



HAL
open science

Equilibria und weiteres Heiteres

Dov Gabbay, Karl Schlechta

► **To cite this version:**

| Dov Gabbay, Karl Schlechta. Equilibria und weiteres Heiteres. 2009. hal-00406635v6

HAL Id: hal-00406635

<https://hal.science/hal-00406635v6>

Preprint submitted on 8 Jul 2010 (v6), last revised 15 Aug 2011 (v10)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Equilibria und weiteres Heiteres

Dov M Gabbay *
King's College, London †
and
Bar-Ilan University, Israel ‡
and
University of Luxembourg §

Karl Schlechta ¶
Laboratoire d'Informatique Fondamentale de Marseille ||

July 8, 2010

Abstract

We investigate several technical and conceptual questions.

Contents

1	Introduction	2
2	Countably many disjoint sets	2
3	Operations on logical structures	3
3.1	Introduction	3
3.1.1	Conditionals	3
3.2	Theory revision	4
3.2.1	“Meta-revision”	4
3.3	Preferential systems	5
3.3.1	Why does \sim not modify itself?	5
3.3.2	(Meta) Operations on logics	5
3.3.3	Implementation	5
3.3.4	Operations on linear and ranked structures	6
3.3.5	Intuitionistic preferential logic	6
3.4	Non-monotonic interpolation	6

*Dov.Gabbay@kcl.ac.uk, www.dcs.kcl.ac.uk/staff/dg

†Department of Computer Science, King's College London, Strand, London WC2R 2LS, UK

‡Department of Computer Science, Bar-Ilan University, 52900 Ramat-Gan, Israel

§Computer Science and Communications, Faculty of Sciences, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg

¶ks@cmi.univ-mrs.fr, karl.schlechta@web.de, <http://www.cmi.univ-mrs.fr/~ks>

||UMR 6166, CNRS and Université de Provence, Address: CMI, 39, rue Joliot-Curie, F-13453 Marseille Cedex 13, France

4 **Boldness and justification**

References

1 Introduction

We present here various small results, which may one day be published in a bigger paper, and which we wish to make already available to the community.

2 Countably many disjoint sets

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D.Makinson (personal communication).

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D.Makinson (personal communication).

Example 2.1

There is a countably infinite set of formulas s.t. the defined model sets are pairwise disjoint.

Let $p_i : i \in \omega$ be propositional variables.

Consider $\phi_i := \bigwedge\{\neg p_j : j < i\} \wedge p_i$ for $i \in \omega$.

Obviously, $M(\phi_i) \neq \emptyset$ for all i .

Let $i < i'$, we show $M(\phi_i) \cap M(\phi_{i'}) = \emptyset$. $M(\phi_{i'}) \models \neg p_i$, $M(\phi_i) \models p_i$.

□

Fact 2.1

Any set X of consistent formulas with pairwise disjoint model sets is at most countable.

Proof

Let such X be given.

(1) We may assume that X consists of conjunctions of propositional variables or their negations.

Proof: Re-write all $\phi \in X$ as disjunctions of conjunctions ϕ_j . At least one of the conjunctions ϕ_j is consistent. Replace ϕ by one such ϕ_j . Consistency is preserved, as is pairwise disjointness.

(2) Let X be such a set of formulas. Let $X_i \subseteq X$ be the set of formulas in X with length i , i.e. a consistent conjunction of i many propositional variables or their negations, $i > 0$.

As the model sets for X are pairwise disjoint, the model sets for all $\phi \in X_i$ have to be disjoint.

(3) It suffices now to show that each X_i is at most countable, we even show that each X_i is finite.

Proof by induction:

Consider $i = 1$. Let $\phi, \phi' \in X_1$. Let ϕ be p or $\neg p$. If ϕ' is not $\neg\phi$, then ϕ and ϕ' have a common model. So one must be p , the other $\neg p$. But these are all possibilities, so $card(X_1)$ is finite.

Let the result be shown for $k < i$.

Consider now X_i . Take arbitrary $\phi \in X_i$. Wlog, $\phi = p_1 \wedge \dots \wedge p_i$. Take arbitrary $\phi' \neq \phi$. As $M(\phi) \cap M(\phi') = \emptyset$, ϕ' must be a conjunction containing one of $\neg p_k$, $1 \leq k \leq i$. Consider now $X_{i,k} := \{\phi' \in X_i : \phi' \text{ contains } \neg p_k\}$. Thus $X_i = \{\phi\} \cup \bigcup\{X_{i,k} : 1 \leq k \leq i\}$. Note that all $\psi, \psi' \in X_{i,k}$ agree on $\neg p_k$, so the situation in $X_{i,k}$ is isomorphic to X_{i-1} . So, by induction hypothesis, $\text{card}(X_{i,k})$ is finite, as all $\phi' \in X_{i,k}$ have to be mutually inconsistent. Thus, $\text{card}(X_i)$ is finite. (Note that we did not use the fact that elements from different $X_{i,k}$, $X_{i,k'}$ also have to be mutually inconsistent, our rough proof suffices.)

□

Note that the proof depends very little on logic. We needed normal forms, and used 2 truth values. Obviously, we can easily generalize to finitely many truth values.

3 Operations on logical structures

3.1 Introduction

In many cases, one wants more than a static structure:

- (1) dynamic theory revision a la Pearl etc.
- (2) revising a preferential logic
- (3) changing the language in interpolation
- (4) intuitionistic preferential logic: arrows are added

etc.

One can ask about such “meta-operations” for instance:

- (1) are properties preserved, e.g. is the result of working on a ranked structure again a ranked structure?
- (2) do we lose properties?
- (3) do we win new properties?
- (4) is it reasonable to require higher operators to follow the same laws as the basic operators, e.g. minimal change, and if so, e.g. minimal change of what?
- (5) what can be a structural semantics for such higher operators?

3.1.1 Conditionals

$A > B$ may mean: A becomes true (in the world), or the agent learns/believes A then: B becomes true, or the agent believes B , or the agent does B , or the agent brings B about, or so. Similarly, ternary conditionals $(A, B) > C$ can have very different meanings, and their formal properties may reflect this.

3.2 Theory revision

3.2.1 “Meta-revision”

AGM left K (A below) constant, and this may have contributed to subsequent confusion.

One sees sometimes a conditional $B > C$ expressing that after revising with B , C will hold.

But this hides the fact that it is in reality a 3-place conditional:

$(A, B) > C$: after revising A with B , C will hold, $A * B \models C$

A, B , etc. are formulas, i.e. partial information.

$(A, B) > C$ is partial information about the revision strategy, it describes just a bit of the whole picture. In the LMS tradition, see [LMS01], a revision strategy is just a distance between models. So $(A, B) > C$ describes one part of the distance.

Thus, $(A, B) > C$ is a partial revision strategy, or a set of distances which are compatible with $(A, B) > C$, just as a formula is a set of models.

But now, we have a perfect analogy:

We had $A * B \models C$ for formulas A, B, C , and now we can revise partial strategies:

$((A, B) > C, (A', B') > C') \gg (A'', B'') > C''$, i.e.

if we revise the partial strategy $(A, B) > C$ with the partial strategy $(A', B') > C'$, then the new strategy gives $(A'', B'') > C''$. \gg is the “meta-conditional”.

It all becomes transparent, and we can iterate the whole thing as often as we want.

In the distance language, we have a set of distances on models which all satisfy $(A, B) > C$, i.e. the B -models closest to the A -models all satisfy C , and another set of distances which all satisfy $(A', B') > C'$, we revise the first with the second using a “meta-distance” (a distance on the set of all distances between models of the base language), and get a new set of distances which all satisfy $(A'', B'') > C''$. If the first two sets are consistent, i.e. there is a distance which satisfies $(A, B) > C$ and $(A', B') > C'$, then the result is the intersection of the two distance sets. This corresponds, as usual, to the respect of 0 by a distance: $d(x, y) = 0$ iff $x = y$. Of course, we can consider here special distances like variants of the Hamming distance, working on a suitable set.

Of course, just as a formula may correspond to exactly 1 model, i.e. a complete consistent theory, we may also work with $*$ (i.e. the full revision strategy) instead of with $(A, B) > C$. So we may have $(*, *) \gg *$. Still, as shown in [LMS01], the distance will usually not be fully determined, so we still work with sets of distance.

Note that we can also construct mixed systems, which allow to evaluate expressions like $(A, (B, C) > D) > E$, where factual information/models are mixed with conditional structures - this might be needed e.g. for natural language. We can go as high as we want, or even go down, evaluating “on the fly”.

One problem is TR (global distances). The CFC approach will not work, as we cannot consider the left hand side individually. We have to express it via quantifiers (modal operators). We have to say $x \in X$ is such that there is $y \in Y$ s.t. for no $x' \in X, y' \in Y$ $d(x', y') < d(x, y)$. We use global modality. Let X be defined by ϕ , Y by ψ , then $\diamond\psi \neg(\diamond\phi \diamond\psi \langle x', y', x, y \rangle)$ where $\langle x', y', x, y \rangle$ expresses that $d(x', y') < d(x, y)$ (axiomatize suitably so it is a distance). Etc, should work?

The distance need not be defined everywhere, so it may return “unknown”.

Pearl et al. For Pearl et al. (see [DP94]), Boutilier (see [Bou94]), and Kern-Isberner (see [Ker99]), an epistemic state \mathcal{E} is a pair $(\mathcal{B}, \mathcal{C})$, where \mathcal{B} is a set of beliefs (classical formulas), and \mathcal{C} a (perhaps partial) revision strategy coded by a set of conditionals, whose elements are classical formulas. In our terminology such an conditional will have the form $(\bigwedge \mathcal{B}, A) > C$, where $\bigwedge \mathcal{B}$ is the conjunction of factual beliefs, A, C are classical formulas, expressing: On the basis of \mathcal{B} , if I were to learn A , then I would believe C .

Pearl’s criticism of the AGM approach was that revising an epistemic state by some factual information A should not only modify factual beliefs, i.e. \mathcal{B} , but also the revision strategy \mathcal{C} . Pearl et al. gave some conditions this modification of \mathcal{C} should satisfy.

In our above notation, we then have $\mathcal{E} * A = (\mathcal{B}, \mathcal{C}) * A = (\mathcal{B}', \mathcal{C}') = \mathcal{E}'$, where \mathcal{C} determines the modification of

\mathcal{B} to \mathcal{C} (by the conditionals $A > X$), but *not* the modification of \mathcal{C} to \mathcal{C}' .

Boutilier, see [Bou94], and Kern-Isberner, see [Ker99], extended this idea to revising epistemic states not only by factual information, but also by conditional information. (Kern-Isberner codes factual information X by the conditional $TRUE > X$, and thus avoids a distinction between the two.)

So we have $\mathcal{E} * C = (\mathcal{B}, \mathcal{C}) * (X > Y) = (\mathcal{B}', \mathcal{C}') = \mathcal{E}'$.

Revising by the factual conditional $X > Y$ imposes restrictions also on the transformation of \mathcal{C} to \mathcal{C}' .

3.3 Preferential systems

3.3.1 Why does $|\sim$ not modify itself?

We saw in Section 3.2 (page 4) how Pearl et al. introduced a revision operator $*$ whose application changes (the conditional part of) $*$ itself.

The question is obvious: Is there a logical formalism \vdash which, applied to some formula ϕ , will not only produce a consequence ψ , but also a new logic \vdash' ?

To the authors' knowledge, this does not exist.

We may use Gabbay's idea of reactivity to build such a logic: applying the logic changes it - this would give a formal motivation to the enterprise, from the other side, so to say.

In more detail: simple arrows obey the fundamental law of preferential structures. Adding higher arrows allows us to restrict from above, and thus describe any set, in a static way, see [GS08b]. So we can describe $M(\phi)$. When we "activate" now $M(\phi)$, we use higher arrows to modify the basic preferential relation.

3.3.2 (Meta) Operations on logics

It is natural to consider the operations of deduction and revision on logical systems.

Given some logical system \vdash , we might deduce a new logic \vdash' from it and some formula ϕ , i.e. $\phi \vdash (\psi, \vdash')$. E.g., \vdash' might be weaker than \vdash (this corresponds to classical logic, which is weakening), or we might deduce a new, bolder logic, corresponding to a more daring reasoning (this corresponds to non-monotonic logic, where we go beyond classical logic, win more conclusions, at the price of less certainty).

Perhaps even more useful, we may see that our logic does not give the desired conclusions, and may want to revise it, by some minimal change which obtains the desired result.

As in the case of theory revision, we can take as arguments the whole logic \vdash , or just one or some pairs (ϕ, ψ) with $\phi \vdash \psi$. For instance, we might want to revise \vdash with some new pair (ϕ, ψ) , and see whether $\phi' \vdash' \psi'$ holds in the new logic \vdash' .

3.3.3 Implementation

Usually, working on the semantic side is easier. There are different ways to do it.

- (1) We can work with canonical structures (if they exist) - this may generate different results when we consider different structures as canonical. (This was a problem with [ALS98-1].)
- (2) We can work with the set of all structures corresponding to the logic, e.g. all preferential structures generating the logic
- (3) We can work with the algebraic semantics, i.e. usually with the smallest set of the filter, corresponding to $\mu(\phi)$, the set of minimal models of ϕ .
- (4) The reactive idea was carried out in [GS08b], where we modified preferential structures by adding higher order arrows. There, the view was static, but we can turn it dynamic to achieve revision and "meta-logic".

Algebraic semantics As the algebraic semantics usually is the most robust notion, this is perhaps the easiest to work with.

For each ϕ , $\mu(\phi)$ is defined. So a natural distance between \sim and \sim' is the set of ϕ where $\mu(\phi) \neq \mu'(\phi)$, and for each such ϕ the symmetrical set distance between $\mu(\phi)$ and $\mu'(\phi)$. This gives a distance based revision of \sim to \sim' .

For a “meta-logic”, we can as usual consider a preference relation between logics (which are now simple objects, just as classical models, given by their $\mu(\phi)$ for all ϕ), and work with the algebraic representation results of the second author, see e.g. [Sch04].

Structural semantics It is natural to define a distance between two preferential structures by looking at the arrow sets, or sets of pairs $\langle m, m' \rangle$, such that $m \prec m'$. Again, some Hamming distance would be a first answer.

We treated one technique of modifying general (and smooth) preferential structures in [GS08b].

3.3.4 Operations on linear and ranked structures

Making a linear or ranked structure simply reactive will usually result in a mess, where the central properties of such structures are destroyed.

It seems more reasonable to investigate operations which leave the structure more intact, and postpone questions about their realization.

We may consider here operations which

- (1) cut the linear or ranked structure in two parts, such that within in each part the structure stays as it was, and the two parts are incomparable (they look a bit like a tree trunk, which was cut with a saw in 2 parts)
- (2) do elementary exchange operations (permutations) in the case of linear orders.
- (3) for ranked structures, we may have an operation $\alpha(x, y)$, which puts x on y 's level, $\beta(x, y)$ which changes the levels of x and y , etc.

3.3.5 Intuitionistic preferential logic

We are not sure about all arrows. Some arrows are definitely there, others definitely out, some come and go.

We have successively better information about arrows, and thus about size. Int. rules about size.

The following seems new: We do not only have $\Box(\phi \sim \psi)$, but also $\Box\neg(\phi \sim \psi)$. This has to be treated, especially for ranked structures.

3.4 Non-monotonic interpolation

We investigated non-monotonic interpolation in [GS09c].

The main property needed can be summarized as follows:

Let X be a set, and $X' \cup X'' = X$ be a disjoint cover of X . Consider $\Sigma \subseteq \Pi X$. Suppose $\mu(\Sigma) \subseteq \Sigma'$, where the variables defining Σ' are all in X'' . We now have to consider $\mu(\Pi X' \times \Sigma'')$, where $\Sigma'' = \Sigma \upharpoonright X''$, the restriction of Σ to X'' . We want $\mu(\Pi X' \times \Sigma'') \subseteq \Sigma'$, this gives the desired interpolation.

Formally:

$$(\mu * 3) \mu(\Pi' \times \Sigma'') \upharpoonright X'' \subseteq \mu(\Sigma) \upharpoonright X''.$$

The point here is that, logically, only the X'' -part matters, as Σ' is the full product on X' : $\Sigma' = \Pi X' \times (\Sigma' \upharpoonright X'')$. When we go from $\mu(\Sigma)$ to $\mu(\Delta)$, where $\Delta := \Pi X' \times \Sigma''$, Δ is bigger than Σ in the X' -part, and identical in the X'' -part. So increasing Σ outside X'' does not increase $\mu(\Sigma)$ inside X'' .

The (set variant) of the Hamming order satisfies this property: If $\sigma = \sigma' \circ \sigma''$, $\tau = \tau' \circ \tau''$ (\circ is concatenation), then $\sigma \prec \tau$ iff $\sigma' \prec \tau'$ and $\sigma'' \prec \tau''$. Thus, if e.g. $\sigma, \tau \in \Sigma$, $\sigma'' \prec \tau''$, but $\tau' \prec \sigma'$, then τ may be minimal, but adding $\rho' \circ \sigma''$ with $\rho' \prec \tau'$ will eliminate τ .

This behaviour motivates the following reflections:

- (1) This condition ($\mu * 3$) points to a weakening of the Hamming condition:

Adding new “branches” in X' will not give new minimal elements in X'' , but may destroy other minimal elements in X'' . This can be achieved by a sort of semi-rankedness: If ρ and σ are different only in the X' -part, then $\tau \prec \rho$ iff $\tau \prec \sigma$, but not necessarily $\rho \prec \tau$ iff $\sigma \prec \tau$.

- (2) In more abstract terms:

When we separate support from attack (support: a branch σ' in X' supports a continuation σ'' in X'' iff $\sigma \circ \sigma''$ is minimal, i.e. not attacked, attack: a branch τ in X' attacks a continuation σ'' in X'' iff it prevents all $\sigma \circ \sigma''$ to be minimal), we see that new branches will not support any new continuations, but may well attack continuations.

More radically, we can consider paths σ'' as positive information, σ' as potentially negative information. Thus, Π' gives maximal negative information, and thus smallest set of accepted models.

The concept of size looks only at the result of support and attack, so it is necessarily somewhat coarse. Future research should also investigate both concepts separately.

- (3) We can interpret this as follows:

- (1) X'' determines the base set.

- (2) X' is the context. The context determines the choice (i.e. a subset of the base set).

- (3) When we compare this to preferential structures, we see that also in preferential structures the bigger the set, the more attacks are possible.

We broaden these considerations:

- (1) Following a tradition begun by Kripke, one has added structure to the set of classical models, reachability, preference, etc. Perhaps one should emphasize a more abstract approach, in the line of [Sch92], and elaborated in [Sch04], see in particular the distinction between structural and algebraic semantics in the latter. So we should separate structure from logic in the semantics, and treat what we called context above by a separate “machinery”. Thus, given a set X of models, we have some abstract function f , which chooses the models where the consequences hold, $f(X)$.
- (2) Now, we can put into this “machinery” whatever we want, e.g. the abstract choice function of preferential structures.
- (3) But we can also investigate non-static f , where f changes in function of what we already did - “reacting” to the past.
- (4) We can also look at properties of f , like complexity, generation by some simple structure like a simple automaton, etc.
- (5) So we advocate the separation of usual, classical semantics, from the additional properties, which are treated “outside”.

4 Boldness and justification

Boldness seems to be a (to our knowledge) new and important concept for logic and reasoning.

If, for instance, we generalize from a limited number of cases to a general formula (Popper's problem of "proving" a theory), we need some "boldness".

A theory might be bold enough to contradict some cases.

Reasoning with the normal cases is bold reasoning.

An approximation is a bold theory.

The more abnormal cases a preferential structure has, the bolder it is, when we use it as above.

But boldness also needs a justification. Without justification, it is foolishness.

Justification may come for efficiency, elegance, parcimony, etc.

If T is bolder than T' , then it needs a stronger justification, to be rational.

These concepts have to be elaborated in future work. We need a natural semantics at least for some cases.

References

- [ALS98-1] L.Audibert, C.Lhoussaine, K.Schlechta: "Distance based revision of preferential logics", Belief Revision Workshop of KR98, Trento, Italy, 1998 (electronic proceedings), and Logic Journal of the Interest Group in Pure and Applied Logics, Vol. 7, No. 4, pp. 429-446, 1999
- [Bou94] C.Boutilier, "Unifying default reasoning and belief revision in a modal framework", Artificial Intelligence, 68 : 33 – 85, 1994
- [DP94] A.Darwiche, J.Pearl, "On the Logic of Iterated Belief Revision", in: "Proceedings of the fifth Conference on Theoretical Aspects of Reasoning about Knowledge", R.Fagin ed., pp. 5-23, Morgan Kaufman, Pacific Grove, CA, 1994
- [GS08b] D.Gabbay, K.Schlechta, "Reactive preferential structures and nonmonotonic consequence", hal-00311940, arXiv 0808.3075, Review of Symbolic Logic, Vol. 2, No. 2, pp. 414-450,
- [GS09c] D.Gabbay, K.Schlechta, "Semantic interpolation", submitted, preliminary version: arXiv.org 0906.4082
- [Ker99] G.Kern-Isberner, "Postulates for conditional belief revision", Proceedings IJCAI 99, T.Dean ed., Morgan Kaufmann, pp.186-191, 1999
- [LMS01] D.Lehmann, M.Magidor, K.Schlechta: "Distance Semantics for Belief Revision", Journal of Symbolic Logic, Vol.66, No. 1, March 2001, p. 295-317
- [Sch04] K.Schlechta: "Coherent Systems", Elsevier, Amsterdam, 2004
- [Sch92] K.Schlechta: "Some results on classical preferential models", Journal of Logic and Computation, Oxford, Vol.2, No.6 (1992), p. 675-686