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Equilibria und weiteres Heiteres

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Abstract

We investigate several technical questions.

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1 Equilibrium logic

We consider Equilibrium Logic, as investigated in [PV09].

Work with n variables.

For a model σ , set

$\sigma(x) = 0$ iff x holds neither here nor there

$\sigma(x) = 1$ iff x holds only there

$\sigma(x) = 2$ iff x holds here and there

So a model σ is a sequence of length n , values in $0, 1, 2$.

A model σ is total iff no value of σ is 1.

$\sigma \prec \tau$ iff T is preserved (here was my mistake), and H goes down. Thus, only changes from 2 to 1 are possible when $\sigma \prec \tau$.

Consequently, we have a sort of “anti-smoothnes”: if a model is not minimal, then any model below it is NOT chosen.

Let Σ be a set of models, let $\mu(\Sigma)$ be the equilibrium models in Σ .

We have for a model $\sigma = (h, t)$:

$\sigma \models a$ iff $h \models a$ and $t \models a$ iff $\sigma(a) = 2$.

$\sigma \models \neg a$ iff $h \models \neg a$ and $t \models \neg a$. $t \models \neg a$ iff $t \not\models a$. $h \models \neg a$ iff $h \not\models a$ and $t \not\models a$. Thus:

$\sigma \models \neg a$ iff $\sigma(a) = 0$.

$\sigma \models \neg\neg a$ iff $h \not\models \neg a$ and $t \not\models \neg a$. By the above: $t \not\models \neg a$ iff $t \models a$. $h \not\models \neg a$ iff $\text{not}(h \not\models a \text{ and } t \not\models a)$ iff $h \models a$ or $t \models a$. Thus:

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$\sigma \models \neg\neg a$ iff $(t \models a \text{ and } (h \models a \text{ or } t \models a))$ iff $t \models a$ iff $\sigma(a) = 1$ or 2 .

1.1 EQ has no interpolation of the form $\phi \vdash \alpha \mid \sim \psi$

Example 1.1

(+++ Orig. No.: Example xcl-xcn +++)

LABEL: Example xcl-xcn

Work with 3 variables, a, b, c .

Consider $\Sigma := \{\langle 0, 2, 2 \rangle, \langle 2, 1, 0 \rangle, \langle 2, 2, 0 \rangle\}$.

By the above, and classical behaviour of “or” and “and”, Σ is definable by $(\neg a \wedge b \wedge c) \vee (a \wedge \neg\neg b \wedge \neg c)$.

Note that $\langle 2, 2, 0 \rangle$ is total, but $\langle 2, 1, 0 \rangle \prec \langle 2, 2, 0 \rangle$, thus $\mu(\Sigma) = \{\langle 0, 2, 2 \rangle\}$.

So $\Sigma \vdash c = 2$ (or $\Sigma \vdash \Box c$). Let $X' := \{a, b\}$, $X'' := \{c\}$.

All possible interpolants Γ must not contain a or b as essential variables, and they must contain Σ . The smallest candidate Γ is $\Pi X' \times \{0, 2\}$. But $\sigma := \langle 0, 0, 0 \rangle \in \Gamma$, σ is total, and there cannot be any $\tau \prec \sigma$, so $\sigma \in \mu(\Gamma)$, so $\Gamma \not\vdash c = 2$.

For completeness' sake, we write all elements of Γ :

$\langle 0, 0, 0 \rangle \langle 0, 0, 2 \rangle$
 $\langle 0, 1, 0 \rangle \langle 0, 1, 2 \rangle$
 $\langle 0, 2, 0 \rangle \langle 0, 2, 2 \rangle$
 $\langle 1, 0, 0 \rangle \langle 1, 0, 2 \rangle$
 $\langle 1, 1, 0 \rangle \langle 1, 1, 2 \rangle$
 $\langle 1, 2, 0 \rangle \langle 1, 2, 2 \rangle$
 $\langle 2, 0, 0 \rangle \langle 2, 0, 2 \rangle$
 $\langle 2, 1, 0 \rangle \langle 2, 1, 2 \rangle$
 $\langle 2, 2, 0 \rangle \langle 2, 2, 2 \rangle$

Recall that no sequence containing 1 is total, and when we go from 2 to 1, we have a smaller model. Thus, $\mu(\Gamma) = \{\langle 0, 0, 0 \rangle, \langle 0, 0, 2 \rangle\}$.

1.2 EQ has no interpolation of the form $\phi \mid \sim \alpha \vdash \psi$

Example 1.2

(+++ Orig. No.: Example xcn-xcl +++)

LABEL: Example xcn-xcl

Consider 2 variables, a, b , and $\Sigma := \{0, 2\} \times \{0, 1, 2\}$

No σ containing 1 can be in $\mu(\Sigma)$, as a matter of fact, $\mu(\Sigma) = \{\langle 0, 0 \rangle, \langle 2, 0 \rangle\}$. Σ is defined by $a \vee \neg a$, $\mu(\Sigma)$ is defined by $(a \vee \neg a) \wedge \neg b$.

So we have $a \vee \neg a \vdash b \vee \neg b$, even $a \vee \neg a \vdash \neg b$.

The only possible interpolants are TRUE or FALSE. $a \vee \neg a \not\vdash FALSE$, and $TRUE \not\vdash \neg b$.

1.3 EQ has interpolation of the form $\phi \mid \sim \alpha \mid \sim \psi$

We give a rough argument for the semantic version, probably the set in question is definable - Pearce will certainly know.

Still to be verified.

We consider $\phi \vdash \psi$.

Let X' be the variables not in ψ , X'' the others.

Fact 1.1

(+++ Orig. No.: Fact xcn-xcn +++)

LABEL: Fact xcn-xcn

Let $\mu(\Sigma) \neq \emptyset$. Let $\Pi' := \Pi X'$.

$\mu(\Pi' \times \mu(\Sigma) \upharpoonright X'') \upharpoonright X'' = \mu(\Sigma) \upharpoonright X''$

Proof

(+++ Orig.: Proof +++)

Let $\sigma \in \mu(\Sigma)$. Consider τ s.t. $\sigma \upharpoonright X'' = \tau \upharpoonright X''$, $\tau(x) = 0$ elsewhere. As $\sigma \in \mu(\Sigma)$, $\tau \in \mu(\Pi' \times \mu(\Sigma) \upharpoonright X'')$.

Conversely, let $\sigma \in \mu(\Pi' \times \mu(\Sigma) \upharpoonright X'')$, then $\sigma \upharpoonright X'' \in \mu(\Sigma) \upharpoonright X''$

The reason for both is that we look at the coordinates independently.

□

Corollary 1.2

(+++ Orig. No.: Corollary xcn-xcn +++)

LABEL: Corollary xcn-xcn

$\Pi' \times \mu(\Sigma) \upharpoonright X''$ is a semantical interpolant for $\phi \sim \psi$.

Is it definable? We have not checked yet.

Proof

(+++ Orig.: Proof +++)

Suppose it is definable by α .

$\mu(\Sigma) \subseteq \Pi' \times \mu(\Sigma) \upharpoonright X''$, so $\phi \sim \alpha$.

By prerequisite, $\mu(\Sigma) \subseteq M(\psi)$, and ψ contains no $x' \in X'$ as relevant variable, so $M(\psi)$ can be written as $\Pi' \times \Gamma$ for some $\Gamma \subseteq \Pi X''$. $\mu(\Pi' \times \mu(\Sigma) \upharpoonright X'') \upharpoonright X'' = \mu(\Sigma) \upharpoonright X''$, so $\mu(\Pi' \times \mu(\Sigma) \upharpoonright X'') \subseteq \Pi X' \times (\mu(\Pi' \times \mu(\Sigma) \upharpoonright X'') \upharpoonright X'') = \Pi' \times (\mu(\Sigma) \upharpoonright X'') \subseteq \Pi' \times \Gamma = M(\psi)$, as $\mu(\Sigma) \subseteq \Pi' \times \Gamma$. Thus $\alpha \sim \psi$.

1.4 Definability with 1 variable

Fact 1.3

(+++ Orig. No.: Fact EQ-Definability +++)

LABEL: Fact EQ-Definability

With 1 variable a are definable - with values for a

$a : 0, 1, 2$

$\neg a : 2, 0, 0$

$\neg\neg a : 0, 2, 2$

$a \rightarrow a : 2, 2, 2$

$\neg(a \rightarrow a) : 0, 0, 0$

$\neg\neg a \rightarrow a : 2, 1, 2$

All other formulas with only a give the same truth values, i.e. above set is closed.

2 Countably many disjoint sets

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D.Makinson (personal communication).

Example 2.1

(+++ Orig. No.: Example Co-Ex-Inf +++)

LABEL: Example Co-Ex-Inf

There is a countably infinite set of formulas s.t. the defined model sets are pairwise disjoint.

Let $p_i : i \in \omega$ be propositional variables.

Consider $\phi_i := \bigwedge\{\neg p_j : j < i\} \wedge p_i$ for $i \in \omega$.

Obviously, $M(\phi_i) \neq \emptyset$ for all i .

Let $i < i'$, we show $M(\phi_i) \cap M(\phi_{i'}) = \emptyset$. $M(\phi_{i'}) \models \neg p_i$, $M(\phi_i) \models p_i$.

□

Fact 2.1

(+++ Orig. No.: Fact Co-Ex-Inf +++)

LABEL: Fact Co-Ex-Inf

Any set X of consistent formulas with pairwise disjoint model sets is at most countable.

Proof

(+++ Orig.: Proof +++)

Let such X be given.

(1) We may assume that X consists of conjunctions of propositional variables or their negations.

Proof: Re-write all $\phi \in X$ as disjunctions of conjunctions ϕ_j . At least one of the conjunctions ϕ_j is consistent. Replace ϕ by one such ϕ_j . Consistency is preserved, as is pairwise disjointness.

(2) Let X be such a set of formulas. Let $X_i \subseteq X$ be the set of formulas in X with length i , i.e. a consistent conjunction of i many propositional variables or their negations, $i > 0$.

As the model sets for X are pairwise disjoint, the model sets for all $\phi \in X_i$ have to be disjoint.

(3) It suffices now to show that each X_i is at most countable, we even show that each X_i is finite.

Proof by induction:

Consider $i = 1$. Let $\phi, \phi' \in X_1$. Let ϕ be p or $\neg p$. If ϕ' is not $\neg\phi$, then ϕ and ϕ' have a common model. So one must be p , the other $\neg p$. But these are all possibilities, so $\text{card}(X_1)$ is finite.

Let the result be shown for $k < i$.

Consider now X_i . Take arbitrary $\phi \in X_i$. Wlog, $\phi = p_1 \wedge \dots \wedge p_i$. Take arbitrary $\phi' \neq \phi$. As $M(\phi) \cap M(\phi') = \emptyset$, ϕ' must be a conjunction containing one of $\neg p_k$, $1 \leq k \leq i$. Consider now $X_{i,k} := \{\phi' \in X_i : \phi' \text{ contains } \neg p_k\}$. Thus $X_i = \{\phi\} \cup \bigcup \{X_{i,k} : 1 \leq k \leq i\}$. Note that all $\psi, \psi' \in X_{i,k}$ agree on $\neg p_k$, so the situation in $X_{i,k}$ is isomorphic to X_{i-1} . So, by induction hypothesis, $\text{card}(X_{i,k})$ is finite, as all $\phi' \in X_{i,k}$ have to be mutually inconsistent. Thus, $\text{card}(X_i)$ is finite. (Note that we did not use the fact that elements from different $X_{i,k}, X_{i,k'}$ also have to be mutually inconsistent, our rough proof suffices.)

□

Note that the proof depends very little on logic. We needed normal forms, and used 2 truth values. Obviously, we can easily generalize to finitely many truth values.

References

[PV09] D.Pearce, A.Valverde, “Interpolation in equilibrium logic and answer set programming”, submitted