (x(x+d-1), 1) & \rightarrow & (x(x+d-1), 0) \end{cases}$$

The d dipaths P_x , $x \in \mathbb{Z}_d$, are clearly vertex-disjoint. Only the vertices noted $(xx, 1)$ are not in these d dipaths. The arcs (2) of H_0 allows us to join the end vertices of the d dipaths through the missing vertices $(xx, 1)$ as follows:

$$\begin{array}{ccccccc} P_x & \leftarrow & ((x+d-1)(x+d-1), 1) & \leftarrow & P_{x+d-1} & \leftarrow & \\ & \leftarrow & ((x+d-2)(x+d-2), 1) & \leftarrow & \dots & \leftarrow & \\ & \leftarrow & ((x+1)(x+1), 1) & \leftarrow & P_{x+1} & \leftarrow & \\ & \leftarrow & (xx, 1) & \leftarrow & P_x & \leftarrow & \end{array}$$

One can easily check that we have defined a Hamilton cycle. The d dipaths are joined through their extremal vertices in a cyclic way, using only arcs (2) of H_0 .

By construction, all the vertices at level 1 are crossed. In order to compute the sign of the vertices at level 1, we can choose to walk along the cycle in the direction $(xx, 0) \rightarrow (x(x+1), 1)$. Therefore, all the vertices $(xy, 1)$ with $x \neq y$ are of type + and have +1 as sign, while the vertices $(xx, 1)$ are of type - and have -1 as sign. So, the sum of the signs is $(d^2 - d) - (d) \equiv 0 \pmod{d}$.

To prove that the second set of rules builds a second 1-crossing Hamilton cycle, it suffices to notice that we can rewrite this rule up to a permutation of the letters x and y as being:

- *Arcs of H_1 (with permutation of x and y):*

$$\begin{cases} \text{if } y \neq x, & (y(x+1), 0) \xrightarrow{\pm 1} (yx, 1) \xrightarrow{\pm 0} (yx, 0), \\ \text{if } y = x, & (yy, 0) \xrightarrow{\pm 0} (yy, 1) \xrightarrow{\pm 1} ((y-1)y, 0). \end{cases}$$

Construction 2 is then clearly similar to construction 1; to be convinced, just exchange x and y , and replace 1 by -1 in the proof for construction (1).

Hence, H_0 and H_1 are two arc-disjoint 1-crossing Hamilton cycles. As the levels are equivalent, the result holds also for level 0. \square

Figure (3) gives a decomposition of $\mathcal{WB}\mathcal{F}_{0,1}(3, 2)$ into two 1-crossing Hamilton cycles.

Proposition 2 *For $n \geq 2$, $\mathcal{WB}\mathcal{F}_{0,1}(d, n)$ is decomposable into 2 l -crossing Hamilton cycles.*

Proof. The proposition is proved for $n = 2$ by the lemma (4). Then, we use lemma (2) with the two compatible cyclic-potent families (2) which use the slopes $\{0, 1\}$ to construct inductively two arcs-disjoint l -crossing Hamilton cycles in $\mathcal{WB}\mathcal{F}_{0,1}(d, n)$. \square

Figure (3) gives the recursive construction of two 1-crossing arc-disjoint Hamilton cycles in $\mathcal{WB}\mathcal{F}_{0,1}(3, 3)$ from two 1-crossing arc-disjoint cycles in $\mathcal{WB}\mathcal{F}(3, 2)$.

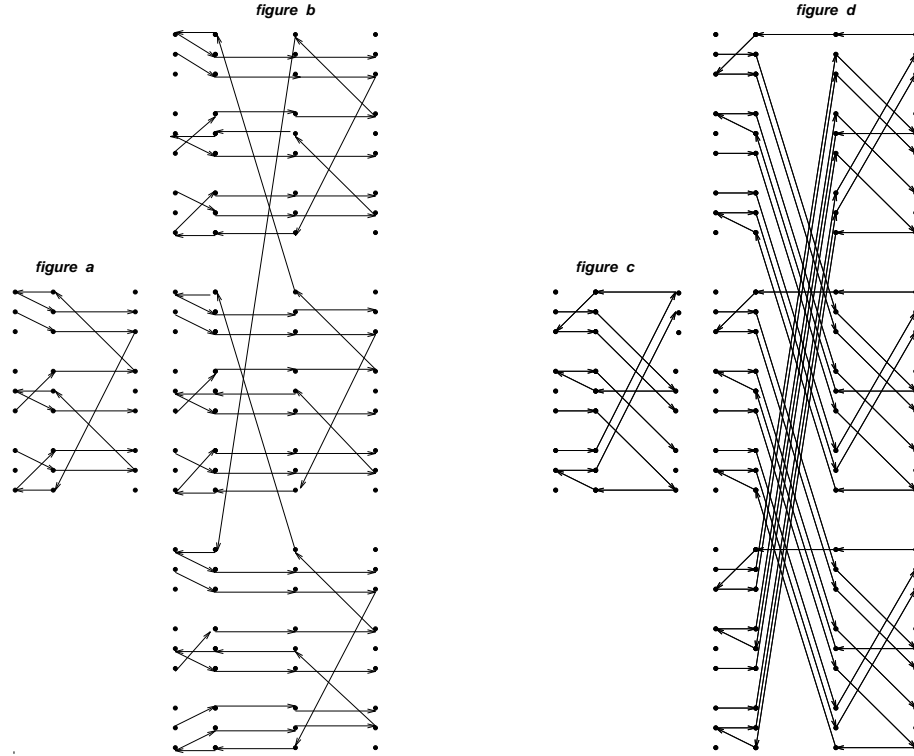


Figure 3: Figures a and c show the two 1-crossing arc-disjoint Hamiltonian cycles of $\mathcal{WBF}_{0,1}(3,2)$. We display on figures b and d, two 1-crossing arc-disjoint Hamiltonian cycles in $\mathcal{WBF}_{0,1}(3,3)$ obtained by applying lemma (2) with the families (2).

3.3. Global decomposition

We are now ready to prove the main result:

Theorem 2 For $n \geq 2$,

- for $d \notin \{3, 4, 6\}$, $\mathcal{WBF}(d, n)$ is decomposable into $d - 2$ Hamiltonian circuits and 2 Hamiltonian cycles,
- for $d \in \{4, 6\}$, $\mathcal{WBF}(d, n)$ is decomposable into d Hamiltonian circuits,
- $\mathcal{WBF}(3, n)$ is decomposable into 1 Hamiltonian circuit and 2 Hamiltonian cycles.

Proof. According to propositions (1) and (2) we have, when $d \notin \{3, 4, 6\}$, $d - 2$ arc-disjoint circuits in $\mathcal{WBF}_{2, \dots, d-1}(d, n)$ and 2 arc-disjoint cycles in $\mathcal{WBF}_{0,1}(d, n)$. So, the result holds in these cases. For $d \in \{4, 6\}$ and $n = 2$, an exhaustive computer

search shows that $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ is decomposable into Hamilton circuits, and so, for $n \geq 2$, $\mathcal{WB}\vec{\mathcal{F}}(4, n)$ and $\mathcal{WB}\vec{\mathcal{F}}(6, n)$ are decomposable into Hamilton circuits. For $d = 3$, we can construct two 1-crossing arc-disjoint Hamilton cycles and one arc-disjoint Hamilton circuit in $\mathcal{WB}\vec{\mathcal{F}}(3, 2)$ (see figure (4)). Then, we can apply lemma (2) with families (1) and the result holds for $\mathcal{WB}\vec{\mathcal{F}}(3, n)$ with $n \geq 2$. \square

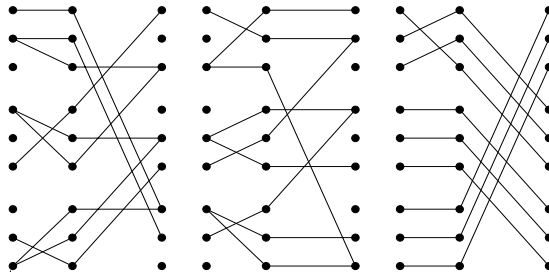


Figure 4: *The decomposition of $\mathcal{WB}\vec{\mathcal{F}}(3, 2)$ into two 1-crossing arc-disjoint Hamilton cycles and one arc-disjoint Hamilton circuit.*

The preceding result implies the conjecture of Barth and Raspaud:

Theorem 3 *For any d and $n \geq 2$, $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ is decomposable into d Hamilton cycles.*

Remark 4 We could also have derived theorem (3) by proving that, if $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ is decomposable into l -crossing Hamilton cycles, then $\mathcal{WB}\vec{\mathcal{F}}(d, n+1)$ is also decomposable into l -crossing Hamilton cycles. This can be done by applying lemma (2) with the families (1). But to start the induction we needed to split the Butterfly digraph into two partial digraphs in order to prove that $\mathcal{WB}\vec{\mathcal{F}}(d, 2)$ is decomposable into l -crossing Hamilton cycles for $n = 2$ and $d \neq 3$.

4. Conclusion

In this paper we have proved that $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ is always decomposable into Hamilton cycles. In the paper [7], we considered the problem of decomposing $\mathcal{WB}\vec{\mathcal{F}}(d, n)$ into Hamilton circuits and conjectured that such a decomposition into d Hamilton circuits exists for $n \geq 2$, except for $(d = 2$ and $(n = 2$ or $n = 3))$ and $(d = 3$ and $n = 2)$. The difficulty in that case was to start the induction; indeed in [7] we were able to reduce the problem to the case $n = 2$ and d prime and to solve it in many cases. Consequently, we proposed as an open problem the following conjecture:

Conjecture 3 ([7]) *For any prime number $p > 3$, $\mathcal{WB}\vec{\mathcal{F}}(p, 2)$ is decomposable into Hamilton circuits.*

Note added in proof

Recently, Helen Verrall^a has informed us that she has been able to prove conjecture (3), thus closing completely the problem of Hamilton decomposition of the Butterfly network.

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