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Elliptic Functions and Transcendence

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Summary. Transcendental numbers form a fascinating subject: so little is known about the nature of analytic constants that more research is needed in this area. Even when one is interested only in numbers like π and e^{π} that are related to the classical exponential function, it turns out that elliptic functions are required (so far, this should not last forever!) to prove transcendence results and get a better understanding of the situation.

First we briefly recall some of the basic transcendence results related to the exponential function (Section 1). Next, in Section 2, we survey the main properties of elliptic functions that are involved in transcendence theory.

We survey transcendence theory of values of elliptic functions in Section 3, linear independence in Section 4, and algebraic independence in Section 5. This splitting is somewhat artificial but convenient. Moreover, we restrict ourselves to elliptic functions, even when many results are only special cases of statements valid for abelian functions. A number of related topics are not considered here (e.g., heights, p-adic theory, theta functions, Diophantine geometry on elliptic curves).

Key words: Transcendental numbers, elliptic functions, elliptic curves, elliptic integrals, algebraic independence, transcendence measures, measures of algebraic independence, Diophantine approximation.

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1 Exponential Function and Transcendence

We start with a very brief list of some of the main transcendence results concerning numbers related to the exponential function. References are, for instance, [13, 83, 89, 120, 192, 207, 211, 233].

Next, we point out some properties of the exponential function, the elliptic analogue of which we shall consider later (Section 2.1).

1.1 Short Survey on the Transcendence of Numbers Related to the Exponential Function

Hermite, Lindemann, and Weierstrass

The first transcendence result for a number related to the exponential function is Hermite's theorem on the transcendence of e.

Theorem 1 (Hermite, 1873). *The number e is transcendental.*

This means that for any nonzero polynomial $P \in \mathbb{Z}[X]$, the number $P(e)$ is not zero. We denote by $\overline{\mathbb{Q}}$ the set of algebraic numbers. Hence Hermite's theorem can be written $e \notin \overline{\mathbb{Q}}$. A complex number is called *transcendental* if it is transcendental over $\mathbb{Q}$, or over $\overline{\mathbb{Q}}$, which is the same. Also we shall say that complex numbers $\theta_1, \ldots, \theta_n$ are *algebraically independent* if they are algebraically independent over $\mathbb{Q}$, which is the same as over $\overline{\mathbb{Q}}$: for any nonzero polynomial P in n variables (and coefficients in $\mathbb{Z}$, $\mathbb{Q}$, or $\overline{\mathbb{Q}}$), the number $P(\theta_1, \ldots, \theta_n)$ is not zero.

The second result in chronological order is Lindemann's theorem on the transcendence of π.

Theorem 2 (Lindemann, 1882). *The number π is transcendental.*

The next result contains the transcendence of both numbers e and π:

Theorem 3 (Hermite–Lindemann, 1882). *For $\alpha \in \overline{\mathbb{Q}}^\times$, any nonzero logarithm $\log \alpha$ of α is transcendental.*

We denote by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \left\{\log \alpha \ ; \ \alpha \in \overline{\mathbb{Q}}^\times\right\} = \left\{\ell \in \mathbb{C} \ ; \ e^\ell \in \overline{\mathbb{Q}}^\times\right\} = \exp^{-1}(\overline{\mathbb{Q}}^\times).$$

Hence Theorem 3 means that $\mathcal{L} \cap \overline{\mathbb{Q}} = \{0\}$. An alternative form is the following:

Theorem 4 (Hermite–Lindemann, 1882). *For any $\beta \in \overline{\mathbb{Q}}^\times$, the number e^β is transcendental.*

The first result of algebraic independence for the values of the exponential function goes back to the end of the nineteenth century.

Theorem 5 (Lindemann–Weierstrass, 1885). *Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent.*

Again, there is an alternative form of Theorem 5: it amounts to a statement of linear independence.

Theorem 6 (Lindemann–Weierstrass, 1885). *Let $\gamma_1, \ldots, \gamma_m$ be distinct algebraic numbers. Then the numbers $e^{\gamma_1}, \ldots, e^{\gamma_m}$ are linearly independent over $\overline{\mathbb{Q}}$.*

It is not difficult to check that Theorem 6 is equivalent to Theorem 5 with the conclusion that $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\overline{\mathbb{Q}}$; since it is equivalent to saying that $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbb{Q}$, one does not lose anything if one changes the conclusion of Theorem 6 by stating that the numbers $e^{\gamma_1}, \ldots, e^{\gamma_m}$ are linearly independent over $\mathbb{Q}$.

Hilbert's Seventh Problem, Gel'fond and Schneider

The solution of Hilbert's seventh problem on the transcendence of α^β was obtained by Gel'fond and Schneider in 1934 (see [89, 207]).

Theorem 7 (Gel'fond–Schneider, 1934). *For α and β algebraic numbers with $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$ and for any choice of $\log \alpha \neq 0$, the number $\alpha^\beta = \exp(\beta \log \alpha)$ is transcendental.*

This means that the two algebraically independent functions e^z and $e^{\beta z}$ cannot take algebraic values at the points $\log \alpha$ (A.O. Gel'fond) and also that the two algebraically independent functions z and $\alpha^z = e^{z \log \alpha}$ cannot take algebraic values at the points $m + n\beta$ with $(m, n) \in \mathbb{Z}^2$ (Th. Schneider).

Examples (quoted by D. Hilbert in 1900) of numbers whose transcendence follows from Theorem 7 are $2^{\sqrt{2}}$ and e^π (recall that $e^{i\pi} = -1$). The transcendence of e^π had already been proved in 1929 by A.O. Gel'fond.

Here is an equivalent statement to Theorem 7:

Theorem 8 (Gel'fond–Schneider, 1934). *Let $\log \alpha_1, \log \alpha_2$ be two nonzero logarithms of algebraic numbers. Assume that the quotient $(\log \alpha_1)/(\log \alpha_2)$ is irrational. Then this quotient is transcendental.*

Linear Independence of Logarithms of Algebraic Numbers

The generalization of Theorem 8 to more than two logarithms, conjectured by A.O. Gel'fond [89], was proved by A. Baker in 1966. His results include not only Theorem 8 but also Theorem 3.

Theorem 9 (Baker, 1966). *Let $\log \alpha_1, \ldots, \log \alpha_n$ be $\mathbb{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

The Six Exponentials Theorem and the Four Exponentials Conjecture

The next result, which does not follow from any of the previously mentioned results, was proved independently in the 1940s by C.L. Siegel (unpublished) and in the 1960s by S. Lang and K. Ramachandra (see [120, 191, 238]; see also Problem 1 in [207] for the four exponentials conjecture). As suggested by K. Ramachandra (see [192] Section 3.1, Theorem 2), Theorem 10 also follows from Schneider's criterion proved in 1949 [206].

Theorem 10 (Six Exponentials Theorem). *Let $x_1, \ldots, x_d$ be $\mathbb{Q}$-linearly independent complex numbers and let $y_1, \ldots, y_\ell$ be $\mathbb{Q}$-linearly independent complex numbers. Assume $\ell d > \ell + d$. Then at least one of the ℓd numbers*

$$e^{x_i y_j} \qquad (1 \le i \le d,\ 1 \le j \le \ell)$$

is transcendental.

Notice that the condition $\ell d > \ell + d$ can be written ($\ell \ge 2$ and $d \ge 3$) or ($\ell \ge 3$ and $d \ge 2$); it suffices to consider the case $\ell d = 6$ (hence the name of the result). Therefore, Theorem 10 can be stated in an equivalent form:

Theorem 11 (Six Exponentials Theorem—logarithmic form). *Let*

$$M = \begin{pmatrix} \log \alpha_1 & \log \alpha_2 & \log \alpha_3 \\ \log \beta_1 & \log \beta_2 & \log \beta_3 \end{pmatrix}$$

be a 2-by-3 matrix whose entries are logarithms of algebraic numbers. Assume that the three columns are linearly independent over $\mathbb{Q}$ and the two rows are also linearly independent over $\mathbb{Q}$. Then the matrix M has rank 2.

It is expected that the condition $d\ell > d + \ell$ in Theorem 10 is too restrictive and that the same conclusion holds in the case $d = \ell = 2$. We state this conjecture in the logarithmic form:

Conjecture 12 (Four exponentials conjecture — logarithmic form). Let

$$M = \begin{pmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \beta_1 & \log \beta_2 \end{pmatrix}$$

be a 2-by-2 matrix whose entries are logarithms of algebraic numbers. Assume that the two columns are linearly independent over $\mathbb{Q}$ and that the two rows are also linearly independent over $\mathbb{Q}$. Then the matrix M has rank 2.

Algebraic Independence

In 1948 and 1949, A.O. Gel'fond extended his solution of Hilbert's seventh problem to a result of algebraic independence [89]. One of his theorems is that the two

numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. His general statements can be seen as extensions of Theorem 10 into a result of algebraic independence (in spite of the fact that Theorem 10 was stated and proved only several years later). In his original work, Gel'fond needed a stronger assumption, namely a measure of linear independence of the x_i's as well as of the y_j's. This assumption was removed in the early 1970s by R. Tijdeman [217] (further references, especially to papers by A.A. Smelev and W.D. Brownawell, are given in [230]; see also [35, 235, 236, 242]).

Theorem 13. *Let $x_1, \ldots, x_d$ be $\mathbb{Q}$-linearly independent complex numbers and let $y_1, \ldots, y_\ell$ be $\mathbb{Q}$-linearly independent complex numbers.*
1. If $d\ell \geq 2(d+\ell)$, then at least two of the $d\ell$ numbers

$$e^{x_i y_j} \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
2. If $d\ell \geq d + 2\ell$, then at least two of the $d\ell + d$ numbers

$$x_i,\ e^{x_i y_j} \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
3. If $d\ell > d + \ell$, then at least two of the $d\ell + d + \ell$ numbers

$$x_i,\ y_j,\ e^{x_i y_j} \qquad (1 \leq i \leq d,\ 1 \leq j \leq \ell)$$

are algebraically independent.
4. If $d = \ell = 2$ and if the two numbers $e^{x_1 y_1}$ and $e^{x_1 y_2}$ are algebraic, then at least two of the six numbers

$$x_1,\ x_2,\ \ y_1,\ y_2,\ \ e^{x_2 y_1},\ \ e^{x_2 y_2}$$

are algebraically independent.

From the last part of Theorem 13, taking $x_1 = y_1 = i\pi$ and $x_2 = y_2 = 1$, one deduces that at least one of the two following statements is true:
(i) *The number e^{π^2} is transcendental.*
(ii) *The two numbers e and π are algebraically independent.*

One expects that both statements are true.

If it were possible to prove that under the assumptions of Theorem 13, at least two of the eight numbers

$$x_1,\ x_2,\ \ y_1,\ y_2,\ \ e^{x_1 y_1},\ \ e^{x_1 y_2},\ \ e^{x_2 y_1},\ \ e^{x_2 y_2}$$

are algebraically independent, one would deduce the algebraic independence of the two numbers π and e^π (take $x_1 = 1$, $x_2 = i$, $y_1 = \pi$, $y_2 = i\pi$; see Corollary 48 below).

For results concerning *large transcendence degree*, see Section 5.3 below.

1.2 The Exponential Function

The exponential function

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C}^\times, \\ z &\mapsto e^z, \end{aligned}$$

satisfies both a differential equation and an addition formula:

$$\frac{d}{dz}e^z = e^z, \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

It is a homomorphism of the additive group $\mathbb{C}$ of complex numbers onto the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers, with kernel

$$\ker \exp = 2\pi i\mathbb{Z}.$$

Hence it yields an isomorphism between the quotient additive group $\mathbb{C}/2\pi i\mathbb{Z}$ and the multiplicative group $\mathbb{C}^\times$.

The group $\mathbb{C}^\times$ is the group of complex points of the multiplicative group $\mathbb{G}_m$; $z \mapsto e^z$ is the exponential function of the multiplicative group $\mathbb{G}_m$. We shall replace this algebraic group by an elliptic curve. We could replace it also by other commutative algebraic groups. As a first example, the exponential function of the additive group $\mathbb{G}_a$ is

$$\begin{aligned} \mathbb{C} &\to \mathbb{C}, \\ z &\mapsto z. \end{aligned}$$

More general examples are commutative linear algebraic groups; over an algebraically closed field, these are nothing else than products of several copies of the additive and multiplicative group. Further examples of algebraic groups are abelian varieties. In full generality, algebraic groups are extensions of abelian varieties by commutative linear algebraic groups. See, for instance, [120, 158, 233].

2 Elliptic Curves and Elliptic Functions

Among many references for this section are the books by S. Lang [127], K. Chandrasekharan [43], and J. Silverman [212, 213]. See also the book by M. Hindry and J. Silverman [99].

2.1 Basic Concepts

An elliptic curve may be defined as

- $y^2 = C(x)$ for a square-free cubic polynomial $C(x)$,
- a connected compact Lie group of dimension 1,
- a complex torus $\mathbb{C}/\Omega$, where Ω is a lattice in $\mathbb{C}$,
- a Riemann surface of genus 1,

– a nonsingular cubic in $\mathbb{P}_2(\mathbb{C})$ (together with a point at infinity),
– an algebraic group of dimension 1, with underlying projective algebraic variety.

We shall use the Weierstrass form

$$E = \{(t : x : y)\ ;\ y^2 t = 4x^3 - g_2 x t^2 - g_3 t^3\} \subset \mathbb{P}_2.$$

Here g_2 and g_3 are complex numbers, with the only assumption $g_2^3 \neq 27 g_3^2$, which means that the discriminant of the polynomial $4X^3 - g_2 X - g_3$ does not vanish.

An analytic parametrization of the complex points $E(\mathbb{C})$ of E is given by means of *the Weierstrass elliptic function* $\wp$, which satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{1}$$

It has a double pole at the origin with principal part $1/z^2$ and also satisfies an addition formula

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4} \cdot \left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right)^2. \tag{2}$$

The exponential map of the Lie group $E(\mathbb{C})$ is

$$\begin{aligned} \exp_E : \mathbb{C} &\to E(\mathbb{C}), \\ z &\mapsto (1 : \wp(z) : \wp'(z)). \end{aligned}$$

The kernel of this map is a *lattice* in $\mathbb{C}$ (that is, a discrete rank-2 subgroup),

$$\Omega = \ker \exp_E = \{\omega \in \mathbb{C}\ ;\ \wp(z + \omega) = \wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Hence $\exp_E$ induces an isomorphism between the quotient additive group $\mathbb{C}/\Omega$ and $E(\mathbb{C})$ with the law given by (2). The elements of Ω are the *periods* of $\wp$. A pair (ω_1, ω_2) of fundamental periods is given by (cf. [244] Section 20.32, Example 1)

$$\omega_i = 2 \int_{e_i}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} \qquad (i = 1, 2),$$

where

$$4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3).$$

Indeed, since $\wp'$ is periodic and odd, it vanishes at $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$; hence the values of $\wp$ at these points are the three distinct complex numbers e_1, e_2, and e_3 (recall that the discriminant of $4x^3 - g_2 x - g_3$ is not 0).

Conversely, given a lattice Ω, there is a unique Weierstrass elliptic function $\wp_\Omega$ whose period lattice is Ω (see Section 2.5). We denote its invariants in the differential equation (1) by $g_2(\Omega)$ and $g_3(\Omega)$.

We shall be interested mainly (but not only) in elliptic curves that are defined over the field of algebraic numbers: they have a Weierstrass equation with algebraic

g_2 and g_3. However, we shall also use the Weierstrass elliptic function associated with the lattice $\lambda\Omega$, where $\lambda \in \mathbb{C}^\times$ may be transcendental; the relations are

$$\wp_{\lambda\Omega}(\lambda z) = \lambda^{-2}\wp_\Omega(z), \qquad g_2(\lambda\Omega) = \lambda^{-4}g_2(\Omega), \qquad g_3(\lambda\Omega) = \lambda^{-6}g_3(\Omega). \quad (3)$$

The lattice $\Omega = \mathbb{Z} + \mathbb{Z}\tau$, where τ is a complex number with positive imaginary part, satisfies

$$g_2(\mathbb{Z} + \mathbb{Z}\tau) = 60G_2(\tau) \quad \text{and} \quad g_3(\mathbb{Z} + \mathbb{Z}\tau) = 140G_3(\tau),$$

where $G_k(\tau)$ (with $k \geq 2$) are the Eisenstein series (see, for instance, [48] Section 3.2, [208] Section 7.2.3, [116] Section 3.2 or [212] Section 6.3 — the normalization in [254] p. 240 is different):

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m + n\tau)^{-2k}. \quad (4)$$

2.2 Morphisms between Elliptic Curves. The Modular Invariant

If Ω and Ω' are two lattices in $\mathbb{C}$ and if $f : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ is an analytic homomorphism, then the map $\mathbb{C} \to \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ factors through a homothecy $\mathbb{C} \to \mathbb{C}$ given by some $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Omega & \xrightarrow[f]{} & \mathbb{C}/\Omega' \end{array}$$

If $f \neq 0$, then $\lambda \in \mathbb{C}^\times$ and f is surjective.

Conversely, if there exists $\lambda \in \mathbb{C}$ such that $\lambda\Omega \subset \Omega'$, then $f_\lambda(x + \Omega) = \lambda x + \Omega'$ defines an analytic surjective homomorphism $f_\lambda : \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$. In this case $\lambda\Omega$ is a subgroup of finite index in Ω'; hence the kernel of f_λ is finite and there exists $\mu \in \mathbb{C}^\times$ with $\mu\Omega' \subset \Omega$: the two elliptic curves $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ are *isogenous*.

If Ω and Ω^* are two lattices, $\wp$ and $\wp^*$ the associated Weierstrass elliptic functions, and g_2, g_3 the invariants of $\wp$, the following statements are equivalent:

(i) There is a 2×2 matrix with rational coefficients that maps a basis of Ω to a basis of Ω^*.
(ii) There exists $\lambda \in \mathbb{Q}^\times$ such that $\lambda\Omega \subset \Omega^*$.
(iii) There exists $\lambda \in \mathbb{Z} \setminus \{0\}$ such that $\lambda\Omega \subset \Omega^*$.
(iv) The two functions $\wp$ and $\wp^*$ are algebraically dependent over the field $\mathbb{Q}(g_2, g_3)$.
(v) The two functions $\wp$ and $\wp^*$ are algebraically dependent over $\mathbb{C}$.

The map f_λ is an isomorphism if and only if $\lambda\Omega = \Omega'$.

The number

$$j = \frac{1728g_2^3}{g_2^3 - 27g_3^2}$$

is the *modular invariant* of the elliptic curve E. Two elliptic curves over $\mathbb{C}$ are isomorphic if and only if they have the same modular invariant.

Set $\tau = \omega_2/\omega_1$, $q = e^{2\pi i\tau}$ and $J(e^{2\pi i\tau}) = j(\tau)$. Then

$$J(q) = q^{-1}\left(1 + 240\sum_{m=1}^{\infty} m^3 \frac{q^m}{1-q^m}\right)^3 \prod_{n=1}^{\infty}(1-q^n)^{-24}$$

$$= \frac{1}{q} + 744 + 196884\,q + 21493760\,q^2 + \cdots .$$

See [142] Section 4.12 or [208] Sections 7.3.3 and 7.4.

2.3 Endomorphisms of an Elliptic Curve; Complex Multiplication

Let Ω be a lattice in $\mathbb{C}$. The set of analytic endomorphisms of $\mathbb{C}/\Omega$ is the subring

$$\mathrm{End}(\mathbb{C}/\Omega) = \{f_\lambda \ ; \ \lambda \in \mathbb{C} \ \text{ with } \ \lambda\Omega \subset \Omega\}$$

of $\mathbb{C}$. We also call it the ring of endomorphisms of the associated elliptic curve, or of the corresponding Weierstrass $\wp$ function, and we identify it with the subring

$$\{\lambda \in \mathbb{C} \ ; \ \lambda\Omega \subset \Omega\}$$

of $\mathbb{C}$. The *field of endomorphisms* is the quotient field $\mathrm{End}(\mathbb{C}/\Omega) \otimes_{\mathbb{Z}} \mathbb{Q}$ of this ring.

If $\lambda \in \mathbb{C}$ satisfies $\lambda\Omega \subset \Omega$, then λ is either a rational integer or an algebraic integer in an imaginary quadratic field. For such a λ, $\wp_\Omega(\lambda z)$ is a rational function of $\wp_\Omega(z)$; the degree of the numerator is λ^2 if $\lambda \in \mathbb{Z}$ and $N(\lambda)$ otherwise (here, N is the norm of the imaginary quadratic field); the degree of the denominator is $\lambda^2 - 1$ if $\lambda \in \mathbb{Z}$ and $N(\lambda) - 1$ otherwise.

Let E be the elliptic curve attached to the Weierstrass $\wp$ function. The ring of endomorphisms $\mathrm{End}(E)$ of E is either $\mathbb{Z}$ or an order in an imaginary quadratic field k. The latter case arises if and only if the quotient $\tau = \omega_2/\omega_1$ of a pair of fundamental periods is a quadratic number; in this case the field of endomorphisms of E is $k = \mathbb{Q}(\tau)$ and the curve E has *complex multiplication*; this is the so-called *CM case*. This means also that the two functions $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent. In this case, the value $j(\tau)$ of the modular invariant j is an algebraic integer whose degree is the class number of the quadratic field $k = \mathbb{Q}(\tau)$.

Remark 14. From Theorem 7 one deduces the transcendence of the number

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,250\,072\,59\ldots .$$

If we set

$$\tau = \frac{1 + i\sqrt{163}}{2}, \quad q = e^{2\pi i\tau} = -e^{-\pi\sqrt{163}},$$

then the class number of the imaginary quadratic field $\mathbb{Q}(\tau)$ is 1, we have $j(\tau) = -(640\,320)^3$, and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Also ([57] Section 2.4)

$$\left(e^{\pi\sqrt{163}} - 744\right)^{1/3} = 640\,319.999\,999\,999\,999\,999\,999\,999\,999\,390\,31\ldots.$$

Let $\wp$ be a Weierstrass elliptic function with field of endomorphisms k. Hence $k = \mathbb{Q}$ if the associated elliptic curve has no complex multiplication, while in the other case k is an imaginary quadratic field, namely $k = \mathbb{Q}(\tau)$, where τ is the quotient of two linearly independent periods of $\wp$. Let $u_1, \ldots, u_d$ be nonzero complex numbers. Then the functions $\wp(u_1 z), \ldots, \wp(u_d z)$ are algebraically independent (over $\mathbb{C}$ or over $\mathbb{Q}(g_2, g_3)$; this is equivalent) if and only if the numbers $u_1, \ldots, u_d$ are linearly independent over k. This generalizes the fact that $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent if and only if the elliptic curve has complex multiplication. Much more general and deeper results of algebraic independence of functions (exponential and elliptic functions, zeta functions, ...) were proved by W.D. Brownawell and K.K. Kubota [37].

If $\wp$ is a Weierstrass elliptic function with algebraic invariants g_2 and g_3, if E is the associated elliptic curve, and if k denotes its field of endomorphisms, then the set

$$\mathcal{L}_E = \Omega \cup \left\{u \in \mathbb{C} \setminus \Omega\ ;\ \wp(u) \in \overline{\mathbb{Q}}\right\}$$

is a k-vector subspace of $\mathbb{C}$: this is the set of *elliptic logarithms of algebraic points on E*. It plays a role with respect to E similar to the role of $\mathcal{L}$ for the multiplicative group $\mathbb{G}_m$.

Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number $h(-d) = h$. There are h nonisomorphic elliptic curves $E_1, \ldots, E_h$ with ring of endomorphisms the ring of integers of k. The numbers $j(E_i)$ are conjugate algebraic integers of degree h; each of them generates the Hilbert class field H of k (maximal unramified abelian extension of k). The Galois group of H/k is isomorphic to the ideal class group of k.

Since the group of roots of units of an imaginary quadratic field is $\{-1, +1\}$ except for $\mathbb{Q}(i)$ and $\mathbb{Q}(\varrho)$, where $\varrho = e^{2\pi i/3}$, it follows that there are exactly two elliptic curves over $\mathbb{Q}$ (up to isomorphism) having an automorphism group bigger than $\{-1, +1\}$. They correspond to Weierstrass elliptic functions $\wp$ for which there exists a complex number $\lambda \neq \pm 1$ with $\lambda^2 \wp(\lambda z) = \wp(z)$.

The first one has $g_3 = 0$ and $j = 1728$. An explicit value for a pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2$$

follows from computations by Legendre using Gauss's lemniscate function ([244] Section 22.8) and yields (see [4], as well as Appendix 1 of [241])

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} \quad \text{and} \quad \omega_2 = i\omega_1. \tag{5}$$

The lattice $\mathbb{Z}[i]$ has $g_2 = 4\omega_1^4$. Thus

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6\cdot 3\cdot 5\cdot \pi^2}.$$

The second one has $g_2 = 0$ and $j = 0$. Again from computations by Legendre ([244] Section 22.81 II) one deduces that a pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4t^3$$

is (see once more [4] and Appendix 1 of [241])

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{3}B(1/6,1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} \quad \text{and} \quad \omega_2 = \varrho\omega_1. \tag{6}$$

The lattice $\mathbb{Z}[\varrho]$ has $g_3 = 4\omega_1^6$. Thus

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\cdot 5\cdot 7\cdot \pi^6}.$$

These two examples involve special values of Euler's gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^z\cdot\frac{dt}{t} = e^{-\gamma z}z^{-1}\prod_{n=1}^\infty\left(1+\frac{z}{n}\right)^{-1}e^{z/n}, \tag{7}$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) = 0.577\,215\,664\,901\,532\,860\,606\,512\,09\ldots$$

is Euler's constant (Section 12.1 in [244]), while Euler's beta function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

More generally, the formula of Chowla and Selberg (1966) [44] (see also [9, 95, 96, 115, 117, 234] for related results) expresses periods of elliptic curves with complex multiplication as products of gamma values: *if k is an imaginary quadratic field and $\mathcal{O}$ an order in k, if E is an elliptic curve with complex multiplication by $\mathcal{O}$, then the corresponding lattice Ω determines a vector space $\Omega\otimes_{\mathbb{Z}}\mathbb{Q}$ that is invariant under the action of k and thus has the form $k\cdot\omega$ for some $\omega\in\mathbb{C}^\times$ defined up to elements in $k^\times$. In particular, if $\mathcal{O}$ is the ring of integers $\mathbb{Z}_k$ of k, then*

$$\omega = \alpha\sqrt{\pi}\prod_{\substack{0<a<d\\(a,d)=1}}\Gamma(a/d)^{w\epsilon(a)/4h},$$

where α is a nonzero algebraic number, w is the number of roots of unity in k, h is the class number of k, and ϵ is the Dirichlet character modulo the discriminant d of k.

2.4 Standard Relations among Gamma Values

Euler's gamma function satisfies the following relations ([244] Chapter XII):
(Translation)

$$\Gamma(z+1) = z\Gamma(z);$$

(Reflection)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)};$$

(Multiplication) For any positive integer n,

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-nz+(1/2)} \Gamma(nz).$$

D. Rohrlich conjectured that any multiplicative relation among gamma values is a consequence of these standard relations, while S. Lang was more optimistic (see [125], [128] I Chapter 2 Appendix p. 66 and [9] Chapter 24):

Conjecture 15 (D. Rohrlich). Any multiplicative relation

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ is a consequence of the standard relations.

Conjecture 16 (S. Lang). Any algebraic dependence relation with algebraic coefficients among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ is in the ideal generated by the standard relations.

2.5 Quasiperiods of Elliptic Curves and Elliptic Integrals of the Second Kind

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The *Weierstrass canonical product* attached to this lattice is the entire function σ_Ω defined by ([244] Section 20.42)

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It has a simple zero at any point of Ω.

Hence the Weierstrass sigma function plays, for the lattice Ω, the role that is played by the function

$$z \prod_{n \geq 1} \left(1 - \frac{z}{n}\right) e^{z/n} = -e^{\gamma z}\Gamma(-z)^{-1}$$

for the set of positive integers $\mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$ (see the infinite product (7) for Euler's gamma function), and also by the function

$$\pi^{-1}\sin(\pi z) = z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{n}\right) e^{z/n}$$

for the set $\mathbb{Z}$ of rational integers ([43] Section 4.2).

The Weierstrass sigma function σ associated with a lattice in $\mathbb{C}$ is an entire function of *order* 2:

$$\limsup_{r\to\infty} \frac{1}{\log r} \cdot \log\log \sup_{|z|=r} |\sigma(z)| = 2;$$

the product $\sigma^2\wp$ is also an entire function of order 2 (this can be checked using infinite products, but it is easier to use the quasiperiodicity of σ, see formula (8) below).

The logarithmic derivative of the sigma function is *the Weierstrass zeta function* $\zeta = \sigma'/\sigma$ whose Laurent expansion at the origin is ([127] Section 18.3, [208] Section 7.2.3 and [212] Section 6.3, Theorem 3.5)

$$\zeta(z) = \frac{1}{z} - \sum_{k\geq 2} s_k z^{2k-1},$$

where for $k \in \mathbb{Z}$, $k \geq 2$,

$$s_k = s_k(\Omega) = \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \omega^{-2k} = \omega_1^{-2k} G_k(\tau)$$

(recall (4); also $\tau = \omega_2/\omega_1$).

The derivative of ζ is $-\wp$. From

$$\wp'' = 6\wp^2 - (g_2/2)$$

one deduces that $s_k(\Omega)$ is a homogeneous polynomial in $\mathbb{Q}[g_2, g_3]$ of weight $2k$ for the graduation of $\mathbb{Q}[g_2, g_3]$ determined by assigning to g_2 the degree 4 and to g_3 the degree 6.

As a side remark, we notice that for any $u \in \mathbb{C} \setminus \Omega$ we have

$$\mathbb{Q}(g_2, g_3) \subset \mathbb{Q}\big(\wp(u), \wp'(u), \wp''(u)\big).$$

Since its derivative is periodic, the function ζ is *quasiperiodic*: for each $\omega \in \Omega$ there is a complex number $\eta = \eta(\omega)$ such that

$$\zeta(z + \omega) = \zeta(z) + \eta.$$

These numbers η are the *quasiperiods* of the elliptic curve. If (ω_1, ω_2) is a pair of fundamental periods and if we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$, then, for $(a, b) \in \mathbb{Z}^2$,

$$\eta(a\omega_1 + b\omega_2) = a\eta_1 + b\eta_2.$$

Returning to the sigma function, one deduces that

$$\sigma(z+\omega_i) = -\sigma(z)\exp(\eta_i(z+(\omega_i/2))) \qquad (i=1,2). \tag{8}$$

The zeta function also satisfies an addition formula:

$$\zeta(z_1+z_2) = \zeta(z_1)+\zeta(z_2)+\frac{1}{2}\cdot\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}.$$

The Legendre relation relating the periods and the quasiperiods

$$\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i,$$

when ω_2/ω_1 has positive imaginary part, can be obtained by integrating $\zeta(z)$ along the boundary of a fundamental parallelogram ([43] Section 4.2, [124] Section 1.6, [244] Section 20.411).

In the case of complex multiplication, if τ is the quotient of a pair of fundamental periods of $\wp$, then the function $\zeta(\tau z)$ is algebraic over the field $\mathbb{Q}(g_2, g_3, z, \wp(z), \zeta(z))$.

Examples ([4, 241]). For the curve $y^2t = 4x^3 - 4xt^2$, the quasiperiods attached to the above-mentioned pair of fundamental periods (5) are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1; \tag{9}$$

it follows that the fields $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ and $\mathbb{Q}\big(\pi, \Gamma(1/4)\big)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree. For the curve $y^2t = 4x^3 - 4t^3$ with periods (6), they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1. \tag{10}$$

In this case the fields $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)$ and $\mathbb{Q}\big(\pi, \Gamma(1/3)\big)$ have the same algebraic closure over $\mathbb{Q}$, hence the same transcendence degree.

2.6 Elliptic Integrals

Let

$$\mathcal{E} = \{(t:x:y) \in \mathbb{P}_2;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\}$$

be an elliptic curve. The field of rational (meromorphic) functions on $\mathcal{E}$ over $\mathbb{C}$ is $\mathbb{C}(\mathcal{E}) = \mathbb{C}(\wp, \wp') = \mathbb{C}(x, y)$, where x and y are related by the cubic equation $y^2 = 4x^3 - g_2x - g_3$. Under the isomorphism $\mathbb{C}/\Omega \to \mathcal{E}(\mathbb{C})$ given by $(1:\wp:\wp')$, the differential form dz is mapped to dx/y. The holomorphic differential forms on $\mathbb{C}/\Omega$ are $\lambda\, dz$ with $\lambda \in \mathbb{C}$.

The differential form $d\zeta = \zeta'/\zeta$ is mapped to $-x\,dx/y$. The differential forms of the second kind on $\mathcal{E}(\mathbb{C})$ are $a\,dz + b\,d\zeta + d\chi$, where a and b are complex numbers and $\chi \in \mathbb{C}(x, y)$ is a meromorphic function on $\mathcal{E}$.

Assume that the elliptic curve $\mathcal{E}$ is defined over $\overline{\mathbb{Q}}$: the invariants g_2 and g_3 are algebraic. We shall be interested in differential forms defined over $\overline{\mathbb{Q}}$. Those of the second kind are $a\,dz + b\,d\zeta + d\chi$, where a and b are algebraic numbers and $\chi \in \overline{\mathbb{Q}}(x, y)$.

An elliptic integral (see [244] Section 22.7; see also [43] Section 1.4 and [212] Section 6.1) is an integral

$$\int R(x, y)dx,$$

where R is a rational function of x and y, while y^2 is a polynomial in x of degree 3 or 4 without multiple roots, with the proviso that the integral cannot be integrated by means of elementary functions. One may transform this integral as follows: one reduces it to an integral of $dx/\sqrt{P(x)}$, where P is a polynomial of third or fourth degree; in case P has degree 4, one replaces it with a degree-3 polynomial by sending one root to infinity; finally, one reduces it to a Weierstrass equation by means of a birational transformation. The value of the integral is not modified.

For transcendence purposes, if the initial differential form is defined over $\overline{\mathbb{Q}}$, then all these transformations involve only algebraic numbers.

We refer to Section 22.7 of [244] for the definition of elliptic integrals of the first, second, and third kinds.

3 Transcendence Results of Numbers Related to Elliptic Functions

3.1 Elliptic Analogue of Lindemann's Theorem on the Transcendence of π and the Hermite–Lindemann Theorem on the Transcendence of $\log \alpha$

The first transcendence result on periods of elliptic functions was proved by C.L. Siegel [210] as early as 1932.

Theorem 17 (Siegel, 1932). *Let $\wp$ be a Weierstrass elliptic function with period lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then at least one of the two numbers ω_1, ω_2 is transcendental.*

One main feature of Siegel's proof is that he used Dirichlet's box principle (the so-called Thue–Siegel lemma, which is included in his 1929 paper) to construct an auxiliary function. This idea turned out to be of fundamental importance for the solution of Hilbert's seventh problem by Gel'fond and Schneider two years later.

In the case of complex multiplication, it follows from Theorem 17 that *any nonzero period of $\wp$ is transcendental.*

From formulas (5) and (6) it follows as a consequence of Siegel's 1932 result [210] that both numbers $\Gamma(1/4)^4/\pi$ and $\Gamma(1/3)^3/\pi$ are transcendental.

Other consequences of Siegel's result concern the transcendence of the length of an arc of an ellipse [207, 211]:

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,dx$$

for algebraic a and b, as well as the transcendence of an arc of the lemniscate $(x^2+y^2)^2 = 2a^2(x^2-y^2)$ with a algebraic.

A further example of application of Siegel's theorem [211] is the transcendence of values of hypergeometric series related to elliptic integrals

$$K(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} = \frac{\pi}{2}\cdot {}_2F_1(1/2,\ 1/2\ ;\ 1 \mid z^2),$$

where ${}_2F_1$ denotes the Gauss hypergeometric series

$${}_2F_1\left(a,\ b\ ;\ c \mid z\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Further results on this topic were obtained by Th. Schneider [203] in 1934 and in joint work by K. Mahler and J. Popken [190] in 1935 using Siegel's method. These results were superseded by Th. Schneider's fundamental memoir [204] in 1936 in which he proved a number of definitive results on the subject, including the following:

Theorem 18 (Schneider, 1936). *Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then for any nonzero period ω of $\wp$, the numbers ω and $\eta(\omega)$ are transcendental.*

It follows from Theorem 18 that any nonzero period of an elliptic integral of the first or second kind is transcendental:

Corollary 19. *Let $\mathcal{E}$ be an elliptic curve over $\overline{\mathbb{Q}}$, p_1 and p_2 two algebraic points on $\mathcal{E}(\overline{\mathbb{Q}})$, w a differential form of the first or second kind on $\mathcal{E}$ that is defined over $\overline{\mathbb{Q}}$, holomorphic at p_1 and p_2, and is not the differential of a rational function. Let γ be a path on $\mathcal{E}$ from p_1 to p_2. In case $p_1 = p_2$ one assumes that γ is not homologous to* 0. *Then the number*

$$\int_\gamma w$$

is transcendental.

Examples. Using Corollary 19 and formulas (9) and (10), one deduces that the numbers

$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$

are transcendental.

The main results of Schneider's 1936 paper [204] are as follows (see also [207]):

Theorem 20 (Schneider, 1936).

1. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let β be a nonzero algebraic number. Then β is not a pole of $\wp$ and $\wp(\beta)$ is transcendental. More generally, if a and b are two algebraic numbers with $(a, b) \neq (0, 0)$, then for any $u \in \mathbb{C} \setminus \Omega$ at least one of the two numbers $\wp(u)$, $au + b\zeta(u)$ is transcendental.*
2. *Let $\wp$ and $\wp^*$ be two algebraically independent elliptic functions with algebraic invariants g_2, g_3, g_2^*, g_3^*. If $t \in \mathbb{C}$ is not a pole of $\wp$ or of $\wp^*$, then at least one of the two numbers $\wp(t)$ and $\wp^*(t)$ is transcendental.*
3. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Then for any $t \in \mathbb{C} \setminus \Omega$, at least one of the two numbers $\wp(t)$, e^t is transcendental.*

It follows from Theorem 20.2 that the quotient of an elliptic integral of the first kind (between algebraic points) by a nonzero period is either in the field of endomorphisms (hence a rational number, or a quadratic number in the field of complex multiplication), or a transcendental number.

Here is another important consequence of Theorem 20.2.

Corollary 21 (Schneider, 1936). *Let $\tau \in \mathcal{H}$ be a complex number in the upper half-plane $\Im m(\tau) > 0$ such that $j(\tau)$ is algebraic. Then τ is algebraic if and only if τ is imaginary quadratic.*

In this connection we quote Schneider's second problem in [207], which is still open (see Wakabayashi's papers [226, 227, 228]):

Open Problem. Prove Corollary 21 without using elliptic functions.

Sketch of proof of Corollary 21 as a consequence of part 2 of Theorem 20. Assume that both $\tau \in \mathcal{H}$ and $j(\tau)$ are algebraic. There exists an elliptic function with algebraic invariants g_2, g_3 and periods ω_1, ω_2 such that

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}.$$

Set $\wp^*(z) = \tau^2 \wp(\tau z)$. Then $\wp^*$ is a Weierstrass function with algebraic invariants g_2^*, g_3^*. For $u = \omega_1/2$ the two numbers $\wp(u)$ and $\wp^*(u)$ are algebraic. Hence the two functions $\wp(z)$ and $\wp^*(z)$ are algebraically dependent. It follows that the corresponding elliptic curve has nontrivial endomorphisms; therefore τ is quadratic. □

A quantitative refinement of Schneider's theorem on the transcendence of $j(\tau)$ given by A. Faisant and G. Philibert in 1984 [74] became useful 10 years later in connection with Nesterenko's result (see Section 5). See also [75].

We will not review the results related to abelian integrals, but only quote the first result on this topic, which involves the Jacobian of a Fermat curve: in 1941 Schneider [205] proved that *for a and b in $\mathbb{Q}$ with a, b and $a + b$ not in $\mathbb{Z}$, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. We notice that in his 1932 paper [210], C.L. Siegel had already announced partial results on the values of the Euler gamma function (see also [19]).

Schneider's above-mentioned results deal with elliptic (and abelian) integrals of the first or second kind. His method can be extended to deal with elliptic (and abelian) integrals of the third kind (this is Schneider's third problem in [207]).

As pointed out by J.-P. Serre in 1979 [233], it follows from the quasiperiodicity of the Weierstrass sigma function (8) that the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)} e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z) e^{\eta_i u - \omega_i \zeta(u)}.$$

Theorem 22. *Let u_1 and u_2 be two nonzero complex numbers. Assume that g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β are algebraic and $\mathbb{Z}u_1 \cap \Omega = \{0\}$. Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\left(\beta - \zeta(u_1)\right)u_2}$$

is transcendental.

From the next corollary, one can deduce that nonzero periods of elliptic integrals of the third kind are transcendental (see [232]).

Corollary 23. *For any nonzero period ω and for any $u \in \mathbb{C} \setminus \Omega$, the number $e^{\omega\zeta(u) - \eta u + \beta\omega}$ is transcendental.*

Further results on elliptic integrals are due to M. Laurent [132]. See also his papers [134, 135, 136, 137].

Ya. M. Kholyavka wrote several papers devoted to the approximation of transcendental numbers related to elliptic functions [106, 107, 108, 109, 110, 111, 112, 113, 114].

Quantitative estimates (measures of transcendence) related to the results of this section were derived by N.I. Fel'dman [76, 77, 78, 79, 80]. See also the papers by S. Lang [119], N.D. Nagaev [165], N. Hirata [101], E. Reyssat [195, 196, 198, 199], M. Laurent [133], R. Tubbs [219], G. Diaz [64], N. Saradha [202], P. Grinspan [94].

3.2 Elliptic Analogues of the Six Exponentials Theorem

Elliptic analogs of the six exponentials theorem (Theorem 10) were considered by S. Lang [120] and K. Ramachandra [191] in the 1960s.

Let d_1, d_2 be nonnegative integers and m a positive integer, let $x_1, \ldots, x_{d_1}$ be complex numbers that are linearly independent over $\mathbb{Q}$, let $y_1, \ldots, y_m$ be complex numbers that are linearly independent over $\mathbb{Q}$, and let $u_1, \ldots, u_{d_2}$ be nonzero

complex numbers. We consider Weierstrass elliptic functions $\wp_1, \ldots, \wp_{d_2}$ and we denote by K_0 the field generated over $\mathbb{Q}$ by their invariants $g_{2,k}$ and $g_{3,k}$ ($1 \leq k \leq d_2$). We assume that the d_2 functions $\wp_1(u_1 z), \ldots, \wp_{d_2}(u_{d_2} z)$ are algebraically independent. We denote by K_1 the field generated over K_0 by the numbers $\exp(x_i y_j)$ ($1 \leq i \leq d_1$, $1 \leq j \leq m$) together with the numbers $\wp_k(u_k y_j)$ ($1 \leq k \leq d_2$, $1 \leq j \leq m$). Next, define

$$K_2 = K_1(y_1, \ldots, y_m), \qquad K_3 = K_1(x_1, \ldots, x_{d_1},\ u_1, \ldots, u_{d_2}),$$

and let K_4 be the compositum of K_2 and K_3:

$$K_4 = K_1(y_1, \ldots, y_m,\ x_1, \ldots, x_{d_1},\ u_1, \ldots, u_{d_2}).$$

The theorems of Hermite–Lindemann (Theorem 3), Gel'fond–Schneider (Theorem 7), the six exponentials theorem, and their elliptic analogues due to Schneider, Lang, and Ramachandra can be stated as follows.

Any one of the four assumptions below will imply $d_1 + d_2 > 0$; the case in which d_1 (respectively d_2) vanishes means that one considers only elliptic (respectively exponential) functions.

Theorem 24.
1. *Assume $(d_1 + d_2)m > m + d_1 + 2d_2$. Then the field K_1 has transcendence degree ≥ 1 over $\mathbb{Q}$.*
2. *Assume either $d_1 \geq 1$ and $m \geq 2$, or $d_2 \geq 1$ and $m \geq 3$. Then K_2 has transcendence degree ≥ 1 over $\mathbb{Q}$.*
3. *Assume $d_1 + d_2 \geq 2$. Then K_3 has transcendence degree ≥ 1 over $\mathbb{Q}$.*
4. *Assume $d_1 + d_2 \geq 1$. Then K_4 has transcendence degree ≥ 1 over $\mathbb{Q}$.*

Parts 3 and 4 of Theorem 24 are consequences of the Schneider–Lang criterion [120], which deals with meromorphic functions satisfying differential equations, while parts 1 and 2 follow from a criterion that involves no differential equations. Such criteria were given by Schneider [206, 207], Lang [120], and Ramachandra [191] (see also [228] and [229]).

Theorem 24 also includes Theorem 20 apart from the case $b \neq 0$ in part 1 of that statement. However, there are extensions of Theorem 24 that include results on Weierstrass zeta functions (and also on Weierstrass sigma functions in connection with elliptic integrals of the third kind). See [132, 134, 135, 136, 137, 199, 232, 233].

Here is a corollary of part 1 of Theorem 24 (take $d_1 = 0$, $d_2 = 3$, $\wp_1 = \wp_2 = \wp_3 = \wp$, $m = 4$, $y_1 = 1$, $y_2 \in \mathrm{End}(E) \setminus \mathbb{Q}$, $y_3 = v_1/u_1$, $y_4 = y_2 y_3$; there is an alternative proof with $d_2 = 2$ and $m = 6$).

Corollary 25. *Let E be an elliptic curve with algebraic invariants g_2, g_3. Assume that E has complex multiplication. Let*

$$M = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

be a 2×3 *matrix whose entries are elliptic logarithms of algebraic numbers, i.e.,* u_i *and* v_i $(i = 1, 2, 3)$ *are in* $\mathcal{L}_E$. *Assume that the three columns are linearly independent over* $\mathrm{End}(E)$ *and the two rows are also linearly independent over* $\mathrm{End}(E)$. *Then the matrix* M *has rank* 2.

In the non-CM case, one deduces from Theorem 24 a similar (but weaker) statement according to which such matrices $\big(u_{ij}\big)$ (where $\wp(u_{ij})$ are algebraic numbers) have rank ≥ 2 if they have size 2×5 (taking $d_1 = 0$, $d_2 = 2$, and $m = 5$) or 3×4 (taking $d_1 = 0$ and either $d_2 = 3$, $m = 4$ or $d_2 = 4$ and $m = 3$) instead of 2×3.

Lower bounds better than 2 for the rank of matrices of larger sizes are known, but we will not discuss this question here. We just mention the fact that higher-dimensional considerations are relevant to a problem discussed by B. Mazur on the density of rational points on varieties [240].

4 Linear Independence of Numbers Related to Elliptic Functions

From Schneider's theorem (Theorem 20) part 1, one deduces the linear independence over the field of algebraic numbers of the three numbers 1, ω, and η, when ω is a nonzero period of a Weierstrass elliptic function (with algebraic invariants g_2 and g_3) and $\eta = \eta(\omega)$ is the associated quasiperiod of the corresponding Weierstrass zeta function. However, the Gel'fond–Schneider method in one variable alone does not yield strong results of linear independence. Baker's method is better suited for this purpose.

4.1 Linear Independence of Periods and Quasiperiods

Baker's method of proof for his theorem (Theorem 9) on linear independence of logarithms of algebraic numbers was used as early as 1969 and 1970 by A. Baker himself [12, 10] when he proved the transcendence of linear combinations with algebraic coefficients of the numbers ω_1, ω_2, η_1, and η_2 associated with an elliptic curve having algebraic invariants g_2 and g_3. His method is effective: it provides quantitative Diophantine estimates [11].

In 1971, J. Coates [52] proved the transcendence of linear combinations with algebraic coefficients of $\omega_1, \omega_2, \eta_1, \eta_2$, and $2\pi i$. Moreover, he proved in [51, 53, 54, 55] that in the non-CM case, the three numbers ω_1, ω_2, and $2\pi i$ are $\overline{\mathbb{Q}}$-linearly independent. Further results including usual logarithms of algebraic numbers are due to T. Harase in 1974 and 1976 [97, 98].

The final result on the question of linear dependence of periods and quasiperiods for a single elliptic function was given by D.W. Masser in 1975 [143, 144].

Theorem 26 (Masser, 1975). *Let* $\wp$ *be a Weierstrass elliptic function with algebraic invariants* g_2 *and* g_3, *denote by* ζ *the corresponding Weierstrass zeta function, let* ω_1, ω_2 *be a basis of the period lattice of* $\wp$, *and let* η_1, η_2 *be the associated quasiperiods of* ζ. *Then the six numbers* 1, ω_1, ω_2, η_1, η_2, *and* $2\pi i$ *span a* $\overline{\mathbb{Q}}$*-vector space of dimension* 6 *in the non-CM case,* 4 *in the CM case:*

$$\dim_{\overline{\mathbb{Q}}}\{1, \omega_1, \omega_2, \eta_1, \eta_2, 2\pi i\} = 2 + 2\dim_{\overline{\mathbb{Q}}}\{\omega_1, \omega_2\}.$$

The fact that the dimension is 4 in the CM case means that there are two independent linear relations among these six numbers. One of them is $\omega_2 = \tau\omega_1$ with $\tau \in \overline{\mathbb{Q}}$. The second one (see [144]; see also [37]) can be written

$$C^2\tau\eta_2 - AC\eta_1 + \gamma\omega_1 = 0,$$

where $A + BX + CX^2$ is the minimal polynomial of τ over $\mathbb{Z}$ and γ is an element in $\mathbb{Q}(g_2, g_3, \tau)$.

In [144], D.W. Masser also produced quantitative estimates (measures of linear independence). In 1976, R. Franklin and D.W. Masser [85, 151] obtained an extension involving a logarithm of an algebraic number.

Further results can be found in papers by P. Bundschuh [40], S. Lang's surveys [121, 122], D.W. Masser [152, 154], M. Anderson [5], and in the joint paper [6] by M. Anderson and D.W. Masser.

4.2 Elliptic Analogue of Baker's Theorem

The elliptic analogue of Baker's theorem on linear independence of logarithms was proved by D.W. Masser in 1974 [143, 144] in the CM case.

His proof also yields quantitative estimates (measures of linear independence of elliptic logarithms of algebraic points on an elliptic curve). Such estimates have a number of applications: this was shown by A.O. Gel'fond for usual logarithms of algebraic numbers [89], and further consequences of such lower bounds in the case of elliptic curves for solving Diophantine equations (integer points on elliptic curves) were derived by S. Lang [126].

Lower bounds for linear combinations of elliptic logarithms in the CM case were obtained by several mathematicians including J. Coates [52], D.W. Masser [145, 149, 150], J. Coates and S. Lang [56], M. Anderson [5]. The work of Yu Kunrui [253] yields similar estimates, but his method is not that of Baker–Masser: instead of using a generalization of Gel'fand's solution to Hilbert's seventh problem, Yu Kunrui uses a generalization in several variables of Schneider's solution to the same problem. Again, this method is restricted to the CM case.

The question of linear independence of elliptic logarithms in the non-CM case was settled only in 1980 by D. Bertrand and D.W. Masser [30, 31]. They found a new proof of Baker's theorem using functions of several variables, and they were able to extend this argument to the situation of elliptic functions, either with or without complex multiplication. The criterion they use is the one that Schneider established in 1949 [205] for his proof of the transcendence of beta values. This criterion (revisited by S. Lang in [120]) deals with Cartesian products. From the several variables point of view, this is a rather degenerate situation; much deeper results are available, including Bombieri's solution in 1970 of Nagata's conjecture [120, 233], which involves Hörmander L^2-estimates for analytic functions of several variables. However Bombieri's theorem does not seem to yield new transcendence results, so far.

So far, these deeper results do not give further transcendence results in our context.

Theorem 27 (D.W. Masser 1974 for the CM case, D. Bertrand and D.W. Masser 1980 for the non-CM case). *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3 and field of endomorphisms k. Let $u_1, \ldots, u_n$ be k-linearly independent complex numbers. Assume, for $1 \le i \le n$, that either $u_i \in \Omega$ or $\wp(u_i) \in \overline{\mathbb{Q}}$. Then the numbers $1, u_1, \ldots, u_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.*

This means that *for an elliptic curve E that is defined over $\overline{\mathbb{Q}}$, if $u_1, \ldots, u_n$ are elements in $\mathcal{L}_E$ that are linearly independent over the field of endomorphisms of E, then the numbers $1, u_1, \ldots, u_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

The method of Bertrand–Masser yields only weak Diophantine estimates (measures of linear independence of logarithms).

4.3 Further Results of Linear Independence

Theorem 26 deals only with periods and quasiperiods associated with one lattice; Theorem 27 deals only with elliptic logarithms of algebraic points on one elliptic curve. A far-reaching generalization of both results was achieved by G. Wüstholz in 1987 [249, 250, 251] when he succeeded in extending Baker's theorem to abelian varieties and integrals, and, more generally, to commutative algebraic groups. If we restrict his general result to products of a commutative linear group, of copies of elliptic curves, and of extensions of elliptic curves by the additive or the multiplicative group, the resulting statement settles the questions of linear independence of logarithms of algebraic numbers and of elliptic logarithms of algebraic points, including periods, quasiperiods, elliptic integrals of the first, second, or third kind. This is a main step toward an answer to the questions of M. Kontsevich and D. Zagier on periods [118].

Wüstholz's method can be extended to yield measures of linear independence of logarithms of algebraic points on an algebraic group. The first effective such lower bounds were given in 1989 [188, 189]. As a special case, they provide the first measures of linear independence for elliptic logarithms that is also valid in the non-CM case. More generally, they give effective lower bounds for any nonvanishing linear combination of logarithms of algebraic points on algebraic groups (including usual logarithms, elliptic logarithms, elliptic integrals of any kind).

Refinements were obtained by N. Hirata-Kohno [100, 101, 102, 103, 104], S. David [60], N. Hirata-Kohno and S. David [62], M. Ably [2, 3], and É. Gaudron [86, 87, 88], who uses not only Hirata's reduction argument, but also the work of J-B. Bost [33] (slope inequalities) involving Arakelov's theory. For instance, thanks to the recent work of David and Hirata-Kohno on the one hand, of Gaudron on the other, one knows that the above-mentioned nonvanishing linear combinations of logarithms of algebraic points are not Liouville numbers.

In the p-adic case there is a paper of G. Rémond and F. Urfels [194] dealing with two elliptic logarithms.

Further applications to elliptic curves of the Baker–Masser–Wüstholz method were derived by D.W. Masser and G. Wüstholz [163, 164].

A survey on questions related to the isogeny theorem is [178]. Other surveys dealing with the questions of *small points*, Bogomolov conjecture, and the André Oort conjecture are [59, 61]. We do not cover these aspects of the theory in the present paper. Other related topics that would deserve more attention are the theory of height and theta functions as well as ultrametric questions.

Extensions of the above-mentioned results to abelian varieties were considered by D.W. Masser [145, 146, 147, 148, 149, 150, 153, 155, 156, 157], S. Lang [123], J. Coates and S. Lang [56], D. Bertrand and Y.Z. Flicker [28], Y.Z. Flicker [84], D. Bertrand [25, 26]. For instance, J. Wolfart and G. Wüstholz [245] have shown that the only linear dependence relations with algebraic coefficients between the values $B(a, b)$ of the Euler beta function at points $(a, b) \in \mathbb{Q}^2$ are those that follow from the Deligne–Koblitz–Ogus relations (see further references in [243]).

5 Algebraic Independence of Numbers Related to Elliptic Functions

5.1 Small Transcendence Degree

We keep the notation and assumptions of Section 3.2.

The following extension of Theorem 24 to a result of algebraic independence containing Gel'fond's 1948 results on the exponential function (see Section 1.1) is a consequence of the work of many mathematicians, including A.O. Gel'fond [89], A.A. Šmelev [214, 215], W.D. Brownawell [35], W.D. Brownawell and K.K. Kubota [37], G. Wüstholz [246], D.W. Masser and G. Wüstholz [160], and others (further references are given in [35, 235, 236]).

Theorem 28.

1. *Assume* $(d_1 + d_2)m \geq 2(m + d_1 + 2d_2)$. *Then the field* K_1 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

2. *Assume* $(d_1 + d_2)m \geq m + 2(d_1 + 2d_2)$. *Then* K_2 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

3. *Assume* $(d_1 + d_2)m \geq 2m + d_1 + 2d_2$. *Then* K_3 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

4. *Assume* $(d_1 + d_2)m > m + d_1 + 2d_2$. *Then* K_4 *has transcendence degree* ≥ 2 *over* $\mathbb{Q}$.

Quantitative estimates (measures of algebraic independence) exist (R. Tubbs [220] and E.M. Jabbouri [105]).

Further related results are due to N.I. Fel'dman [81, 82], R. Tubbs [219, 220, 221, 222, 223, 224], É. Reyssat [201], M. Toyoda and T. Yasuda [218]. See also the measure of algebraic independence given by M. Ably in [1] and by S.O. Shestakov in [209].

A survey on results related to small transcendence degree is given in [236] (see also Chapter 13 of [177]).

Again, as for Theorem 24, there are extensions of Theorem 28 that include results on Weierstrass zeta functions as well as on functions of several variables, with a number of consequences related to abelian functions [237].

5.2 Algebraic Independence of Periods and Quasiperiods

In the 1970s, G.V. Chudnovsky proved strong results of algebraic independence (small transcendence degree) related to elliptic functions. One of his most spectacular contributions was obtained in 1976 [45] (see also [48] and [50]):

Theorem 29 (G.V. Chudnovsky, 1976). *Let $\wp$ be a Weierstrass elliptic function with invariants g_2, g_3. Let (ω_1, ω_2) be a basis of the lattice period of $\wp$ and $\eta_1 = \eta(\omega_1)$, $\eta_2 = \eta(\omega_2)$ the associated quasiperiods of the associated Weierstrass zeta function. Then at least two of the numbers $g_2,\ g_3,\ \omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ are algebraically independent.*

A more precise result ([50] Chapter 7, Theorem 3.1) is that for any nonzero period ω, at least two of the four numbers $g_2,\ g_3,\ \omega/\pi,\ \eta/\omega$ (with $\eta = \eta(\omega)$) are algebraically independent.

In the case that g_2 and g_3 are algebraic, one deduces from Theorem 29 that two among the four numbers $\omega_1,\ \omega_2,\ \eta_1,\ \eta_2$ are algebraically independent; this statement is also a consequence of the next result (Theorem 4 of [48]; see also [50, 235]):

Theorem 30 (G.V. Chudnovsky, 1981). *Assume that g_2 and g_3 are algebraic. Let ω be a nonzero period of $\wp$, set $\eta = \eta(\omega)$, and let u be a complex number that is not a period such that u and ω are $\mathbb{Q}$-linearly independent: $u \notin \mathbb{Q}\omega \cup \Omega$. Assume $\wp(u) \in \overline{\mathbb{Q}}$. Then the two numbers*

$$\zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

From Theorem 29 or Theorem 30 one deduces the following result:

Corollary 31. *Let ω be a nonzero period of $\wp$ and $\eta = \eta(\omega)$. If g_2 and g_3 are algebraic, then the two numbers π/ω and η/ω are algebraically independent.*

The following consequence of Corollary 31 shows that in the CM case, Chudnovsky's results are sharp:

Corollary 32. *Assume that g_2 and g_3 are algebraic and the elliptic curve has complex multiplication. Let ω be a nonzero period of $\wp$. Then the two numbers ω and π are algebraically independent.*

As a consequence of formulas (5) and (6), one deduces the following corollary:

Corollary 33. *The numbers π and $\Gamma(1/4)$ are algebraically independent. Also the numbers π and $\Gamma(1/3)$ are algebraically independent.*

In connection with these results let us quote a conjecture of S. Lang from 1971 [121] p. 652.

Conjecture 34. If $j(\tau)$ is algebraic with $j'(\tau) \neq 0$, then $j'(\tau)$ is transcendental.

According to Siegel's relation (see [121] p. 652 and [66] Section 1.2.5 p. 165),

$$j'(\tau) = 18\frac{\omega_1^2}{2\pi i} \cdot \frac{g_3}{g_2} \cdot j(\tau).$$

Conjecture 34 amounts to the transcendence of ω^2/π. Hence Corollary 32 implies that Conjecture 34 is true at least in the CM case (see [22]):

Corollary 35. *If $\tau \in \mathcal{H}$ is quadratic and $j'(\tau) \neq 0$, then π and $j'(\tau)$ are algebraically independent.*

A quantitative refinement (measure of algebraic independence) of Corollary 31 due to G. Philibert [181] turns out to be useful in connection with Nesterenko's work in 1996 (further references on this topic are given in [239]).

A transcendence measure for $\Gamma(1/4)$ was obtained by P. Philippon [186, 187] and S. Bruiltet [39]:

Theorem 36. *For $P \in \mathbb{Z}[X, Y]$ with degree d and height H,*

$$\log|P(\pi, \Gamma(1/4)| > -10^{326}\big((\log H + d\log(d+1)\big)d^2\big(\log(d+1)\big)^2.$$

Corollary 37. *The number $\Gamma(1/4)$ is not a Liouville number:*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}.$$

Further algebraic independence results can be found in papers including those of D. Bertrand [20, 23], G.V. Chudnovsky [49] (however, see Zbl 0456.10016) and E. Reyssat [197, 200] (see also the Bourbaki lecture [231] and the book of E.B. Burger and R. Tubbs [42]). Among Chudnovsky's other contributions are results dealing with G-functions (see [50]; see also Y. André's work [7, 8]).

Au: Please verify.

We conclude this section with the following open problem, which simultaneously generalizes Theorems 29 and 30 of G.V. Chudnovsky.

Conjecture 38. Let $\wp$ be a Weierstrass elliptic function with invariants g_2, g_3, let ω be a nonzero period of $\wp$, set $\eta = \eta(\omega)$, and let $u \in \mathbb{C} \setminus \{\mathbb{Q}\omega \cup \Omega\}$. Then at least two of the five numbers

$$g_2, \quad g_3, \quad \wp(u), \quad \zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

Chudnovsky's method was extended by K.G. Vasil'ev [225] and P. Grinspan [94], who proved that at least two of the three numbers π, $\Gamma(1/5)$, and $\Gamma(2/5)$ are algebraically independent. Their proof involves the Jacobian of the Fermat curve $X^5 + Y^5 = Z^5$, which contains an abelian variety of dimension 2 as a factor. See also Pellarin's papers [179, 180].

5.3 Large Transcendence Degree

Another important (and earlier) contribution of G.V. Chudnovsky goes back to 1974, when he worked on extending Gel'fond's method in order to prove results on large transcendence degree (see references in [50, 231]).

Chudnovsky proved that three of the numbers

$$\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}} \tag{11}$$

are algebraically independent if α is a nonzero algebraic number, $\log \alpha$ a nonzero logarithm of α, and β an algebraic number of degree $d \geq 7$. The same year, with a much more difficult and highly technical proof, he made the first substantial progress toward a proof that there exist at least n algebraically independent numbers in the set (11), provided that $d \geq 2^n - 1$. This was a remarkable achievement since no such result providing a lower bound for the transcendence degree was known (see [235] Section 2.1). Later, thanks to the work of several mathematicians, including P. Philippon (see [182] for his trick involving the introduction of redundant variables) and Yu. V. Nesterenko [166, 167, 168], the proof was completed and the exponential lower bound for d was reduced to a polynomial bound, until G. Diaz [63] obtained the best known results so far: the transcendence degree is at least $[(d+1)/2]$.

During a short time, thanks to the work of Philippon, the elliptic results dealing with large transcendence degree where stronger than the exponential ones (see [235] p. 561).

Further results of algebraic independence related to elliptic functions are given in [46, 48, 50, 169, 170, 171, 197, 231].

In 1980, G.V. Chudnovsky [47] proved the Lindemann–Weierstrass theorem for $n = 2$ and $n = 3$ (small transcendence degree) by means of a clever variation of Gel'fond's method. At the same time he obtained the elliptic analogue in the CM case of the Lindemann–Weierstrass theorem for $n = 2$ and $n = 3$ in [46] and [47]. Also in [46] he announces further results of small transcendence degree (algebraic independence of four numbers).

This method was extended to large transcendence degree by P. Philippon [183, 184, 185] and G. Wüstholz [247, 248], who also succeeded in 1982 to prove the elliptic analogue of the Lindemann–Weierstrass theorem on the algebraic independence of $e^{\alpha_1}, \ldots, e^{\alpha_n}$ in the CM case:

Theorem 39. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3 and complex multiplication. Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers that are linearly independent over the field of endomorphisms of E. Then the numbers $\wp(\alpha_1), \ldots, \wp(\alpha_n)$ are algebraically independent.*

The same conclusion should also hold in the non-CM case; so far, only the algebraic independence of at least $n/2$ of these numbers is known.

Further results on large transcendence degree are due to D.W. Masser and G. Wüstholz [161, 162], W.D. Brownawell [36], W.D. Brownawell and R. Tubbs [38], M. Takeuchi [216].

A survey on algebraic independence was written in 1979 by W.D. Brownawell [35]. The period prior to 1984 is covered by [235] (see also [236]), while [242] gives references for the period 1984–1997. A more recent reference is [177] Chapter 14.

5.4 Modular Functions and Ramanujan Functions

Ramanujan [193] introduced the following functions:

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$

$$R(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

They are special cases of Fourier expansions of Eisenstein series. Recall the Bernoulli numbers B_k defined by

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{z^{2k}}{(2k)!},$$

$$B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42.$$

For $k \geq 1$ the normalized Eisenstein series of weight k is ([116] Section 3.2, Proposition 6, [208] Section 7.4.2)

$$E_{2k}(q) = 1 + (-1)^k\frac{4k}{B_k}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}.$$

The connection with (4) is

$$E_{2k}(q) = \frac{1}{2\zeta(2k)}\cdot G_k(\tau),$$

for $k \geq 2$, where $q = e^{2\pi i\tau}$ ([48] Section 3.2, Proposition 6). In particular,

$$G_2(\tau) = \frac{\pi^4}{3^2\cdot 5}\cdot E_4(q), \qquad G_3(\tau) = \frac{2\pi^6}{3^3\cdot 5\cdot 7}\cdot E_6(q).$$

With Ramanujan's notation we have

$$P(q) = E_2(q), \quad Q(q) = E_4(q), \quad R(q) = E_6(q).$$

The discriminant Δ and the modular invariant J are related to these functions by Jacobi's product formula ([127] Section 18.4 and [208] Sections 7.2.3, 7.3.3, 7.4.4)

$$\Delta = \frac{(2\pi)^{12}}{12^3} \cdot (Q^3 - R^2) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

and

$$J = \frac{(2\pi)^{12} Q^3}{\Delta} = \frac{(2^4 3^2 5 G_2)^3}{\Delta}.$$

Let q be a complex number, $0 < |q| < 1$. There exists τ in the upper half-plane $\mathcal{H}$ such that $q = e^{2\pi i \tau}$. Select any twelfth root ω of $\Delta(q)$. The invariants g_2 and g_3 of the Weierstrass $\wp$ function attached to the lattice $(\mathbb{Z} + \mathbb{Z}\tau)\omega$ satisfy $g_2^3 - 27 g_3^2 = 1$ and (see [66] Section 1.2.2 p. 163, [127], Section 4.2, Proposition 4 and Section 18.3)

$$P(q) = 3\frac{\omega}{\pi} \cdot \frac{\eta}{\pi}, \quad Q(q) = \frac{3}{4}\left(\frac{\omega}{\pi}\right)^4 g_2, \quad R(q) = \frac{27}{8}\left(\frac{\omega}{\pi}\right)^6 g_3.$$

According to formulas (5) and (6), here are a few special values (see, for instance, [4], [177] Section 3.1 and [241]).

– For $\tau = i$, $q = e^{-2\pi}$,

$$P(e^{-2\pi}) = \frac{3}{\pi}, \quad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4, \quad R(e^{-2\pi}) = 0, \quad \text{and}$$
$$\Delta(e^{-2\pi}) = 2^6 \omega_1^{12}, \tag{12}$$

with

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots.$$

– For $\tau = \varrho$, $q = -e^{-\pi\sqrt{3}}$,

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \; Q(-e^{-\pi\sqrt{3}}) = 0, \; R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6,$$
$$\Delta(-e^{-\pi\sqrt{3}}) = -2^4 3^3 \omega_1^{12}, \tag{13}$$

with

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots.$$

5.5 Mahler–Manin Problem on $J(q)$

After Schneider's theorem (Corollary 21) on the transcendence of the values of the modular function $j(\tau)$, the first results on Eisenstein series (cf. Section 5.6) go back to 1977 with D. Bertrand's work [21, 19]. See also his papers [18, 20, 23, 24], his

work [29] with M. Laurent on values of theta functions, and Yanchenko's paper [252]. Further related results are Theorems 5 and 6 (p. 344) and Theorem 4 (p. 347) in Chudnovsky's lecture at the Helsinki ICM in 1978 [48].

The first transcendence proof using modular forms is due to a team from St Étienne (K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert), whence the nickname *théorème stéphanois* for the next result; see [16] (see also [91, 92, 90] and Chapter 2 of [177]). Theorem 40 answers a conjecture of K. Mahler [139, 140] in the complex case and of Yu. V. Manin [142] in the p-adic case. Manin's question on the arithmetic nature of the p-adic number $J(q)$ is motivated by Mazur's theory, but he also asked "an obvious analogue" in the complex case; see Conjecture 43 below). We state the result only in the complex case; the paper [16] solves both cases.

Theorem 40 (K. Barré, G. Diaz, F. Gramain, G. Philibert, 1996). *Let $q \in \mathbb{C}$, $0 < |q| < 1$. If q is algebraic, then $J(q)$ is transcendental.*

The solution of Manin's problem in the p-adic case has several consequences. It is a tool both for R. Greenberg in his study of zeros of p-adic L functions and for H. Hida, J. Tilouine, and É. Urban in their solution of the main conjecture for the Selmer group of the symmetric square of an elliptic curve with multiplicative reduction at p (references are given in [239]).

The proof of Theorem 40 involves upper bounds for the growth of the coefficients of the modular function $J(q)$. Such estimates were produced first by K. Mahler [141] Section 3. A refined estimate, due to N. Brisebarre and G. Philibert [34], for the coefficients $c_k(m)$ (which are nonnegative rational integers) in

$$\big(qJ(q)\big)^k = \sum_{m=0}^{\infty} c_k(m) q^m$$

is

$$c_k(m) \le e^{4\sqrt{km}}.$$

According to a remark by D. Bertrand (Lemma 1 in [241] and Lemma 2.4 p. 17 in [177]; see also Lemma 2 in [15] and Lemma 1 in [27]), the upper bound

$$|\tilde{c}_{N,k}(m)| \le C^N m^{12N}$$

($0 \le k \le N$, $N \ge 1$, $m \ge 1$, with an absolute constant C) for the coefficients in the Taylor development at the origin of $\Delta^{2N} J^k$,

$$\Delta(q)^{2N} J(q)^k = \sum_{m=1}^{\infty} \tilde{c}_{Nk}(m) q^m,$$

is sufficient for the proof of Theorem 40 and is an easy consequence of a theorem of Hecke ([208] Section 7.4.3, Theorem 5), together with the fact that Δ^2 and $\Delta^2 J$ are parabolic modular forms of weight 24.

One of the main tools involved in the proof of Theorem 40 is an estimate for the degrees and height of $J(q^n)$ in terms of $J(q)$ (which is assumed to be algebraic) and $n \geq 1$. There exists a symmetric polynomial $\Phi_n \in \mathbb{Z}[X, Y]$, of degree

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

in each variable, such that $\Phi_n(J(q), J(q^n)) = 0$. Again, K. Mahler was the first to investigate the coefficients of the polynomial $\Phi_n(X, Y)$: in [141] he proved that its length (sum of the absolute values of the coefficients) satisfies

$$L(\Phi_n) \leq e^{cn^{3/2}}$$

with an absolute constant c. In the special case $n = 2^m$ he had an earlier stronger result in [140], namely

$$L(\Phi_n) \leq 2^{57n} n^{36n},$$

and he claimed (see [140] p. 97) that if the sharper upper bound

$$L(\Phi_n) \leq 2^{Cn}$$

with a positive absolute constant $C > 0$ were true for $n = 2^m$, he could prove Theorem 40. However, in 1984, P. Cohen [58] produced asymptotic estimates that show that Mahler's expectation was too optimistic:

$$\lim_{\substack{n=2^m \\ m\to\infty}} \frac{1}{n \log n} \log L(\Phi_n) = 9.$$

In fact she proved more precise results, without the condition $n = 2^m$, which imply, for instance, $\log L(\Phi_n) \sim 6\psi(n) \log n$ for $n \to \infty$.

Further related results are given in [67] (G. Diaz and G. Philibert) for the j-function and [159] (D.W. Masser) for the $\wp$-function.

The proof of [16] can be adapted to yield quantitative estimates [14, 15].

A reformulation of Theorem 40 on the transcendence of $J(q)$ is the following mixed analogue of the four exponentials conjecture (Conjecture 12):

Corollary 41. *Let* $\log \alpha$ *be a logarithm of a nonzero algebraic number. Let* $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *be a lattice with algebraic invariants* g_2, g_3. *Then the determinant*

$$\begin{vmatrix} \omega_1 & \log \alpha \\ \omega_2 & 2\pi i \end{vmatrix}$$

does not vanish.

The four exponentials conjecture for the product of an elliptic curve by the multiplicative group is the following more general open problem:

Conjecture 42. Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let E be the corresponding elliptic curve, u_1 and u_2 two elements in $\mathcal{L}_E$, and $\log \alpha_1$, $\log \alpha_2$ two logarithms of algebraic numbers. Assume further that the two rows of the matrix

$$M = \begin{pmatrix} u_1 & \log \alpha_1 \\ u_2 & \log \alpha_2 \end{pmatrix}$$

are linearly independent over $\mathbb{Q}$. Then the determinant of M does not vanish.

Another special case of Conjecture 42, stronger than Corollary 41, is the next question of Yu. V. Manin, who asks in Section 4.2 of [142] to determine the nature of the invariant of the complex elliptic curve having periods 1 and a quotient $(\log \alpha_1)/(\log \alpha_2)$ of two logarithms of algebraic numbers:

Conjecture 43 (Yu.V. Manin). Let $\log \alpha_1$ and $\log \alpha_2$ be two nonzero logarithms of algebraic numbers and let $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2 and g_3. Then

$$\frac{\omega_1}{\omega_2} \neq \frac{\log \alpha_1}{\log \alpha_2}.$$

In this direction let us quote some of the open problems raised by G. Diaz [65, 66].

Conjecture 44 (G. Diaz).
1. For any $z \in \mathbb{C}$ with $|z| = 1$ and $z \neq \pm 1$, the number $e^{2\pi i z}$ is transcendental.
2. If q is an algebraic number with $0 < |q| < 1$ such that $J(q) \in [0, 1728]$, then $q \in \mathbb{R}$.
3. The function J is injective on the set of algebraic numbers α with $0 < |\alpha| < 1$.

Remark (G. Diaz). Part 3 of Conjecture 44 implies the other two and also follows from the four exponentials conjecture. It also follows from the next conjecture of D. Bertrand.

Conjecture 45 (D. Bertrand). If α_1 and α_2 are two multiplicatively independent algebraic numbers in the domain $\{q \in \mathbb{C}\,;\, 0 < |q| < 1\}$, then the two numbers $J(\alpha_1)$ and $J(\alpha_2)$ are algebraically independent.

Conjecture 45 (see [27], where Section 5 is devoted to conjectural statements inspired by a conjecture of Oort and André) implies the special case of the four exponentials conjecture, where two of the algebraic numbers are roots of unity and the two others have modulus $\neq 1$.

5.6 Nesterenko's Theorem

In 1976 [18], D. Bertrand pointed out that Schneider's theorem the transcendence of ω/π implies the following statement:

For any $q \in \mathbb{C}$ with $0 < |q| < 1$, at least one of the two numbers $Q(q)$, $R(q)$ is transcendental.

He also proved in [18] the p-adic analogue by means of a new version of the Schneider–Lang criterion for meromorphic functions (he allows one essential singularity), which he applied to Jacobi–Tate elliptic functions (see also [252]). Two years later [20], he noticed that Theorem 29 yields the following:
For any $q \in \mathbb{C}$ with $0 < |q| < 1$, at least two of the numbers $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.

The following result of Yu. V. Nesterenko [172, 173] (see also [175, 176, 239, 241, 32] as well as Chapters 3 and 4 of [177]) goes one step further:

Theorem 46 (Nesterenko, 1996). *For any $q \in \mathbb{C}$ with $0 < |q| < 1$, three of the four numbers q, $P(q)$, $Q(q)$, $R(q)$ are algebraically independent.*

Among the tools used by Nesterenko in his proof is the following result due to K. Mahler [138] (see also Chapter 1 of [177]):

The functions P, Q, R are algebraically independent over $\mathbb{C}(q)$.

Also he uses the fact that they satisfy a system of differential equations for $D = q\ d/dq$ discovered by S. Ramanujan in 1916 [193] (see also Chapters 1 and 3 of [177]):

$$12\frac{DP}{P} = P - \frac{Q}{P}, \quad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \quad 2\frac{DR}{R} = P - \frac{Q^2}{R}.$$

One of the main steps in his original proof [172, 173] is the following zero estimate:

Theorem 47 (Nesterenko's zero estimate). *Let L_0 and L be positive integers, $A \in \mathbb{C}[q, X_1, X_2, X_3]$ a nonzero polynomial in four variables of degree $\leq L_0$ in q and $\leq L$ in each of the three other variables X_1, X_2, X_3. Then the multiplicity at the origin of the analytic function $A\big(q, P(q), Q(q), R(q)\big)$ is at most $2 \cdot 10^{45} L_0 L^3$.*

In the special case in which $J(q)$ is algebraic, P. Philippon [187] produced an alternative proof for Nesterenko's result in which this zero estimate is not used; instead of it, he used Philibert's measure of algebraic independence for ω/π and η/π (see [181] and Section 5.2 above). However, Philibert's proof requires a zero estimate for algebraic groups.

Using (12) one deduces from Theorem 46 (see [177] Section 3.1, Corollary 1.2) the following corollary:

Corollary 48. *The three numbers π, e^{π}, $\Gamma(1/4)$ are algebraically independent.*

Using (13) one deduces (see [177] Section 1.3.1, Corollary 3.2, Remark (ii)) the following:

Corollary 49. *The three numbers π, $e^{\pi\sqrt{3}}$, $\Gamma(1/3)$ are algebraically independent.*

Consequences of Corollary 48 are the transcendence of the numbers

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

and (P. Bundschuh [41])

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}.$$

D. Duverney, K. and K. Nishioka, and I. Shiokawa [68, 69, 70, 71, 72, 73] as well as D. Bertrand [27] derived from Nesterenko's theorem a number of interesting corollaries, including the following ones ([177] Chapter 3).

Corollary 50. *The Rogers–Ramanujan continued fraction*

$$RR(\alpha) = 1 + \cfrac{\alpha}{1 + \cfrac{\alpha^2}{1 + \cfrac{\alpha^3}{1 + \ddots}}}$$

is transcendental for any algebraic α with $0 < |\alpha| < 1$.

Corollary 51. *Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Then the number*

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}$$

is transcendental.

Jacobi theta series ([43] Chapter V, [244] Chapter XXI and [177] Section 3.1.3) are defined by

$$\theta_2(q) = 2q^{1/4} \sum_{n\geq 0} q^{n(n+1)} = 2q^{1/4} \prod_{n=1}^{\infty} (1-q^{4n})(1+q^{2n}),$$

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})^2,$$

$$\theta_4(q) = \theta_3(-q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2.$$

Corollary 52. *Let i, j and $k \in \{2,3,4\}$ with $i \neq j$. Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Then each of the two fields*

$$\mathbb{Q}\big(q, \theta_i(q), \theta_j(q), D\theta_k(q)\big) \quad \textit{and} \quad \mathbb{Q}\big(q, \theta_k(q), D\theta_k(q), D^2\theta_k(q)\big)$$

has transcendence degree ≥ 3 over $\mathbb{Q}$.

As an example, *for an algebraic number* $q \in \mathbb{C}$ *with* $0 < |q| < 1$, *the three numbers*

$$\sum_{n\geq 0} q^{n^2}, \quad \sum_{n\geq 1} n^2 q^{n^2}, \quad \sum_{n\geq 1} n^4 q^{n^2}$$

are algebraically independent. In particular, *the number*

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

is transcendental. The number $\theta_3(q)$ was explicitly considered by Liouville as far back as 1851 (see [174] p. 295 and [177] p. 30).

The proof by Yu. V. Nesterenko is effective and yields quantitative refinements (measures of algebraic independence): [93, 174, 187].

5.7 Further Open Problems

Among many open problems, we mention

- the algebraic independence of the three numbers π, $\Gamma(1/3)$, $\Gamma(1/4)$.
- the algebraic independence of three numbers among π, $\Gamma(1/5)$, $\Gamma(2/5)$, $e^{\pi\sqrt{5}}$.
- the algebraic independence of the four numbers e, π, e^{π} and $\Gamma(1/4)$.

The main conjectures in this domain are due to S. Schanuel, A. Grothendieck, Y. André [9], and C. Bertolin [17]. Chudnovsky's proof of the algebraic independence of π and $\Gamma(1/4)$ involves elliptic functions; Nesterenko's proof of the algebraic independence of π and e^{π} requires modular functions. One may expect that higher-dimensional objects (abelian varieties, motives) may be required in order to go further. In this respect we conclude by alluding to the remarkable progress that has been achieved recently in finite characteristic (after the work by Jing Yu, G.W. Anderson and D. Thakur, L. Denis, W.D. Brownawell, J.F. Voloch, M. Papanikolas, among others).

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