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Incidence coloring of the Cartesian product of two cycles

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Abstract

We prove that the incidence chromatic number of the Cartesian product $C_m \square C_n$ of two cycles equals 5 when $m, n \equiv 0 \pmod{5}$ and 6 otherwise.

Key words: Incidence coloring, Cartesian product of cycles.

2000 Mathematics Subject Classification: 05C15

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *incidence* in G is a pair (v, e) with $v \in V(G)$ and $e \in E(G)$, such that v and e are incident. We denote by $I(G)$ the set of all incidences in G . Two incidences (v, e) and (w, f) are *adjacent* if one of the following holds: (i) $v = w$, (ii) $e = f$ or (iii) the edge vw equals e or f .

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An *incidence k -coloring* of G is a mapping from $I(G)$ to a set of k colors such that adjacent incidences are assigned distinct colors. The *incidence chromatic number* $\chi_i(G)$ of G is the smallest k such that G admits an incidence k -coloring.

Incidence colorings have been introduced by Brualdi and Massey in [1]. In this paper, the authors also conjectured that the relation $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph G , where $\Delta(G)$ denotes the maximum degree of G . In [2], Guiduli disproved this *Incidence Coloring Conjecture* (ICC for short). However, the ICC conjecture has been proved for several graph classes [2, 3, 4, 5, 6, 7, 8, 10, 11].

Let G and H be graphs. The *Cartesian product* $G \square H$ of G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. Let P_n and C_n denote respectively the path and the cycle on n vertices. We will denote by $G_{m,n} = P_m \square P_n$ the *grid* with m rows and n columns and by $T_{m,n} = C_m \square C_n$ the *toroidal grid* with m rows and n columns.

In this paper, we determine the incidence chromatic number of toroidal grids and prove that this class of graphs satisfies the ICC:

Theorem 1 *For every $m, n \geq 3$, $\chi_i(T_{m,n}) = 5$ if $m, n \equiv 0 \pmod{5}$ and $\chi_i(T_{m,n}) = 6$ otherwise.*

In [5], Huang *et al.* proved that $\chi_i(G_{m,n}) = 5$ for every m, n . Since every toroidal graph $T_{m,n}$ contains the grid $G_{m,n}$ as a subgraph, we get that $\chi_i(T_{m,n}) \geq 5$ for every m, n .

The paper is organized as follows. In Section 2 we give basic properties and illustrate the techniques we shall use in the proof of our main result, which is given in Section 3.

2 Preliminaries

Let G be a graph, u a vertex of G with maximum degree and v a neighbour of u . Since in any incidence coloring of G all the incidences of the form (u, e) have to get distinct colors and since all of them have to get a color distinct from the color of (v, vu) , we have:

Proposition 2 *For every graph G , $\chi_i(G) \geq \Delta(G) + 1$.*

The *square* G^2 of a graph G is given by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or there exists $w \in V(G)$ such that $uw, vw \in E(G)$. In other words, any two vertices within distance at most two in G are linked by an edge in G^2 . Let now c be a proper vertex coloring of G^2 and μ be the mapping defined by $\mu(u, uv) = c(v)$ for every incidence (u, uv) in $I(G)$. It is not difficult to check that μ is indeed an incidence coloring of G (see Example 8 below). Therefore we have:

Proposition 3 *For every graph G , $\chi_i(G) \leq \chi(G^2)$.*

In [9], we studied the chromatic number of the squares of toroidal grids and proved the following:

Theorem 4 *Let $T_{m,n} = C_m \square C_n$. Then $\chi(T_{m,n}^2) \leq 7$ except $\chi(T_{3,3}^2) = 9$ and $\chi(T_{3,5}^2) = \chi(T_{4,4}^2) = 8$.*

By Proposition 3, this result provides upper bounds on the incidence chromatic number of toroidal grids.

In [9], we also proved the following:

Theorem 5 *For every $m, n \geq 3$, $\chi(T_{m,n}^2) \geq 5$. Moreover, $\chi(T_{m,n}^2) = 5$ if and only if $m, n \equiv 0 \pmod{5}$.*

In [10], the second author proved the following:

Theorem 6 *If G is regular, then $\chi_i(G) = \Delta(G) + 1$ if and only if $\chi(G^2) = \Delta(G) + 1$.*

Since toroidal graphs are 4-regular, by combining Proposition 2, Theorem 5 and Proposition 6 we get the following:

Corollary 7 *For every $m, n \geq 3$, $\chi_i(T_{m,n}) \geq 5$. Moreover, $\chi_i(T_{m,n}) = 5$ if and only if $m, n \equiv 0 \pmod{5}$.*

Note here that this corollary is part of our main result.

Any vertex coloring of the square of a toroidal grid $T_{m,n}$ can be given as an $m \times n$ matrix whose entries correspond in an obvious way to the colors of the vertices. Such a matrix will be called a *pattern* in the following.

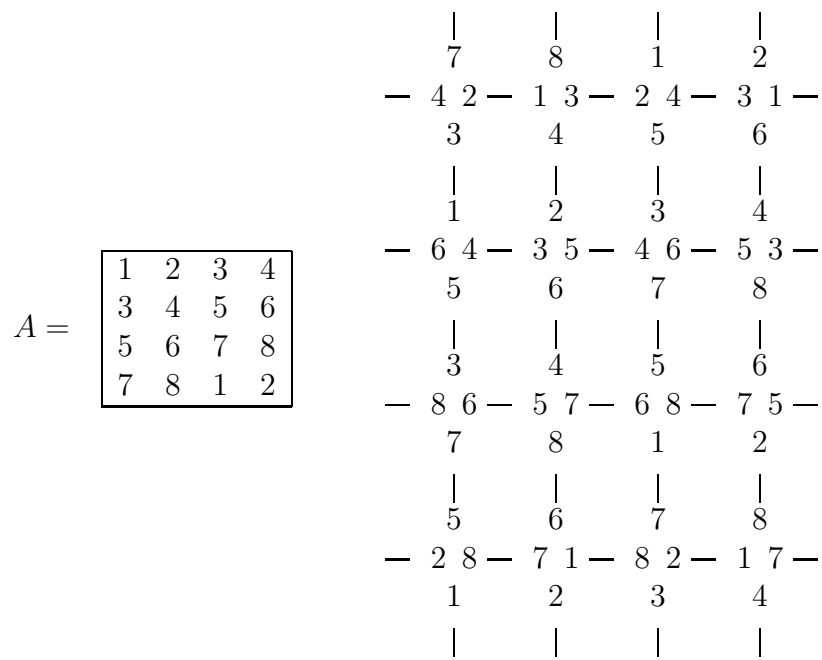


Figure 1: A pattern A and the corresponding incidence coloring of $T_{4,4}$

Example 8 Figure 1 shows a 4×4 pattern A , which defines a vertex coloring of $T_{4,4}^2$, and the incidence coloring of $T_{4,4}$ induced by this pattern, according to the discussion before Proposition 3. Note for instance that the four incidences of the form (u, uv) , for v being the second vertex in the third row, have color 6, which corresponds to the entry in row 3, column 2, of pattern A .

If A and B are patterns of size $m \times n$ and $m \times n'$ respectively, we shall denote by $A+B$ the pattern of size $m \times (n+n')$ obtained by “gluing” together the patterns A and B . Moreover, we shall denote by ℓA , $\ell \geq 2$, the pattern of size $m \times \ell n$ obtained by gluing together ℓ copies of the pattern A .

We now shortly describe the technique we shall use in the next section. The main idea is to use a pattern for coloring the square of a toroidal grid in order to get an incidence coloring of this toroidal grid. However, as shown in [9], the squares of toroidal grids are not all 6-colorable. Therefore, we shall use the notion of a *quasi-pattern* which corresponds to a vertex 6-coloring of the square of a *subgraph* of a toroidal grid obtained by deleting some edges. We can then use such a quasi-pattern in the same way as before to obtain a *partial* incidence coloring of the toroidal grid. Finally, we shall prove that such a partial incidence coloring can be extended to the whole toroidal grid without using any additional color.

We shall also use the following:

Observation 9 For every $m, n \geq 3$, $p, q \geq 1$, if $\chi_i(T_{m,n}) \leq k$ then $\chi_i(T_{pm,qn}) \leq k$.

To see that, it is enough to observe that every incidence k -coloring c of $T_{m,n}$ can be extended to an incidence k -coloring $c_{p,q}$ of $T_{pm,qn}$ by “repeating” the pattern given by c , p times “vertically” and q times “horizontally”.

3 Proof of Theorem 1

According to Corollary 7 above, we only need to prove that $\chi_i(T_{m,n}) \leq 6$ for every $m, n \geq 3$. The proof will be declined in a series of Lemmas, according to different values of m and n .

We first consider the case when $m \equiv 0 \pmod{3}$. We have proved in [9] the following:

Proposition 10 For every $k \geq 1$, $n \geq 3$, n even, $\chi(T_{3k,n}^2) \leq 6$.

$$\begin{aligned}
B &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} & C &= \begin{bmatrix} 1 & 4 & 2 & 5 \\ 2 & 5 & 3 & 6 \\ 3 & 6 & 1 & 4 \end{bmatrix} & D &= \begin{bmatrix} 1 & 4 & 3 & 6 \\ 2 & 5 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} & E &= \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 1 & 4 \end{bmatrix} \\
B + C &= \begin{bmatrix} 3 & 1 & 4 & 2 & 5 \\ 1 & 2 & 5 & 3 & 6 \\ 2 & 3 & 6 & 1 & 4 \end{bmatrix} & B + D + E &= \begin{bmatrix} 3 & 1 & 4 & 3 & 6 & 2 & 5 \\ 1 & 2 & 5 & 1 & 4 & 3 & 6 \\ 2 & 3 & 6 & 2 & 5 & 1 & 4 \end{bmatrix}
\end{aligned}$$

Figure 2: Patterns and quasi-patterns for Lemma 11

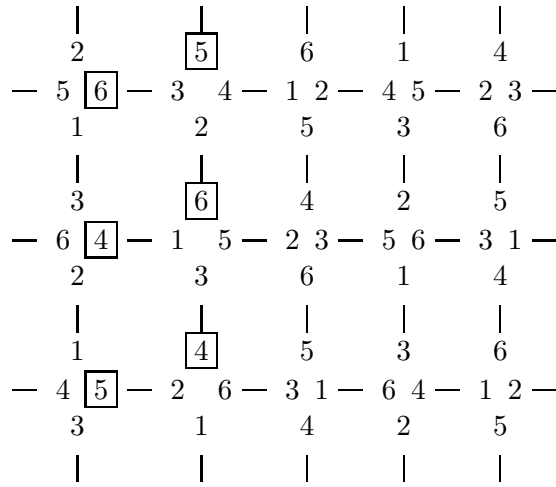


Figure 3: Incidence coloring for Lemma 11

Here we prove:

Lemma 11 For every $k \geq 1$, $n \geq 3$, $\chi_i(T_{3k,n}) \leq 6$.

Proof. If n is even, the result follows from Propositions 3 and 10.

We thus assume that n is odd, and we let first $k = 1$. We consider three cases.

1. $n = 3$.

We can easily get an incidence 6-coloring by coloring the incidences of one dimension with $\{1, 2, 3\}$ and the incidences of the other dimension with $\{4, 5, 6\}$.

$$\begin{array}{c}
F = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \quad G = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 & 6 \\ \hline 3 & 4 & 5 & 6 \\ \hline \end{array} \\
\\
H = 2F + 2G = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 1 & 2 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 4 & 1 & 2 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ \hline 3 & 5 & 6 & 3 & 5 & 6 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 \\ \hline 3 & 5 & 6 & 3 & 5 & 6 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 \\ \hline \end{array}
\end{array}$$

Figure 4: Quasi-patterns for Lemma 13

2. $n = 4\ell + 1$.

Let B and C be the patterns depicted in Figure 2 and consider the quasi-pattern $B + \ell C$ (the quasi-pattern $B + C$ is depicted in Figure 2). This quasi-pattern provides a 6-coloring of $T_{m,n}^2$ if we delete all the edges linking vertices in the first column to vertices in the second column. We can use this quasi-pattern to obtain an incidence 6-coloring of $T_{m,n}$ by modifying six incidence colors, as shown in Figure 3 (modified colors are in boxes).

3. $n = 4\ell + 3$.

Let B , D and E be the patterns depicted in Figure 2 and consider the quasi-pattern $B + \ell D + E$ (the quasi-pattern $B + D + E$ is depicted in Figure 2). As in the previous case, we can use this quasi-pattern to obtain an incidence 6-coloring of $T_{m,n}$ by modifying the same six incidence colors.

For $k \geq 2$, the result now directly follows from Observation 9. ■

We now consider the case when $m \equiv 0 \pmod{4}$. For $m \equiv 0 \pmod{5}$, we have proved in [9] the following:

Proposition 12 *For every $k \geq 1$, $n \geq 5$, $n \neq 7$, $\chi(T_{5k,n}^2) \leq 6$.*

Here we prove:

Lemma 13 *For every $k \geq 1$, $n \geq 3$, $\chi_i(T_{4k,n}) \leq 6$, except for $k = 1$ and $n = 5$.*

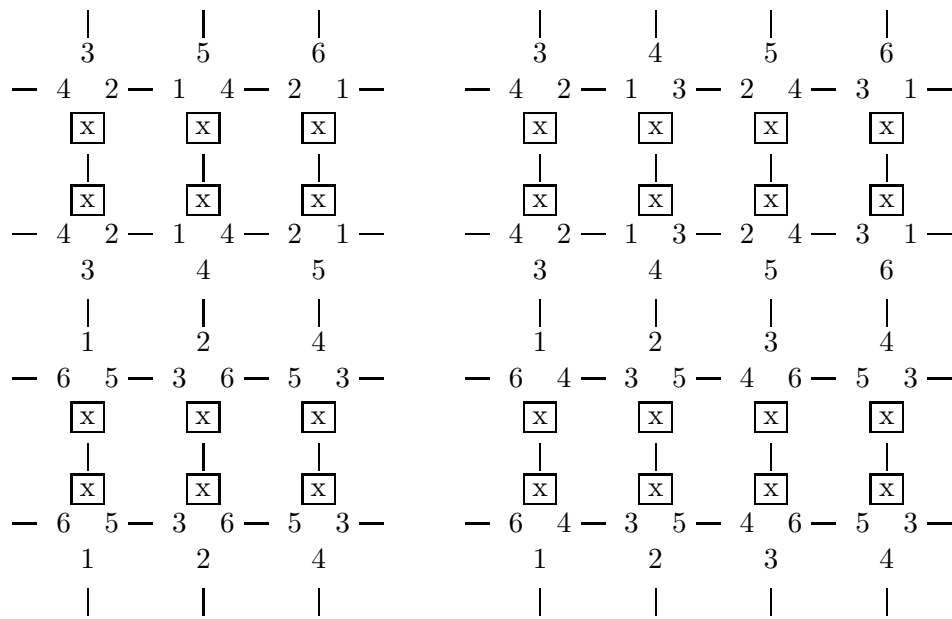


Figure 5: Partial incidence colorings for Lemma 13

$$I = \begin{array}{|c|c|c|c|c|c|} \hline 6 & 1 & 2 & 3 & 4 & 5 \\ \hline 3 & 4 & 5 & 6 & 1 & 2 \\ \hline 5 & 6 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 & 6 & 1 \\ \hline 4 & 5 & 6 & 1 & 2 & 3 \\ \hline \end{array} \qquad I' = \begin{array}{|c|c|c|c|c|c|} \hline 6 & 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 1 & 2 & 3 & 4 & 5 \\ \hline 3 & 4 & 5 & 6 & 1 & 2 \\ \hline 5 & 6 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 & 6 & 1 \\ \hline 4 & 5 & 6 & 1 & 2 & 3 \\ \hline \end{array}$$

Figure 6: Patterns for Lemma 14

Proof. For $n = 5$, the result holds by Proposition 12, except for $k = 1$.

Assume now $k = 1$ and $n \neq 5$ and consider the quasi-patterns F and G depicted in Figure 4. From these patterns, we can derive a partial incidence 6-coloring of $T_{4,3}$ and $T_{4,4}$, respectively, as shown in Figure 5, where the uncolored incidences are denoted by x . It is easy to check that every such incidence has only four forbidden colors and that only incidences belonging to a same edge have to be distinct. Therefore, these partial incidence colorings can be extended to incidence 6-colorings of $T_{4,3}$ and $T_{4,4}$.

For $n \geq 6$, we shall use the quasi-pattern $H = pF + qG$ where p and q are such that $n = 3p + 4q$ (recall that every integer except 1,2 and 5 can be written in this form). The quasi-pattern $H = 2F + 2G$ is depicted in Figure 4. As in the previous case, this quasi-pattern provides a partial incidence 6-coloring of $T_{4,n}$ that can be extended to an incidence 6-coloring of $T_{4,n}$.

For $k \geq 2$, the result now directly follows from Observation 9. ■

We now consider the remaining cases.

Lemma 14 For every $m, n \geq 5$, $m \neq 6, 8$, $n \neq 7$, $\chi_i(T_{m,n}) \leq 6$.

Proof. Assume $m, n \geq 5$, $m \neq 6, 8$ and $n \neq 7$. By Proposition 12, $\chi_i(T_{5k,n}) \leq 6$ for $n \neq 7$. Hence, there exists a vertex 6-coloring of $T_{5k,n}^2$ which corresponds to some pattern M of size $5k \times n$. We claim that each row of pattern M can be repeated one or three times to get quasi-patterns that can be extended to incidence 6-colorings of the corresponding toroidal grids.

Let for instance M' be the quasi-pattern obtained from M by repeating the first row of M three times. The quasi-pattern M' has thus size $(5k+2) \times n$. The quasi-pattern M' induces a partial incidence coloring of $T_{5k+2,n}$ in which

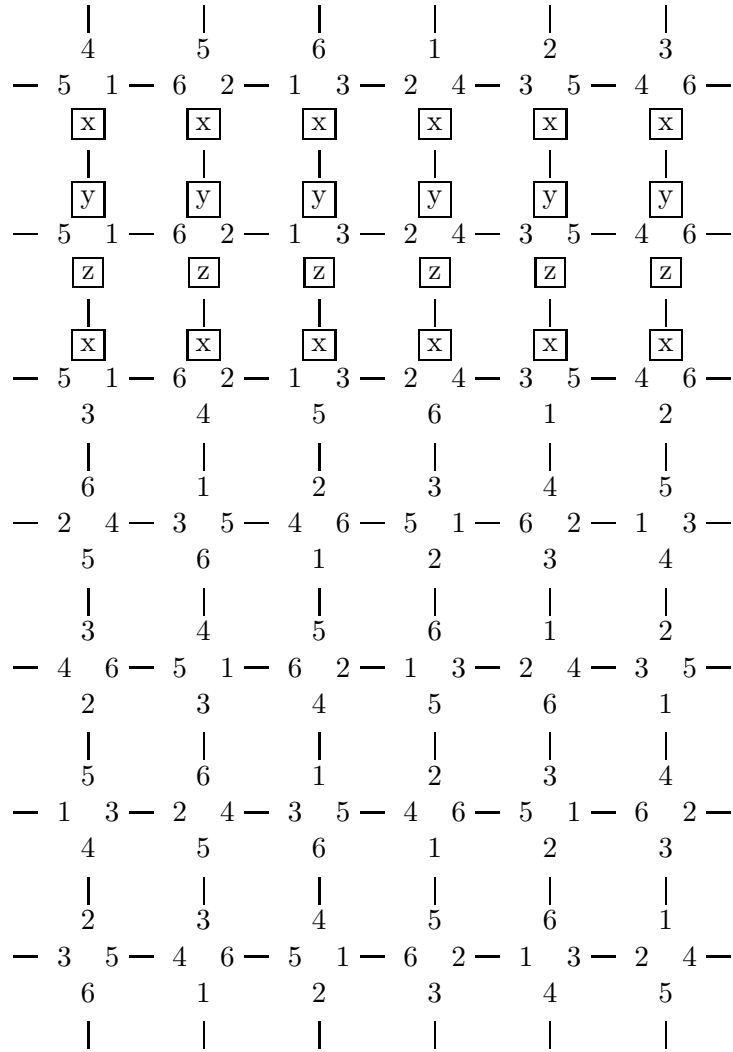


Figure 7: A partial incidence coloring of $T_{7,6}$

the only uncolored incidences are those lying on the edges linking vertices in the first row to vertices in the second row and on the edges linking vertices in the second row to vertices in the third row.

We illustrate this in Figure 6 with a pattern I of size 5×6 (this pattern induces a vertex 6-coloring of $T_{5,6}^2$) and its associated pattern I' of size 7×6 . The partial incidence coloring of $T_{7,6}$ obtained from I' is then given in Figure 7, where uncolored incidences are denoted by x , y and z .

Observe now that in each column, the two incidences denoted by x have three forbidden colors in common and each of them has four forbidden colors in total. Therefore, we can assign them the same color. Now, in each column, the incidences denoted by y and z have four forbidden colors in common (the color assigned to x is one of them) and each of them has five forbidden colors in total. They can be thus colored with distinct colors. Doing that, we extend the partial incidence coloring of $T_{7,6}$ to an incidence 6-coloring of $T_{7,6}$.

The same technique can be used for obtaining an incidence 6-coloring of $T_{5k+2,n}$ since all the columns are “independent” in the quasi-pattern M' , with respect to uncolored incidences.

If we repeat three times several distinct rows of pattern M , each repeated row will produce a chain of four uncolored incidences, as before, and any two such chains in the same column will be “independent”, since they will be separated by an edge whose incidences are both colored. Hence, we will be able to extend to corresponding quasi-pattern to an incidence 6-coloring of the toroidal grid, by assigning available colors to each chain as we did above.

Starting from a pattern M of size $5k \times n$, we can thus obtain quasi-patterns of size $(5k + 2) \times n$, $(5k + 4) \times n$, $(5k + 6) \times n$ and $(5k + 8) \times n$, by repeating respectively one, two, three or four lines from M . Using these quasi-patterns, we can produce incidence 6-colorings of the toroidal grid $T_{m,n}$, $m, n \geq 5$, $n \neq 7$, for every m except $m = 6, 8$. ■

The only remaining cases are $m = 4$, $n = 5$ and $m = n = 7$. Then we have:

Lemma 15 $\chi_i(T_{4,5}) \leq 6$ and $\chi_i(T_{7,7}) \leq 6$.

Proof. Let $m = 4$ and $n = 5$. Consider the pattern C of size 3×4 depicted in Figure 2. As in the proof of Lemma 14, we can repeat the first row of C three times to get a quasi-pattern C' that can be extended to an incidence 6-coloring of $T_{5,4}$. We then exchange m and n to get an incidence 6-coloring of $T_{4,5}$, depicted in Figure 10 (the colors assigned to uncolored incidences are drawn in boxes).

$$J = \begin{array}{|cccccc|} \hline 3 & 5 & 6 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 4 & 1 & 2 & 3 & 4 \\ \hline 3 & 5 & 6 & 3 & 4 & 5 & 6 \\ \hline 3 & 5 & 6 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 4 & 1 & 2 & 3 & 4 \\ \hline \end{array}$$

Figure 8: A quasi-pattern for Lemma 15

Let now $m = n = 7$ and consider the quasi-pattern J depicted in Figure 8. This quasi-pattern provides the partial incidence coloring of $T_{7,7}$ given in Figure 9, where incidences with modified colors are in boxes and uncolored incidences are denoted by x and y . Observe now that the incidences denoted by y have five forbidden colors while the incidences denoted by x have four forbidden colors. Therefore, this partial coloring can be extended to an incidence 6-coloring of $T_{7,7}$. ■

We are now able to prove our main result:

Proof of Theorem 1. By Corollary 7, we get that $\chi_i(T_{m,n}) \geq 5$ for every m, n , and that equality holds if and only if $m, n \equiv 0 \pmod{5}$. For $m = 3$ or 6 , the result follows from Lemma 11 and for $m = 4$ or 8 , the result follows from Lemma 13, except the case $m = 4$ and $n = 5$ which follows from Lemma 15. Assume now that $m > 6, m \neq 7$. If $n = 7$, the results follows from Lemma 15 if $m = 7$ and by Lemma 14 otherwise, by exchanging m and n . Finally, if $n \neq 7$, the results follows from Lemma 14. ■

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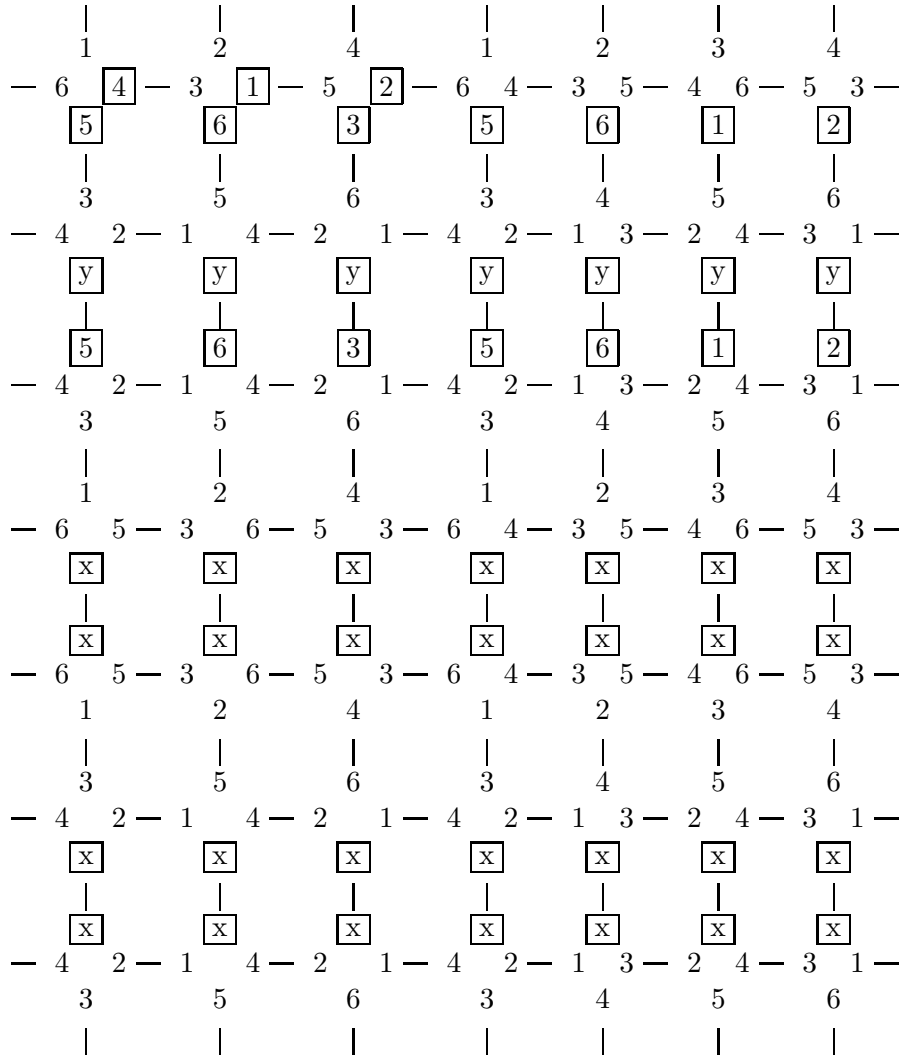


Figure 9: A partial incidence 6-coloring of $T_{7,7}$

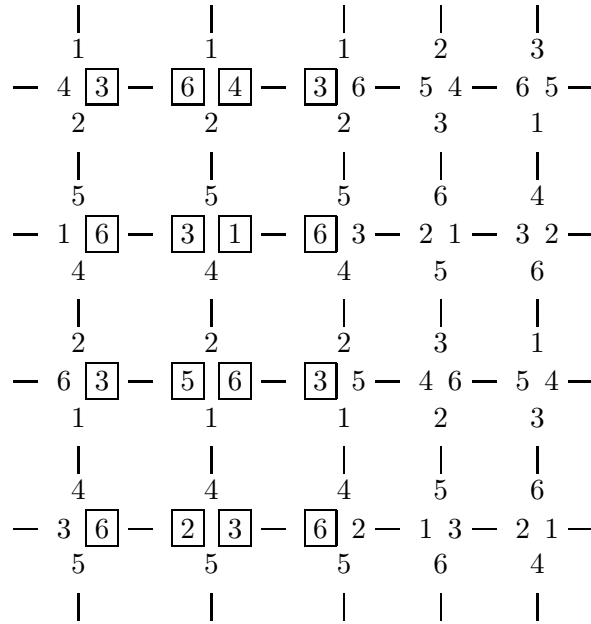


Figure 10: An incidence 6-coloring of $T_{4,5}$

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