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Equilibria und weiteres Heiteres

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Abstract

We investigate several technical questions.

Contents

1	Introduction	3
2	Equilibrium logic	3
2.1	Introduction	3
2.1.1	Overview	3
2.2	Semantics and interpolation of many valued logics	4
2.2.1	Generalization of model sets and (in)essential variables	4
2.2.2	General remark about multi-valued interpolation	5
2.3	Some logical results for the base logic Here/There HT	6
2.3.1	Basic definitions and results	6
2.3.2	Normal form	7
2.3.3	Interpolation	8
2.4	The equilibrium logic EQ	11
2.4.1	Introduction and outline	11
2.4.2	Definability of minimal models	12
2.4.3	EQ has no interpolation of the form $\phi \vdash \alpha \sim \psi$	12

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2.4.4	EQ has no interpolation of the form $\phi \vdash \alpha \vdash \psi$	13
2.4.5	EQ has interpolation of the form $\phi \vdash \alpha \vdash \psi$	13
3	Countably many disjoint sets	14
4	Operations on logical structures	15
4.1	Introduction	15
4.1.1	Conditionals	16
4.2	Theory revision	16
4.2.1	“Meta-revision”	16
4.3	Preferential systems	17
4.3.1	Why does \vdash not modify itself?	17
4.3.2	(Meta) Operations on logics	17
4.3.3	Implementation	17
4.3.4	Operations on linear and ranked structures	18
4.3.5	Intuitionistic preferential logic	18
4.4	Non-monotonic interpolation	19
5	Independence and multiplication of abstract size	20
5.1	Definition of independence	20
5.1.1	Discussion	21
5.1.2	Independence and multiplication of abstract size	21
5.2	Introduction to abstract size	21
5.2.1	Notation	22
5.2.2	The groupes of rules	22
5.2.3	Table	24
5.3	Multiplication of size for subsets	29
5.3.1	Properties	29
5.4	Multiplication of size for subspaces	30
5.4.1	Properties	30
5.4.2	Size multiplication and corresponding preferential relations	32
5.4.3	Interpolation	34
5.4.4	Language change	36
5.4.5	A relevance problem	36
5.4.6	Small subspaces	36
5.5	Revision and distance relations	37
5.5.1	Interpolation for distance based revision	39
5.6	Summary of properties	39
6	General semantic interpolation	41
6.1	Introduction	41
6.1.1	Generalization of model sets and (in)essential variables	41

6.2	Many-valued propositional interpolation	41
6.3	Many-valued propositional modal interpolation	43
6.4	Many-valued first order interpolation	43
7	Repairing finite Goedel logics	44
7.1	Introduction	44
7.2	Cyclic addition of 1	44
References		45

1 Introduction

We present here various small results, which may one day be published in a bigger paper, and which we wish to make already available to the community.

2 Equilibrium logic

2.1 Introduction

Define $M(\phi) := \{m : v_m(\phi) = 2\}$.

Define $\phi \sim \psi$ iff $\mu(M(\phi)) \subseteq M(\psi)$.

$\mu(M(\phi))$ is definable by some ϕ' , i.e. $\mu(M(\phi)) = M(\phi')$. See 3.1 (now 4.1).

Consider $\Pi J \times M(\phi') \upharpoonright J' \times \Pi J'' = M$. By Table 2 “neglecting” and Remark directly above Remark 2.5/3.5 (integrate into table!) there is ψ' s.t. $M = M(\psi')$.

We have $\mu(M(\phi)) \subseteq M$, so $\phi \sim \psi'$. We still need $\psi' \sim \psi$, i.e. $\mu(M(\psi')) \subseteq M(\psi)$. This results from independence of definition of \prec (see Sect. 4.4, approx.), and the remark below Diagram 4.1

2.1.1 Overview

We first give an improvement of our semantic interpolation result in [GS09c], which is now adapted to many valued logics.

We then give a short introduction to the well-known 3-valued intuitionistic logic HT (Here/There), with some results also for similar logics with more than 3 values. (Many of these properties were found and checked with a small computer program.) In particular, we show the existence of a normal form, similar to classical propositional logic, but \rightarrow cannot be eliminated. Consequently, we cannot always separate propositional variables easily.

Our main result here (which is probably well known, we claim no priority) is that “forgetting” a variable preserves definability in the following sense: Let, e.g., $\phi = a \wedge b$, and $M(\phi)$ be the set of models where ϕ has maximal truth value (2 here), then there is ϕ' such that the set of models where ϕ' has value 2 is the set of all models which agree with a model of ϕ on, e.g., b . We “forget” about a . Our ϕ' is here, of course, b . Here, the problem is trivial, it is a bit less so when \rightarrow is involved, as we cannot always separate the two parts. For example, the result of “forgetting” about a in the formula $a \wedge (a \rightarrow b)$ is b , in the formula $a \rightarrow b$ it is TRUE. Thus, forgetting about a variable preserves definability, and the abovementioned semantical interpolation property carries over to the syntactic side.

This opens the way to our main results, about Equilibrium Logic, EQ in short, as introduced by Pearce et al., see [PV09].

Here, we are only interested in certain models of HT formulas, which can be seen as minimal models under a suitable order \prec (which, by the way, is *not* smooth, so some general results from [GS09f] are not applicable). We pose again the interpolation question. But, first, we have to make it precise. Suppose then $\phi \sim \psi$ in EQ. What does this mean? Does this mean

(a) let $M_2(\phi) := \{m : m(\phi) = 2\}$ be the set of models where ϕ has value 2, and $\mu_2(\phi)$ be the set of \prec -minimal elements of $M_2(\phi)$, and finally $\phi \sim \psi$ means that $\mu_2(\phi) \subseteq M_2(\psi)$?

or

(b) in analogous definitions $\mu_1(\phi) \subseteq M_1(\psi)$ and $\mu_2(\phi) \subseteq M_2(\psi)$?

or, still more complicated,

(c) shall \prec also compare elements from $M_2(\phi)$ with those from $M_1(\phi)$?

We decide for the first variant, so let \sim be defined as above.

Let \vdash be the consequence relation of HT, defined by:

$\forall m. m(\phi) \leq m(\psi)$.

Our problem is now:

Given $\phi \sim \psi$, is there α using only variables common to ϕ and ψ such that

(a) $\phi \vdash \alpha \sim \psi$

or

(b) $\phi \sim \alpha \vdash \psi$

or

(c) $\phi \sim \alpha \sim \psi$?

We show that (a) and (b) fail in general, but that (c) will hold.

The semantical version is easy, again, the syntactic version is more difficult. For this, we show:

(1) $\mu_2(\phi) = M_2(\phi')$ for some ϕ' , i.e., $\mu_2(X)$ is definable, if X is definable (both for value 2),

(2) there is not only a semantical interpolant for $\mu_2(\phi)$ and $M_2(\psi)$, but there is α such that $M_2(\alpha)$ is such a semantical interpolant, i.e., there is a definable (again by value 2) interpolant. For this, we use the definability results for HT.

2.2 Semantics and interpolation of many valued logics

We generalize the concepts and ideas of [GS09c].

2.2.1 Generalization of model sets and (in)essential variables

In classical 2 valued logic, the models of ϕ is a set, $M(\phi)$. In many valued logic, we do not have this correspondence between value and model set. Rather, given ϕ , we have a value function f_ϕ assigning to each model a truth value, the value the formula has in this model. (Of course, we could also assign to each model m a function f_m , giving a value to each formula, or take a binary function, etc. - this is a matter of taste.) In classical logic, $M(\phi) = \{m : f_\phi(m) = TRUE\}$.

More precisely,

Definition 2.1

(1) We assume V , the set of truth values, to be a totally ordered (by \leq), finite set, with minimal value FALSE, and maximal value TRUE.

(2) We assume a propositional language \mathcal{L} to be given, defined by a set of variables L , and the usual operators \neg, \wedge , etc., and perhaps others - where the semantics of \neg etc. still has to be defined. Given \mathcal{L} and V , we denote by M the set of models for this logic - context will tell what \mathcal{L} and V are, and how validity is defined. Any $m \in M$ will be a function from L to V - there are no further restrictions.

- (3) For each formula ϕ (or set of formulas T), we define a function f_ϕ (f_T) from the set of models M to V by $f_\phi(m) :=$ the truth value of ϕ in m (analogously $f_T(m) := \min\{f_\phi(m) : \phi \in T\}$)
- (4) In classical logic, we also may have arbitrary, not necessarily definable, model sets, these correspond now to arbitrary functions $f : M \rightarrow V$. Such f are called definable iff there is ϕ with $f = f_\phi$.

In the logics which interest us, $f_{\phi \rightarrow \psi}(m) = TRUE$ iff $f_\phi(m) \leq f_\psi(m)$, this motivates the following definition:

Definition 2.2

- (1) For $J \subseteq L$, $m, m' \in M$, let $m \sim_J m' :\Leftrightarrow \forall x \in J. m(x) = m'(x)$.
- (2) Given $f : M \rightarrow V$ and $x \in L$, we say that x is irrelevant for f iff $\forall m, m' \in M (m \sim_{L-\{x\}} m' \Rightarrow f(m) = f(m'))$, i.e., the value of m under f does not depend on $m(x)$. $I(f)$ will be the set of $x \in L$, which are irrelevant for f .
- (3) Given $f, g, h : M \rightarrow V$, we say that h is a semantic interpolant for f and g iff
- (3.1) $\forall m \in M (f(m) \leq h(m) \leq g(m))$,
- (3.2) $I(f) \cup I(g) \subseteq I(h)$
- (4) Given ϕ, ψ , we say that α is a syntactic interpolant for ϕ and ψ iff
- (4.1) $\forall m \in M (f_\phi(m) \leq f_\alpha(m) \leq f_\psi(m))$,
- (4.2) all variables occurring in α occur also in ϕ and ψ .
- (5) The following will be central for constructing a semantical interpolant:

Let $J \subseteq L$, $m \in M$, $m \upharpoonright J : J \rightarrow V$ be the restriction of a model m to J , $f : M \rightarrow V$, then $f^+(m \upharpoonright J) := \max\{f(m') : m \sim_J m'\}$, the maximal value for any m' which agrees with m on J .

2.2.2 General remark about multi-valued interpolation

Lemma 2.1

Let \mathcal{L}, V, \leq be given, $f, g : M \rightarrow V$.

Then f, g have a semantical interpolant iff $\forall m \in M. f(m) \leq g(m)$. If the condition holds, and $L = J \cup J' \cup J''$, $J = I(f)$, $J'' = I(g)$, then $h : M \rightarrow V$ defined by $h(m) := f^+(m \upharpoonright J')$ is a semantical interpolant for f and g .

Proof

“ \Rightarrow ” is trivial.

“ \Leftarrow ”: We have to show that for all $m \in M$ $f(m) \leq h(m) \leq g(m)$. $f(m) \leq h(m)$ is trivial by definition. We have to show $h(m) \leq g(m)$.

Note that we can combine parts of different models arbitrarily, we may, e.g., take $m(x), m'(y)$ etc., and combine them to a new model. There is no coherence between the values for different $x \in L$. We will use this property now. $m \upharpoonright J + m' \upharpoonright (J' \cup J'')$ will, e.g., denote the model combined from m on J , and m' on $J' \cup J''$.

Fix m . $h(m) := f^+(m \upharpoonright J')$, let m' be such that $m \upharpoonright J' = m' \upharpoonright J'$, and $f(m') = f^+(m \upharpoonright J')$, thus, we have to show $f(m') \leq g(m)$. As $J = I(f)$, $f(m') = f(m \upharpoonright J + m' \upharpoonright (J' \cup J'')) = f(m \upharpoonright J + m \upharpoonright J' + m' \upharpoonright J'')$, so by prerequisite $f \leq g$ we have $f(m \upharpoonright J + m \upharpoonright J' + m' \upharpoonright J'') \leq g(m \upharpoonright J + m \upharpoonright J' + m' \upharpoonright J'') = g(m)$, as $J'' = I(g)$.

□

This shows semantical interpolation for many valued logics, and we turn to the syntactic side. For this, we make first a leisurely introduction to the three valued intuitionistic logic of Here/There.

2.3 Some logical results for the base logic Here/There HT

2.3.1 Basic definitions and results

There are 3 truth values, 0, 1, 2, so any model is a function from the set of propositional variables into $\{0, 1, 2\}$.

For a model σ , and a propositional variable x , set

$\sigma(x) = 0$ iff x holds neither here nor there

$\sigma(x) = 1$ iff x holds only there

$\sigma(x) = 2$ iff x holds here and there

We introduce the operators $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ by their semantic matrices.

		b		0	1	2		0	1	2		0	1	2		0	1	2
a	$\neg a$		$a \rightarrow b$				$a \wedge b$				$a \vee b$				$a \leftrightarrow b$			
0	2			2	2	2		0	0	0		0	1	2		2	0	0
1	0			0	2	2		0	1	1		1	1	2		0	2	1
2	0			0	1	2		0	1	2		2	2	2		0	1	2

Thus, \wedge is the binary minimum, \vee the binary maximum operator, as in classical logic.

For comparison, we also introduce \neg and \rightarrow for the analogous 4- and 6-valued intuitionistic logics. (\wedge and \vee are again the minimum or maximum.)

		b		0	1	2	3
a	$\neg a$		$a \rightarrow b$				
0	3			3	3	3	3
1	0			0	3	3	3
2	0			0	1	3	3
3	0			0	1	2	3

		b		0	1	2	3	4	5
a	$\neg a$		$a \rightarrow b$						
0	5			5	5	5	5	5	5
1	0			0	5	5	5	5	5
2	0			0	1	5	5	5	5
3	0			0	1	2	5	5	5
4	0			0	1	2	3	5	5
5	0			0	1	2	3	4	5

The following results were checked with a small computer program:

Fact 2.2

(1)

The following formulas are definable with 1 variable a (values given for a). This means, that all formulas ϕ generated with a single variable a have a semantic function f_ϕ which is exactly the same as the semantic function of one of the 6 formulas below. Thus, this set is semantically closed. (And, of course, the 6 formulas below all have different semantic functions.)

a	0	1	2
a	0	1	2
$\neg a$	2	0	0
$\neg\neg a$	0	2	2
$a \rightarrow a$	2	2	2
$\neg(a \rightarrow a)$	0	0	0
$\neg\neg a \rightarrow a$	2	1	2

(2) With 2 variables a, b are definable, using the operators $\neg, \rightarrow, \wedge, \vee$, 174 semantically different formulas.

\vee is not needed, i.e. with or without \vee we have the same set of definable formulas. We have, e.g., $a \vee b \leftrightarrow \left((b \rightarrow (\neg\neg a \rightarrow a)) \rightarrow ((\neg a \rightarrow b) \wedge (\neg\neg a \rightarrow a)) \right)$.

(3) With the operators \neg, \wedge, \vee only are 120 semantically different formulas definable. Thus, \rightarrow cannot be expressed by the other operators.

Fact 2.3

The following semantic equivalences hold:

(Note: all except (14) hold also for 4 and 6 truth values, so probably for arbitrarily many truth values, but this is not checked so far.)

Triple negation can be simplified:

(1) $\neg\neg\neg a \leftrightarrow \neg a$

Disjunction and conjunction combine classically:

(2) $\neg(a \vee b) \leftrightarrow \neg a \wedge \neg b$

(3) $\neg(a \wedge b) \leftrightarrow \neg a \vee \neg b$

(4) $a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c)$

(5) $a \vee (b \wedge c) \leftrightarrow (a \vee b) \wedge (a \vee c)$

Implication can be eliminated from combined negation and implication:

(6) $\neg(a \rightarrow b) \leftrightarrow \neg\neg a \wedge \neg b$

(7) $(a \rightarrow \neg b) \leftrightarrow (\neg a \vee \neg b)$

(8) $(\neg a \rightarrow b) \leftrightarrow (\neg\neg a \vee b)$

Implication can be put inside when combined with \wedge and \vee :

(9) $(a \vee b \rightarrow c) \leftrightarrow ((a \rightarrow c) \wedge (b \rightarrow c))$

(10) $(a \wedge b \rightarrow c) \leftrightarrow ((a \rightarrow c) \vee (b \rightarrow c))$

(11) $(a \rightarrow b \wedge c) \leftrightarrow ((a \rightarrow b) \wedge (a \rightarrow c))$

(12) $(a \rightarrow b \vee c) \leftrightarrow ((a \rightarrow b) \vee (a \rightarrow c))$

Nested implication can be flattened:

(13) $(a \rightarrow (b \rightarrow c)) \leftrightarrow ((a \wedge b \rightarrow c) \wedge (a \wedge \neg c \rightarrow \neg b))$

(14) $((a \rightarrow b) \rightarrow c) \leftrightarrow ((\neg a \rightarrow c) \wedge (b \rightarrow c) \wedge (a \vee \neg b \vee c))$

(Thanks to D.Pearce for (14) and other help.)

2.3.2 Normal form

Fact 2.4

Every formula ϕ can be transformed into a semantically equivalent formula ψ of the following form:

(1) ψ has the form $\phi_1 \vee \dots \vee \phi_n$

(2) every ϕ_i has the form $\phi_{i,1} \wedge \dots \wedge \phi_{i,m}$

(3) every $\phi_{i,m}$ has one of the following forms:

p , or $\neg p$, or $\neg\neg p$, or $p \rightarrow q$ - where p and q are propositional variables.

Proof

The numbers refer to Fact 2.3 (page 7).

We first push \neg downward, towards the interior:

- $\neg(\phi \wedge \psi)$ is transformed to $\neg\phi \vee \neg\psi$ by (3).
- $\neg(\phi \vee \psi)$ is transformed to $\neg\phi \wedge \neg\psi$ by (2).
- $\neg(\phi \rightarrow \psi)$ is transformed to $\neg\neg\phi \wedge \neg\psi$ by (6).

We next eliminate any $\phi \rightarrow \psi$ where ϕ and ψ are not propositional variables:

- $\neg\phi \rightarrow \psi$ is transformed to $\neg\neg\phi \vee \psi$ by (8).
- $\phi \wedge \phi' \rightarrow \psi$ is transformed to $(\phi \rightarrow \psi) \vee (\phi' \rightarrow \psi)$ by (10).
- $\phi \vee \phi' \rightarrow \psi$ is transformed to $(\phi \rightarrow \psi) \wedge (\phi' \rightarrow \psi)$ by (9).
- $(\phi \rightarrow \phi') \rightarrow \psi$ is transformed to $(\neg\phi \rightarrow \psi) \wedge (\phi' \rightarrow \psi) \wedge (\phi \vee \neg\phi' \vee \psi)$ by (14).
- $\phi \rightarrow \neg\psi$ is transformed to $\neg\phi \vee \neg\psi$ by (7).
- $\phi \rightarrow \psi \wedge \psi'$ is transformed to $(\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi')$ by (11).
- $\phi \rightarrow \psi \vee \psi'$ is transformed to $(\phi \rightarrow \psi) \vee (\phi \rightarrow \psi')$ by (12).
- $\phi \rightarrow (\psi \rightarrow \psi')$ is transformed to $(\phi \wedge \psi \rightarrow \psi') \wedge (\phi \wedge \neg\psi' \rightarrow \neg\psi)$ by (13).

Finally, we push \wedge inside:

$\phi \wedge (\psi \vee \psi')$ is transformed to $(\phi \wedge \psi) \vee (\phi \wedge \psi')$ by (4).

□

2.3.3 Interpolation

We will now show syntactic interpolation. For this purpose, we show that, if f is definable in Lemma 2.1 (page 5), i.e. there is ϕ with $f = f_\phi$, then h in the same Lemma is also definable. Recall that $h(m)$ was defined as the maximal $f(m')$ for $m' \upharpoonright J' = m \upharpoonright J'$. We use the normal form just shown, to show that conjuncts and disjuncts can be treated separately.

Our aim is to find a formula which characterizes the max. More precisely, if there is ϕ , and $M(\phi) := \{m \in M : f_\phi(m) = TRUE\}$, and $J \subseteq L$, then we look for some ϕ' such that $f_{\phi'}(m) = TRUE$ iff there is $m' \in M$, $m \upharpoonright (L - J) = m' \upharpoonright (L - J)$, and $f_\phi(m') = TRUE$. Thus, $f_{\phi'}(m) = \max\{f_\phi(m') : m' \in M, m \upharpoonright (L - J) = m' \upharpoonright (L - J)\}$.

First, a trivial fact, which shows that we can treat the elements of J one after the other: $\max\{g(x, y) : x \in X, y \in Y\} = \max\{\max\{g(x, y) : x \in X\} : y \in Y\}$. (Proof: The interior max on the right hand side range over subsets of $X \times Y$, so they are all \leq than the left hand side. Conversely, the left hand max is assumed for some $\langle x, y \rangle$, which also figures on the right hand side. A full proof would be an induction.)

Next, we show that we can treat disjunctions separately for one $x \in L$, and also conjunctions, as long as x occurs only in one of the conjuncts. Again, a full proof would be by induction, we only show the crucial arguments. First, some notation:

Notation 2.1

- (1) We write $m =_{(x)} m'$ as shorthand for $m \upharpoonright (L - \{x\}) = m' \upharpoonright (L - \{x\})$
- (2) Let $f : M \rightarrow V$, $x \in L$, then $f_{(x)}(m) := \max\{f(m') : m' \in M, m =_{(x)} m'\}$.
- (3) Let $f_\phi : M \rightarrow V$, and $f_{\phi, (x)} = f_{\phi'}$ for some ϕ' , then we write $\phi_{(x)}$ for (some such) ϕ' .

Fact 2.5

- (1) If $\phi = \phi' \vee \phi''$, and $\phi'_{(x)}, \phi''_{(x)}$ both exist, then so does $\phi_{(x)}$, and $\phi_{(x)} = \phi'_{(x)} \vee \phi''_{(x)}$.
- (2) If $\phi = \phi' \wedge \phi''$, $\phi'_{(x)}$ exists, and ϕ'' does not contain x , then $\phi_{(x)}$ exists, and $\phi_{(x)} = \phi'_{(x)} \wedge \phi''$.

Proof

(1) We have to show $f_{\phi(x)} = f_{(\phi'(x) \vee \phi''(x))}$.

By definition of validity of \vee , we have $f_{(\phi'(x) \vee \phi''(x))}(m) = \max\{f_{\phi'(x)}(m), f_{\phi''(x)}(m)\}$. $f_{\phi(x)}(m) := \max\{f_{\phi}(m') : m' =_{(x)} m\}$, so $f_{(\phi'(x) \vee \phi''(x))}(m) = \max\{\max\{f_{\phi'}(m') : m' =_{(x)} m\}, \max\{f_{\phi''}(m') : m' =_{(x)} m\}\} = \max\{\max\{f_{\phi'}(m'), f_{\phi''}(m')\} : m' =_{(x)} m\} =$ (again by definition of validity of \vee) $\max\{f_{\phi' \vee \phi''}(m') : m' =_{(x)} m\} = \max\{f_{\phi}(m') : m' =_{(x)} m\} = f_{(\phi(x))}(m)$.

(2) We have to show $f_{\phi(x)} = f_{(\phi'(x) \wedge \phi''(x))}$. By definition of validity of \wedge , we have $f_{(\phi'(x) \wedge \phi''(x))}(m) = \min\{f_{\phi'(x)}(m), f_{\phi''(x)}(m)\}$. So $f_{(\phi'(x) \wedge \phi''(x))}(m) = \min\{\max\{f_{\phi'}(m') : m' =_{(x)} m\}, \max\{f_{\phi''}(m') : m' =_{(x)} m\}\} =$ (as ϕ'' does not contain x) $\min\{\max\{f_{\phi'}(m') : m' =_{(x)} m\}, f_{\phi''}(m)\} = \max\{\min\{f_{\phi'}(m'), f_{\phi''}(m)\} : m' =_{(x)} m\} =$ (again by definition of validity of \wedge , and by the fact that ϕ'' does not contain x) $\max\{f_{\phi' \wedge \phi''}(m') : m' =_{(x)} m\} = \max\{f_{\phi}(m') : m' =_{(x)} m\} = f_{(\phi)}(m)$.

□

Thus, we can calculate disjunctions separately, and also conjunctions, as long as they have no variables in common. In classical logic, we are finished, as we can break down conjunctions into parts which have no variables in common. The problem here are formulas of the type $a \rightarrow b$, as they may have variables in common with other conjuncts, and, as we saw in Fact 2.2 (page 6) (2) and (3), they cannot be eliminated.

Thus, we have to consider situations like $(a \rightarrow b) \wedge (b \rightarrow c)$, $a \wedge (a \rightarrow b)$, etc.

Table 2.3.3 (page 10) “Neglecting a variable” presents some results, calculated by computer. The left hand column gives the formula ϕ , the center column the “forgotten” or “neglected” variable x , and the right hand column $\phi(x)$, or $\phi(x, y)$, when we neglect two variables. We tested that these projections also hold for 3, 4, 6 truth values. Those where a number is given in the last column were checked formally. The number refers to the following proof. “+” in the column “TRUE preserved” indicates, that if the value of the formula in the first column is TRUE in a model, so is the value of the formula in the third column in the same model. Moreover, the formula in the third column does not contain the neglected variable any more, so this formula will also have value TRUE in any model which differs at most in this neglected variable.

We privileged systematization over simplicity, e.g. $(b \rightarrow c) \wedge c$ can be further simplified to c .

To do as many cases together as possible, it is useful to use Fact 2.3 (page 7) (9) and (11) backwards, to obtain general formulas. We then see that the cases to examine are of the form:

$$\phi = ((b_1 \vee \dots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \dots \wedge c_m)) \wedge \sigma a \wedge \tau a,$$

where n, m may be 0, and σ, τ are absence (\emptyset , no a), $1a = a$, $\neg a$, or $\neg\neg a$.

First, we may simplify, as $\sigma a \wedge \tau a$ is equivalent to some $\sigma' a$, where $\sigma' a$ may also be FALSE. Moreover, and $a \rightarrow a$ may be replaced by TRUE, i.e. omitted.

Proof

Note that the b and c need not be propositional variables, it suffices that they are expressions independent from a which can take any value, e.g. disjunctions or conjunctions of propositional variables.

We first show that the maximum is reached for a as indicated.

(1) trivial.

(2) (a) If $a < c$, then $a \rightarrow c$ is not better than for $a = c$, but a is worse. (b) If $a > c$, then $a \rightarrow c \leq c$, so $a \rightarrow c * a \leq c * a = c$.

(3), (7), (11) $\neg a \neq 0$ only if $a = 0$.

(4) (a) If $a < c$, then $a \rightarrow c$ is not better than for $a = c$, but a and then $\neg\neg a$ might be worse. (b) If $a > c$, then $a \rightarrow c \leq c$, so $a \rightarrow c * \neg\neg a \leq c * \neg\neg a = c$.

(5), (6), (8) trivial.

(9) If $b \leq c$, then $a = c$ will give the maximal value. If $b > c$, and $a < c$, then $a \rightarrow c$ will be TRUE, as it is for

Table 1: Table Neglecting a variable

Neglecting a variable (TRUE = max. value, FALSE = min. value)					
ϕ	neglected variable(s)	$\phi(x)$	max. assumed for $a =$	TRUE preserved	Proof
$a \rightarrow c$	a	TRUE	c	+	(1)
$(a \rightarrow c) \wedge a$	a	c	c	+	(2)
$(a \rightarrow c) \wedge \neg a$	a	TRUE	FALSE	+	(3)
$(a \rightarrow c) \wedge \neg \neg a$	a	$\neg \neg c$	c	+	(4)
$b \rightarrow a$	a	TRUE	TRUE	+	(5)
$(b \rightarrow a) \wedge a$	a	TRUE	TRUE	+	(6)
$(b \rightarrow a) \wedge \neg a$	a	$\neg b$	FALSE	+	(7)
$(b \rightarrow a) \wedge \neg \neg a$	a	TRUE	TRUE	+	(8)
$b \vee d \rightarrow a$	a	TRUE			
$(b \vee d \rightarrow a) \wedge a$	a	TRUE			
$(b \vee d \rightarrow a) \wedge \neg a$	a	$\neg(b \vee d)$			
$(b \vee d \rightarrow a) \wedge \neg \neg a$	a	TRUE			
$a \rightarrow c \wedge e$	a	TRUE			
$(a \rightarrow c \wedge e) \wedge a$	a	$c \wedge e$			
$(a \rightarrow c \wedge e) \wedge \neg a$	a	TRUE			
$(a \rightarrow c \wedge e) \wedge \neg \neg a$	a	$\neg \neg(c \wedge e)$			
$(a \rightarrow b) \wedge (b \rightarrow c)$	a	$b \rightarrow c$			
$(a \rightarrow b) \wedge (b \rightarrow c)$	b	$a \rightarrow c$			
$(a \rightarrow b) \wedge (b \rightarrow c)$	c	$a \rightarrow b$			
$(a \rightarrow b) \wedge (b \rightarrow c) \wedge (b \rightarrow d)$	b	$(a \rightarrow c) \wedge (a \rightarrow d)$			
$(a \rightarrow b) \wedge (b \rightarrow c) \wedge b$	b	$(a \rightarrow c) \wedge c$			
$(a \rightarrow b) \wedge (b \rightarrow c) \wedge (c \rightarrow d)$	a, b	$a \rightarrow d$			
$(c \rightarrow a) \wedge (a \rightarrow b) \wedge (b \rightarrow d) \wedge b \wedge \neg a$	a, b	$(c \rightarrow d) \wedge d \wedge \neg c$			
$(b \rightarrow a) \wedge (a \rightarrow c) \wedge (d \rightarrow a) \wedge (a \rightarrow e)$	a	$b \vee d \rightarrow c \wedge e$			
$(b \vee d \rightarrow a) \wedge (a \rightarrow c \wedge e)$	a	$b \vee d \rightarrow c \wedge e$			
$(b \vee d \rightarrow a) \wedge (a \rightarrow c \wedge e) \wedge a$	a	$(b \vee d \rightarrow c \wedge e) \wedge (c \wedge e)$			
$(b \vee d \rightarrow a) \wedge (a \rightarrow c \wedge e) \wedge \neg a$	a	$(b \vee d \rightarrow c \wedge e) \wedge \neg(b \vee d)$			
$(b \vee d \rightarrow a) \wedge (a \rightarrow c \wedge e) \wedge \neg \neg a$	a	$(b \vee d \rightarrow c \wedge e) \wedge \neg \neg(c \wedge e)$			
$(b \rightarrow a) \wedge (a \rightarrow c)$	a	$b \rightarrow c$	c	+	(9)
$(b \rightarrow a) \wedge (a \rightarrow c) \wedge a$	a	$(b \rightarrow c) \wedge c$	c	+	(10)
$(b \rightarrow a) \wedge (a \rightarrow c) \wedge \neg a$	a	$(b \rightarrow c) \wedge \neg b$	FALSE	+	(11)
$(b \rightarrow a) \wedge (a \rightarrow c) \wedge \neg \neg a$	a	$(b \rightarrow c) \wedge \neg \neg c$	c	+	(12)

$a = c$, so we do not win anything, but we might loose for $b \rightarrow a$. So $a \geq c$. But when we go down, the value of $x \rightarrow y$ is determined by the destination, i.e. one of the factors will be determined by c . (This holds for 4 and 6 values, too.)

(10) and (12): as for (9).

We turn to the equivalences.

(1), (2), (3), (4) trivial.

(5), (6), (8) trivial.

(7) $b \rightarrow FALSE = \neg b$, inspection of tables.

(9) trivial, the second factor is TRUE, and $a = c$.

(10), (12) like (9).

(11) $p \rightarrow FALSE = \neg b$. $FALSE \rightarrow c$ and $\neg a$ are TRUE. $b \rightarrow c \geq f \rightarrow FALSE$, so $b \rightarrow c * \neg b = \neg b$.

□

Note that if in the first set e.g. $a \wedge (a \rightarrow b)$ has value TRUE in a model, then in all new models, b will have value TRUE, too. Thus, validity TRUE is preserved: The new formula holds in the old model, but as all new models coincide with the old model in the relevant variables, the new formula holds there with value TRUE too.

Remark 2.6

We cannot improve the value of $\phi \rightarrow \psi$ by taking a detour $\phi \rightarrow \alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow \psi$ because the destination determines the value: in any column of \rightarrow , there is only max and a constant value. And if we go further down than needed, we get only worse, going from right to left deteriorates the values in the lines. □

We can achieve the same result by first closing under the following rules, and then erasing all formulas containing a :

(1) \rightarrow under transitivity, i.e.

$$((b_1 \vee \dots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \dots \wedge c_m)) \Rightarrow ((b_1 \vee \dots \vee b_n) \rightarrow (c_1 \wedge \dots \wedge c_m))$$

(2) $\sigma'a$ and \rightarrow as follows:

$$((b_1 \vee \dots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \dots \wedge c_m)), a \Rightarrow ((b_1 \vee \dots \vee b_n) \rightarrow (c_1 \wedge \dots \wedge c_m)) \wedge c_1 \wedge \dots \wedge c_m$$

$$((b_1 \vee \dots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \dots \wedge c_m)), \neg a \Rightarrow ((b_1 \vee \dots \vee b_n) \rightarrow (c_1 \wedge \dots \wedge c_m)) \wedge \neg c_1 \wedge \dots \wedge \neg c_m$$

$$((b_1 \vee \dots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \dots \wedge c_m)), \neg a \Rightarrow ((b_1 \vee \dots \vee b_n) \rightarrow (c_1 \wedge \dots \wedge c_m)) \wedge \neg b_1 \wedge \dots \wedge \neg b_n$$

In summary: the semantical interpolant constructed in Section 2.2 (page 4) is definable, so the HT logic has also syntactic interpolation. This result is well-known, but we need the techniques for the next section, and will use it also in forthcoming papers.

2.4 The equilibrium logic EQ

2.4.1 Introduction and outline

We turn to the Equilibrium Logic introduced by Pearce et al., see [PV09] - see there for background. We first repeat the basic definitions.

Fix a language \mathcal{L} .

Definition 2.3

(1) A model m is total iff for no $a \in L$ $m(a) = 1$.

(2) $m \prec m'$ iff

(2.1) for all $a \in L$ $m(a) = 0 \Leftrightarrow m'(a) = 0$ and

(2.2) $\{a \in L : m(a) = 2\} \subset \{a \in L : m'(a) = 2\}$

($\sigma \prec \tau$ iff T is preserved, and H goes down. Thus, only changes from 2 to 1 are possible when $\sigma \prec \tau$.)

(3) $m \in X$ is an equilibrium model of X iff m is total, and there is no $m' \prec m$, $m' \in X$. (We can add $m \prec m$ if $m(x) = 1$ for some $x \in L$, so we can define equilibrium models as minimal HT models for some relation \prec .)

(4) $\mu(X)$ will be the set of equilibrium models of X .

Remark 2.7

Consequently, we have a sort of “anti-smoothnes”: if a model is not minimal, then any model below it is NOT chosen. Consequently, we cannot use general results based on smoothness.

We now repeat from the introduction our plans.

Definition 2.4

(1) Set $M_2(\phi) := \{m \in M : m(\phi) = 2 = TRUE\}$.

(2) Set $\mu_2(\phi) := \{m \in M : m(\phi) = 2, \text{ and } m \text{ is an equilibrium model}\}$.

(3) Set $\phi \sim \psi$ iff $\mu_2(\phi) \subseteq M_2(\psi)$.

(4) Set $\phi \vdash \psi$ iff $\forall m \in M. m(\phi) \leq m(\psi)$.

The problem is now to find a semantic and syntactic interpolant for $\phi \sim \psi$. We first show that, if $X = M_2(\phi)$, i.e., X is definable, then $\mu_2(\phi)$ is also definable. Then, we show that we can construct from $\mu_2(\phi)$ a suitable semantical interpolant, in a way similar to the HT case. Thus, we can use the techniques and results developed there (“neglecting” some variables), and see that the semantical interpolant is definable, so we have also syntactical interpolation.

2.4.2 Definability of minimal models

Fact 2.8

$\mu_2(\phi)$ is definable.

Proof

Note: We do not need a uniform way to find the defining formula, we just need the formula - even if it is “handcrafted”.

By prerequisite, $M_2(\phi) := \{m \in M : f_\phi(m) = TRUE\}$ is definable by ϕ . Suppose there are $m, m' \in M_2(\phi)$ such that $m \prec m'$. Then $\{a \in L : m(a) = 0\} = \{a \in L : m'(a) = 0\}$. We can define all models n which have $n(a) = 0$ at the same place as m has, by (an individually chosen) formula: Let $m^- := \bigwedge \{-b : m(b) = 0\}$ and $m^+ := \bigwedge \{\neg b : m(b) \geq 0\}$, then any model n satisfying $m^- \wedge m^+$, will have value 2 exactly iff it has $n(a) = 0$ for the same a as m has. So $\neg(m^- \wedge m^+)$ will exclude all these models. (Note that \neg gives only value 0 or 2.) $\neg(m^- \wedge m^+)$ excludes all models which have 0 at the same place as m has, and somewhere 1 or 2 - as desired: Let m' have 0 for the same b 's as m does. Then neither m nor m' can be minimal models of ϕ : Any m' , which has $m'(b) = 1$ for some b , is not total. But if m' has never value 1, then $m \prec m'$, so m' is not minimal, either.

We add such $\neg(m^- \wedge m^+)$ for all such pairs m, m' to ϕ , and have a suitable definition of $\mu_2(\phi)$.

□

Before we show interpolation of the form $\vdash \circ \vdash$, we show that EQ has no interpolation of the form $\vdash \circ \vdash$ or $\vdash \circ \vdash$.

2.4.3 EQ has no interpolation of the form $\phi \vdash \alpha \mid \sim \psi$

Example 2.1

Work with 3 variables, a, b, c .

Consider $\Sigma := \{\langle 0, 2, 2 \rangle, \langle 2, 1, 0 \rangle, \langle 2, 2, 0 \rangle\}$.

By the above, and classical behaviour of “or” and “and”, Σ is definable by $(\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c)$.

Note that $\langle 2, 2, 0 \rangle$ is total, but $\langle 2, 1, 0 \rangle \prec \langle 2, 2, 0 \rangle$, thus $\mu(\Sigma) = \{\langle 0, 2, 2 \rangle\}$.

So $\Sigma \vdash c = 2$ (or $\Sigma \vdash \square c$). Let $X' := \{a, b\}$, $X'' := \{c\}$.

All possible interpolants Γ must not contain a or b as essential variables, and they must contain Σ . The smallest candidate Γ is $\Pi X' \times \{0, 2\}$. But $\sigma := \langle 0, 0, 0 \rangle \in \Gamma$, σ is total, and there cannot be any $\tau \prec \sigma$, so $\sigma \in \mu(\Gamma)$, so $\Gamma \not\vdash c = 2$.

For completeness' sake, we write all elements of Γ :

$\langle 0, 0, 0 \rangle \langle 0, 0, 2 \rangle$
 $\langle 0, 1, 0 \rangle \langle 0, 1, 2 \rangle$
 $\langle 0, 2, 0 \rangle \langle 0, 2, 2 \rangle$
 $\langle 1, 0, 0 \rangle \langle 1, 0, 2 \rangle$
 $\langle 1, 1, 0 \rangle \langle 1, 1, 2 \rangle$
 $\langle 1, 2, 0 \rangle \langle 1, 2, 2 \rangle$
 $\langle 2, 0, 0 \rangle \langle 2, 0, 2 \rangle$
 $\langle 2, 1, 0 \rangle \langle 2, 1, 2 \rangle$
 $\langle 2, 2, 0 \rangle \langle 2, 2, 2 \rangle$

Recall that no sequence containing 1 is total, and when we go from 2 to 1, we have a smaller model. Thus, $\mu(\Gamma) = \{\langle 0, 0, 0 \rangle, \langle 0, 0, 2 \rangle\}$.

2.4.4 EQ has no interpolation of the form $\phi \mid \sim \alpha \vdash \psi$

Example 2.2

Consider 2 variables, a, b , and $\Sigma := \{0, 2\} \times \{0, 1, 2\}$

No σ containing 1 can be in $\mu(\Sigma)$, as a matter of fact, $\mu(\Sigma) = \{\langle 0, 0 \rangle, \langle 2, 0 \rangle\}$. Σ is defined by $a \vee \neg a$, $\mu(\Sigma)$ is defined by $(a \vee \neg a) \wedge \neg b$.

So we have $a \vee \neg a \mid \sim b \vee \neg b$, even $a \vee \neg a \mid \sim \neg b$.

The only possible interpolants are TRUE or FALSE. $a \vee \neg a \not\vdash FALSE$, and $TRUE \not\vdash \neg b$.

2.4.5 EQ has interpolation of the form $\phi \mid \sim \alpha \mid \sim \psi$

Let $\phi \mid \sim \psi$, i.e., $\mu_2(\phi) \subseteq M_2(\psi)$. We have to find α such that $\mu_2(\phi) \subseteq M_2(\alpha)$, and $\mu_2(\alpha) \subseteq M_2(\psi)$.

Let $J = I(\phi)$, $J'' = I(\psi)$. Consider $X := \Pi J \times (\mu_2(\phi) \upharpoonright J') \times \Pi J''$. By the same arguments ("neglecting" J and J''), X is definable as $M_2(\alpha)$ for some α .

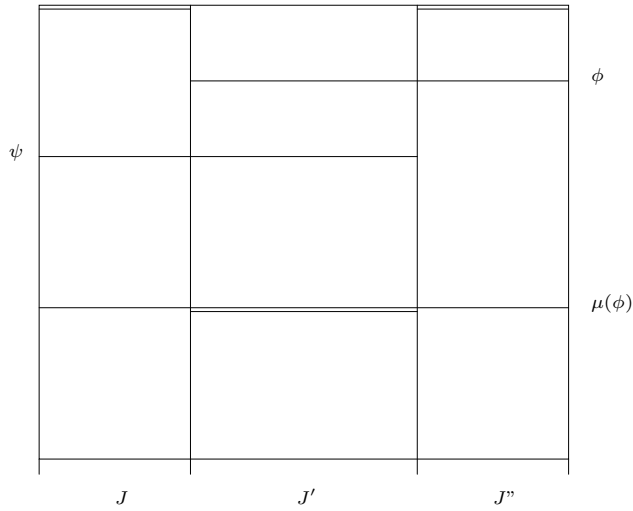
Obviously, $\mu_2(\phi) \subseteq M_2(\alpha)$. Consider now $\mu_2(\alpha)$, we have to show $\mu_2(\alpha) \subseteq M_2(\psi)$. If $\mu_2(\alpha) = \emptyset$, we are done, so suppose there is $m \in \mu_2(\alpha)$. Suppose $m \notin M_2(\psi)$. There is $m' \in \mu_2(\phi)$, $m' \upharpoonright J' = m \upharpoonright J'$.

Consider $m'' = (m \upharpoonright J) + m' \upharpoonright (J' \cup J'')$. As $m' \in \mu_2(\phi) \subseteq M_2(\phi)$, and $M_2(\phi) = \Pi J \times M_2(\phi) \upharpoonright (J' \cup J'')$, $m'' \in M_2(\phi)$. $m \upharpoonright (J \cup J') = m'' \upharpoonright (J \cup J')$, thus by $J'' \in I(\psi)$, $m'' \notin M_2(\psi)$. Thus, $m'' \notin \mu_2(\phi)$. So either there is $n \in M_2(\phi)$ such that $n(y) = 0$ iff $m''(y) = 0$ and $\{y : n(y) = 2\} \subset \{y : m''(y) = 2\}$ or $m''(y) = 1$ for some $y \in L$. Suppose $m''(y) = 1$ for some y . y cannot be in $J' \cup J''$, as $m'' \upharpoonright (J' \cup J'') = m' \upharpoonright (J' \cup J'')$, and $m' \in \mu_2(\phi)$. y cannot be in J , as $m'' \upharpoonright J = m \upharpoonright J$, and $m \in \mu_2(X)$.

So there must be $n \in M_2(\phi)$ as above. Case 1: $\{y \in J' \cup J'' : n(y) = 2\} \subset \{y \in J' \cup J'' : m''(y) = 2\}$. Then $n' = m' \upharpoonright J + n \upharpoonright (J' \cup J'')$ would eliminate m' from $\mu_2(\phi)$, so this cannot be. Thus, $n \upharpoonright (J' \cup J'') = m'' \upharpoonright (J' \cup J'')$. So $\{y \in J : n(y) = 2\} \subset \{y \in J : m''(y) = 2\} = \{y \in J : m(y) = 2\}$ by $m'' \upharpoonright J = m \upharpoonright J$. Consider now $n' = n \upharpoonright J + m \upharpoonright (J' \cup J'')$. $n' \in \Pi J \times \mu_2(\phi) \upharpoonright J' \times \Pi J''$. $n'(y) = 0$ iff $m(y) = 0$ by construction of n' and n . So $n' \prec m$, and $m \notin \mu_2(\Pi J \times (\mu_2(\phi) \upharpoonright J') \times \Pi J'')$, contradiction.

□

Diagram 2.1



Non-monotonic interpolation
Double lines: interpolant

3 Countably many disjoint sets

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D.Makinson (personal communication).

Example 3.1

There is a countably infinite set of formulas s.t. the defined model sets are pairwise disjoint.

Let $p_i : i \in \omega$ be propositional variables.

Consider $\phi_i := \bigwedge\{\neg p_j : j < i\} \wedge p_i$ for $i \in \omega$.

Obviously, $M(\phi_i) \neq \emptyset$ for all i .

Let $i < i'$, we show $M(\phi_i) \cap M(\phi_{i'}) = \emptyset$. $M(\phi_{i'}) \models \neg p_i$, $M(\phi_i) \models p_i$.

□

Fact 3.1

Any set X of consistent formulas with pairwise disjoint model sets is at most countable.

Proof

Let such X be given.

(1) We may assume that X consists of conjunctions of propositional variables or their negations.

Proof: Re-write all $\phi \in X$ as disjunctions of conjunctions ϕ_j . At least one of the conjunctions ϕ_j is consistent. Replace ϕ by one such ϕ_j . Consistency is preserved, as is pairwise disjointness.

(2) Let X be such a set of formulas. Let $X_i \subseteq X$ be the set of formulas in X with length i , i.e. a consistent conjunction of i many propositional variables or their negations, $i > 0$.

As the model sets for X are pairwise disjoint, the model sets for all $\phi \in X_i$ have to be disjoint.

(3) It suffices now to show that each X_i is at most countable, we even show that each X_i is finite.

Proof by induction:

Consider $i = 1$. Let $\phi, \phi' \in X_1$. Let ϕ be p or $\neg p$. If ϕ' is not $\neg\phi$, then ϕ and ϕ' have a common model. So one must be p , the other $\neg p$. But these are all possibilities, so $\text{card}(X_1)$ is finite.

Let the result be shown for $k < i$.

Consider now X_i . Take arbitrary $\phi \in X_i$. Wlog, $\phi = p_1 \wedge \dots \wedge p_i$. Take arbitrary $\phi' \neq \phi$. As $M(\phi) \cap M(\phi') = \emptyset$, ϕ' must be a conjunction containing one of $\neg p_k$, $1 \leq k \leq i$. Consider now $X_{i,k} := \{\phi' \in X_i : \phi' \text{ contains } \neg p_k\}$. Thus $X_i = \{\phi\} \cup \bigcup \{X_{i,k} : 1 \leq k \leq i\}$. Note that all $\psi, \psi' \in X_{i,k}$ agree on $\neg p_k$, so the situation in $X_{i,k}$ is isomorphic to X_{i-1} . So, by induction hypothesis, $\text{card}(X_{i,k})$ is finite, as all $\phi' \in X_{i,k}$ have to be mutually inconsistent. Thus, $\text{card}(X_i)$ is finite. (Note that we did not use the fact that elements from different $X_{i,k}$, $X_{i,k'}$ also have to be mutually inconsistent, our rough proof suffices.)

□

Note that the proof depends very little on logic. We needed normal forms, and used 2 truth values. Obviously, we can easily generalize to finitely many truth values.

4 Operations on logical structures

4.1 Introduction

In many cases, one wants more than a static structure:

- (1) dynamic theory revision a la Pearl etc.
- (2) revising a preferential logic
- (3) changing the language in interpolation
- (4) intuitionistic preferential logic: arrows are added

etc.

One can ask about such “meta-operations” for instance:

- (1) are properties preserved, e.g. is the result of working on a ranked structure again a ranked structure?
- (2) do we lose properties?
- (3) do we win new properties?

- (4) is it reasonable to require higher operators to follow the same laws as the basic operators, e.g. minimal change, and if so, e.g. minimal change of what?
- (5) what can be a structural semantics for such higher operators?

4.1.1 Conditionals

$A > B$ may mean: A becomes true (in the world), or the agent learns/believes A then: B becomes true, or the agent believes B , or the agent does B , or the agent brings B about, or so. Similarly, ternary conditionals $(A, B) > C$ can have very different meanings, and their formal properties may reflect this.

4.2 Theory revision

4.2.1 “Meta-revision”

AGM left K (A below) constant, and this may have contributed to subsequent confusion.

One sees sometimes a conditional $B > C$ expressing that after revising with B , C will hold.

But this hides the fact that it is in reality a 3-place conditional:

$(A, B) > C$: after revising A with B , C will hold, $A * B \models C$

A, B , etc. are formulas, i.e. partial information.

$(A, B) > C$ is partial information about the revision strategy, it describes just a bit of the whole picture. In the LMS tradition, see [LMS01], a revision strategy is just a distance between models. So $(A, B) > C$ describes one part of the distance.

Thus, $(A, B) > C$ is a partial revision strategy, or a set of distances which are compatible with $(A, B) > C$, just as a formula is a set of models.

But now, we have a perfect analogy:

We had $A * B \models C$ for formulas A, B, C , and now we can revise partial strategies:

$((A, B) > C, (A', B') > C') \gg (A'', B'') > C''$, i.e.

if we revise the partial strategy $(A, B) > C$ with the partial strategy $(A', B') > C'$, then the new strategy gives $(A'', B'') > C''$. \gg is the “meta-conditional”.

It all becomes transparent, and we can iterate the whole thing as often as we want.

In the distance language, we have a set of distances on models which all satisfy $(A, B) > C$, i.e. the B -models closest to the A -models all satisfy C , and another set of distances which all satisfy $(A', B') > C'$, we revise the first with the second using a “meta-distance” (a distance on the set of all distances between models of the base language), and get a new set of distances which all satisfy $(A'', B'') > C''$. If the first two sets are consistent, i.e. there is a distance which satisfies $(A, B) > C$ and $(A', B') > C'$, then the result is the intersection of the two distance sets. This corresponds, as usual, to the respect of 0 by a distance: $d(x, y) = 0$ iff $x = y$. Of course, we can consider here special distances like variants of the Hamming distance, working on a suitable set.

Of course, just as a formula may correspond to exactly 1 model, i.e. a complete consistent theory, we may also work with $*$ (i.e. the full revision strategy) instead of with $(A, B) > C$. So we may have $(*, *) \gg *'$. Still, as shown in [LMS01], the distance will usually not be fully determined, so we still work with sets of distance.

Note that we can also construct mixed systems, which allow to evaluate expressions like $(A, (B, C) > D) > E$, where factual information/models are mixed with conditional structures - this might be needed e.g. for natural language. We can go as high as we want, or even go down, evaluating “on the fly”.

One problem is TR (global distances). The CFC approach will not work, as we cannot consider the left hand side individually. We have to express it via quantifiers (modal operators). We have to say $x \in X$ is such that there is $y \in Y$ s.t. for no $x' \in X, y' \in Y$ $d(x', y') < d(x, y)$. We use global modality. Let X be defined by ϕ , Y by ψ , then $\diamond\psi \neg(\diamond\phi \diamond\psi \langle x', y', x, y \rangle)$ where $\langle x', y', x, y \rangle$ expresses that $d(x', y') < d(x, y)$ (axiomatize suitably so it is a distance). Etc, should work?

The distance need not be defined everywhere, so it may return “unknown”.

Pearl et al. For Pearl et al. (see [DP94]), Boutilier (see [Bou94]), and Kern-Isberner (see [Ker99]), an epistemic state \mathcal{E} is a pair $(\mathcal{B}, \mathcal{C})$, where \mathcal{B} is a set of beliefs (classical formulas), and \mathcal{C} a (perhaps partial) revision strategy coded by a set of conditionals, whose elements are classical formulas. In our terminology such an conditional will have the form $(\bigwedge \mathcal{B}, A) > C$, where $\bigwedge \mathcal{B}$ is the conjunction of factual beliefs, A, C are classical formulas, expressing: On the basis of \mathcal{B} , if I were to learn A , then I would believe C .

Pearl’s criticism of the AGM approach was that revising an epistemic state by some factual information A should not only modify factual beliefs, i.e. \mathcal{B} , but also the revision strategy \mathcal{C} . Pearl et al. gave some conditions this modification of \mathcal{C} should satisfy.

In our above notation, we then have $\mathcal{E} * A = (\mathcal{B}, \mathcal{C}) * A = (\mathcal{B}', \mathcal{C}') = \mathcal{E}'$, where \mathcal{C} determines the modification of \mathcal{B} to \mathcal{C} (by the conditionals $A > X$), but *not* the modification of \mathcal{C} to \mathcal{C}' .

Boutilier, see [Bou94], and Kern-Isberner, see [Ker99], extended this idea to revising epistemic states not only by factual information, but also by conditional information. (Kern-Isberner codes factual information X by the conditional $TRUE > X$, and thus avoids a distinction between the two.)

So we have $\mathcal{E} * C = (\mathcal{B}, \mathcal{C}) * (X > Y) = (\mathcal{B}', \mathcal{C}') = \mathcal{E}'$.

Revising by the factual conditional $X > Y$ imposes restrictions also on the transformation of \mathcal{C} to \mathcal{C}' .

4.3 Preferential systems

4.3.1 Why does \sim not modify itself?

We saw in Section 4.2 (page 16) how Pearl et al. introduced a revision operator $*$ whose application changes (the conditional part of) $*$ itself.

The question is obvious: Is there a logical formalism \sim which, applied to some formula ϕ , will not only produce a consequence ψ , but also a new logic \sim' ?

To the authors’ knowledge, this does not exist.

We may use Gabbay’s idea of reactivity to build such a logic: applying the logic changes it - this would give a formal motivation to the enterprise, from the other side, so to say.

In more detail: simple arrows obey the fundamental law of preferential structures. Adding higher arrows allows us to restrict from above, and thus describe any set, in a static way, see [GS08b]. So we can describe $M(\phi)$. When we “activate” now $M(\phi)$, we use higher arrows to modify the basic preferential relation.

4.3.2 (Meta) Operations on logics

It is natural to consider the operations of deduction and revision on logical systems.

Given some logical system \sim , we might deduce a new logic \sim' from it and some formula ϕ , i.e. $\phi \sim (\psi, \sim')$. E.g., \sim' might be weaker than \sim (this corresponds to classical logic, which is weakening), or we might deduce a new, bolder logic, corresponding to a more daring reasoning (this corresponds to non-monotonic logic, where we go beyond classical logic, win more conclusions, at the price of less certainty).

Perhaps even more useful, we may see that our logic does not give the desired conclusions, and may want to revise it, by some minimal change which obtains the desired result.

As in the case of theory revision, we can take as arguments the whole logic \sim , or just one or some pairs (ϕ, ψ) with $\phi \sim \psi$. For instance, we might want to revise \sim with some new pair (ϕ, ψ) , and see whether $\phi' \sim' \psi'$ holds in the new logic \sim' .

4.3.3 Implementation

Usually, working on the semantic side is easier. There are different ways to do it.

- (1) We can work with canonical structures (if they exist) - this may generate different results when we consider different structures as canonical. (This was a problem with [ALS98-1].)
- (2) We can work with the set of all structures corresponding to the logic, e.g. all preferential structures generating the logic
- (3) We can work with the algebraic semantics, i.e. usually with the smallest set of the filter, corresponding to $\mu(\phi)$, the set of minimal models of ϕ .
- (4) The reactive idea was carried out in [GS08b], where we modified preferential structures by adding higher order arrows. There, the view was static, but we can turn it dynamic to achieve revision and “meta-logic”.

Algebraic semantics As the algebraic semantics usually is the most robust notion, this is perhaps the easiest to work with.

For each ϕ , $\mu(\phi)$ is defined. So a natural distance between \sim and \sim' is the set of ϕ where $\mu(\phi) \neq \mu'(\phi)$, and for each such ϕ the symmetrical set distance between $\mu(\phi)$ and $\mu'(\phi)$. This gives a distance based revision of \sim to \sim' .

For a “meta-logic”, we can as usual consider a preference relation between logics (which are now simple objects, just as classical models, given by their $\mu(\phi)$ for all ϕ), and work with the algebraic representation results of the second author, see e.g. [Sch04].

Structural semantics It is natural to define a distance between two preferential structures by looking at the arrow sets, or sets of pairs $\langle m, m' \rangle$, such that $m \prec m'$. Again, some Hamming distance would be a first answer.

We treated one technique of modifying general (and smooth) preferential structures in [GS08b].

4.3.4 Operations on linear and ranked structures

Making a linear or ranked structure simply reactive will usually result in a mess, where the central properties of such structures are destroyed.

It seems more reasonable to investigate operations which leave the structure more intact, and postpone questions about their realization.

We may consider here operations which

- (1) cut the linear or ranked structure in two parts, such that within in each part the structure stays as it was, and the two parts are incomparable (they look a bit like a tree trunk, which was cut with a saw in 2 parts)
- (2) do elementary exchange operations (permutations) in the case of linear orders.
- (3) for ranked structures, we may have an operation $\alpha(x, y)$, which puts x on y 's level, $\beta(x, y)$ which changes the levels of x and y , etc.

4.3.5 Intuitionistic preferential logic

We are not sure about all arrows. Some arrows are definitely there, others definitely out, some come and go.

We have successively better information about arrows, and thus about size. Int. rules about size.

The following seems new: We do not only have $\Box(\phi \sim \psi)$, but also $\Box\neg(\phi \sim \psi)$. This has to be treated, especially for ranked structures.

4.4 Non-monotonic interpolation

We investigated non-monotonic interpolation in [GS09c].

The main property needed can be summarized as follows:

Let X be a set, and $X' \cup X'' = X$ be a disjoint cover of X . Consider $\Sigma \subseteq \Pi X$. Suppose $\mu(\Sigma) \subseteq \Sigma'$, where the variables defining Σ' are all in X'' . We now have to consider $\mu(\Pi X' \times \Sigma'')$, where $\Sigma'' = \Sigma \upharpoonright X''$, the restriction of Σ to X'' . We want $\mu(\Pi X' \times \Sigma'') \subseteq \Sigma'$, this gives the desired interpolation.

Formally:

$$(\mu * 3) \mu(\Pi' \times \Sigma'') \upharpoonright X'' \subseteq \mu(\Sigma) \upharpoonright X''.$$

The point here is that, logically, only the X'' -part matters, as Σ' is the full product on X' : $\Sigma' = \Pi X' \times (\Sigma' \upharpoonright X'')$. When we go from $\mu(\Sigma)$ to $\mu(\Delta)$, where $\Delta := \Pi X' \times \Sigma''$, Δ is bigger than Σ in the X' -part, and identical in the X'' -part. So increasing Σ outside X'' does not increase $\mu(\Sigma)$ inside X'' .

The (set variant) of the Hamming order satisfies this property: If $\sigma = \sigma' \circ \sigma''$, $\tau = \tau' \circ \tau''$ (\circ is concatenation), then $\sigma \prec \tau$ iff $\sigma' \prec \tau'$ and $\sigma'' \prec \tau''$. Thus, if e.g. $\sigma, \tau \in \Sigma$, $\sigma'' \prec \tau''$, but $\tau' \prec \sigma'$, then τ may be minimal, but adding $\rho' \circ \sigma''$ with $\rho' \prec \tau'$ will eliminate τ .

This behaviour motivates the following reflections:

- (1) This condition $(\mu * 3)$ points to a weakening of the Hamming condition:

Adding new “branches” in X' will not give new minimal elements in X'' , but may destroy other minimal elements in X'' . This can be achieved by a sort of semi-rankedness: If ρ and σ are different only in the X' -part, then $\tau \prec \rho$ iff $\tau \prec \sigma$, but not necessarily $\rho \prec \tau$ iff $\sigma \prec \tau$.

- (2) In more abstract terms:

When we separate support from attack (support: a branch σ' in X' supports a continuation σ'' in X'' iff $\sigma \circ \sigma''$ is minimal, i.e. not attacked, attack: a branch τ in X' attacks a continuation σ'' in X'' iff it prevents all $\sigma \circ \sigma''$ to be minimal), we see that new branches will not support any new continuations, but may well attack continuations.

More radically, we can consider paths σ'' as positive information, σ' as potentially negative information. Thus, Π' gives maximal negative information, and thus smallest set of accepted models.

The concept of size looks only at the result of support and attack, so it is necessarily somewhat coarse. Future research should also investigate both concepts separately.

- (3) We can interpret this as follows:

- (1) X'' determines the base set.

- (2) X' is the context. The context determines the choice (i.e. a subset of the base set).

- (3) When we compare this to preferential structures, we see that also in preferential structures the bigger the set, the more attacks are possible.

We broaden these considerations:

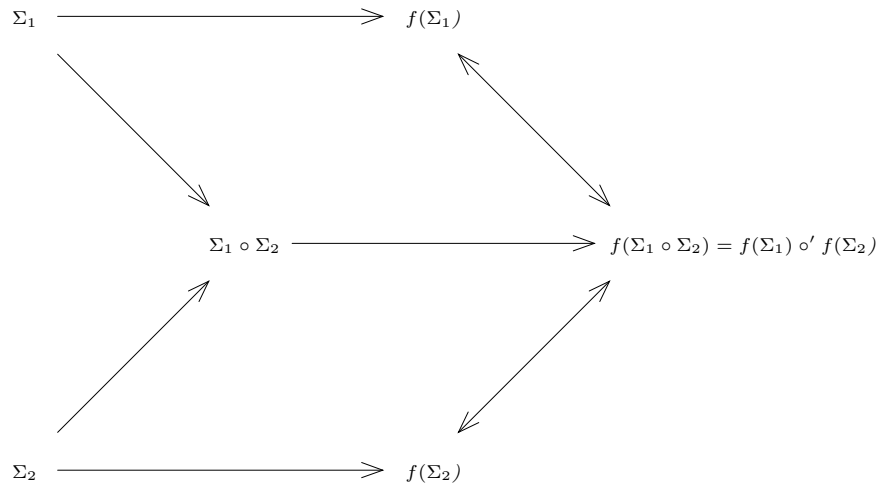
- (1) Following a tradition begun by Kripke, one has added structure to the set of classical models, reachability, preference, etc. Perhaps one should emphasize a more abstract approach, in the line of [Sch92], and elaborated in [Sch04], see in particular the distinction between structural and algebraic semantics in the latter. So we should separate structure from logic in the semantics, and treat what we called context above by a separate “machinery”. Thus, given a set X of models, we have some abstract function f , which chooses the models where the consequences hold, $f(X)$.
- (2) Now, we can put into this “machinery” whatever we want, e.g. the abstract choice function of preferential structures.
- (3) But we can also investigate non-static f , where f changes in function of what we already did - “reacting” to the past.

- (4) We can also look at properties of f , like complexity, generation by some simple structure like a simple automaton, etc.
- (5) So we advocate the separation of usual, classical semantics, from the additional properties, which are treated “outside”.

5 Independence and multiplication of abstract size

5.1 Definition of independence

Diagram 5.1



Note that \circ and \circ' might be different

Independence

The right notion of independence in our context seems to be:

We have compositions \circ and \circ' , and operation f . We can calculate $f(\Sigma_1 \circ \Sigma_2)$ from $f(\Sigma_1)$ and $f(\Sigma_2)$, but also conversely, given $f(\Sigma_1 \circ \Sigma_2)$ we can calculate $f(\Sigma_1)$ and $f(\Sigma_2)$. Of course, in other contexts, other notions of independence might be adequate. More precisely:

Definition 5.1

Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be any function from domain \mathcal{D} to co-domain \mathcal{C} . Let \circ be a “composition function” $\circ : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$, likewise for $\circ' : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$.

We say that $\langle f, \circ, \circ' \rangle$ are independent iff for any $\Sigma_i \in \mathcal{D}$

- (1) $f(\Sigma_1 \circ \Sigma_2) = f(\Sigma_1) \circ' f(\Sigma_2)$,
- (2) we can recover $f(\Sigma_i)$ from $f(\Sigma_1 \circ \Sigma_2)$, provided we know how $\Sigma_1 \circ \Sigma_2$ splits into the Σ_i .

5.1.1 Discussion

- (1) Ranked structures satisfy it:

Let $\circ = \circ' = \cup$. Let f be the minimal model operator μ of preferential logic. Let $X, Y \subseteq X \cup Y$ have (at least) medium size. Then $\mu(X \cup Y) = \mu(X) \cup \mu(Y)$, and $\mu(X) = \mu(X \cup Y) \cap X$, $\mu(Y) = \mu(X \cup Y) \cap Y$.

- (2) Consistent classical formulas and their interpretation satisfy it:

Let \circ be conjunction in the composed language, \circ' be model set intersection, $f(\phi) = M(\phi)$. Let ϕ, ψ be classical formulas, defined on disjoint language fragments $\mathcal{L}, \mathcal{L}'$ of some language \mathcal{L}'' . Then $f(\phi \wedge \psi) = M(\phi) \cap M(\psi)$, and $M(\phi)$ is the projection of $M(\phi) \cap M(\psi)$ onto the (models of) language \mathcal{L} , likewise for $M(\psi)$. This is due to the way validity is defined, using only variables which occur in the formula.

As a consequence, monotonic logic has interpolation - see [GS09c].

- (3) It does not hold for inconsistent classical formulas: We cannot recover $M(a \wedge \neg a)$ and $M(b)$ from $M(a \wedge \neg a \wedge b)$, as we do not know where the inconsistency came from. The basic reason is trivial: One empty factor suffices to make the whole product empty, and we do not know which factor was the culprit. See Section 5.4.5 (page 36) for the discussion of a remedy.
- (4) Preferential logic satisfies it under certain conditions:
If $\mu(X \times Y) = \mu(X) \times \mu(Y)$ holds for model products and \vdash , then it holds by definition. An important consequence is that such a logic has interpolation of the form $\vdash \circ \vdash$, see Section 5.4.3 (page 34).
- (5) Modular revision a la Parikh is based on a similar idea.

5.1.2 Independence and multiplication of abstract size

We are mainly interested in nonmonotonic logic. In this domain, independence is strongly connected to multiplication of abstract size, and much of the present paper treats this connection and its repercussions.

We have at least two scenarios for multiplication, one is described in Diagram 5.2 (page 29), the second in Diagram 5.3 (page 30). In the first scenario, we have nested sets, in the second, we have set products. In the first scenario, we consider subsets which behave as the big set does, in the second scenario we consider subspaces, and decompose the behaviour of the big space into behaviour of the subspaces. In both cases, this results naturally in multiplication of abstract sizes. When we look at the corresponding relation properties, they are quite different (rankedness vs. some kind of modularity). But this is perhaps to be expected, as the two scenarios are quite different.

We do not know whether there are still other, interesting, scenarios to consider in our framework.

5.2 Introduction to abstract size

To put our work more into perspective, we repeat in this section material from [GS09a]. Essentially, we give here the main definitions, and the main overview table from [GS09a].

5.2.1 Notation

- (1) $\mathcal{P}(X)$ is the power set of X , \subseteq is the subset relation, \subset the strict part of \subseteq , i.e. $A \subset B$ iff $A \subseteq B$ and $A \neq B$. The operators $\wedge, \neg, \vee, \rightarrow$ and \vdash have their usual, classical interpretation.
- (2) $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ and $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ are dual abstract notions of size, $\mathcal{I}(X)$ is the set of “small” subsets of X , $\mathcal{F}(X)$ the set of “big” subsets of X . They are dual in the sense that $A \in \mathcal{I}(X) \Leftrightarrow X - A \in \mathcal{F}(X)$. “ \mathcal{I} ” evokes “ideal”, “ \mathcal{F} ” evokes “filter” though the full strength of both is reached only in $(< \omega * s)$. “s” evokes “small”, and “ $(x * s)$ ” stands for “ x small sets together are still not everything”.
- (3) If $A \subseteq X$ is neither in $\mathcal{I}(X)$, nor in $\mathcal{F}(X)$, we say it has medium size, and we define $\mathcal{M}(X) := \mathcal{P}(X) - (\mathcal{I}(X) \cup \mathcal{F}(X))$. $\mathcal{M}^+(X) := \mathcal{P}(X) - \mathcal{I}(X)$ is the set of subsets which are not small.
- (4) $\nabla x\phi$ is a generalized first order quantifier, it is read “almost all x have property ϕ ”. $\nabla x(\phi : \psi)$ is the relativized version, read: “almost all x with property ϕ have also property ψ ”. To keep the table “Rules on size” simple, we write mostly only the non-relativized versions. Formally, we have $\nabla x\phi := \Leftrightarrow \{x : \phi(x)\} \in \mathcal{F}(U)$ where U is the universe, and $\nabla x(\phi : \psi) := \Leftrightarrow \{x : (\phi \wedge \psi)(x)\} \in \mathcal{F}(\{x : \phi(x)\})$. Soundness and completeness results on ∇ can be found in [Sch95-1].
- (5) Analogously, for propositional logic, we define:

$$\alpha \sim \beta := \Leftrightarrow M(\alpha \wedge \beta) \in \mathcal{F}(M(\alpha)),$$
 where $M(\phi)$ is the set of models of ϕ .
- (6) In preferential structures, $\mu(X) \subseteq X$ is the set of minimal elements of X . This generates a principal filter by $\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$. Corresponding properties about μ are not listed systematically.
- (7) The usual rules (*AND*) etc. are named here (*AND $_{\omega}$*), as they are in a natural ascending line of similar rules, based on strengthening of the filter/ideal properties.
- (8) For any set of formulas T , and any consequence relation \sim , we will use $\overline{T} := \{\phi : T \vdash \phi\}$, the set of classical consequences of T , and $\overline{\overline{T}} := \{\phi : T \sim \phi\}$, the set of consequences of T under the relation \sim .
- (9) We say that a set X of models is definable by a formula (or a theory) iff there is a formula ϕ (a theory T) such that $X = M(\phi)$, or $X = M(T)$, the set of models of ϕ or T , respectively.
- (10) Most rules are explained in the table “Logical rules”, and “RW” stands for Right Weakening.

5.2.2 The groupes of rules

The rules concern properties of $\mathcal{I}(X)$ or $\mathcal{F}(X)$, or dependencies between such properties for different X and Y . All X, Y , etc. will be subsets of some universe, say V . Intuitively, V is the set of all models of some fixed propositional language. It is not necessary to consider all subsets of V , the intention is to consider subsets of V , which are definable by a formula or a theory. So we assume all X, Y etc. taken from some $\mathcal{Y} \subseteq \mathcal{P}(V)$, which we call the domain. In the former case, \mathcal{Y} is closed under set difference, in the latter case not necessarily so. (We will mention it when we need some particular closure property.)

The rules are divided into 5 groups:

- (1) (*Opt*), which says that “All” is optimal - i.e. when there are no exceptions, then a soft rule \sim holds.
- (2) 3 monotony rules:
 - (2.1) (*iM*) is inner monotony, a subset of a small set is small,
 - (2.2) (*eMT*) external monotony for ideals: enlarging the base set keeps small sets small,
 - (2.3) (*eMF*) external monotony for filters: a big subset stays big when the base set shrinks.

These three rules are very natural if “size” is anything coherent over change of base sets. In particular, they can be seen as weakening.

- (3) (\approx) keeps proportions, it is here mainly to point the possibility out.
- (4) a group of rules $x * s$, which say how many small sets will not yet add to the base set. The notation “ $(< \omega * s)$ ” is an allusion to the full filter property, that filters are closed under *finite* intersections.
- (5) Rational monotony, which can best be understood as robustness of \mathcal{M}^+ , see $(\mathcal{M}^{++})(3)$.

We will assume all base sets to be non-empty in order to avoid pathologies and in particular clashes between (Opt) and $(1 * s)$.

Note that the full strength of the usual definitions of a filter and an ideal are reached only in line $(< \omega * s)$.

Regularities

- (1) The group of rules $(x * s)$ use ascending strength of \mathcal{I}/\mathcal{F} .
- (2) The column (\mathcal{M}^+) contains interesting algebraic properties. In particular, they show a strengthening from $(3 * s)$ up to Rationality. They are not necessarily equivalent to the corresponding (I_x) rules, not even in the presence of the basic rules. The examples show that care has to be taken when considering the different variants.
- (3) Adding the somewhat superflous (CM_2) , we have increasing cautious monotony from (wCM) to full (CM_ω) .
- (4) We have increasing “or” from (wOR) to full (OR_ω) .
- (5) The line $(2 * s)$ is only there because there seems to be no (\mathcal{M}_2^+) , otherwise we could begin $(n * s)$ at $n = 2$.

Explanation of Table 2 (page 24), “Logical rules, definitions and connections Part I” and Table 3 (page 26), “Logical rules, definitions and connections Part II”:

The tables are split in two, as they would not fit onto a page otherwise. The difference between the first two columns is that the first column treats the formula version of the rule, the second the more general theory (i.e., set of formulas) version.

The first column “Corr.” is to be understood as follows:

Let a logic \vdash satisfy (LLE) and (CCL) , and define a function $f : \mathbf{D}_{\mathcal{L}} \rightarrow \mathbf{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp) , and $\overline{\overline{T}} = Th(f(M(T)))$.

If \vdash satisfies a rule in the left hand side, then - provided the additional properties noted in the middle for \Rightarrow hold, too - f will satisfy the property in the right hand side.

Conversely, if $f : \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\mathbf{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic \vdash by $\overline{\overline{T}} := Th(f(M(T)))$, then \vdash satisfies (LLE) and (CCL) . If f satisfies (μdp) , then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then - provided the additional properties noted in the middle for \Leftarrow hold, too - \vdash will satisfy the property in the left hand side.

We use the following abbreviations for those supplementary conditions in the “Correspondence” columns: “ $T = \phi$ ” means that, if one of the theories (the one named the same way in Definition 5.2.2 (page 23)) is equivalent to a formula, we do not need (μdp) . $-(\mu dp)$ stands for “without (μdp) ”.

$A = B \parallel C$ will abbreviate $A = B$, or $A = C$, or $A = B \cup C$.

Summary We can obtain all rules except $(RatM)$ and (\approx) from (Opt) , the monotony rules - (iM) , (eMT) , (eMF) -, and $(x * s)$ with increasing x .

Table 2: Logical rules, definitions and connections Part I

Logical rules, definitions and connections Part I					
Logical rule	Corr.	Model set	Corr.	Size Rules	
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \sim \beta$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	($\mu \subseteq$) $f(X) \subseteq X$	trivial	(Opt)	
(REF) Reflexivity $T \cup \{\alpha\} \sim \alpha$					
(LLE) Left Logical Equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \sim \beta \Rightarrow \alpha' \sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$				
(RW) Right Weakening $\alpha \sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow \alpha \sim \beta'$	(RW) $T \sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow T \sim \beta'$		trivial	(iM)	
(wOR) $\alpha \sim \beta, \alpha' \vdash \beta \Rightarrow \alpha \vee \alpha' \sim \beta$	(wOR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	\Leftrightarrow	(eMI)	
(disjOR) $\alpha \vdash \neg \alpha', \alpha \sim \beta, \alpha' \sim \beta \Rightarrow \alpha \vee \alpha' \sim \beta$	(disjOR) $\neg Con(T \cup T') \Rightarrow \overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	($\mu disjOR$) $X \cap Y = \emptyset \Rightarrow f(X \cup Y) \subseteq f(X) \cup f(Y)$	\Leftrightarrow	(I \cup disj)	
(CP) Consistency Preservation $\alpha \sim \perp \Rightarrow \alpha \vdash \perp$	(CP) $T \sim \perp \Rightarrow T \vdash \perp$	($\mu \emptyset$) $f(X) = \emptyset \Rightarrow X = \emptyset$	trivial	(I ₁)	
		($\mu \emptyset fin$) $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X		(I ₁)	
	(AND ₁) $\alpha \sim \beta \Rightarrow \alpha \not\sim \neg \beta$			(I ₂)	
	(AND _n) $\alpha \sim \beta_1, \dots, \alpha \sim \beta_{n-1} \Rightarrow \alpha \not\sim (\neg \beta_1 \vee \dots \vee \neg \beta_{n-1})$			(I _n)	
(AND) $\alpha \sim \beta, \alpha \sim \beta' \Rightarrow \alpha \sim \beta \wedge \beta'$	(AND) $T \sim \beta, T \sim \beta' \Rightarrow T \sim \beta \wedge \beta'$		trivial	(I _w)	
(CCL) Classical Closure $\overline{\overline{T}}$ classically closed	(CCL)		trivial	(iM) + (I _w)	
(OR) $\alpha \sim \beta, \alpha' \sim \beta \Rightarrow \alpha \vee \alpha' \sim \beta$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	\Leftrightarrow	(eMI) + (I _w)	
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\alpha}} \cup \{\alpha'\}$	$\overline{\overline{T \cup T'}} \subseteq \overline{\overline{T}} \cup \overline{\overline{T'}}$	(μPR) $X \subseteq Y \Rightarrow f(Y) \cap X \subseteq f(X)$	\Leftrightarrow	(eMI) + (I _w)	
		$\Leftrightarrow (\mu dp) + (\mu \subseteq) \neq -(\mu dp)$			
		$\Leftrightarrow (\mu \subseteq) T' = \emptyset$			
		($\mu PR'$) $f(X) \cap Y \subseteq f(X \cap Y)$			
(CUT) $T \sim \alpha; T \cup \{\alpha\} \sim \beta \Rightarrow T \sim \beta$	(CUT) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow \overline{\overline{T}} \subseteq \overline{\overline{T}}$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow f(X) \subseteq f(Y)$	\Leftrightarrow	(eMI) + (I _w)	
			\neq		

5.2.3 Table

The following table is split in two, as it is too big for printing in one page.

(See Table 4 (page 27), "Rules on size - Part I'" and Table 5 (page 28), "Rules on size - Part II".

Logical rules, definitions and connections Part II					
Logical rule	Corr.	Model set	Corr.	Size-Rule	
	Cumulativity				
(wCM) $\alpha \sim \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \sim \beta$			trivial	(eMF)	
(CM_2) $\alpha \sim \beta, \alpha \sim \beta' \Rightarrow \alpha \wedge \beta \not\sim \beta'$				(I_2)	
(CM_n) $\alpha \sim \beta_1, \dots, \alpha \sim \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \dots \wedge \beta_{n-1} \not\sim \beta_n$				(I_n)	
(CM) Cautious Monotony $\alpha \sim \beta, \alpha \sim \beta' \Rightarrow$ $\alpha \wedge \beta \sim \beta'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{T'}$	\Rightarrow \Leftarrow	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$	\Leftrightarrow	$(\mathcal{M}_\omega^+)(4)$
or $(ResM)$ Restricted Monotony $T \vdash \alpha, \beta \Rightarrow T \cup \{\alpha\} \vdash \beta$		\Rightarrow \Leftarrow	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$		
(CUM) Cumulativity $\alpha \sim \beta \Rightarrow$ $(\alpha \sim \beta' \Leftrightarrow \alpha \wedge \beta \sim \beta')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{T'}$	\Rightarrow \Leftarrow	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$	\Leftarrow \neq	$(eMI) + (I_\omega) + (\mathcal{M}_\omega^+)(4)$
	$(\subseteq \supseteq)$ $T \subseteq \overline{T'}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	\Rightarrow \Leftarrow	$(\mu \subseteq \supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$	\Leftarrow \neq	$(eMI) + (I_\omega) + (eMF)$
Rationality					
$(RatM)$ Rational Monotony $\alpha \sim \beta, \alpha \not\sim \beta' \Rightarrow$ $\alpha \wedge \beta' \sim \beta$	$(RatM)$ $Con(\overline{T \cup \overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{T'}} \cup T$	\Rightarrow $\Leftarrow (\mu dp)$ $\neq -(\mu dp)$ $\Leftarrow T = \emptyset$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$	\Leftrightarrow	(\mathcal{M}^{++})
	$(RatM =)$ $Con(\overline{T \cup \overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}} \cup T$	\Rightarrow $\Leftarrow (\mu dp)$ $\neq -(\mu dp)$ $\Leftarrow T = \emptyset$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$		
	$(Log =')$ $Con(\overline{T'} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{T'}} \cup T$	\Rightarrow $\Leftarrow (\mu dp)$ $\neq -(\mu dp)$ $\Leftarrow T = \emptyset$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$		
(DR) $\alpha \vee \beta \sim \gamma \Rightarrow$ $\alpha \sim \gamma$ or $\beta \sim \gamma$	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}},$ or $\overline{\overline{T'}},$ or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	\Rightarrow \Leftarrow	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X), f(Y)$ or $f(X) \cup f(Y)$		
	$(Log \cup)$ $Con(\overline{\overline{T}} \cup T), -Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $-Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\neq -(\mu dp)$	$(\mu \cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$		
	$(Log \cup')$ $Con(\overline{\overline{T'}} \cup T), -Con(\overline{\overline{T}} \cup \overline{\overline{T'}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\neq -(\mu dp)$	$(\mu \cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$		
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$		

Table 3: Logical rules, definitions and connections Part II

Table 4: Rules on size - Part I

Rules on size - Part I				
	"Ideal"	"Filter"	\mathcal{M}^+	∇
Optimal proportion				
(Opt)	$\emptyset \in \mathcal{I}(X)$	$X \in \mathcal{F}(X)$		$\forall x\alpha \rightarrow \nabla x\alpha$
Monotony (Improving proportions), (iM): internal monotony, (eMI): external monotony for ideals, (eMF): external monotony for filters				
(iM)	$A \subseteq B \in \mathcal{I}(X)$ \Rightarrow $A \in \mathcal{I}(X)$	$A \in \mathcal{F}(X)$, $A \subseteq B \subseteq X$ $\Rightarrow B \in \mathcal{F}(X)$		$\nabla x\alpha \wedge \forall x(\alpha \rightarrow \alpha')$ $\rightarrow \nabla x\alpha'$
(eMI)	$X \subseteq Y \Rightarrow$ $\mathcal{I}(X) \subseteq \mathcal{I}(Y)$			$\nabla x(\alpha : \beta) \wedge$ $\forall x(\alpha' \rightarrow \beta) \rightarrow$ $\nabla x(\alpha \vee \alpha' : \beta)$
(eMF)		$X \subseteq Y \Rightarrow$ $\mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq$ $\mathcal{F}(X)$		$\nabla x(\alpha : \beta) \wedge$ $\forall x(\beta \wedge \alpha \rightarrow \alpha') \rightarrow$ $\nabla x(\alpha \wedge \alpha' : \beta)$
Keeping proportions				
(\approx)	$(\mathcal{I} \cup disj)$ $A \in \mathcal{I}(X)$, $B \in \mathcal{I}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{I}(X \cup Y)$	$(\mathcal{F} \cup disj)$ $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{F}(X \cup Y)$	$(\mathcal{M}^+ \cup disj)$ $A \in \mathcal{M}^+(X)$, $B \in \mathcal{M}^+(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{M}^+(X \cup Y)$	$\nabla x(\alpha : \beta) \wedge$ $\nabla x(\alpha' : \beta) \wedge$ $\neg \exists x(\alpha \wedge \alpha') \rightarrow$ $\nabla x(\alpha \vee \alpha' : \beta)$
Robustness of proportions: $n * small \neq All$				
(1 * s)	(\mathcal{I}_1) $X \notin \mathcal{I}(X)$	(\mathcal{F}_1) $\emptyset \notin \mathcal{F}(X)$		(∇_1) $\nabla x\alpha \rightarrow \exists x\alpha$
(2 * s)	(\mathcal{I}_2) $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \neq X$	(\mathcal{F}_2) $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \neq \emptyset$		(∇_2) $\nabla x\alpha \wedge \nabla x\beta$ $\rightarrow \exists x(\alpha \wedge \beta)$
(n * s) (n \geq 3)	(\mathcal{I}_n) $A_1, \dots, A_n \in \mathcal{I}(X)$ \Rightarrow $A_1 \cup \dots \cup A_n \neq X$	(\mathcal{F}_n) $A_1, \dots, A_n \in \mathcal{F}(X)$ \Rightarrow $A_1 \cap \dots \cap A_n \neq \emptyset$	(\mathcal{M}_n^+) $X_1 \in \mathcal{F}(X_2), \dots,$ $X_{n-1} \in \mathcal{F}(X_n) \Rightarrow$ $X_1 \in \mathcal{M}^+(X_n)$	(∇_n) $\nabla x\alpha_1 \wedge \dots \wedge \nabla x\alpha_n$ \rightarrow $\exists x(\alpha_1 \wedge \dots \wedge \alpha_n)$
(< ω * s)	(\mathcal{I}_ω) $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \in \mathcal{I}(X)$	(\mathcal{F}_ω) $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \in \mathcal{F}(X)$	(\mathcal{M}_ω^+) (1) $A \in \mathcal{F}(X), X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (2) $A \in \mathcal{M}^+(X), X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (3) $A \in \mathcal{F}(X), X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{F}(Y)$ (4) $A, B \in \mathcal{I}(X) \Rightarrow$ $A - B \in \mathcal{I}(X - B)$	(∇_ω) $\nabla x\alpha \wedge \nabla x\beta \rightarrow$ $\nabla x(\alpha \wedge \beta)$
Robustness of \mathcal{M}^+				
(\mathcal{M}^{++})			(1) $A \in \mathcal{I}(X), B \notin \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{I}(X - B)$ (2) $A \in \mathcal{F}(X), B \notin \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{F}(X - B)$ (3) $A \in \mathcal{M}^+(X),$ $X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$	

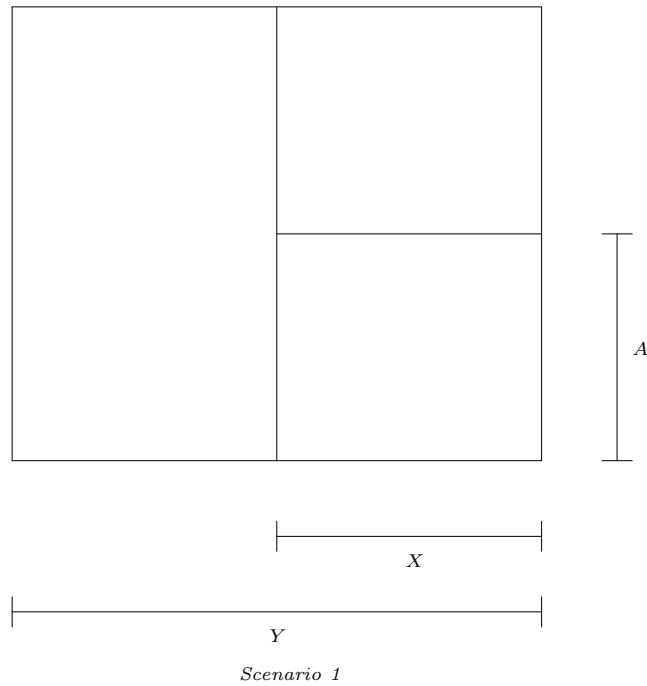
Table 5: Rules on size - Part II

Rules on size - Part II				
	various rules	AND	OR	Caut./Rat.Mon.
Optimal proportion				
(<i>Opt</i>)	(<i>SC</i>) $\alpha \vdash \beta \Rightarrow \alpha \sim \beta$			
Monotony (Improving proportions)				
(<i>iM</i>)	(<i>RW</i>) $\alpha \sim \beta, \beta \vdash \beta' \Rightarrow$ $\alpha \sim \beta'$			
(<i>eMI</i>)	(<i>PR'</i>) $\alpha \sim \beta, \alpha \vdash \alpha',$ $\alpha' \wedge \neg \alpha \vdash \beta \Rightarrow$ $\alpha' \sim \beta$ (μPR) $X \subseteq Y \Rightarrow$ $\mu(Y) \cap X \subseteq \mu(X)$		(<i>wOR</i>) $\alpha \sim \beta, \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \sim \beta$ (μwOR) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	
(<i>eMF</i>)				(<i>wCM</i>) $\alpha \sim \beta, \alpha' \vdash \alpha,$ $\alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \sim \beta$
Keeping proportions				
(\approx)	(<i>NR</i>) $\alpha \sim \beta \Rightarrow$ $\alpha \wedge \gamma \sim \beta$ or $\alpha \wedge \neg \gamma \sim \beta$		(<i>disjOR</i>) $\alpha \sim \beta, \alpha' \sim \beta'$ $\alpha \vdash \neg \alpha', \Rightarrow$ $\alpha \vee \alpha' \sim \beta \vee \beta'$ ($\mu disjOR$) $X \cap Y = \emptyset \Rightarrow$ $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	
Robustness of proportions: $n * small \neq All$				
($1 * s$)	(<i>CP</i>) $\alpha \sim \perp \Rightarrow \alpha \vdash \perp$	(<i>AND₁</i>) $\alpha \sim \beta \Rightarrow \alpha \not\vdash \neg \beta$		
($2 * s$)		(<i>AND₂</i>) $\alpha \sim \beta, \alpha \sim \beta' \Rightarrow$ $\alpha \not\vdash \neg \beta \vee \neg \beta'$	(<i>OR₂</i>) $\alpha \sim \beta \Rightarrow \alpha \not\vdash \neg \beta$	(<i>CM₂</i>) $\alpha \sim \beta \Rightarrow \alpha \not\vdash \neg \beta$
($n * s$) ($n \geq 3$)		(<i>AND_n</i>) $\alpha \sim \beta_1, \dots, \alpha \sim \beta_n$ \Rightarrow $\alpha \not\vdash \neg \beta_1 \vee \dots \vee \neg \beta_n$	(<i>OR_n</i>) $\alpha_1 \sim \beta, \dots, \alpha_{n-1} \sim \beta$ \Rightarrow $\alpha_1 \vee \dots \vee \alpha_{n-1} \not\vdash \neg \beta$	(<i>CM_n</i>) $\alpha \sim \beta_1, \dots, \alpha \sim \beta_{n-1}$ \Rightarrow $\alpha \wedge \beta_1 \wedge \dots \wedge \beta_{n-2} \not\vdash$ $\neg \beta_{n-1}$
($< w * s$)		(<i>AND_w</i>) $\alpha \sim \beta, \alpha \sim \beta' \Rightarrow$ $\alpha \sim \beta \wedge \beta'$	(<i>OR_w</i>) $\alpha \sim \beta, \alpha' \sim \beta \Rightarrow$ $\alpha \vee \alpha' \sim \beta$ (μOR) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	(<i>CM_w</i>) $\alpha \sim \beta, \alpha \sim \beta' \Rightarrow$ $\alpha \wedge \beta \sim \beta'$ (μCM) $\mu(X) \subseteq Y \subseteq X \Rightarrow$ $\mu(Y) \subseteq \mu(X)$
Robustness of \mathcal{M}^+				
(\mathcal{M}^{++})				(<i>RatM</i>) $\alpha \sim \beta, \alpha \not\vdash \neg \beta' \Rightarrow$ $\alpha \wedge \beta' \sim \beta$ ($\mu RatM$) $X \subseteq Y,$ $X \cap \mu(Y) \neq \emptyset \Rightarrow$ $\mu(X) \subseteq \mu(Y) \cap X$

5.3 Multiplication of size for subsets

Here we have nested sets, $A \subseteq X \subseteq Y$, A is a certain proportion of X , and X of Y , resulting in a multiplication of relative size or proportions. This is a classical subject of nonmonotonic logic, see the last section, taken from [GS09a], it is partly repeated here to stress the common points with the other scenario.

Diagram 5.2



5.3.1 Properties

Diagram 5.2 (page 29) is to be read as follows: The whole set Y is split in X and $Y - X$, X is split in A and $X - A$. X is a small/medium/big part of Y , A is a small/medium/big part of X . The question is: is A a small/medium/big part of Y ?

Note that the relation of A to X is conceptually different from that of X to A , as we change the base set by going from X to A , but not when going from A to X . Thus, in particular, when we read the diagram as expressing multiplication, commutativity is not necessarily true.

We looked at this scenario already in [GS09a], but there from an additive point of view, using various basic properties like (iM), (eMT), (eMF). Here, we use just multiplication - except sometimes for motivation.

We examine different rules:

If $Y = X$ or $X = A$, there is nothing to show, so 1 is the neutral element of multiplication.

If $X \in \mathcal{I}(Y)$ or $A \in \mathcal{I}(X)$, then we should have $A \in \mathcal{I}(Y)$. (Use for motivation (iM) or (eMT) respectively.)

So it remains to look at the following cases, with the “natural” answers given already:

- (1) $X \in \mathcal{F}(Y), A \in \mathcal{F}(X) \Rightarrow A \in \mathcal{F}(Y)$,
- (2) $X \in \mathcal{M}^+(Y), A \in \mathcal{F}(X) \Rightarrow A \in \mathcal{M}^+(Y)$,
- (3) $X \in \mathcal{F}(Y), A \in \mathcal{M}^+(X) \Rightarrow A \in \mathcal{M}^+(Y)$,
- (4) $X \in \mathcal{M}^+(Y), A \in \mathcal{M}^+(X) \Rightarrow A \in \mathcal{M}^+(Y)$.

But (1) is case (3) of (\mathcal{M}_ω^+) in [GS09a], see Table “Rules on size” in the last section.

(2) is case (2) of (\mathcal{M}_ω^+) there,

(3) is case (1) of (\mathcal{M}_ω^+) there, finally,

(4) is (\mathcal{M}^{++}) there.

So the first three correspond to various expressions of (AND_ω) , (OR_ω) , (CM_ω) , the last one to $(RatM)$.

But we can read them also the other way round, e.g.:

- (1) corresponds to: $\alpha \vdash \beta, \alpha \wedge \beta \vdash \gamma \Rightarrow \alpha \vdash \gamma$,
- (2) corresponds to: $\alpha \not\vdash \neg\beta, \alpha \wedge \beta \vdash \gamma \Rightarrow \alpha \not\vdash \neg(\beta \wedge \gamma)$,
- (3) corresponds to: $\alpha \vdash \beta, \alpha \wedge \beta \not\vdash \neg\gamma \Rightarrow \alpha \not\vdash \neg(\beta \wedge \gamma)$.

All these rules might be seen as too idealistic, so just as we did in [GS09a], we can consider milder versions: We might for instance consider a rule which says that $big * \dots * big$, n times, is not small. Consider for instance the case $n = 2$. So we would conclude that A is not small in Y . In terms of logic, we then have: $\alpha \vdash \beta, \alpha \wedge \beta \vdash \gamma \Rightarrow \alpha \not\vdash (\neg\beta \vee \neg\gamma)$. We can obtain the same logic property from $3 * small \neq all$.

Fact 5.1

The two rules $Small + small = small$ and $1 * big = big$ entail the rule $big * big = big$

Proof

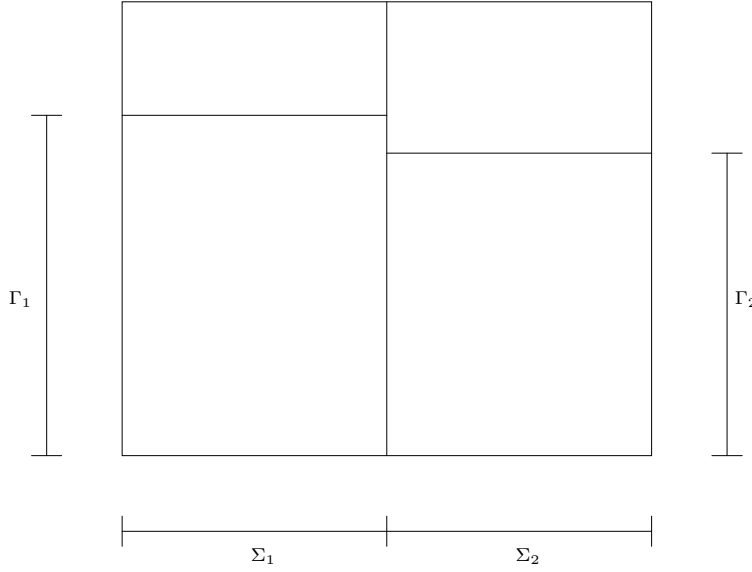
$$1 * big \cap big * 1 = big * big \quad \square$$

Finally, it is well-known that rankedness corresponds to the rule (\mathcal{M}^{++}) , see, e.g., [GS09a], and we will not discuss this here any further.

5.4 Multiplication of size for subspaces

5.4.1 Properties

Diagram 5.3



Scenario 2

In this scenario, Σ_i are sets of sequences. (Corresponding, intuitively, to a set of models in language \mathcal{L}_i , Σ_i will be the set of α_i -models, and the subsets Γ_i are to be seen as the “best” models, where β_i will hold. The languages are supposed to be disjoint sublanguages of a common language \mathcal{L} .)

In this scenario, the Σ_i have symmetrical roles, so there is no intuitive reason for multiplication not to be commutative.

We can interpret the situation twofold:

First, we work separately in sublanguage \mathcal{L}_1 and \mathcal{L}_2 , and, say, α_i and β_i are both defined in \mathcal{L}_i , and we look at $\alpha_i \sim \beta_i$ in the sublanguage \mathcal{L}_i , or, we consider both α_i and β_i in the big language \mathcal{L} , and look at $\alpha_i \sim \beta_i$ in \mathcal{L} . These two ways are a priori completely different. Speaking in preferential terms, it is not at all clear why the orderings on the submodels should have anything to do with the orderings on the whole models. It seems a very desirable property, but we have to postulate it, which we do now (an overview is given in Table 6 (page 40)):

(*big * 1 = big*) Let $\Gamma_1 \subseteq \Sigma_1$, then $\Gamma_1 \times \Sigma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$ if $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, (and the dual rule for Σ_2 and Γ_2).

This property preserves proportions, so it seems intuitively quite uncontested, whenever we admit coherence over products. (Recall that there was nothing to show in the first scenario.)

When we re-consider above case: suppose $\alpha \sim \beta$ in the sublanguage, so $M(\beta) \in \mathcal{F}(M(\alpha))$, so by (*big * 1 = big*), $M(\beta) \in \mathcal{F}(M(\alpha))$ in the big language \mathcal{L} .

We obtain the dual rule for small (and likewise, medium size) sets:

(*small * 1 = small*) Let $\Gamma_1 \subseteq \Sigma_1$, then $\Gamma_1 \times \Sigma_2 \in \mathcal{I}(\Sigma_1 \times \Sigma_2)$ if $\Gamma_1 \in \mathcal{I}(\Sigma_1)$, (and the dual rule for Σ_2 and Γ_2), establishing *All = 1* as the neutral element for multiplication.

We look now at other, plausible rules:

(*small * x = small*) $\Gamma_1 \in \mathcal{I}(\Sigma_1), \Gamma_2 \subseteq \Sigma_2 \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{I}(\Sigma_1 \times \Sigma_2)$

(*big * big = big*) $\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{F}(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$

$(big * medium = medium) \Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{M}(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{M}(\Sigma_1 \times \Sigma_2)$

$(medium * medium = medium) \Gamma_1 \in \mathcal{M}^+(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$

When we accept all, we can invert $(big * big)$, as a big product must be composed of big components. Likewise, at least one component of a small product has to be small.

We see that these properties give a lot of modularity. We can calculate the consequences of α and α' separately - provided α, α' use disjoint alphabets - and put the results together afterwards. Such properties are particularly interesting for classification purposes, where subclasses are defined with disjoint alphabets.

5.4.2 Size multiplication and corresponding preferential relations

We turn to those conditions which provide the key to non-monotonic interpolation theorems - see Section 5.4.3 (page 34). We quote from [GS09c] the following pairwise equivalent conditions for a principal filter generated by μ :

Definition 5.2

$(S * 1) \Delta \subseteq \Sigma' \times \Sigma''$ is big iff there is $\Gamma = \Gamma' \times \Gamma'' \subseteq \Delta$ s.t. $\Gamma' \subseteq \Sigma'$ and $\Gamma'' \subseteq \Sigma''$ are big

$(\mu * 1) \mu(\Sigma' \times \Sigma'') = \mu(\Sigma') \times \mu(\Sigma'')$

$(S * 2) \Gamma \subseteq \Sigma$ is big $\Rightarrow \Gamma \upharpoonright X' \subseteq \Sigma \upharpoonright X'$ is big - where Σ is not necessarily a product.

$(\mu * 2) \mu(\Sigma) \subseteq \Gamma \Rightarrow \mu(\Sigma \upharpoonright X') \subseteq \Gamma \upharpoonright X'$

Definition 5.3

Call a relation \prec a GH (= general Hamming) relation iff the following two conditions hold:

(GH1) $\sigma \preceq \tau \wedge \sigma' \preceq \tau' \wedge (\sigma \prec \tau \vee \sigma' \prec \tau') \Rightarrow \sigma\sigma' \prec \tau\tau'$

(where $\sigma \preceq \tau$ iff $\sigma \prec \tau$ or $\sigma = \tau$)

(GH2) $\sigma\sigma' \prec \tau\tau' \Rightarrow \sigma \prec \tau \vee \sigma' \prec \tau'$

(GH2) means that some compensation is possible, e.g., $\tau \prec \sigma$ might be the case, but $\sigma' \prec \tau'$ wins in the end.

We use (GH) for $(GH1) + (GH2)$.

Example 5.1

The following are example of GH relations:

Define on all components X_i a relation \prec_i .

(1) The set variant Hamming relation:

Then the relation \prec defined on $\Pi\{X_i : i \in I\}$ by $\sigma \prec \tau$ iff for all $j \sigma_j \preceq_j \tau_j$, and there is at least one i s.t. $\sigma_i \prec_i \tau_i$.

(2) The counting variant Hamming relation:

Then the relation \prec defined on $\Pi\{X_i : i \in I\}$ by $\sigma \prec \tau$ iff the number of i such that $\sigma_i \prec_i \tau_i$ is bigger than the number of i such that $\tau_i \prec_i \sigma_i$.

(3) The weighed counting Hamming relation:

Like the counting relation, but we give different (numerical) importance to different i . E.g., $\sigma_1 \prec \tau_1$ may count 1, $\sigma_2 \prec \tau_2$ may count 2, etc.

□

Proposition 5.2

Let $\sigma \prec \tau \Leftrightarrow \tau \notin \mu(\{\sigma, \tau\})$ and \prec be smooth. Then μ satisfies $(\mu * 1)$ iff \prec is a GH relation.

Proof

(1) $(\mu * 1)$ entails the GH relation conditions

(GH1): Suppose $\sigma \prec \tau$ and $\sigma' \preceq \tau'$. Then $\mu(\{\sigma, \tau\}) = \{\sigma\}$, and $\mu(\{\sigma', \tau'\}) = \{\sigma'\}$ (either $\sigma' \prec \tau'$ or $\sigma' = \tau'$, so in both cases $\mu(\{\sigma', \tau'\}) = \{\sigma'\}$). As $\tau \notin \mu(\{\sigma, \tau\})$, $\tau\tau' \notin \mu(\{\sigma, \tau\} \times \{\sigma', \tau'\}) =_{(\mu * 1)} \mu(\{\sigma, \tau\}) \times \mu(\{\sigma', \tau'\}) = \{\sigma\} \times \{\sigma'\} = \{\sigma\sigma'\}$, so by smoothness $\sigma\sigma' \prec \tau\tau'$.

(GH2): Let $X := \{\sigma, \tau\}$, $Y := \{\sigma', \tau'\}$, so $X \times Y = \{\sigma\sigma', \sigma\tau', \tau\sigma', \tau\tau'\}$. Suppose $\sigma\sigma' \prec \tau\tau'$, so $\tau\tau' \notin \mu(X \times Y) =_{(\mu * 1)} \mu(X) \times \mu(Y)$. If $\sigma \not\prec \tau$, then $\tau \in \mu(X)$, likewise if $\sigma' \not\prec \tau'$, then $\tau' \in \mu(Y)$, so $\tau\tau' \in \mu(X \times Y)$, contradiction.

(2) The GH relation conditions generate $(\mu * 1)$.

$\mu(X \times Y) \subseteq \mu(X) \times \mu(Y)$: Let $\tau \in X$, $\tau' \in Y$, $\tau\tau' \notin \mu(X) \times \mu(Y)$, then $\tau \notin \mu(X)$ or $\tau' \notin \mu(Y)$. Suppose $\tau \notin \mu(X)$, let $\sigma \in X$, $\sigma \prec \tau$, so by condition (GH1) $\sigma\tau' \prec \tau\tau'$, so $\tau\tau' \notin \mu(X \times Y)$.

$\mu(X) \times \mu(Y) \subseteq \mu(X \times Y)$: Let $\tau \in X$, $\tau' \in Y$, $\tau\tau' \notin \mu(X \times Y)$, so there is $\sigma\sigma' \prec \tau\tau'$, $\sigma \in X$, $\sigma' \in Y$, so by (GH2) either $\sigma \prec \tau$ or $\sigma' \prec \tau'$, so $\tau \notin \mu(X)$ or $\tau' \notin \mu(Y)$, so $\tau\tau' \notin \mu(X) \times \mu(Y)$.

□

Fact 5.3

Let $\Gamma \subseteq \Sigma$, $\Gamma' \subseteq \Sigma'$, $\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ be small, let (GH2) hold, then $\Gamma \subseteq \Sigma$ is small or $\Gamma' \subseteq \Sigma'$ is small.

Proof

Suppose $\Gamma \subseteq \Sigma$ is not small, so there is $\gamma \in \Gamma$ and no $\sigma \in \Sigma$ with $\sigma \prec \gamma$. Fix this γ . Consider $\{\gamma\} \times \Gamma'$. As $\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ is small, there is for each $\gamma\gamma'$, $\gamma' \in \Gamma'$ some $\sigma\sigma' \in \Sigma \times \Sigma'$, $\sigma\sigma' \prec \gamma\gamma'$. By (GH2) $\sigma \prec \gamma$ or $\sigma' \prec \gamma'$, but $\sigma \prec \gamma$ was excluded, so for all $\gamma' \in \Gamma'$ there is $\sigma' \in \Sigma'$ with $\sigma' \prec \gamma'$, so $\Gamma' \subseteq \Sigma'$ is small. □

Fact 5.4

Let $\Gamma \subseteq \Sigma$ be small, $\Gamma' \subseteq \Sigma'$, let (GH1) hold, then $\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ is small.

Proof

Let $\gamma \in \Gamma$, so there is $\sigma \in \Sigma$ and $\sigma \prec \gamma$. By (GH1), for any $\gamma' \in \Gamma'$ $\sigma\gamma' \prec \gamma\gamma'$, so no $\gamma\gamma' \in \Gamma \times \Gamma'$ is minimal. □

Proposition 5.5

$(\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ is small iff at least one of $\Gamma \subseteq \Sigma$, $\Gamma' \subseteq \Sigma'$ is small) entails (GH1) and (GH2).

Proof

(GH1): Let $\sigma \prec \tau$, $\sigma' \preceq \tau'$, we have to show $\sigma\sigma' \prec \tau\tau'$. $\sigma \prec \tau \Rightarrow \{\tau\} \subseteq \{\sigma, \tau\}$ is small $\Rightarrow \{\tau\} \times \{\sigma', \tau'\} \subseteq \{\sigma, \tau\} \times \{\sigma', \tau'\}$ is small, so some element has to be smaller than $\tau\tau'$, by smoothness, there has to be a minimal element smaller than $\tau\tau'$, so $\sigma\sigma' \prec \tau\tau'$ or $\sigma\tau' \prec \tau\tau'$. Case 1: $\sigma' = \tau'$. Then $\sigma\sigma' \prec \tau\tau'$. Case 2: $\sigma' \prec \tau'$. Then $\{\sigma, \tau\} \times \{\tau'\}$ is small, so $\sigma\tau'$ is not minimal. Thus, again by smoothness, $\sigma\sigma' \prec \tau\tau'$.

(GH2): Let $\sigma\sigma' \prec \tau\tau'$, we have to show $\sigma \prec \tau$ or $\sigma' \prec \tau'$. Suppose $\sigma \not\prec \tau$, $\sigma' \not\prec \tau'$, then $\{\tau\} \subseteq \{\sigma, \tau\}$ is not small, $\{\tau'\} \subseteq \{\sigma', \tau'\}$ is not small, so $\{\tau\tau'\} \subseteq \{\sigma\tau\} \times \{\sigma'\tau'\}$ is not small, so $\sigma\sigma' \not\prec \tau\tau'$, contradiction.

□

To complete our picture, we repeat from [GS09c] the following very (perhaps too much so - see the discussion there) strong definition and two results (the reader is referred there for proofs):

Definition 5.4

$$(GH+) \sigma \preceq \tau \wedge \sigma' \preceq \tau' \wedge (\sigma \prec \tau \vee \sigma' \prec \tau') \Leftrightarrow \sigma\sigma' \prec \tau\tau'$$

Fact 5.6

$(\mu * 1)$ and $(\mu * 2)$ and the usual axioms for smooth relations characterize relations satisfying $(GH+)$.

Proposition 5.7

Interpolation of the form $\phi \vdash \alpha \sim \psi$ exists, if $(\mu * 1)$ and $(\mu * 2)$ hold.

Note

Note that already $(\mu * 1)$ results in a strong independence result in the second scenario: Let $\sigma\rho' \prec \tau\rho'$, then $\sigma\rho'' \prec \tau\rho''$ for all ρ'' . Thus, whether $\{\rho''\}$ is small, or medium size (i.e. $\rho'' \in \mu(\Sigma')$), the behaviour of $\Sigma \times \{\rho''\}$ is the same. This we do not have in the first scenario, as small sets may behave very differently from medium size sets. (But, still, their internal structure is the same, only the minimal elements change.) When $(\mu * 2)$ holds, then if $\sigma\sigma' \prec \tau\tau'$ and $\sigma \neq \tau$, then $\sigma \prec \tau$, i.e. we need not have $\sigma' = \tau'$.

5.4.3 Interpolation

Proposition 5.8

$(\mu * 1)$ entails interpolation of the form $\phi \sim \alpha \sim \psi$ in 2-valued non-monotonic logic generated by minimal model sets.

Proof

Let the product be defined on $J \cup J' \cup J''$ (i.e., $J \cup J' \cup J''$ is the set of propositional variables in the intended application). See Diagram 5.4 (page 35).

We abuse notation and write $\phi \sim \Sigma$ if $\mu(\phi) \subseteq \Sigma$. As usual, $\mu(\phi)$ abbreviates $\mu(M(\phi))$.

For clarity, even if it clutters up notation, we will be precise about where μ is formed. Thus, we write $\mu_{J \cup J' \cup J''}(X)$ when we take the minimal elements in the full product, $\mu_J(X)$ when we consider only the product on J , etc.

Let $\phi \sim \psi$, i.e., $\mu_{J \cup J' \cup J''}(\phi) \subseteq M(\psi)$, ϕ be defined on $J' \cup J''$, ψ on $J \cup J'$.

As $\mu_{J \cup J' \cup J''}(\phi) \subseteq X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}$, $\phi \sim X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}$.

We show $\mu_{J \cup J' \cup J''}(X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}) \subseteq M(\psi)$.

(1) As $M(\phi) = X_J \times M(\phi) \upharpoonright (J' \cup J'')$, $\mu_{J \cup J' \cup J''}(\phi) = \mu_J(X_J) \times \mu_{J' \cup J''}(M(\phi) \upharpoonright (J' \cup J''))$ by $(\mu * 1)$.

(2) By $(\mu * 1)$ again, $\mu_{J \cup J' \cup J''}(X_J \times \mu_{J \cup J' \cup J''}(\phi) \upharpoonright J' \times X_{J''}) \subseteq \mu_J(X_J) \times \mu_{J'}((\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J')) \times \mu_{J''}(X_{J''})$.

So it suffices to show $\mu_J(X_J) \times \mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times \mu_{J''}(X_{J''}) \models \psi$.

Proof: Let $\sigma = \sigma_J \sigma_{J'} \sigma_{J''} \in \mu_J(X_J) \times \mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times \mu_{J''}(X_{J''})$, so $\sigma_J \in \mu_J(X_J)$.

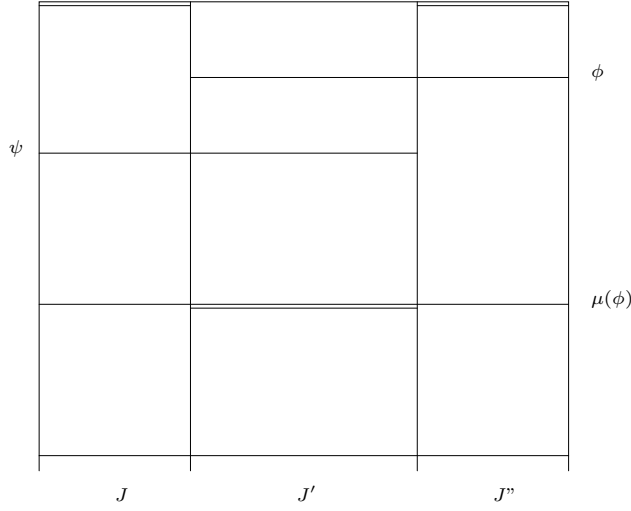
By definition and $\mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \subseteq \mu_{J \cup J' \cup J''}(\phi) \upharpoonright J'$, there is $\sigma' = \sigma'_J \sigma'_{J'} \sigma'_{J''} \in \mu_{J \cup J' \cup J''}(\phi)$ s.t. $\sigma'_{J'} = \sigma_{J'}$, i.e. $\sigma' = \sigma'_J \sigma_{J'} \sigma'_{J''}$. As $\sigma' \in \mu_{J \cup J' \cup J''}(\phi)$, $\sigma' \models \psi$.

By (1) and $\sigma_J \in \mu_J(X_J)$ also $\sigma_J \sigma_{J'} \sigma'_{J''} \in \mu_{J \cup J' \cup J''}(\phi)$, so also $\sigma_J \sigma_{J'} \sigma'_{J''} \models \psi$.

But ψ does not depend on J'' , so also $\sigma = \sigma_J \sigma_{J'} \sigma_{J''} \models \psi$.

□

Diagram 5.4



Non-monotonic interpolation
Double lines: interpolant

Remarks for the converse: from interpolation to $(\mu * 1)$

Example 5.2

We show here in (1) and (2) that half of the condition $(\mu * 1)$ is not sufficient for interpolation, and in (3) that interpolation may hold, even if $(\mu * 1)$ fails. When looking closer, the latter is not surprising: μ of sub-products may be defined in a funny way, which has nothing to do with the way μ on the big product is defined.

Consider the language based on p, q, r .

For (1) and (2) define the order \prec on sequences of length 3 by $\neg p \neg q \neg r \prec p \neg q \neg r$, leave all other 3-sequences incomparable.

Let $\phi = \neg q \wedge \neg r$, $\psi = \neg p \wedge \neg q$, so $\mu(\phi) = \neg p \wedge \neg q \wedge \neg r$, and $\phi \sim \psi$. Suppose there is α , $\phi \vdash \alpha \vdash \psi$, α written with q only, so α is equivalent to FALSE, TRUE, q , or $\neg q$. $\phi \not\vdash \text{FALSE}$, $\phi \not\vdash q$. $\text{TRUE} \not\vdash \psi$, $\neg q \not\vdash \psi$. Thus, there is no such α , and \vdash has no interpolation. We show in (1) and (2) that we can make both directions of $(\mu * 1)$ true separately, so they do not suffice to obtain interpolation.

(1) We make $\mu(X \times Y) \subseteq \mu(X) \times \mu(Y)$ true, but not the converse.

Do not order any sequences of length 2 or 1, i.e. μ is there always identity. Thus, $\mu(X \times Y) \subseteq X \times Y = \mu(X) \times \mu(Y)$ holds trivially.

For (2) and (3), consider the following ordering $<$ between sequences: $\sigma < \tau$ iff there is $\neg x$ in σ , x in τ , but for no y in σ , $\neg y$ in τ . E.g., $\neg p < p$, $\neg pq < pq$, $\neg p \neg q < pq$, but $\neg pq \not< p \neg q$.

(2) We make $\mu(X \times Y) \supseteq \mu(X) \times \mu(Y)$ true, but not the converse.

We order all sequences of length 1 or 2 by $<$.

Suppose $\sigma \in X \times Y - \mu(X \times Y)$. Case 1: $X \times Y$ consists of sequences of length 2. Then, by definition, $\sigma \notin \mu(X) \times \mu(Y)$. Case 2: $X \times Y$ consists of sequences of length 3. Then $\sigma = p \neg q \neg r$, and there is $\tau = \neg p \neg q \neg r \in X \times Y$. So $\{p, \neg p\} \subseteq X$ or $\{p \neg q, \neg p \neg q\} \subseteq X$, but in both cases $\sigma \upharpoonright X \notin \mu(X)$.

Finally, note that $\mu(TRUE) \not\subseteq \{\neg p \neg q \neg r\}$, so full $(\mu * 1)$ does not hold.

(3) We make interpolation hold, but $\mu(X) \times \mu(Y) \not\subseteq \mu(X \times Y)$:

We order all sequences of length 3 by $<$. Shorter sequences are made incomparable, so for shorter sequences $\mu(X) = X$.

Obviously, in general $\mu(X) \times \mu(Y) \not\subseteq \mu(X \times Y)$.

But the proof of Proposition 5.8 (page 34) goes through as above, only directly, without the use of factorizing and taking μ of the factors.

□

5.4.4 Language change

Independence of language fragments gives us the following perspectives:

- (1) it makes independent and parallel treatment of fragments possible, and offers thus efficient treatment in applications (descriptive logics etc.).
- (2) it results in new rules similar to the classical ones like AND, OR, Cumulativity, etc. We can thus obtain postulates about reasonable behaviour, but also classification by those rules, see Table 6 (page 40), Scenario 2, Logical property.
- (3) it sheds light on notions like “ceteris paribus”, which we saw in the context of obligations, see [GS08g].
- (4) it clarifies notions like “normal with respect to ϕ , but not ψ ”
- (5) it helps to understand e.g. inheritance diagrams where arrows make other information accessible, and we need an underlying mechanism to combine bits of information, given in different languages.

5.4.5 A relevance problem

Consider the formula $\phi := a \wedge \neg a \wedge b$. Then $M(\phi) = \emptyset$. But we cannot recover where the problem came from, and this results in the EFQ rule. We now discuss one, purely algebraic, approach to remedy.

Consider 3 valued models, with a new value b for both, in addition to t and f . Above formula would then have the model $m(a) = b, m(b) = t$. So there is a model, EFQ fails, and we can recover the culprit.

To have the usual behaviour of \wedge as intersection, it might be good to change the definition so that $m(x) = b$ is always a model. Then $M(b) = \{m(b) = t, m'(b) = b\}$, $M(\neg b) = \{m(b) = f, m'(b) = b\}$, and $M(b \wedge \neg b) = \{m'(b) = b\}$.

It is not yet clear which version to choose, and we have no syntactic characterization.

Other idea:

Use meaningless models. Take a conjunction of literals. $m(a) = t$ and $m(a) = x$ is a model if there is only a in the conjunction, $m(a) = f$ and $m(a) = x$ if there is only $\neg a$ in the conjunction, $m(a) = *$ if both are present, and all models, if none is present. Thus there is always a model, and we can isolate the contradictory parts: there, only $m(a) = x$ is present.

5.4.6 Small subspaces

When considering small subsets in nonmonotonic logic, we neglect small subsets of models. What is the analogue when considering small subspaces, i.e. when $J = J' \cup J''$, with J'' small in J in nonmonotonic logic?

It is perhaps easiest to consider the relation based approach first. So we have an order on $\Pi J'$ and one on $\Pi J''$, J'' is small, and we want to know how to construct a corresponding order on ΠJ . Two solutions come to mind:

- a less radical one: we make a lexicographic ordering, where the one on $\Pi J'$ has precedence over the one on $\Pi J''$,
- a more radical one: we totally forget about the ordering of $\Pi J''$, i.e. we do as if the ordering on $\Pi J''$ were the empty set, i.e. $\sigma'\sigma'' \prec \tau'\tau''$ iff $\sigma' \prec \tau'$ and $\sigma'' = \tau''$.

We call this condition *forget*(J'').

The more radical one is probably more interesting. Suppose ϕ' is written in language J' , ϕ'' in language J'' , we then have

$$\phi' \wedge \phi'' \vdash \psi' \wedge \psi'' \text{ iff } \phi' \vdash \psi' \text{ and } \phi'' \vdash \psi''.$$

This approach, is of course the same as considering on the small coordinate only ALL as a big subset, (see the lines $x * 1/1 * x$ in Table 6 (page 40)), so, in principle, we get nothing new.

5.5 Revision and distance relations

We will look here into distance based theory revision a la AGM, see [AGM85] and [LMS01], and also [Sch04] for more details. First, we introduce some notation, and give a result taken from [GS09c] (slightly modified).

Definition 5.5

Let d be a distance on some product space $X \times Y$, and its components. (We require of distances only that they are comparable, that $d(x, y) = 0$ iff $x = y$, and that $d(x, y) \geq 0$.)

d is called a generalized Hamming distance (GHD) iff it satisfies the following two properties:

$$(GHD1) \ d(\alpha, \tau) \leq d(\alpha, \beta) \text{ and } d(\sigma', \tau') \leq d(\alpha', \beta') \text{ and } (d(\sigma, \tau) < d(\alpha, \beta) \text{ or } d(\sigma', \tau') < d(\alpha', \beta')) \Rightarrow d(\sigma\sigma', \tau\tau') < d(\alpha\alpha', \beta\beta')$$

$$(GHD2) \ d(\sigma\sigma', \tau\tau') < d(\alpha\alpha', \beta\beta') \Rightarrow d(\sigma, \tau) < d(\alpha, \beta) \text{ or } d(\sigma', \tau') < d(\alpha', \beta')$$

Definition 5.6

Given a distance d , define for two sets X, Y

$$X \mid Y := \{y \in Y : \exists x \in X (\neg \exists x' \in X, y' \in Y. d(x', y') < d(x, y))\}.$$

We assume that $X \mid Y \neq \emptyset$ if $X, Y \neq \emptyset$. Note that this is related to the consistency axiom of AGM theory revision: revising by a consistent formula gives a consistent result. The assumption may be wrong due to infinite descending chains of distances.

Definition 5.7

Given \mid on models, we can define an AGM revision operator $*$ as follows:

$$T * \phi := Th(M(T) \mid M(\phi))$$

where T is a theory, and $Th(X)$ is the set of formulas which hold in all $x \in X$.

It was shown in [LMS01] that a revision operator thus defined satisfies the AGM revision postulates.

We have a result analogous to the relation case:

Fact 5.9

Let \mid be defined by a generalized Hamming distance, then \mid satisfies

$$(\mid *) \ (\Sigma_1 \times \Sigma'_1) \mid (\Sigma_2 \times \Sigma'_2) = (\Sigma_1 \mid \Sigma_2) \times (\Sigma'_1 \mid \Sigma'_2).$$

Proof

“ \subseteq ”:

Suppose $d(\sigma\sigma', \tau\tau')$ is minimal. If there is $\alpha \in \Sigma_1, \beta \in \Sigma_2$ s.t. $d(\alpha, \beta) < d(\sigma, \tau)$, then $d(\alpha\sigma', \beta\tau') < d(\sigma\sigma', \tau\tau')$ by (GHD1), so $d(\sigma, \tau)$ and $d(\sigma', \tau')$ have to be minimal.

“ \supseteq ”:

For the converse, suppose $d(\sigma, \tau)$ and $d(\sigma', \tau')$ are minimal, but $d(\sigma\sigma', \tau\tau')$ is not, so $d(\alpha\alpha', \beta\beta') < d(\sigma\sigma', \tau\tau')$ for some $\alpha\alpha', \beta\beta'$, then $d(\alpha, \beta) < d(\sigma, \tau)$ or $d(\alpha', \beta') < d(\sigma', \tau')$ by (GHD2), contradiction.

□

These properties translate to logic as follows:

Corollary 5.10

If ϕ and ψ are defined on a separate language from that of ϕ' and ψ' , and the distance satisfies (GHD1) and (GHD2), then for revision holds:

$$(\phi \wedge \phi') * (\psi \wedge \psi') = (\phi * \psi) \wedge (\phi' * \psi').$$

5.5.1 Interpolation for distance based revision

The limiting condition (consistency) imposes a strong restriction: Even for $\phi * TRUE$, the result may need many variables (those in ϕ).

Lemma 5.11

Let $|$ satisfy $(| *)$.

Let $J \subseteq L$, ρ be written in sublanguage J , let ϕ, ψ be written in $L-J$, let ϕ', ψ' be written in $J' \subseteq J$.

Let $(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$, then $\phi' * \psi' \vdash \rho$.

(This is suitable interpolation, but we also need to factorize the revision construction.)

Proof

$(\phi \wedge \phi') * (\psi \wedge \psi') = (\phi * \psi) \wedge (\phi' * \psi')$ by $(| *)$. So $(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$ iff $(\phi * \psi) \wedge (\phi' * \psi') \vdash \rho$, but $(\phi * \psi) \wedge (\phi' * \psi') \vdash \rho$ iff $(\phi' * \psi') \vdash \rho$, as ρ contains no variables of ϕ' or ψ' . \square

5.6 Summary of properties

$pr(b)$ means: the projection of a big set on one of its coordinates is big again.

Note that $A \times B \subseteq X \times Y$ big $\Rightarrow A \subseteq X$ big etc. is intuitively better justified than the other direction, as the proportion might increase in the latter, decrease in the former. Cf. [GS09a], the Size Table, increasing proportions.

Table 6: Multiplication laws

Multiplication laws							
Multiplication law	Scenario 1 (Diagram 5.2 (page 29))			Scenario 2 (* symmetrical, only 1 side shown) (Diagram 5.3 (page 30))			
	Corresponding algebraic addition property	Logical property	Relation property	Algebraic property ($\Gamma_i \subseteq \Sigma_i$)	Logical property α, β in $\mathcal{L}_1, \alpha', \beta'$ in \mathcal{L}_2 $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ (disjoint)	Relation property	Interpolation
Non-monotonic logic							
$x * 1 = x$	trivial			$\Gamma_1 \in \mathcal{F}(\Sigma_1) \Rightarrow$	$\alpha \sim_{\mathcal{L}_1} \beta \Rightarrow \alpha \sim_{\mathcal{L}} \beta$		
$1 * x = x$	trivial			$\Gamma_1 \times \Sigma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$	$\alpha \sim_{\mathcal{L}} \beta$		
$x * s = s$	(iM) $A \subseteq B \in \mathcal{I}(X) \Rightarrow A \in \mathcal{I}(X)$	$\alpha \sim \neg\beta \Rightarrow \alpha \sim \neg\beta \vee \gamma$	-	dual to $x * 1 = 1$	$\alpha \sim_{\mathcal{L}_1} \beta \Rightarrow \alpha \sim_{\mathcal{L}} \beta$		
$s * x = s$	(eML) $X \subseteq Y \Rightarrow \mathcal{I}(X) \subseteq \mathcal{I}(Y),$ $X \subseteq Y \Rightarrow \mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq \mathcal{F}(X)$	$\alpha \wedge \beta \sim \neg\gamma \Rightarrow$ $\alpha \sim \neg\beta \vee \neg\gamma$	-				
$b * b = b$	($\langle \omega * s \rangle, (\mathcal{M}_\omega^+)$ (3)) $A \in \mathcal{F}(X), X \in \mathcal{F}(Y) \Rightarrow$ $A \in \mathcal{F}(Y)$	$\alpha \sim \beta, \alpha \wedge \beta \sim \gamma \Rightarrow$ $\alpha \sim \gamma$	(Filter)	$\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{F}(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$	$\alpha \sim_{\mathcal{L}_1} \beta, \alpha' \sim_{\mathcal{L}_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \sim_{\mathcal{L}} \beta \wedge \beta'$	(GH)	$\sim \circ \sim$
$b * m = m$	($\langle \omega * s \rangle, (\mathcal{M}_\omega^+)$ (2)) $A \in \mathcal{M}^+(X), X \in \mathcal{F}(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	$\alpha \not\sim \neg\beta, \alpha \wedge \beta \sim \gamma \Rightarrow$ $\alpha \not\sim \neg\beta \vee \neg\gamma$	(Filter)	$\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$	$\alpha \not\sim_{\mathcal{L}_1} \neg\beta, \alpha' \sim_{\mathcal{L}_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \not\sim_{\mathcal{L}} \neg\beta \vee \beta'$		
$m * b = m$	($\langle \omega * s \rangle, (\mathcal{M}_\omega^+)$ (1)) $A \in \mathcal{F}(X), X \in \mathcal{M}^+(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	$\alpha \sim \beta, \alpha \wedge \beta \not\sim \neg\gamma \Rightarrow$ $\alpha \not\sim \neg\beta \vee \neg\gamma$	(Filter)				
$m * m = m$	(\mathcal{M}^{++}) $A \in \mathcal{M}^+(X), X \in \mathcal{M}^+(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	Rational Monotony	ranked	$\Gamma_1 \in \mathcal{M}^+(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$	$\alpha \not\sim_{\mathcal{L}_1} \neg\beta, \alpha' \sim_{\mathcal{L}_2} \neg\beta' \Rightarrow$ $\alpha \wedge \alpha' \not\sim_{\mathcal{L}} \neg\beta \vee \neg\beta'$		
$b * b = b +$ $pr(b) = b$					$\alpha \sim \beta \Rightarrow \alpha \uparrow \mathcal{L}_1 \sim \beta \uparrow \mathcal{L}_1$ and $\alpha \sim_{\mathcal{L}_1} \beta, \alpha' \sim_{\mathcal{L}_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \sim_{\mathcal{L}} \beta \wedge \beta'$	(GH+)	$\vdash \circ \sim$
J' small					$\alpha \wedge \alpha' \sim \beta \wedge \beta' \Rightarrow$ $\alpha \wedge \beta, \alpha' \vdash \beta'$	forget(J')	
Theory revision							
				($ *$)		GHD	$(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho \Rightarrow$ $\phi' * \psi' \vdash \rho$

160

6 General semantic interpolation

6.1 Introduction

We assume that \rightarrow respects the order \leq on truth values.

6.1.1 Generalization of model sets and (in)essential variables

See Table 7 (page 42) for notation and definitions.

Example 6.1

This example shows that 2 different formulas ϕ and ϕ' may define the same $f_\phi = f_{\phi'}$, but neglecting a certain variable should give different results.

Define two new unary operators $K(x) := 1$ (constant), $M(x) := \min\{1, x\}$. Consider 3 truth values.

a	b	$\phi = K(a) \wedge b$	$\phi' = K(a) \wedge M(b)$
0	0	0	0
0	1	1	1
0	2	1	1
1	0	0	0
1	1	1	1
1	2	1	1
2	0	0	0
2	1	1	1
2	2	1	1

So they define the same model function $f : M_{\mathcal{L}} \rightarrow Val$. But when we forget about a , the first should just be b , but the second should be $M(b)$.

□

6.2 Many-valued propositional interpolation

Definition 6.1

Let M be the set of models for some language \mathcal{L} with set L of propositional variables. Let (V, \leq) be a finite, totally ordered set (of values). Let $\Gamma \subseteq M$. m, n etc. will be elements of M .

- (1) Let $J \subseteq L$, $f : \Gamma \rightarrow V$. Define $f^+(m \upharpoonright J) := \max\{f(m') : m \upharpoonright J = m' \upharpoonright J\}$ and $f^-(m \upharpoonright J) := \min\{f(m') : m \upharpoonright J = m' \upharpoonright J\}$. (Similarly, if m is defined only on J , the condition is $m' \upharpoonright J = m$.)
- (2) Call Γ rich iff for all $m, m' \in \Gamma$, $J \subseteq L$ $(m \upharpoonright J) \cup (m' \upharpoonright (L - J)) \in \Gamma$. (I.e., we may cut and paste models.)
- (3) Call $f : \Gamma \rightarrow V$ insensitive to $J \subseteq L$ iff for all $m, n \in \Gamma$ $(m \upharpoonright (L - J) = n \upharpoonright (L - J))$ implies $f(m) = f(n)$ - i.e., the values of m on J have no importance for f .

Notation and definitions			
		2-valued $\{0, 1\}$	many-valued (V, \leq)
propositional $L' \subseteq L$	language L	propositional variables s, \dots	
	model m	$m : L \rightarrow \{0, 1\}$	$m : L \rightarrow V$
	M set of all L -models		
	$m \upharpoonright L'$	like m , but restricted to L'	
	$m \sim_{L'} m'$	$m \sim_{L'} m'$ iff $\forall s \in L'. m(s) = m'(s)$	
	model set of formula ϕ	$M(\phi) \subseteq M, f_\phi : M \rightarrow \{0, 1\}$	$f_\phi : M \rightarrow V$
	general model set	$M \subseteq M, f : M \rightarrow \{0, 1\}$	$f : M \rightarrow V$
	f insensitive to L'	$\forall m, m' \in M. (m \sim_{L-L'} m' \Rightarrow f(m) = f(m'))$	
	$f^+(m \upharpoonright L'), f^-(m \upharpoonright L')$	$f^+(m \upharpoonright L') = \max\{f(m') : m' \in M, m \sim_{L'} m'\}$ $f^-(m \upharpoonright L') = \min\{f(m') : m' \in M, m \sim_{L'} m'\}$	
$f \leq g$	$\forall m \in M. f(m) \leq g(m)$		
propositional modal $L' \subseteq L$	language L	propositional variables s, \dots	
	(Kripke) structure $m = \langle U_m, R_m, I_m \rangle$ U_m : points, R_m rel., I_m interpret. M set of all L -structures with desired R_m -properties	$I_m : U_m \times L \rightarrow \{0, 1\}$	$I_m : U_m \times L \rightarrow V$
	$m \upharpoonright L'$	like m , but I_m restricted to L'	
	$m \sim_{L'} m'$	$m \sim_{L'} m'$ iff $U_m = U_{m'}$ and $R_m = R_{m'}$ and $\forall u \in U_m \forall s \in L'. I_m(u, s) = I_{m'}(u, s)$	
	model set of formula ϕ in M	$M_m(\phi) \subseteq U_m, f_{m,\phi} : U_m \rightarrow \{0, 1\}$	$f_{m,\phi} : U_m \rightarrow V$
	general model set in m	$M_m \subseteq U_m, f_m : U_m \rightarrow \{0, 1\}$	$f_m : U_m \rightarrow V$
	f_m insensitive to L'	$\forall m, m' \in M (m \sim_{L-L'} m' \Rightarrow f_m = f_{m'})$	
	$f_m^+(m \upharpoonright L'), f_m^-(m \upharpoonright L')$	$f_m^+(m \upharpoonright L') = \max\{f_{m'}(u) : m' \in M, m \sim_{L'} m'\}$ $f_m^-(m \upharpoonright L') = \min\{f_{m'}(u) : m' \in M, m \sim_{L'} m'\}$	
	$f_m \leq g_m$	$\forall u \in U_m. f_m(u) \leq g_m(u)$	
first order L_1 unary pred., L_2 binary, ... $L' \subseteq L$	language $L = L_1 \cup L_2, \dots$	predicates $p(\cdot), q(\cdot, \cdot)$	
	structure $m = \langle U_m, I_m \rangle$ M set of all L -structures	$I_m(p(\cdot)) \subseteq U_m,$ $I_m(q(\cdot, \cdot)) \subseteq U_m \times U_m$	$I_m : U_m \times L_1 \rightarrow V,$ $I_m : (U_m \times U_m) \times L_2 \rightarrow V$
	$m \upharpoonright L'$	like m , but I_m restricted to L'	
	$m \sim_{L'} m'$	$m \sim_{L'} m'$ iff $U_m = U_{m'}$ and $\forall s \in L' \forall \langle u \rangle \in U_m^n. I_m(\langle u \rangle, s) = I_{m'}(\langle u \rangle, s)$	
	model set of formula ϕ with n free variables in m	$M_m(\phi(x_1, \dots, x_n)) \subseteq U_m^n$ $f_{m,\phi} : U_m^n \rightarrow \{0, 1\}$	$f_{m,\phi} : U_m^n \rightarrow V$
	gen. model set, n var. in m	$f_m : U_m^n \rightarrow \{0, 1\}$	$f_m : U_m^n \rightarrow V$
	f insensitive to L'	$\forall m, m' \in M (m \sim_{L-L'} m' \Rightarrow f_m = f_{m'})$	
	$f_m^+(m \upharpoonright L'), f_m^-(m \upharpoonright L')$	$f_m^+(m \upharpoonright L') = \max\{f_{m'}(\langle u \rangle) : m' \in M, m \sim_{L'} m'\}$ $f_m^-(m \upharpoonright L') = \min\{f_{m'}(\langle u \rangle) : m' \in M, m \sim_{L'} m'\}$	
	$f_m \leq g_m$	$\forall \langle u_1, \dots, u_n \rangle \in U_m^n. f_m(u_1, \dots, u_n) \leq g_m(u_1, \dots, u_n)$	
for all cases	semantic equivalence of ϕ, ψ	$f_\phi = f_\psi$ (or for all $m f_{m,\phi} = f_{m,\psi}$)	
	definability of f	$\exists \phi : f_\phi = f$ (or for all $m f_{m,\phi} = f_m$)	
	$\Gamma \upharpoonright L'$	(for $\Gamma \subseteq M$) $\Gamma \upharpoonright L' := \{m \upharpoonright L' : m \in \Gamma\}$	

Table 7: Notation and Definitions

Let $L = J \cup J' \cup J''$ be a disjoint union. If $f : M \rightarrow V$ is insensitive to $J \cup J''$, we can define for $m_{J'} : J' \rightarrow V$ $f(m_{J'})$ as any $f(m')$ such that $m' \upharpoonright J' = m_{J'}$.

Fact 6.1

Let Γ be rich, $f, g : \Gamma \rightarrow V$, $f(m) \leq g(m)$ for all $m \in \Gamma$.

Then $f^+(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in \Gamma \upharpoonright J'$, and any $h : \Gamma \upharpoonright J' \rightarrow V$ which is insensitive to $J \cup J''$ is an interpolant iff

$$f^+(m_{J'}) \leq h(m_{J'}) \leq g^-(m_{J'}) \text{ for all } m_{J'} \in \Gamma \upharpoonright J'.$$

Proof

Let $L = J \cup J' \cup J''$ be a pairwise disjoint union. Let f be insensitive to J , g be insensitive to J'' .

$h : \Gamma \rightarrow V$ will have to be insensitive to $J \cup J''$, so we will have to define h on $\Gamma \upharpoonright J'$.

Fix $m_{J'} : J' \rightarrow V$, $m_{J'} = m \upharpoonright J'$ for some $m \in \Gamma$. Choose $m_{J''}$ such that $f^+(m_{J'}) = f(m_J m_{J'} m_{J''})$ for any m_J . Recall that f is insensitive to J . Likewise, choose m_J such that $g^-(m_{J'}) = g(m_J m_{J'} m_{J''})$ for any $m_{J''}$.

We have $f^+(m_{J'}) \leq g^-(m_{J'})$.

Proof: Choose $m_{J''}$ such that $f(m_J m_{J'} m_{J''})$ is maximal for $m_{J'}$ and any m_J . Let $n_{J''}$ be one such $m_{J''}$. Choose m_J such that $g(m_J m_{J'} m_{J''})$ is minimal for $m_{J'}$ and any $m_{J''}$. Let n_J be one such m_J . Consider $n_J m_{J'} n_{J''} \in \Gamma$ (recall that Γ is rich). By definition, $f^+(m_{J'}) := f(n_J m_{J'} n_{J''})$ and $g^-(m_{J'}) := g(n_J m_{J'} n_{J''})$, but by prerequisite $f(n_J m_{J'} n_{J''}) \leq g(n_J m_{J'} n_{J''})$, so $f^+(m_{J'}) \leq g^-(m_{J'})$.

Thus, any h such that h is insensitive to $J \cup J''$ and

$$(Int) \quad f^+(m_{J'}) \leq h(m_{J'}) \leq g^-(m_{J'})$$

is an interpolant for f and g .

But (Int) is also a necessary condition.

Proof:

Suppose h is an interpolant and $h(m_{J'}) < f^+(m_{J'})$. Let $n_{J''}$ be as above, i.e., $f(m_J m_{J'} n_{J''}) = f^+(m_{J'})$ for any m_J . Then $h(m_J m_{J'} n_{J''}) = h(m_{J'}) < f^+(m_{J'}) = f(m_J m_{J'} n_{J''})$, so h is not an interpolant.

The proof that $h(m_{J'})$ has to be $\leq g^-(m_{J'})$ is analogous.

We summarize:

f and g have an interpolant h , and h is an interpolant for f and g iff h is insensitive to $J \cup J''$ and $f^+(m_{J'}) \leq h(m_{J'}) \leq g^-(m_{J'})$.

□

6.3 Many-valued propositional modal interpolation

We work for all $m \in M$ separately, so we do the construction for the propositional case individually in all structures:

Fix $m \in M$. By prerequisite, $f_m \leq g_m$. Find an interpolant h_m , $f_m \leq h_m \leq g_m$ as above in the propositional case, Then $\{h_m : m \in M\}$ is an interpolant for all $f_m, g_m, m \in M$.

6.4 Many-valued first order interpolation

In the first order case, we have fully quantified formulas, and those with free variables. The latter may be true for some elements or tuples, the former will always be true or false - respectively have the same truth value in

the many-valued case. The easiest way seems to be to give fully quantified formulas the same truth value for all elements, so we can treat them together with formulas with free variables.

Then, for each model m , we have $f_{m,\phi}$, which assigns each element of the universe of m a truth value for ϕ , if ϕ has one free variable, each pair of elements of the universe a truth value for ϕ , if ϕ has two free variables, etc., so $f_{m,\phi} : U_m^n \rightarrow V$ in general.

Thus, apart from the fully quantified case, we are in the same situation as for modal logic. Our propositional variables are now replaced by unary, binary, etc., predicates.

Fix $m \in M$. Let $f_m \leq g_m$ be defined for $\langle u_1, \dots, u_n \rangle \in U_m^n$. Find an interpolant h_m , $f_m \leq h_m \leq g_m$ as above in the propositional case, then $\{h_m : m \in M\}$ is an interpolant for all f_m, g_m .

7 Repairing finite Goedel logics

7.1 Introduction

Define for a Goedel logic with truth values $0 \dots n$ the usual operators \neg, \wedge, \vee as follows:

- (1) $v(\neg\phi) := 0$ iff $v(\phi) \neq 0$, and n otherwise
- (2) $v(\phi \vee \psi) := \max\{v(\phi), v(\psi)\}$
- (3) $v(\phi \wedge \psi) := \min\{v(\phi), v(\psi)\}$

7.2 Cyclic addition of 1

Definition 7.1

Z is a new operator, addition of 1, modulo n :

$v(Z(\phi)) := v(\phi) + 1$ iff $v(\phi) < n$, 0 otherwise.

We introduce the following, derived, auxiliary, operators:

$S_i(\phi) := n$ iff $v(\phi) = i$, and 0 otherwise, for $i = 0, \dots, n$

$K_i(\phi) := i$ for any ϕ , for $i = 0, \dots, n$

Example 7.1

We give here the example for $n = 5$.

	a	0	1	2	3	4	5
Z		1	2	3	4	5	0
	a	0	1	2	3	4	5
K_0		0	0	0	0	0	0
K_1		1	1	1	1	1	1
K_2		2	2	2	2	2	2
K_3		3	3	3	3	3	3
K_4		4	4	4	4	4	4
K_5		5	5	5	5	5	5
	a	0	1	2	3	4	5
S_0		5	0	0	0	0	0
S_1		0	5	0	0	0	0
S_2		0	0	5	0	0	0
S_3		0	0	0	5	0	0
S_4		0	0	0	0	5	0
S_5		0	0	0	0	0	5

Fact 7.1

- (1) We can define $S_i(\phi)$ and $K_i(\phi)$ for $0 \leq i \leq n$ from \neg, \wedge, \vee, Z .
- (2) We can define any m -ary truth function from \neg, \wedge, \vee, Z .

Proof

(1)

$$S_i(\phi) = \neg Z^{n-i}(\phi), K_i(\phi) = Z^{i+1}(\neg(\phi \wedge \neg\phi))$$

(2)

Suppose $\langle i_1, \dots, i_m \rangle$ should have value i , $\langle i_1, \dots, i_m \rangle \mapsto i$, we can express this by

$$S_{i_1}(x_1) \wedge \dots \wedge S_{i_m}(x_m) \wedge K_i.$$

We then take the disjunction of all such expressions:

$$\bigvee \{ S_{i_1}(x_1) \wedge \dots \wedge S_{i_m}(x_m) \wedge K_i : \langle i_1, \dots, i_m \rangle \mapsto i \}$$

Corollary 7.2

Any model function is definable, so any semantical interpolant is also a syntactic one. \square

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