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# COVERING THE WHOLE SPACE WITH POISSON RANDOM BALLS

HERMINE BIERMÉ AND ANNE ESTRADE

ABSTRACT. We consider Poisson random balls models generalizing Boolean models with balls for grains. We investigate necessary and sufficient conditions for covering the whole space with these balls. Our arguments are based on both stochastic geometry and martingale theory. Following the framework of percolation theory, a critical regime is exhibited and bounds for critical intensities are obtained.

## 1. INTRODUCTION AND SETTING

Let  $d \geq 1$ . Let  $\Phi$  be a Poisson point process in  $\mathbb{R}^d \times (0, +\infty)$  of intensity

$$\nu(dx, dr) = dx\mu(dr),$$

where  $dx$  denotes the Lebesgue measure on  $\mathbb{R}^d$  and  $\mu$  is a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ . We consider a family of grains  $x + B(0, r)$  in  $\mathbb{R}^d$  with  $(x, r) \in \Phi$ , where  $B(x, r)$  is the open Euclidean ball in  $\mathbb{R}^d$  with center at  $x$  and radius  $r$ .

Equivalently, we let  $N(dx, dr)$  be a Poisson random measure on  $\mathbb{R}^d \times (0, +\infty)$  with intensity measure given by  $\nu(dx, dr)$  and associate with each random point  $(x, r) \in \mathbb{R}^d \times (0, +\infty)$  the random ball  $B(x, r)$ . For measurable sets  $A \subset \mathbb{R}^d \times (0, +\infty)$  we let  $N(A)$  count the number of points  $(x, r)$  in  $A$  and consider the map  $A \mapsto N(A)$  with values viewed as integer-valued random variables on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We recall the basic facts that  $N(A)$  is Poisson distributed with mean  $\nu(A) = \int_A dx\mu(dr)$  (if the integral diverges then  $N(A)$  is countably infinite with probability one) and if  $A_1, \dots, A_n$  are disjoint sets then  $N(A_1), \dots, N(A_n)$  are independent.

We consider the random set

$$\Xi = \bigcup_{(x,r) \in \Phi} B(x, r).$$

Since balls are open, the random set  $\Xi$  is an open set, in which each point is covered at least once by a ball. In this way, the space  $\mathbb{R}^d$  is partitioned into two regions, the occupied region  $\Xi$  and the vacant region  $\mathbb{R}^d \setminus \Xi$ .

When considering closed balls  $\overline{B(x, r)}$  and assuming that  $\lambda = \int_0^{+\infty} \mu(dr) < +\infty$  then  $\mu_0 = \frac{1}{\lambda}\mu$  is a probability measure and

$$\tilde{\Xi} = \bigcup_{(x,r) \in \Phi} \overline{B(x, r)}$$

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is a Poisson Boolean model as usually considered in stochastic geometry and named as germ-grain model (see [23, 21, 19] for instance). Actually, in this case, the centers form a homogeneous Poisson process  $\chi$  in  $\mathbb{R}^d$  of intensity  $\lambda$ . Moreover, for any  $x \in \chi$  there exists a unique  $r(x)$  such that  $(x, r(x)) \in \Phi$ . Then  $(r(x))_{x \in \chi}$  is a family of iid copies of  $R$ , where  $R$  is a random variable with law  $\mu_0$ , and independent of the Poisson process  $\chi$ . Several authors studied continuum percolation for Boolean models (see [19] for an exhaustive survey).

In this paper we mainly focus on the question whether or not the space  $\mathbb{R}^d$  is almost surely completely covered by  $\Xi$  or  $\tilde{\Xi}$ . It is an old question initiated in the sixties (see [6, 7] and [16] for an historical survey of this problem). Up to our knowledge, there are only two situations for which the problem is totally solved, in the sense that a necessary and sufficient condition for almost sure coverage is known.

The first situation concerns the dimension  $d = 1$ . Shepp [22], following Mandelbrot [18], solved the problem in one dimension giving an if and only if condition for  $\mathbb{R}$  being almost surely completely covered by  $\bigcup_{(x,r) \in \Phi} (x, x+r)$ . In our setting, with  $B(x, r) = (x-r, x+r)$ , this iff condition is

$$(1) \quad \int_0^1 \exp\left(2 \int_u^{+\infty} (r-u) \mu(dr)\right) du = +\infty.$$

The second case is concerned with the germ-grain model, ie when the measure  $\mu$  is finite. In that case, it is known (see [12, 19]) that  $\mathbb{R}^d$  is a.s. covered if and only if

$$(2) \quad \int_0^{+\infty} r^d \mu(dr) = +\infty,$$

which is equivalent to saying that the balls have an infinite mean volume. In particular, this forbids the coverage of  $\mathbb{R}^d$  with balls of same radius. Actually (2) is still a necessary condition for coverage even when  $\mu$  is no longer a finite measure.

Among the numerous recent contributions on related topics, let us mention a result of Kahane [15], partially inspired by ideas of Janson [13]. The covering problem is solved in dimension  $d > 1$  in a general setting where  $B(x, r)$  is replaced by  $x + r\mathcal{C}$ , with  $\mathcal{C}$  an open bounded convex set of  $\mathbb{R}^d$  that cannot be a ball. Actually, the restriction imposed on the convex set  $\mathcal{C}$  is only required for the sufficient condition of a.s. coverage. We also mention two more recent papers. In [20], Molchanov and Scherbakov are concerned with an inhomogeneous framework where the radii are random variables that depend on the centers locations. In a 1 dimensional setting, Barral and Fan consider the asymptotic behavior of the number of Poisson intervals which cover a point in [2].

In this paper we give a sufficient condition ensuring a.s. coverage when  $d > 1$  and  $\mu$  is not a finite measure and recall a necessary one. Combining these sufficient and necessary conditions, we derive a criteria for coverage or non-coverage. It relies on the compared asymptotics of  $\int_\varepsilon^1 r^d \mu(dr)$  and  $|\ln \varepsilon|$  as  $\varepsilon$  goes to  $0^+$ .

Moreover, when the whole space is not completely covered, percolation arises as a natural question. Actually, the occupied and the vacant regions consist of connected components. Denoting  $W$  the connected component of  $\Xi$  that contains the origin, with  $W = \emptyset$  if  $0 \notin \Xi$ , one can wonder whether  $W$  is bounded or unbounded. It is said (see [19] for instance) that percolation occurs if  $W$  is not bounded with positive probability.

This question is closely linked with the coverage question. For instance, it is proved in [8] that when  $\mu$  is a finite measure,  $W$  is bounded a.s. for  $\lambda$  small enough if and only if (2) fails. Results obtained in [9], when  $\mu$  is no more finite, are compared with presented ones in the last section.

The paper is organized as follows. In Section 2 we start with general principles, in particular with a zero-one law for the probability of coverage. A sort of dichotomy result is established: when coverage holds, it can be due to the contribution of either the small balls (high frequency coverage) or the large balls (low frequency coverage). Section 3 is devoted to the presentation of sufficient and necessary conditions for a.s. coverage. It is quite easy to obtain that (2) remains an iff condition for low frequency coverage. The problem of high frequency coverage is more delicate. Our main result, namely Theorem 3.4, consists in getting a sufficient condition in this case. Our proof is inspired by the geometrical arguments of [10], originally developed for a germ-grain model. We also recall the necessary condition given in [15] coming from martingale theory. At the end of Section 3, a simple criteria (see Proposition 3.5) for coverage or non-coverage is exhibited. The particular case of a power law distribution for  $\mu$  is also discussed. In the last section, we introduce a notion of coverage critical intensity. A critical regime is described and the links with percolation (sub)critical regime as studied in [9] are explored.

## 2. GENERAL PRINCIPLES

Let us consider  $\mu$  a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$  and  $\Phi$  a Poisson point process in  $\mathbb{R}^d \times (0, +\infty)$  of intensity  $dx\mu(dr)$ . We set

$$\Xi = \bigcup_{(x,r) \in \Phi} B(x, r),$$

the occupied region. Our main goal is to find assumptions on  $\mu$  ensuring  $\mathbb{R}^d \subset \Xi$  or  $\mathbb{R}^d \not\subset \Xi$ . As in [19] where a percolation function is introduced, one can consider

$$\psi(\mu) = \mathbb{P}_\mu(\mathbb{R}^d \subset \Xi)$$

the probability that  $\mathbb{R}^d$  is completely covered by  $\Xi$ . We call  $\psi$  the coverage function.

**Proposition 2.1.** *If  $\mu_1, \mu_2$  are  $\sigma$ -finite non-negative measures on  $(0, +\infty)$  with  $\mu_1 \leq \mu_2$  then  $\psi(\mu_1) \leq \psi(\mu_2)$ .*

*Proof.* Let us consider  $\Phi_1$  a Poisson point process in  $\mathbb{R}^d \times (0, +\infty)$  of intensity  $dx\mu_1(dr)$  and  $\Xi_1 = \bigcup_{(x,r) \in \Phi_1} B(x, r)$ . On the other hand, one can consider  $\Psi$  a Poisson point process in  $\mathbb{R}^d \times (0, +\infty)$  of intensity  $dx(\mu_2 - \mu_1)(dr)$  independant from  $\Phi_1$  and  $\Sigma = \bigcup_{(x,r) \in \Psi} B(x, r)$ .

Then  $\Phi_2$ , the superposition of the two independent Poisson point processes  $\Phi_1$  and  $\Psi$ , is a Poisson point process of intensity  $dx\mu_2(dr)$  and one can consider  $\Xi_2 = \Xi_1 \cup \Sigma$ . It is then obvious that if  $\mathbb{R}^d \subset \Xi_1$  then  $\mathbb{R}^d \subset \Xi_2$  such that  $\psi(\mu_1) \leq \psi(\mu_2)$ .  $\square$

The next lemma states a zero-one law for the coverage function, due to ergodicity of the Poisson point process of the centers of the balls. It allows to focus on the coverage of an arbitrary compact set with positive Lebesgue measure instead of the whole space.

**Lemma 2.2.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ . Then,*

- i)  $\psi(\mu) = \mathbb{P}_\mu(\mathbb{R}^d \subset \Xi) = \text{zero or one.}$*
- ii) If there exists  $K$  a compact set of  $\mathbb{R}^d$  such that  $\mathbb{P}_\mu(K \subset \Xi) < 1$  then  $\psi(\mu) = 0$ .*
- iii) If there exists  $K$  a compact set of  $\mathbb{R}^d$  with positive Lebesgue measure such that  $\mathbb{P}_\mu(K \subset \Xi) = 1$  then  $\psi(\mu) = 1$ .*

*Proof.* *i)* We can for instance mimic the proof of [22] given in dimension 1. For any  $n \in \mathbb{Z}$ , let us denote

$$C_n = \left\{ (x_1, \dots, x_d, r) \in \mathbb{R}^d \times (0, +\infty); x_1 \leq n \right\},$$

and  $B_n = C_n \setminus C_{n-1}$ . Since  $(B_n)_{n \in \mathbb{Z}}$  are disjoint in  $\mathbb{R}^d \times (0, +\infty)$  the  $\sigma$ -fields  $\mathcal{B}_n = \sigma((x_j, r_j)_{j \in J}; (x_j, r_j) \in \Phi \cap B_n)$  are independent.

Let us consider the event  $A_n = \{(n, +\infty) \times \mathbb{R}^{d-1} \subset \Xi\}$  and remark that  $(A_n)_{n \in \mathbb{Z}}$  is an increasing sequence such that  $A_n \in \sigma(\mathcal{B}_k; k \geq n)$  for any  $n$ . Therefore,  $A_\infty = \bigcap_{n \in \mathbb{Z}} A_n$  is a tail event of  $\bigcap_{n \in \mathbb{Z}} \sigma(\mathcal{B}_k; k \geq n)$  and  $\mathbb{P}_\mu(A_\infty) = 0$  or  $1$  by zero-one law (see [17] Theorem 2.13 p.30 for instance). Moreover, by stationarity of  $\Xi$  the probability  $\mathbb{P}_\mu(A_n)$  does not depend on  $n$  and we get

$$\mathbb{P}_\mu(\mathbb{R}^d \subset \Xi) = \lim_{n \rightarrow -\infty} \mathbb{P}_\mu(A_n) = \mathbb{P}_\mu(A_\infty) = 0 \text{ or } 1.$$

*ii)* It is enough to remark that  $\mathbb{P}_\mu(\mathbb{R}^d \subset \Xi) \leq \mathbb{P}_\mu(K \subset \Xi) < 1$  and to use *i)* to conclude.

*iii)* Since  $K$  is a compact set of  $\mathbb{R}^d$  with positive Lebesgue measure, one has  $\mathbb{R}^d = \bigcup_{q \in \mathbb{Q}^d} (q + K)$ . Moreover, by stationarity of  $\Xi$  one has  $\mathbb{P}_\mu(K + q \not\subset \Xi) = \mathbb{P}_\mu(K \not\subset \Xi) = 0$  by assumption. Then,  $\mathbb{P}_\mu(\mathbb{R}^d \not\subset \Xi) \leq \sum_{q \in \mathbb{Q}^d} \mathbb{P}_\mu(K + q \not\subset \Xi) = 0$  and  $\psi(\mu) = 1$ .  $\square$

One can split the random set  $\Xi$  into two independent sets, one made of “small balls” (radius less than 1), and the other one made of “large balls” (radius larger than 1). For this purpose, following [18] and [22], we introduce the next definition.

**Definition.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$  and let us write  $\mu = \mu_H + \mu_L$  with*

$$\mu_H(dr) = \mathbf{1}_{(0,1]}(r)\mu(dr) \quad \text{and} \quad \mu_L(dr) = \mathbf{1}_{(1,+\infty)}(r)\mu(dr) .$$

*The measure  $\mu$  is said to give a high frequency covering if  $\psi(\mu_H) = 1$  and a low frequency covering if  $\psi(\mu_L) = 1$ .*

In order to ensure that  $\mathbb{R}^d$  is totally covered by  $\Xi$  it is enough to ensure that high or low frequency holds, so

$$\psi(\mu) \geq \max(\psi(\mu_H), \psi(\mu_L)) .$$

In one dimension, using the if and only if covering condition (1), one can obtain the reverse inequality (see Proposition 4.2 of [18]):

$$\text{when } d = 1, \quad \psi(\mu) = \max(\psi(\mu_H), \psi(\mu_L)).$$

In that case, covering is equivalent to high or low frequency covering.

### 3. COVERING CONDITIONS

First, we consider hereafter a general necessary covering condition on the measure  $\mu$ . It again illustrates the fact that covering is only possible either under a specific behavior of the small balls or a specific behavior of the large balls. A straightforward generalization of the germ-grain case of [19] applies for low frequency covering. The general result is obtained from martingale's theory used in [15], whose proof is recalled for sake of completeness.

**Theorem 3.1.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ .*

*If  $\psi(\mu) = 1$  ie  $\mathbb{R}^d$  is a.s. covered by  $\Xi$ , then*

$$\int_1^{+\infty} r^d \mu(dr) = +\infty$$

or

$$(3) \quad \int_0^1 u^{d-1} \exp\left(v_d \int_u^1 r^{d-1}(r-u)\mu(dr)\right) du = +\infty$$

where  $v_d$  is the volume of the unit ball. In particular it implies that  $\int_0^{+\infty} r^d \mu(dr) = +\infty$ .

*Proof.* Let us prove the opposite implication and assume that  $\int_1^{+\infty} r^d \mu(dr) < +\infty$  and

$$\int_0^1 u^{d-1} \exp\left(v_d \int_u^1 r^{d-1}(r-u)\mu(dr)\right) du < +\infty.$$

Let us denote  $\Xi_\varepsilon = \bigcup_{(x,r) \in \Phi; r \geq \varepsilon} B(x,r)$  for  $\varepsilon > 0$ . The Boolean model  $\Xi_\varepsilon$  is associated with the Poisson point process  $\Phi_\varepsilon = \Phi \cap (\mathbb{R}^d \times [\varepsilon, +\infty))$  with intensity measure  $\mathbf{1}_{r \geq \varepsilon} dx \mu(dr)$ .

**Lemma 3.2.** *The Boolean random field  $Y_\varepsilon = \{\mathbf{1}_{\Xi_\varepsilon}(y), y \in \mathbb{R}^d\}$  satisfies the following*

- $\mathbb{E}(Y_\varepsilon(y)) = e^{-\kappa_\varepsilon}$
- $\mathbb{E}(Y_\varepsilon(x)Y_\varepsilon(y)) \leq e^{-2\kappa_\varepsilon} \exp\left(v_d \int_\varepsilon^{+\infty} r^{d-1}(r-b|x-y|)_+ \mu(dr)\right)$

where  $\kappa_\varepsilon := v_d \int_\varepsilon^{+\infty} r^d \mu(dr)$  and  $b$  is some positive constant less than 1/2.

*Proof of the lemma.* The first statement follows from

$$\mathbb{E}(Y_\varepsilon(y)) = \mathbb{P}(y \notin \Xi_\varepsilon) = \exp\left(-\int_{\mathbb{R}^d} \int_\varepsilon^{+\infty} \mathbf{1}_{B(x,r)}(y) dx \mu(dr)\right) .$$

For the second one, we write

$$\begin{aligned}
\mathbb{E}(Y_\varepsilon(x)Y_\varepsilon(y)) &= \mathbb{P}(x \notin \Xi_\varepsilon, y \notin \Xi_\varepsilon) \\
&= \exp\left(\int_{\mathbb{R}^d} \int_\varepsilon^{+\infty} \mathbf{1}_{B(x,r) \cup B(y,r)}(\xi) d\xi \mu(dr)\right) \\
&= \exp\left(-2\kappa_\varepsilon + \int_{\mathbb{R}^d} \int_\varepsilon^{+\infty} \mathbf{1}_{B(x,r) \cap B(y,r)}(\xi) d\xi \mu(dr)\right) \\
&= e^{-2\kappa_\varepsilon} \exp\left(\int_\varepsilon^{+\infty} \gamma(|x-y|, r) \mu(dr)\right)
\end{aligned}$$

where for  $u, r > 0$ ,  $\gamma(u, r)$  denotes the Lebesgue measure of the intersection of two balls in  $\mathbb{R}^d$  with common radius  $r$  and whose centers are at distance  $u$  (in [15],  $\gamma$  is called ‘‘pagode’’ function). We will prove that  $\gamma$  satisfies the following: for any  $u, r > 0$

$$(4) \quad \gamma(u, r) \leq v_d r^{d-1} (r - bu)_+$$

for a constant  $b \in (0, 1/2]$ . First let us assume that  $d = 1$  and remark that  $\gamma(u, r) = 2(r - u/2)_+ = v_1 (r - u/2)_+$  such that (4) is satisfied with  $b = 1/2$ . In the general case  $d \geq 2$ , on the one hand let us remark that  $\gamma(u, r) = 0$  for all  $u \geq 2r$  such that (4) holds in this case whatever the constant  $b \leq 1/2$ . On the other hand, for  $u < 2r$ , let us write for  $e \in \mathbb{R}^d$  a fixed direction

$$\begin{aligned}
\gamma(u, r) &= r^d \int_{\mathbb{R}^d} \mathbf{1}_{B((u/r)e, 1) \cap B(0, 1)}(\xi) d\xi \\
&= r^d \int_{z \in \mathbb{R}^{d-1}; |z| < 1} 2\sqrt{1 - |z|^2} \left(1 - u/(2r\sqrt{1 - |z|^2})\right)_+ dz \\
&= v_d r^d - \delta(u, r),
\end{aligned}$$

where

$$\begin{aligned}
\delta(u, r) &= 2r^d \left( \int_{z \in \mathbb{R}^{d-1}; 2\sqrt{1 - |z|^2} \leq u/r} \sqrt{1 - |z|^2} dz + \frac{u}{r} \int_{z \in \mathbb{R}^{d-1}; 2\sqrt{1 - |z|^2} > u/r} dz \right) \\
&\geq bv_d u r^{d-1},
\end{aligned}$$

for some  $b \in (0, 1/2]$ . Then (4) is proven.  $\square$

We come back to the proof of Theorem 3.1. Let  $K$  be a compact subset of  $\mathbb{R}^d$  with positive Lebesgue measure and diameter less than  $1/b$ . Let us consider

$$M_\varepsilon = \int_{\mathbb{R}^d} e^{\kappa_\varepsilon} Y_\varepsilon(x) \sigma_K(dx)$$

where  $\sigma_K$  is the probability measure  $\sigma_K(dx) = \text{Leb}(K)^{-1} \mathbf{1}_K(x) dx$ . Since for  $0 < \varepsilon < \varepsilon'$ ,

$$Y_\varepsilon(y) = \left( \prod_{(x,r) \in \Phi; \varepsilon \leq r < \varepsilon'} \mathbf{1}_{B(x,r)^c}(y) \right) Y_{\varepsilon'}(y),$$

$(M_\varepsilon)_{\varepsilon>0}$  is a positive martingale. Following [14] we will prove that  $(M_\varepsilon)_{\varepsilon>0}$  converges in  $L^2$  to a non degenerate limit by establishing it is bounded in  $L^2$ . We write  $\mathbb{E}(M_\varepsilon^2)$  as

$$\mathbb{E}(M_\varepsilon^2) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{2\kappa_\varepsilon} \mathbb{E}(Y_\varepsilon(x)Y_\varepsilon(y)) \sigma_K(dx) \sigma_K(dy) .$$

Hence by Lemma 3.2

$$\begin{aligned} \mathbb{E}(M_\varepsilon^2) &\leq \text{Leb}(K)^{-1} \int_{\mathbb{R}^d} \mathbf{1}_{|z| \leq \text{diam}(K)} \exp \left( v_d \int_\varepsilon^{+\infty} r^{d-1} (r - b|z|)_+ \mu(dr) \right) dz \\ &\leq \text{Leb}(K)^{-1} v_d b^{-d} \int_0^{b \text{diam}(K)} u^{d-1} \exp \left( v_d \int_\varepsilon^{+\infty} r^{d-1} (r - bu)_+ \mu(dr) \right) du \\ &\leq \text{Leb}(K)^{-1} v_d b^{-d} e^{v_d \int_1^{+\infty} r^d \mu(dr)} \left( \int_0^1 u^{d-1} \exp \left( v_d \int_u^1 r^{d-1} (r - bu) \mu(dr) \right) du \right), \end{aligned}$$

which is finite by hypothesis. Therefore the convergence of the martingale  $(M_\varepsilon)_{\varepsilon>0}$  in  $L^2$  to a non degenerate limit is established. Once noticed the following sequence of increasing events:

$$\{K \subset \Xi\} \subset \{K \subset \limsup_{\varepsilon \rightarrow 0} \Xi_\varepsilon\} \subset \{\forall x \in K, \exists \eta > 0, \varepsilon < \eta \Rightarrow Y_\varepsilon(x) = 0\} \subset \{\lim_{\varepsilon \rightarrow 0} M_\varepsilon = 0\},$$

we obtain  $\mathbb{P}(K \subset \Xi) < 1$ . Lemma 2.2 yields the conclusion.  $\square$

A converse result can be obtained in the case of low frequency covering when there exist some balls which are large enough to cover  $\Xi$  at once.

**Theorem 3.3.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ . If*

$$(5) \quad \int_1^{+\infty} r^d \mu(dr) = +\infty,$$

*then  $\psi(\mu) = 1$  and  $\mathbb{R}^d$  is almost surely low frequency covered by  $\Xi$ .*

*Proof.* We prove that (5) implies that  $B(0, 1)$  is a.s. covered by a single ball of  $\Xi$ . Indeed, this will happen as soon as there exists a point  $(x, r) \in \Phi$  such that  $r > |x| + 1$ . The probability of this last event is equal to  $1 - e^{-\alpha}$  with

$$\alpha = \int_{\mathbb{R}^d} \int_0^{+\infty} \mathbf{1}_{r > |x| + 1} dx \mu(dr) = v_d \int_1^{+\infty} (r - 1)^d \mu(dr) = +\infty .$$

$\square$

We now deal with the case of high frequency covering, i.e. when  $\psi(\mu) = \psi(\mu_H) = 1$  where  $\mu_H = \mathbf{1}_{(0,1]} \mu$ . Since we are looking for sufficient conditions ensuring that  $\mathbb{R}^d$  is a.s. high frequency covered by  $\Xi$ , in view of Theorem 3.1 we will assume that  $\int_0^1 r^d \mu(dr) = +\infty$  and  $\mu$  will certainly not be a finite measure.

**Theorem 3.4.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ . If*

$$(6) \quad \limsup_{\varepsilon \rightarrow 0} \left( \left( \int_\varepsilon^1 r^d \mu(dr) \right)^{d-1} \int_\varepsilon^1 \mu(dr) \right)^{-1} \exp \left( \frac{v_d}{2^d} \int_\varepsilon^1 r^d \mu(dr) \right) = +\infty,$$

*for  $v_d$  the volume of the unit ball in  $\mathbb{R}^d$ , then  $\psi(\mu) = 1$  and  $\mathbb{R}^d$  is almost surely high frequency covered by  $\Xi$ .*



*Proof.* The proof of this result is mainly inspired by [10]. We assume  $d \geq 2$  (the case  $d = 1$  may be written in a more simple way).

First let us remark that one has

$$\int_{\varepsilon}^1 r^d \mu(dr) \leq \int_{\varepsilon}^1 \mu(dr),$$

and (6) implies that  $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 r^d \mu(dr) = +\infty$ . Without loss of generality one can replace all the integrals  $\int_{\varepsilon}^1$  by  $\int_{\varepsilon}^b$  in (6) for some  $b \leq 1/4$  and choose  $\varepsilon > 0$  small enough such that  $\varepsilon < b$  and  $\int_{\varepsilon}^1 r^d \mu(dr) \geq \frac{2^d}{v_d} \geq 1$ . Let us consider

$$\Xi_{\varepsilon} = \bigcup_{(x,r) \in \Phi; x \in [0,1]^d, \varepsilon \leq r \leq b} B(x,r) \subset \Xi.$$

Then  $\Xi_{\varepsilon}$  is a random Boolean model associated with a Poisson process of germs in  $\mathbb{R}^d$  of intensity

$$\lambda_{\varepsilon} = \int_{\varepsilon}^b \mu(dr),$$

and open spherical grains whose random radii are distributed as

$$\mu_{\varepsilon}(dr) = \lambda_{\varepsilon}^{-1} \mathbf{1}_{[\varepsilon,b]}(r) \mu(dr).$$

Let us get an upper bound for  $\mathbb{P}(V_{\varepsilon} \neq \emptyset)$ , where

$$V_{\varepsilon} = [0,1]^d \cap \left( \mathbb{R}^d \setminus \Xi_{\varepsilon} \right)$$

is the vacant region inside the cube  $[0,1]^d$ . Note that  $N_{\varepsilon}^{\Phi}$ , the number of centers thrown on  $[0,1]^d$  follows a Poisson law of parameter  $\lambda_{\varepsilon}$ .

Let  $n \geq d$ , conditionally to  $\{N_{\varepsilon}^{\Phi} = n\}$ , let us denote  $X_1, \dots, X_n$  the independent centers of the balls, uniformly distributed on  $[0,1]^d$  and  $R_1, \dots, R_n$  the independent radii with common law  $\mu_{\varepsilon}$  independent from the centers. Let us restrict our attention to the cube  $[0,1]^d$  with the induced topology. We consider the open set

$$\Xi_{\varepsilon}(n) = \left( \bigcup_{j=1}^n B(X_j, R_j) \right) \cap [0,1]^d,$$

and its complementary, the vacant region inside the cube  $[0,1]^d$

$$V_{\varepsilon}(n) = [0,1]^d \setminus \Xi_{\varepsilon}(n),$$

such that

$$\mathbb{P}(V_{\varepsilon} \neq \emptyset / N_{\varepsilon}^{\Phi} = n) = \mathbb{P}(V_{\varepsilon}(n) \neq \emptyset).$$

Let us remark that, since  $[0,1]^d$  is connected  $V_{\varepsilon}(n) \neq \emptyset$  if and only if

$$\partial \Xi_{\varepsilon}(n) = \overline{\Xi_{\varepsilon}(n)} \setminus \Xi_{\varepsilon}(n) \neq \emptyset.$$

For any  $1 \leq j \leq n$ , we write

$$\mathcal{S}(j) = S(X_j, R_j) \cap [0,1]^d = \left\{ x \in [0,1]^d; \|x - X_j\| = R_j \right\},$$

such that

$$\partial \Xi_{\varepsilon}(n) = \left( \bigcup_{j=1}^n \mathcal{S}(j) \right) \cap \left( \bigcup_{j=1}^n B(X_j, R_j) \right)^c.$$

Let us assume for a while that  $d \geq 2$ . We introduce the number of isolated balls

$$N_1(n) = \sum_{j=1}^n \mathbf{1}_{\mathcal{S}(j) \subset \partial \Xi_\varepsilon(n)},$$

and, as in [10], the number of intersections of the surfaces of two balls in  $[0, 1]^d$  which are not completely covered by  $\Xi_\varepsilon(n)$

$$N_h(n) = \sum_{i < j} \mathbf{1}_{\mathcal{S}(j) \cap \mathcal{S}(i) \cap \partial \Xi_\varepsilon(n) \neq \emptyset}.$$

By definition it is clear that if one of these numbers is not zero then  $\partial \Xi_\varepsilon(n) \neq \emptyset$ . Conversely, let us assume that  $\partial \Xi_\varepsilon(n) \neq \emptyset$ , then there exists  $1 \leq i \leq n$  with  $\mathcal{S}(i) \cap \partial \Xi_\varepsilon(n) \neq \emptyset$ . Let us assume that  $N_h(n) = 0$ . Then, for any  $j \neq i$  we have  $\mathcal{S}(j) \cap \mathcal{S}(i) \cap \partial \Xi_\varepsilon(n) = \emptyset$ . This means that we can write

$$\mathcal{S}(i) = \left( \mathcal{S}(i) \cap \overline{\bigcup_{j \neq i} B(X_j, R_j)} \right) \cup \left( \mathcal{S}(i) \cap \left( \bigcup_{j \neq i} B(X_j, R_j) \right)^c \right),$$

as an union of two disjoint closed sets. Since  $\mathcal{S}(i)$  is connected this means that  $\mathcal{S}(i) \subset \left( \bigcup_{j \neq i} B(X_j, R_j) \right)^c$ . Therefore  $\mathcal{S}(i) \subset \partial \Xi_\varepsilon(n)$  and  $N_1 \geq 1$ . Consequently,

$$\{V_\varepsilon(n) \neq \emptyset\} = \{\partial \Xi_\varepsilon(n) \neq \emptyset\} = \{N_1(n) \geq 1\} \cup \{N_h(n) \geq 1\}.$$

Let us first consider  $\mathbb{P}_n(N_1(n) \geq 1)$ , where  $\mathbb{P}_n$  denotes the conditional probability with respect to  $\{N_\varepsilon^\Phi = n\}$ . By Markov's inequality

$$\mathbb{P}_n(N_1(n) \geq 1) \leq \mathbb{E}_n(N_1(n)) = n \mathbb{P}_n(\mathcal{S}(1) \subset \partial \Xi_\varepsilon(n)).$$

Now, let us write  $C(X_j, R_j) = B(X_j, R_j) \cap [0, 1]^d$  and  $\overline{C(X_j, R_j)} = \overline{B(X_j, R_j)} \cap [0, 1]^d$  and remark that  $C(X_j, R_j)$  is connected for any  $1 \leq j \leq n$ . Therefore we get

$$\begin{aligned} \mathbb{P}_n(\mathcal{S}(1) \subset \partial \Xi_\varepsilon(n)) &= \mathbb{P}_n\left(\forall 2 \leq j \leq n, C(X_j, R_j) \subset \overline{C(X_1, R_1)}^c \text{ or } C(X_j, R_j) \subset C(X_1, R_1)\right) \\ &= \int_{[0, 1]^d} \int_0^{+\infty} (1 - p_n(x_1, r_1))^{n-1} \mu_\varepsilon(dr_1) dx_1, \end{aligned}$$

where, for  $x_1 \in [0, 1]^d$  and  $r_1 \in [\varepsilon, b]$ ,

$$\begin{aligned} p_n(x_1, r_1) &= \mathbb{P}_n\left(C(X_2, R_2) \cap \overline{B(x_1, r_1)} \neq \emptyset \text{ and } C(X_2, R_2) \cap B(x_1, r_1)^c \neq \emptyset\right) \\ &\geq \mathbb{P}_n\left(X_2 \in C(x_1, r_1 + R_2) \setminus \overline{C(x_1, (r_1 - R_2)_+)}\right), \end{aligned}$$

where, as usual,  $(r_1 - R_2)_+ = \max(r_1 - R_2, 0)$ . By a change of variables in polar coordinates, for all  $x_1 \in [0, 1]^d$  and  $r_1 \in [\varepsilon, b] \subset [0, 1/4]$ ,

$$\begin{aligned} &\mathbb{P}_n\left(X_2 \in C(x_1, r_1 + R_2) \setminus \overline{C(x_1, (r_1 - R_2)_+)}\right) \\ &= \int_0^{+\infty} \int_{(t, \theta) \in ((r_1 - r_2)_+, r_1 + r_2) \times S^{d-1}} \mathbf{1}_{\{x_1 + t\theta \in [0, 1]^d\}} t^{d-1} dt d\theta \mu_\varepsilon(dr_2) \\ &\geq \frac{v_d}{2^d} \int_0^{+\infty} \left( (r_1 + r_2)^d - (r_1 - r_2)_+^d \right) \mu_\varepsilon(dr_2). \end{aligned}$$

Note that

$$\begin{aligned} \int_{r_2 \leq r_1} \left( (r_1 + r_2)^d - (r_1 - r_2)_+^d \right) \mu_\varepsilon(dr_2) &= \int_{r_2 \leq r_1} r_1^d \left( \left( 1 + \frac{r_2}{r_1} \right)^d - \left( 1 - \frac{r_2}{r_1} \right)^d \right) \mu_\varepsilon(dr_2) \\ &\geq d \int_{r_2 \leq r_1} r_1^{d-1} r_2 \mu_\varepsilon(dr_2) \\ &\geq d \int_{r_2 \leq r_1} r_2^d \mu_\varepsilon(dr_2). \end{aligned}$$

Therefore, for all  $x_1 \in [0, 1]^d$  and  $r_1 \in [\varepsilon, b]$ ,

$$p_n(x_1, r_1) \geq \frac{v_d}{2^d} \int_0^{+\infty} r_2^d \mu_\varepsilon(dr_2).$$

This allows us to conclude that

$$(7) \quad \mathbb{P}_n(N_1(n) \geq 1) \leq n \left( 1 - \frac{v_d}{2^d} \int_0^{+\infty} r_2^d \mu_\varepsilon(dr_2) \right)^{n-1}.$$

Now let us be concerned with  $N_h(n)$ .

First we assume that  $d = 2$ . In this case the intersection of two spheres is reduced at most at two points in  $[0, 1]^2$ . Let  $x_1, x_2 \in [0, 1]^2$ ,  $r_1, r_2 \in [\varepsilon, b]$ , with  $|x_1 - x_2| \leq r_1 + r_2$  and write

$$T_2(x_1, r_1, x_2, r_2) = S(x_1, r_1) \cap S(x_2, r_2) \cap [0, 1]^2.$$

Again, by Markov's inequality

$$\mathbb{P}_n(N_h(n) \geq 1) \leq \mathbb{E}_n(N_h(n)).$$

Then, with the same arguments as previously,

$$\mathbb{E}_n(N_h(n)) = \frac{n(n-1)}{2} \int_0^{+\infty} \int_0^{+\infty} \int_{[0,1]^2 \times [0,1]^2} p_{n,2}(x_1, r_1, x_2, r_2) dx_1 dx_2 \mu_\varepsilon(dr_1) \mu_\varepsilon(dr_2),$$

where  $p_{n,2}(x_1, r_1, x_2, r_2)$  denotes the probability that  $n-2$  balls do not completely envelop  $T_2(x_1, r_1, x_2, r_2)$  if  $T_2(x_1, r_1, x_2, r_2) \neq \emptyset$ , and is 0 otherwise. If  $T_2(x_1, r_1, x_2, r_2) \neq \emptyset$ , then  $|x_1 - x_2| \leq r_1 + r_2$  and there exist  $z_1, z_2 \in [0, 1]^2$  with possibly  $z_1 = z_2$  such that  $T_2(x_1, r_1, x_2, r_2) = \{z_1, z_2\}$ . Therefore,

$$p_{n,2}(x_1, r_1, x_2, r_2) \leq \mathbb{P}_n \left( z_1 \notin \bigcup_{i=3}^n B(X_i, R_i) \right) + \mathbb{P}_n \left( z_2 \notin \bigcup_{i=3}^n B(X_i, R_i) \right).$$

For any  $z \in [0, 1]^2$  let us denote

$$(8) \quad \pi_n(z) = \mathbb{P}_n(z \in B(X_3, R_3)) = \int_0^{+\infty} \int_{[0,1]^2} \mathbf{1}_{|x_3 - z| \leq r_3} dx_3 \mu_\varepsilon(dr_3).$$

Note that

$$\frac{v_2}{2^2} \int_0^{+\infty} r^2 \mu_\varepsilon(dr) \leq \pi_n(z) \leq v_2 \int_0^{+\infty} r^2 \mu_\varepsilon(dr),$$

such that

$$p_{n,2}(x_1, r_1, x_2, r_2) \leq 2 \left( 1 - \frac{v_2}{2^2} \int_0^{+\infty} r^2 \mu_\varepsilon(dr) \right)^{n-2}.$$

Therefore

$$\mathbb{E}_n(N_h(n)) \leq v_2 n(n-1) \left(1 - \frac{v_2}{2^2} \int_0^{+\infty} r^2 \mu_\varepsilon(dr)\right)^{n-2} \int_0^{+\infty} \int_0^{+\infty} (r_1 + r_2)^2 \mu_\varepsilon(dr_1) \mu_\varepsilon(dr_2)$$

such that

$$(9) \quad \mathbb{P}_n(N_h(n) \geq 1) \leq 2^{2+1} v_2 n(n-1) \left(1 - \frac{v_2}{2^2} \int_0^{+\infty} r^2 \mu_\varepsilon(2r)\right)^{n-2} \int_0^{+\infty} r^2 \mu_\varepsilon(dr).$$

Now, let us assume that  $d \geq 3$  and let us remark that

$$N_h(n) \leq N_2(n) + \sum_{i < j < k} \mathbf{1}_{\mathcal{S}(i) \cap \mathcal{S}(j) \cap \mathcal{S}(k) \cap \partial \Xi_\varepsilon(n) \neq \emptyset},$$

where

$$N_2(n) = \sum_{i < j} \mathbf{1}_{\mathcal{S}(i) \cap \mathcal{S}(j) \subset \partial \Xi_\varepsilon(n) \neq \emptyset}.$$

Therefore for any  $k \leq d-1$  let us denote  $N_k(n)$  the number of non empty intersections of  $k$  spheres such that none of the other  $n-k$  balls even intersect these spheres and  $N_d(n)$  the number of non empty intersections of  $d$  spheres (reduced at most at two points) which are not completely enveloped by the other  $n-d$  balls. The same arguments as previously lead to:  $\forall k = 1, \dots, d$

$$(10) \quad \mathbb{P}_n(N_k(n) \geq 1) \leq c_k 2^{(k-1)d} k! v_d^{k-1} \binom{n}{k} \left(1 - \frac{v_d}{2^d} \int_0^{+\infty} r^d \mu_\varepsilon(dr)\right)^{n-k} \left(\int_0^{+\infty} r^d \mu_\varepsilon(dr)\right)^{k-1},$$

where  $c_k = 1$  for  $1 \leq k \leq d-1$  and  $c_d = 2$ . Note that if  $d = 1$ , there are at most  $2n$  points in  $\partial \Xi_\varepsilon(n)$  such that  $\mathbb{P}_n(V_\varepsilon(n) \neq \emptyset)$  is also bounded by (10) in this case. Then, for any  $d \geq 1$ , one can find a positive constant  $C(d) > 0$  such that

$$\mathbb{P}(\{V_\varepsilon \neq \emptyset\} \cap N_\varepsilon^\Phi \geq d) \leq C(d) \lambda_\varepsilon \left(\int_\varepsilon^b r^d \mu(dr)\right)^{d-1} e^{-\frac{v_d}{2^d} \int_\varepsilon^b r^d \mu(dr)}.$$

Therefore, one can find  $C(d) > 0$  such that

$$\mathbb{P}(V_\varepsilon \neq \emptyset) \leq C(d) \left(\lambda_\varepsilon \left(\int_\varepsilon^b r^d \mu(dr)\right)^{d-1} e^{-\frac{v_d}{2^d} \int_\varepsilon^b r^d \mu(dr)} + \lambda_\varepsilon^{d-1} e^{-\lambda_\varepsilon}\right).$$

Now, according to (6), one can choose a sequence  $(\varepsilon_n)_{n \geq 0}$  such that  $\varepsilon_n \xrightarrow{n \rightarrow +\infty} 0$  and  $\mathbb{P}(V_{\varepsilon_n} \neq \emptyset) \xrightarrow{n \rightarrow +\infty} 0$  such that

$$\mathbb{P}([0, 1]^d \subset \Xi) \geq \lim_{n \rightarrow +\infty} \mathbb{P}(V_{\varepsilon_n} = \emptyset) = 1.$$

By Lemma 2.2 this allows us to conclude that  $\mathbb{R}^d$  is almost surely covered by  $\Xi$ .  $\square$

With Theorem 3.4, a sufficient condition for a.s. high frequency coverage has been established. Note that this assumption is stronger than the iff condition (1) of dimension

1. Let us remark that the sufficient covering assumption (6) is implied by the following stronger one

$$(11) \quad \limsup_{\varepsilon \rightarrow 0} \varepsilon^d \left( \int_{\varepsilon}^1 r^d \mu(dr) \right)^{-d} \exp \left( \frac{v_d}{2^d} \int_{\varepsilon}^1 r^d \mu(dr) \right) = +\infty .$$

On the other hand, assumption (3) implies

$$(12) \quad \exists \alpha > 1, \limsup_{\varepsilon \rightarrow 0} \varepsilon^d |\ln \varepsilon|^\alpha \exp \left( v_d \int_{\varepsilon}^1 r^d \mu(dr) \right) = +\infty ,$$

which is a sufficient condition for coverage. Hence, putting together Theorem 3.1 and Theorem 3.4 yields the next chain rule:

$$(11) \Rightarrow \text{a.s. coverage} \Rightarrow (12).$$

Actually, the previous covering and non-covering assumptions can be reformulated regarding the asymptotic behavior of  $\int_{\varepsilon}^1 r^d \mu(dr)$  compared with  $|\ln \varepsilon|$ . A simple criteria for coverage or non coverage is established in the following Proposition.

**Proposition 3.5.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, 1]$ .*

- If  $\limsup_{\varepsilon \rightarrow 0} \left( |\ln \varepsilon|^{-1} \int_{\varepsilon}^1 r^d \mu(dr) \right) > 2^d d/v_d$  then  $\psi(\mu) = 1$
- If  $\limsup_{\varepsilon \rightarrow 0} \left( |\ln \varepsilon|^{-1} \int_{\varepsilon}^1 r^d \mu(dr) \right) < d/v_d$  then  $\psi(\mu) = 0$

In a former work [4], we studied random balls models with the  $\sigma$ -finite measure  $\mu$  prescribing the distribution of the balls radius specified to be asymptotically of a power law type

$$\mu(dr) = r^{-\beta-1} dr.$$

Our concern was to take advantage of the power law in order to exhibit self-similar properties of the associated shot-noise field. The case  $\beta = d$  is of particular interest since it ensures scale invariance of multiplicative cascade (see [3] or [5] for instance).

We turn now to the question of coverage for those models and summarize all the covering and non covering conditions of the previous sections in this special case. We look separately at low and large frequency coverages, focusing on the parameter  $\beta$ .

**Corollary 3.6.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ .*

(i) *Assume that  $\mu(dr) = f(r) \mathbf{1}_{[1, +\infty)}(r) dr$  with  $f(r) \underset{r \rightarrow +\infty}{\sim} r^{-\beta-1}$  for some  $\beta \in \mathbb{R}$ .*

- If  $\beta \leq d$  then  $\forall \lambda > 0, \psi(\lambda\mu) = 1$ .
- If  $\beta > d$  then  $\forall \lambda > 0, \psi(\lambda\mu) = 0$ .

(ii) *Assume that  $\mu(dr) = f(r) \mathbf{1}_{(0, 1]}(r) dr$  with  $f(r) \underset{r \rightarrow 0}{\sim} r^{-\beta-1}$  for some  $\beta \in \mathbb{R}$ .*

- If  $\beta > d$  then  $\forall \lambda > 0, \psi(\lambda\mu) = 1$ .
- If  $\beta < d$  then  $\forall \lambda > 0, \psi(\lambda\mu) = 0$ .
- If  $\beta = d$  then  $\forall \lambda < \frac{d}{v_d}, \psi(\lambda\mu) = 0$  while  $\forall \lambda > \frac{2^d d}{v_d}, \psi(\lambda\mu) = 1$ .

The case  $\mu$  supported by  $(0, 1]$  and  $\beta = d$  is particularly interesting since a critical behavior is observed. For an intensity  $\lambda$  small enough, coverage never occurs whereas for  $\lambda$  large enough, coverage always occurs. Therefore, the whole space coverage clearly

depends on the intensity  $\lambda$  chosen. Let us emphasize that this result can only be obtained using not finite measures. Actually, for finite measures and associated Boolean models, such a behavior can also be observed considering *eventual coverage* [1]. However this holds only for dimension  $d = 1$ .

In our setting, the result stands whatever the dimension  $d$  is. As for many percolation questions, the exact value of the critical intensity is unknown. In dimension one, since an iff condition for a.s. coverage is available, the exact value is known (and equal to  $1/2$ , see [16] where the fact is mentioned). In the next section, critical behaviors will be studied and linked with continuum percolation.

#### 4. CRITICAL REGIME AND LINK WITH CONTINUUM PERCOLATION

We still consider the occupied region  $\Xi = \bigcup_{(x,r) \in \Phi} B(x,r)$ , with  $\Phi$  a Poisson point process in  $\mathbb{R}^d \times (0, +\infty)$  of intensity  $dx\mu(dr)$  for  $\mu$  a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$ . We set  $W$  the connected component of  $\Xi$  which contains 0. In percolation theory one wonders whether  $W$  is unbounded with a positive probability. As [19] one can consider a percolation function defined by

$$\theta(\mu) = \mathbb{P}_\mu(W \text{ is unbounded}).$$

Note that the percolation function  $\theta$  satisfies Proposition 2 as the coverage function  $\psi$  does. Therefore both functions  $\lambda \mapsto \psi(\lambda\mu)$  and  $\lambda \mapsto \theta(\lambda\mu)$  are increasing and then one can define the coverage and percolation critical intensities as

$$(13) \quad \lambda_c(\mu) = \inf\{\lambda \geq 0; \psi(\lambda\mu) > 0\} \text{ and } \lambda_p(\mu) = \inf\{\lambda \geq 0; \theta(\lambda\mu) > 0\},$$

where as usual we set  $+\infty$  when the sets are empty. Let us remark that  $\lambda_p(\mu) \leq \lambda_c(\mu)$ , since we clearly have  $\psi(\mu) \leq \theta(\mu)$ . Let us also recall that by Lemma 2.2, when  $\lambda_c(\mu) \in (0, +\infty)$ ,

$$\lambda < \lambda_c(\mu) \Rightarrow \psi(\lambda\mu) = 0 \quad \text{and} \quad \lambda > \lambda_c(\mu) \Rightarrow \psi(\lambda\mu) = 1,$$

while the second statement is replaced by  $\lambda > \lambda_p(\mu) \Rightarrow \theta(\lambda\mu) > 0$  for the percolation function, when  $\lambda_p(\mu) \in (0, +\infty)$ .

When  $\mu$  is a finite measure on  $(0, +\infty)$  and  $R$  is a random variable of law  $\mu_0 = \frac{1}{\mu((0, +\infty))}\mu$ , one can prove that  $\lambda_p(\mu) < +\infty$  when  $R \neq 0$  a.s. (see Remark of [19] p.52). Moreover, when  $R$  is bounded almost surely, there exists a sharp transition corresponding to a critical value  $\lambda_p(\mu) \in (0, +\infty)$  of the intensity (see [24, 25] for example):

- when  $\lambda < \lambda_p(\mu)$ , then  $W$  is bounded almost surely;
- when  $\lambda > \lambda_p(\mu)$ , then  $W$  is unbounded with positive probability.

The situation when  $\lambda_p(\mu) \in (0, +\infty)$  is at the heart of percolation theory.

When  $R$  is not bounded, Hall [11] proved that if  $\mathbb{E}(R^{2d-1}) < +\infty$ , then  $W$  contains almost surely a finite number of balls for  $\lambda$  small enough such that we still have  $\lambda_p(\mu) \in (0, +\infty)$ . In fact [8] proved that  $\lambda_p(\mu) \in (0, +\infty)$  if and only if  $\mathbb{E}(R^d) < +\infty$ , which is also equivalent to the fact that  $\lambda_c(\mu) = +\infty$ .

When  $\mu$  is not a finite measure on  $(0, +\infty)$  and  $d \geq 2$ , [9] proved that  $\lambda_p(\mu) \in (0, +\infty)$  if and only if

$$(14) \quad \sup_{\varepsilon > 0} \varepsilon^d \mu([\varepsilon, 1)) < +\infty \text{ and } \int_1^{+\infty} r^d \mu(dr) < +\infty.$$

In the same spirit, using very similar arguments as for Proposition 3.5, we can get an iff condition for the existence of a non degenerate coverage critical intensity. Furthermore we exhibit upper and lower bounds for this critical intensity.

**Proposition 4.1.** *Let  $\mu$  be a  $\sigma$ -finite non-negative measure on  $(0, +\infty)$  and let us denote*

$$\ell(\mu) = \limsup_{\varepsilon \rightarrow 0} \left( |\ln \varepsilon|^{-1} \int_{\varepsilon}^1 r^d \mu(dr) \right) \in [0, +\infty]. \text{ Then}$$

$$\lambda_c(\mu) \in (0, +\infty) \Leftrightarrow \ell(\mu) \in (0, +\infty) \text{ and } \int_1^{+\infty} r^d \mu(dr) < +\infty.$$

Moreover if  $\lambda_c(\mu) \in (0, +\infty)$  then

$$\frac{d}{\ell(\mu)v_d} \leq \lambda_c(\mu) \leq \frac{2^d d}{\ell(\mu)v_d} \text{ and } \lambda_p(\mu) \leq \frac{2^d d}{\ell(\mu)v_d}.$$

A typical example when this situation occurs is given for the measure

$$\mu(dr) = r^{-d-1} \mathbf{1}_{(0,1]}(r) dr.$$

In this case  $\ell(\mu) = 1$  and following Proposition 4.1, we obtain an upper bound for the percolation critical density:

$$\lambda_p(\mu) \leq \lambda_c(\mu) \leq \frac{2^d d}{v_d}.$$

It is worth to emphasize that such a universal bound is very useful for applications. Actually, percolation thresholds are usually estimated using numerical simulations (see [26] for a theoretical basis in the framework of 2 dimensional Boolean models).

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