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BERNSTEIN VON MISES THEOREM FOR LINEAR FUNCTIONALS OF THE DENSITY

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In this paper, we study the asymptotic posterior distribution of linear functionals of the density. In particular, we give general conditions to obtain a semiparametric version of the Bernstein-Von Mises theorem. We then apply this general result to nonparametric priors based on infinite dimensional exponential families. As a byproduct, we also derive adaptive nonparametric rates of concentration of the posterior distributions under these families of priors on the class of Sobolev and Besov spaces.

1. Introduction. The Bernstein-Von Mises property, in Bayesian analysis, concerns the asymptotic form of the posterior distribution of a quantity of interest, and more specifically it corresponds to the asymptotic normality of the posterior distribution centered at some kind of maximum likelihood estimator with variance being equal to the asymptotic frequentist variance of the centering point. Such results are well know in parametric frameworks, see for instance [14] where general conditions are given. This is an important property for both practical and theoretical reasons. In particular the asymptotic normality of the posterior distributions allows us to construct approximate credible regions and the duality between the behaviour of the posterior distribution and the frequentist distribution of the asymptotic centering point of the posterior implies that credible regions will have also good frequentist properties. These results are given in many Bayesian textbooks see for instance [17] or [1].

In a frequentist perspective the Bernstein-Von Mises property enables the construction of confidence regions since under this property a Bayesian credi-

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ble region will be asymptotically a frequentist confidence region as well. This is even more important in complex models, since in such models the construction of confidence regions can be difficult whereas, the Markov Chain Monte Carlo algorithms usually make the construction of a Bayesian credible region feasible. However the more complex the model the harder it is to derive Bernstein - Von Mises theorems. In infinite dimensional setups, the mechanisms are even more complex.

Semi-parametric and non parametric models are widely popular both from a theoretical and practical perspective and have been used by frequentists as well as Bayesians although their theoretical asymptotic properties have been mainly studied in the frequentist literature. The use of Bayesian non parametric or semi-parametric approaches is more recent and has been made possible mainly by the development of algorithms such as Markov Chain Monte-Carlo algorithms but has grown rapidly over the past decade.

However, there is still little work on asymptotic properties of Bayesian procedures in semi-parametric models or even in nonparametric models. Most of existing works on the asymptotic posterior distributions deal with consistency or rates of concentration of the posterior. In other words it consists in controlling objects in the form $\mathbb{P}^{\pi}[U_n|X^n]$ where $\mathbb{P}^{\pi}[.|X^n]$ denotes the posterior distribution given a n vector of observationsX^n and U_n denotes either a fixed neighbourhood (consistency) or a sequence of shrinking neighbourhoods (rates of concentration). As remarked by [6] consistency is an important condition since it is not possible to construct subjective prior in a nonparametric framework. Obtaining concentration rates of the posterior helps in understanding the impact of the choice of a specific prior and allows for a comparison between priors to some extent. However, to obtain a Bernstein-Von Mises theorem it is necessary not only to bound $\mathbb{P}^{\pi}[U_n|X^n]$ but to determine an equivalent of $\mathbb{P}^{\pi}[U_n|X^n]$ for some specific types of sets U_n. This difficulty explains that there is up to now very little work on Bernstein Von Mises theorems in infinite dimensional models. The most well known results are negative results and are given in [7]. Some positiv e e results are provided by [8] on the asymptotic normality of the posterior distribution of the parameter in an exponential family with increasing number of parameters. In a discrete setting [2] derive Bernstein-Von Mises results, in particular satisfied by Dirichlet priors. Nice positive results are obtained in [12] and [13], however they rely heavily on a conjugacy type of property of the family of priors they consider and on the fact that their priors put mass one on discrete probabilities which makes the comparison with the empirical distribution more tractable.

In a semi-parametric framework, where the parameter can be separated

into a parametric part, which is the parameter of interest and a non parametric part, which is the nuisance parameter, [3] obtains interesting conditions leading to a Bernstein - Von Mises theorem on the parametric part, clarifying an earlier work of [18].

In this paper we are interested in studying the existence of a Bernstein-Von Mises property in semi-parametric models where the parameter of interest is a functional of the nuisance parameter, which is the density of the observations. The estimation of functionals of infinite dimensional parameters such as the cumulative distribution function at a specific point, is a widely studied problem both in the frequentist literature and in the Bayesian literature. There is a vast literature on the rates of convergence and on the asymptotic distribution of frequentist estimates of functionals of unknown curves and of finite dimensional functionals of curves in particular, see for instance [21] for an excellent presentation of a general theory on such problems.

One of the most common functional considered in the literature is the cumulative distribution function calculated at a given point, say $F(x)$. The empirical cumulative distribution function, $F_n(x)$ is a natural frequentist estimator and its asymptotic distribution is Gaussian with mean $F(x)$ and variance $F(x)(1-F(x))/n$.

The Bayesian counterpart of this estimator is the one derived from a Dirichlet process prior and it is well known to be asymptotically equivalent to $F_n(x)$, see for instance [10].This result is obtained using the conjugate nature of the Dirichlet prior, leading to an explicit posterior distribution. Other frequentist estimators, based on frequentist estimates of the density have also been studied in the frequentist literature, in particular estimates based on kernel estimators. Hence a natural question arises. Can we generalize the Bernstein - Von Mises theorem of the Dirichlet estimator to other Bayesian estimators? What happens if the prior has support on distributions absolutely continuous with respect to Lebesgue Measure?

In this paper we provide an answer to these questions by establishing conditions under which a Bernstein-Von Mises theorem can be obtained for linear functional of the density of f, such as the cumulative distribution function $F(x)$, with centering its empirical counterpart, for instance $F_n(x)$ the empirical cumulative distribution function, when the prior puts positive mass on absolutely continuous densities with respect to Lebesgue measures. We also study cases where the asymptotic posterior distribution of the functional is not asymptotically Gaussian but is asymptotically a mixture of Gaussian distributions with different centering points.

1.1. *Notations and aim .* In this paper, we assume that given a distribution $\mathbb{P}$ with a compactly supported density f with respect to the Lebesgue measure, $X_1, ..., X_n$ are independent and identically distributed by $\mathbb{P}$. We set $X^n = (X_1, ..., X_n)$ and denote F the cumulative distribution function associated with f. Without loss of generality we assume that for any i, $X_i \in [0,1]$ and we set

$$\mathcal{F} = \left\{ f : [0,1] \to \mathbb{R}^+, \ \int_0^1 f(x)dx = 1 \right\}.$$

We now define other notations that will be used throughout the paper. Denote $l_n(f)$ the log-likelihood associated with the density f and if it is parametrized by a finite dimensional parameter θ, $l_n(\theta) = l_n(f_\theta)$. For an integrable function g, we sometimes use the notation $F(g) = \int_0^1 f(u)g(u)du$. We denote by $< ., . >_f$ the inner product in

$$\mathbb{L}_2(F) = \left\{ g : \quad \int g^2(x)f(x)dx < +\infty \right\}$$

and by $\|.\|_f$ the corresponding norm.

We also consider the inner product in $\mathbb{L}_2[0,1]$ denoted $< ., . >_2$ and $\|.\|_2$ the corresponding norm. When there is no ambiguity we note $< ., . >_{f_0}$ by $< ., . >$ and $\|.\|_{f_0}$ by $\|.\|$.

Let $K(f, f')$ and $h(f, f')$ respectively the Kullback-Leibler divergence and the Hellinger distance between two densities f and f', where we recall that

$$K(f, f') = F\left(\log(f/f')\right), \quad h(f, f') = \left[\int \left(\sqrt{f(x)} - \sqrt{f'(x)} \right)^2 dx \right]^{1/2},$$

and define

$$V(f, f') = F\left((\log(f/f'))^2 \right).$$

Finally, let $\mathbb{P}_0$ the true distribution of the observations X_i. f_0 is the associated density and F_0 the associated cumulative distribution function. We consider the usual notations on the empirical process, namely

$$P_n(g) = \frac{1}{n} \sum_{i=1}^n g(X_i), \quad G_n(g) = \frac{1}{\sqrt{n}} \sum_{i=1}^n [g(X_i) - F_0(g)],$$

and F_n the empirical distribution function.

Consider a prior Π on the set $\mathcal{F}$. The aim of this paper is to study the posterior distribution of $\Psi(f)$, where Ψ is a continuous linear form on $\mathbb{L}_2[0,1]$ (a typical example is $\Psi(f) = F(x_0) = \mathbb{P}[X \leq x_0]$ for $x_0 \in \mathbb{R}$) and to derive conditions under which

$$\mathbb{P}^\pi \left[\sqrt{n}(\Psi(f) - \Psi(P_n)) \leq z | X^n \right] \to \Phi_{V_0}(z) \quad \text{in } \mathbb{P}_0 \text{ Probability},$$

where V_0 is the variance of $\sqrt{n}\Psi(P_n)$ under $\mathbb{P}_0$ and for any V, $\Phi_V(z)$ is the cumulative distribution function of a Gaussian random variable centered at 0 with variance V.

1.2. *Organization of the paper.* In Section 2 we present the general Bernstein Von Mises theorem, which is given in the formal way in the case where linear submodels are adapted to the prior. We then apply, in Section 3, this general theorem to the case where the prior is based on infinite dimensional exponential families. In this section, we first give general results giving the asymptotic posterior distribution of $\Psi(f)$ which can be either Gaussian or a mixture of Gaussian distributions. We also provide a theorem describing the posterior concentration rate under such priors (see Section 3.2). Finally, in Section 3.4, using an example, we explain how bad phenomenons can occur. The proofs are postponed in Section 4.

2. Bernstein Von Mises theorems.

2.1. *Some heuristics for proving Bernstein Von Mises theorems.* We first define some notions that are useful in the study of asymptotic properties of semi-parametric models. These notions can be found for instance in [21].

As in Chapter 25 of [21], to study the asymptotic behaviour of semi-parametric models we consider 1-dimensional differentiable paths locally around the true parameter f_0, that is submodels of the form: $u \to f_u$ for $0 < u < u_0$, for some $u_0 > 0$ such that for each path there exists a measurable function g called the score function for the submodel $\{f_u, , 0 < u < u_0\}$ at $u = 0$ satisfying

$$\lim_{u\to 0} \int_{\mathbb{R}} \left(\frac{f_u^{1/2}(x) - f_0^{1/2}(x)}{u} - \frac{1}{2} g(x) f_0^{1/2}(x) \right)^2 dx = 0. \tag{2.1}$$

We denote by $\mathcal{F}_{f_0}$ the tangent set, i.e. the collection of score functions g associated with these differentiable paths. Using (2.1), $\mathcal{F}_{f_0}$ can be identified with a subset of $\{g \in \mathbb{L}_2(F_0) : \ F_0(g) = 0\}$. For instance, when considering all probability laws, the most usual collection of differentiable paths is given by

$$f_u(x) = c(t) f_0(x) e^{ug(x)} \tag{2.2}$$

with $\|g\|_\infty < \infty$ and c such that $c(0) = 1$ and $c'(0) = 0$. In this case, g is the score function. Note that as explained in [21], the collection of differentiable paths of the form $f_u(x) = 2c(u) f_0(x)(1 + \exp(-2ug(x)))^{-1}$ (with previous

conditions on c), leads to the tangent space given by $\{g \in \mathbb{L}_2(F_0) : \ F_0(g) = 0\}$.

Now, consider a continuous linear form Ψ on $\mathbb{L}_2$. We can identify such a functional by a function $\psi \in \mathbb{L}_2$ such that for all $f \in \mathbb{L}_2$

$$\Psi(f) = \int f(x)\psi(x)dx. \tag{2.3}$$

Then for any differentiable path $t \to f_t$ with score function g, if the function ψ is bounded on $\mathbb{R}$ (or on the support of f_u for all $0 \leq u < u_0$),

$$\begin{aligned}\frac{\Psi(f_u) - \Psi(f_0)}{t} &= \int \psi(x)g(x)f_0(x)dx + \int \frac{\left(f_u^{1/2}(x) - f_0^{1/2}(x)\right)^2}{u}\psi(x)dx \\ &\quad +2\int \psi(x)\left(\frac{f_u^{1/2}(x) - f_0^{1/2}(x)}{u} - \frac{1}{2}g(x)f_0^{1/2}(x)\right)f_0^{1/2}(x)dx \\ &= < \psi, g > +o(1).\end{aligned}$$

Then, we can define the efficient influence function $\tilde{\psi}$ belonging to $\overline{\text{lin}}(\mathcal{F}_{f_0})$ (the closure of the linear space generated by $\mathcal{F}_{f_0}$) that satisfies for any $g \in \mathcal{F}_{f_0}$,

$$\int \tilde{\psi}(u)g(x)f_0(x)dx = \int \psi(x)g(x)f_0(x)dx.$$

This implies:

$$\lim_{u\to 0} \frac{\Psi(f_u) - \Psi(f_0)}{u} =< \tilde{\psi}, g > . \tag{2.4}$$

The efficient influence function will play an important role for our purpose. The efficient influence function is also a key notion to characterize asymptotically efficient estimators (see Section 25.3 of [21]).

Now, let us provide some examples by specifying different types of continuous linear forms that can be considered.

EXAMPLE 2.1. *An important example is provided by the cumulative distribution function. If $x_0 \in \mathbb{R}$ is fixed, consider for any density function $f \in \mathbb{L}_2$ whose cdf is F,*

$$\Psi(f) = \int \mathbb{1}_{x\leq x_0} f(x)dx = F(x_0)$$

so that in this case, $\psi(u) = \mathbb{1}_{x\leq x_0}$, which is a bounded function and if $\mathcal{F}_{f_0}$ is the subspace of $\mathbb{L}_2(F_0)$ of functions g satisfying $F_0(g) = 0$ then $\tilde{\psi}(x) = \mathbb{1}_{x\leq x_0} - F_0(x_0)$.

EXAMPLE 2.2. *More generally, for any measurable set A consider $\psi(x) = \mathbb{1}_{x\in A}$ and for any density function $f \in \mathbb{L}_2$*

$$\Psi(f) = \int \mathbb{1}_{x\in A} f(u)du$$

satisfies the above conditions and $\tilde{\psi}(x) = \mathbb{1}_{x\in A} - \int_A f_0(x)dx$.

EXAMPLE 2.3. *If f_0 has bounded support, say on $[0,1]$ then the functional*

$$\Psi(f) = \mathbb{E}_f[X] = \int_0^1 xf(x)dx$$

satisfies the above conditions, $\psi(x) = x$ and $\tilde{\psi}(x) = x - \mathbb{E}_{f_0}[X]$.

In this framework, the Bernstein Von Mises theorem could be derived from the convergence of the following Laplace transform defined for any $t \in \mathbb{R}$ by

$$\begin{aligned} L_n(t) &= \mathbb{E}^{\pi}[\exp(t\sqrt{n}(\Psi(f) - \Psi(P_n)))|X^n] \\ &= \frac{\int \exp\left(t\sqrt{n}(\Psi(f) - \Psi(P_n)) + l_n(f) - l_n(f_0)\right) d\pi(f)}{\int \exp\left(l_n(f) - l_n(f_0)\right) d\pi(f)}. \end{aligned}$$

Now, let us set $f_{g,n} = f_u$ if $u = n^{-\frac{1}{2}}$. We have:

$$\begin{aligned} \sqrt{n}\left(\Psi(f_{g,n}) - \Psi(P_n)\right) &= \sqrt{n}\int \psi(x)(f_{g,n}(x) - f_0(x))dx - G_n(\tilde{\psi}) \\ &= < \tilde{\psi}, g > -G_n(\tilde{\psi}) + \Delta_n(g). \end{aligned}$$

Furthermore,

$$l_n(f_{g,n}) - l_n(f_0) = R_n(g) + G_n(g) - \frac{F_0(g^2)}{2},$$

with

$$R_n(g) = nP_n\left(\log\left(\frac{f_{g,n}}{f_0}\right)\right) - G_n(g) + \frac{F_0(g^2)}{2}.$$

So,

$$\begin{aligned} &t\sqrt{n}\left(\Psi(f_{g,n}) - \Psi(P_n)\right) + l_n(f_{g,n}) - l_n(f_0) \\ &\qquad = R_n(g) - \frac{F_0(g^2)}{2} + G_n(g - t\tilde{\psi}) + t\Delta_n(g) + t < \tilde{\psi}, g > \\ &\qquad = R_n(g - t\tilde{\psi}) + G_n(g - t\tilde{\psi}) - \frac{F_0((g - t\tilde{\psi})^2)}{2} + \frac{t^2F_0(\tilde{\psi}^2)}{2} + U_n, \end{aligned}$$

with

$$U_n = t\Delta_n(g) + R_n(g) - R_n(g - t\tilde{\psi}).$$

Lemma 25.14 of [21] shows that under (2.1), $R_n(g) = o(1)$ and (2.4) yields $\Delta_n(g) = o(1)$ for a fixed g. It is not enough however to derive a Bernstein-Von Mises theorem. Nonetheless if we can choose a prior distribution π adapted to the previous framework to obtain uniformly $U_n = o(1)$,

$$\sqrt{n}\,(\Psi(f_{g,n}) - \Psi(f)) + l_n(f_{g,n}) - l_n(f) = o(1)$$

and the equalities

$$\begin{aligned}\frac{\int e^{R_n(g-t\tilde{\psi})+G_n(g-t\tilde{\psi})-\frac{F_0((g-t\tilde{\psi})^2)}{2}}d\pi(f)}{\int e^{R_n(g)+G_n(g)-\frac{F_0(g^2)}{2}}d\pi(f)} &= \frac{\int \exp\left(l_n(f) - l_n(f_0)\right)d\pi(f_{g+t\tilde{\psi}})}{\int \exp\left(l_n(f) - l_n(f_0)\right)d\pi(f)} \\ &= 1 + o(1),\end{aligned}$$

then

$$L_n(t) = \exp\left(\frac{t^2 F_0(\tilde{\psi}^2)}{2}\right)(1 + o(1)).$$

In this case, our goal is reached. However, it is not obvious that a given prior π satisfies all these properties. In particular, in a nonparametric framework, the property $R_n(g) = o(1)$ uniformly over a set whose posterior probability goes to 1, is usually not satisfied. We thus consider an alternative approach based on linear submodels.

2.2. *Bernstein Von Mises under linear submodels.* In this section we study the case where linear local models are adapted to the prior. More precisely, we assume that $\|\log(f_0)\|_\infty < \infty$ so, for each density function f, we define h such that for any x,

$$h(x) = \sqrt{n}\log\left(\frac{f(x)}{f_0(x)}\right) \quad \text{or equivalently} \quad f(x) = f_0(x)\exp\left(\frac{h(x)}{\sqrt{n}}\right).$$

For the sake of clarity, we sometime write f_h instead of f and h_f instead of h to underline the relationship between f and h. Note that in this context h is not the score function since $F_0(h) \neq 0$. It would be equivalent to consider local models of the form $f = f_0(1 + h/\sqrt{n})$, except that we would have to impose constraints on h for f to be positive. We consider a continuous linear form Ψ on $\mathbb{L}_2$ such that for any $f \in \mathbb{L}_2$, we consider ψ such that (2.3) is satisfied and we set for any x,

$$\psi_c(x) = \psi(x) - F_0(\psi). \tag{2.5}$$

Note that ψ_c coincides with the influence function $\tilde{\psi}$ associated with the tangent set $\{g \in \mathbb{L}_2(F_0); F_0(g) = 0\}$. Then we consider the following assumptions.

(A1) The posterior distribution concentrates around f_0. More precisely, there exists $u_n = o(1)$ such that if $A^1_{u_n} = \{f \in \mathcal{F} : \quad V(f_0, f) \leq u_n^2\}$ the posterior distribution of $A^1_{u_n}$ satisfies

$$\mathbb{P}^\pi \left\{ A^1_{u_n} | X^n \right\} = 1 + o_{\mathbb{P}_0}(1).$$

(A2) The posterior distribution of the subset $A_n \subset A^1_{u_n}$ of densities such that

$$\int \left| \log \left(\frac{f(x)}{f_0(x)} \right) \right|^3 (f_0(x) + f(x))\, dx = o(1) \tag{2.6}$$

satisfies

$$\mathbb{P}^\pi \left[A_n | X^n \right] = 1 + o_{\mathbb{P}_0}(1).$$

(A3) Let

$$R_n(h) = \sqrt{n} F_0(h) + \frac{F_0(h^2)}{2}$$

and for any x,

$$\bar{\psi}_{t,n}(x) = \psi_c(x) + \frac{\sqrt{n}}{t} \log \left(F_0 \left[\exp \left(\frac{h}{\sqrt{n}} - \frac{t\psi_c}{\sqrt{n}} \right) \right] \right).$$

We have

$$\frac{\int_{A_n} \exp \left(-\frac{F_0((h_f - t\bar{\psi}_{t,n})^2)}{2} + G_n(h_f - t\bar{\psi}_{t,n}) + R_n(h_f - t\bar{\psi}_{t,n}) \right) d\pi(f)}{\int_{A_n} \exp \left(-\frac{F_0(h_f^2)}{2} + G_n(h_f) + R_n(h_f) \right) d\pi(f)}$$
$$= 1 + o_{\mathbb{P}_0}(1). \tag{2.7}$$

Before stating our main result, let us discuss these assumptions. Condition (A1) concerns concentration rates of the posterior distribution and there exists now a large literature on such results. See for instance [20] or [9] for general results. The difficulty here comes from the use of V instead of the Hellinger or the $\mathbb{L}_1$-distance. However since u_n does not need to be optimal, deriving rates in terms of V from those in terms of the Hellinger distance is often not a problem (see below).

Condition (A2) is a refinement of (A1) but can often be derived from (A1) as illustrated below.

The main difficulty comes from condition (A3). To prove it, we need to be able to construct a transformation T such that $Tf_h = f_{h-t\bar{\psi}_{t,n}}$ exists and such that the prior is hardly modified by this transformation. In parametric setups, continuity of the prior near the true value is enough to ensure that the prior would hardly be modified by such a transform and this remains true in semi-parametric setups where we can write the parameter as (θ, η) where θ is the parameter of interest and is finite dimensional. Indeed as shown in [3] under certain conditions the transformations can be transferred to transformations on θ which is finite dimensional. Here this is more complex since T is a transformation on f which is infinite dimensional so that a condition of the form $d\pi(Tf) = d\pi(f)(1 + o(1))$ does not necessarily make sense. We study this aspect in more details in Section 3.

Now, we can state the main result of this section.

THEOREM 2.1. *Let f_0 be a density on $\mathcal{F}$ such that $\|\log(f_0)\|_\infty < \infty$ and $\|\psi\|_\infty < \infty$. Assume that (A1), (A2) and (A3) are true. Then, if*

$$\Psi(P_n) = P_n(\psi) = \frac{\sum_{i=1}^n \psi(X_i)}{n}$$

we have for any z, in probability with respect to $\mathbb{P}_0$,

$$\mathbb{P}^\pi \left\{ \sqrt{n}(\Psi(f) - \Psi(P_n)) \leq z | X^n \right\} - \Phi_{F_0(\psi_c^2)}(z) \to 0.$$

The proof of Theorem 2.1 is given in Section 4.1.

Sieve priors lead to interesting behaviours of the posterior distribution as illustrated in the following section. Indeed they have a behaviour which is half way between parametric and non parametric. We illustrate these features in the following two sections.

3. Bernstein Von Mises theorem under infinite dimensional exponential families. In this section, we study a specific class of priors based on infinite dimensional exponential families on the following class of densities supported by $[0, 1]$:

$$\mathcal{F} = \left\{ f \geq 0 : \quad f \text{ is 1-periodic, } \int_0^1 f(x)dx = 1, \ \log(f) \in \mathbb{L}_2([0,1]) \right\}.$$

We assume that $f_0 \in \mathcal{F}$ and we consider two types of orthonormal bases defined in the following section, namely the Fourier and wavelet bases.

3.1. *Orthonormal bases.* Fourier bases constitute unconditional bases of periodized Sobolev spaces W^γ where γ is the smoothness parameter. Our results are also valid for a wide range of Besov spaces. In this case, we consider wavelet bases which allow for the following expansions:

$$f(x) = \theta_{-10}\mathbb{1}_{[0,1]}(x) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\theta_{jk}\Psi_{jk}(x), \quad x \in [0,1]$$

where $\theta_{-10} = \int_0^1 f(x)dx$ and $\theta_{jk} = \int_0^1 f(x)\Psi_{jk}(x)dx$. We recall that the functions Ψ_{jk} are obtained by periodizing dilations and translations of a mother wavelet Ψ that can be assumed to be supported by the compact set $[-A, A]$:

$$\Psi_{jk}(x) = 2^{\frac{j}{2}}\sum_{l=-\infty}^{+\infty}\Psi(2^j x - k + 2^j l), \quad x \in [0,1].$$

If Ψ belongs to the Hölder space C^r and has r vanishing moments then the wavelet basis constitutes an unconditional basis of the Besov space $\mathcal{B}^\gamma_{p,q}$ for $1 \le p, q \le +\infty$ and $\max\left(0, \frac{1}{p} - \frac{1}{2}\right) < \gamma < r$. In this case, $\mathcal{B}^\gamma_{p,q}$ is the set of functions f of $\mathbb{L}_2[0,1]$ such that $\|f\|_{\gamma,p,q} < \infty$ where

$$\|f\|_{\gamma,p,q} = \begin{cases} |\theta_{-10}| + \left(\sum_{j=0}^{+\infty} 2^{jq(\gamma+\frac{1}{2}-\frac{1}{p})}\left(\sum_{k=0}^{2^j-1}|\theta_{jk}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ |\theta_{-10}| + \sup_{j\ge 0}\left\{2^{j(\gamma+\frac{1}{2}-\frac{1}{p})}\left(\sum_{k=0}^{2^j-1}|\theta_{jk}|^p\right)^{\frac{1}{p}}\right\} & \text{if } q = \infty. \end{cases}$$

We refer the reader to [16] for a good review of wavelets and Besov spaces. We just mention that Besov spaces include in particular Sobolev spaces ($W^\gamma = \mathcal{B}^\gamma_{2,2}$) and, when γ is not an integer, Hölder spaces ($C^\gamma = \mathcal{B}^\gamma_{\infty,\infty}$). To shorten notations, the orthonormal basis will be denoted $(\phi_\lambda)_{\lambda\in\mathbb{N}}$, where $\phi_0 = \mathbb{1}_{[0,1]}$ and

- for the Fourier basis, for $\lambda \ge 1$,

$$\phi_{2\lambda-1}(x) = \sqrt{2}\sin(2\pi\lambda x), \quad \phi_{2\lambda}(x) = \sqrt{2}\cos(2\pi\lambda x).$$

- for the wavelet basis, if $\lambda = 2^j + k$, with $j \in \mathbb{N}$ and $k \in \{0, \dots, 2^j - 1\}$,

$$\phi_\lambda = \Psi_{jk}.$$

Now, the decomposition of each periodized function $f \in \mathbb{L}_2[0,1]$ on $(\phi_\lambda)_{\lambda\in\mathbb{N}}$ is written as follows:

$$f(x) = \sum_{\lambda\in\mathbb{N}}\theta_\lambda\phi_\lambda(x), \quad x \in [0,1],$$

where $\theta_\lambda = \int_0^1 f(x)\phi_\lambda(x)dx$. Recall that when the Fourier basis is used, f lies in W^γ for $\gamma > 0$ if and only if $\|f\|_\gamma < \infty$, where

$$\|f\|_\gamma = \left(\theta_0^2 + \sum_{\lambda \in \mathbb{N}^*} |\lambda|^{2\gamma}\theta_\lambda^2\right)^{\frac{1}{2}}.$$

We respectively use $\|.\|_\gamma$ and $\|.\|_{\gamma,p,q}$ to define the radius of the balls of W^γ and $\mathcal{B}_{p,q}^\gamma$ respectively. We now present the general result on posterior concentration rates associated with such prior models.

3.2. *Posterior rates.* Assume that $f_0 \in \mathcal{F}$ and let Φ be one of the orthonormal basis introduced in Section 3.1, then

$$\log(f_0) - \int_0^1 \log(f_0(x))dx = \sum_{\lambda \in \mathbb{N}^*} \theta_{0\lambda}\phi_\lambda.$$

Set $\theta_0 = (\theta_{0\lambda})_{\lambda \in \mathbb{N}^*}$ and define $c(\theta_0) = -\int_0^1 \log(f_0(x))dx$, we have

$$f_0(x) = \exp\left(\sum_{\lambda \in \mathbb{N}^*} \theta_{0\lambda}\phi_\lambda(x) - c(\theta_0)\right).$$

We consider the following family of models: for any $k \in \mathbb{N}^*$, we set

$$\mathcal{F}_k = \left\{ f_\theta = \exp\left(\sum_{\lambda=1}^k \theta_\lambda\phi_\lambda - c(\theta)\right) : \quad \theta \in \mathbb{R}^k \right\},$$

where

$$c(\theta) = \log\left(\int_0^1 \exp\left(\sum_{\lambda=1}^k \theta_\lambda\phi_\lambda(x)\right)dx\right). \tag{3.1}$$

So, we define a prior π on the set $\mathcal{F} = \cup_k \mathcal{F}_k$ by defining a prior p on $\mathbb{N}^*$ and then, once k is chosen, we fix a prior π_k on $\mathcal{F}_k$. Such priors are often considered in the Bayesian non parametric literature. See for instance [19]. The special case of log-spline priors has been studied by [9] and [11], whereas the prior considered by [22] is based on Legendre polynomials. For the wavelet case, [11] considered the special case of the Haar basis.

Since one of the key conditions needed to obtain a Bernstein Von Mises theorem is a concentration rate of the posterior distribution of order ϵ_n, we first give two general results on concentration rates of posterior distributions based on the two different setups of orthonormal bases: the Fourier basis and

the wavelet basis. These results have their own interest since we obtain in such contexts optimal adaptive rates of convergence. In a similar spirit [19] considers infinite dimensional exponential families and derives minimax and adaptive posterior concentration rates. Her work differs from the following theorem in two main aspects. Firstly she restricts her attention to the case of Sobolev spaces and Fourier basis, whereas we consider Besov spaces and secondly she obtains adaptivity by putting a prior on the smoothness of the Sobolev class whereas we obtain adaptivity by constructing a prior on the size k of the parametric spaces, which to our opinion is a more natural approach. Moreover [19] merely considers Gaussian priors. Also related to this problem is the work of [11] who derives a general framework to obtain adaptive posterior concentration rates and apply her results to the Haar basis case. The limitation in her case, apart from the fact that she considers the Haar basis and no other wavelet basis is that she constraints the θ_j's in each k dimensional model to belong to a ball with fixed radius.

Now, we specify the conditions on the prior π:

DEFINITION 3.1. *Let $1 > \beta > 1/2$ be fixed and let g be a continuous and positive density on $\mathbb{R}$ bounded (up to a contant) by the function $M_{p_*}(x) = \exp(-c|x|^{p_*})$ for positive constants c, p_* and assume that for all $M > 0$ there exists a, b such that*

$$g(y+u) \geq a \exp\{-b(|y|^{p_*} + |u|^{p_*})\}, \quad \forall |y| \leq M, \quad \forall u \in \mathbb{R}$$

The prior p on k satisfies one of the following conditions:

[Case (PH)] *There exist two positive constants c_1 and c_2 such that for any $k \in \mathbb{N}^*$,*

$$\exp(-c_1 k L(k)) \leq p(k) \leq \exp(-c_2 k L(k)), \tag{3.2}$$

where L is the function that can be either $L(x) = 1$ or $L(x) = \log(x)$.

[Case (D)] *If $k_n^* = n^{1/(2\beta+1)}$,*

$$p(k) = \delta_{k_n^*}(k).$$

Conditionally on k we define the prior on $\mathcal{F}_k$ by assuming that the prior distribution π_k on $\theta = (\theta_\lambda)_{1 \leq \lambda \leq k}$ is given by

$$\frac{\theta_\lambda}{\sqrt{\tau_\lambda}} \sim g, \quad \tau_\lambda = \tau_0 \lambda^{-2\beta} \quad \textit{i.i.d.}$$

where $\beta < 1/2 + p_*/2$ if $p_* \leq 2$ and $\beta < 1/2 + 1/p_*$ if $p_* > 2$.

Observe that we do not necessarily consider Gaussian priors since we allow for densities g to have different tails. The prior on k can be non random, which corresponds to the Dirac case (D). For the case (PH), $L(x) = \log(x)$ corresponds typically to a Poisson prior on k and the case $L(x) = 1$ corresponds typically to hypergeometric priors. Now, we have the the following result.

THEOREM 3.1. *Assume that* $\|\log(f_0)\|_\infty < \infty$ *and that there exists* $\gamma > 1/2$ *such that* $\log(f_0) \in \mathcal{B}^\gamma_{p,q}$, *with* $p \geq 2$ *and* $1 \leq q \leq \infty$. *Then,*

$$\mathbb{P}^\pi \left\{ f_\theta : \quad h(f_0, f_\theta) \leq \frac{\log n}{L(n)} \epsilon_n | X^n \right\} = 1 + o_P(1), \tag{3.3}$$

and

$$\mathbb{P}^\pi \left\{ f_\theta : \quad \|\theta_0 - \theta\|_2 \leq \frac{(\log n)^2}{L(n)} \epsilon_n | X^n \right] = 1 + o_P(1), \tag{3.4}$$

where in case (PH),

$$\epsilon_n = \epsilon_0 \left(\frac{\log n}{n} \right)^{\frac{\gamma}{2\gamma+1}},$$

in case (D),

$$\epsilon_n = \epsilon_0 \log n n^{-\frac{\beta}{2\beta+1}}, \quad \text{if } \gamma \geq \beta$$

$$\epsilon_n = \epsilon_0 n^{-\frac{\gamma}{2\beta+1}}, \quad \text{if } \gamma < \beta$$

and ϵ_0 *is a constant large enough.*

The proof of Theorem 3.1 is given in Section 4.2.

REMARK 1. *If the density* g *only satisfies a tail condition of the form*

$$g(x) \leq C_g |x|^{-p_*}, \quad |x| \quad \text{large enough}$$

with $p_* > 1$, *then, in case (PH), if* $\gamma > 1$ *the rates defined by (3.3) and (3.4) remain valid.*

REMARK 2. *Note that in the case (PH) the posterior concentration is, up to a* $\log n$ *term, the minimax rate of convergence on the collection of spaces with smoothness* $\gamma > 1/2$, *whereas in the case (D) the minimax rate is achieved only when* $\gamma = \beta$.

3.3. *Bernstein Von Mises under these models.* In this section, we apply Theorem 2.1 of Section 2.2 to establish the following Bernstein Von Mises-type result. For this purpose, let us expand the function ψ_c defined in (2.5) on the basis $(\phi_\lambda)_{\lambda\in\mathbb{N}}$:

$$\psi_c = \sum_{\lambda\in\mathbb{N}} \psi_{c,\lambda}\phi_\lambda.$$

We denote $\Pi_{f_0,k}$ the projection operator on the vector space generated by $(\phi_\lambda)_{0\leq\lambda\leq k}$ for the scalar product $< f, g >= F_0(fg)$ and $\Delta_\psi = \psi_c - \Pi_{f_0,k}\psi_c$. So we can write for any $x \in [0,1]$,

$$\Pi_{f_0,k}\psi_c(x) = \psi_{\Pi,c,0} + \sum_{\lambda=1}^{k} \psi_{\Pi,c,\lambda}\phi_\lambda(x),$$

since $\phi_0(x) = 1$. We denote $B_{n,k}$ the renormalized sequence of coefficients that appear in the above sum:

$$B_{n,k} = \frac{\psi_{\Pi,c,[k]}}{\sqrt{n}}, \quad \psi_{\Pi,c,[k]} = (\psi_{\Pi,c,\lambda})_{1\leq\lambda\leq k}.$$

Such quantities will play a key role in the sequel. Let $l_0 > 0$ be large enough so that

$$\mathbb{P}^\pi\left[k > \frac{l_0 n\epsilon_n^2}{L(n)}|X^n\right] \leq e^{-cn\epsilon_n^2},$$

for some positive $c > 0$, where ϵ_n is the posterior concentration rate defined in Theorem 3.1 and define $l_n = l_0 n\epsilon_n^2/L(n)$. In the case (D) we set $l_n = k_n^*$. In the following, in the case (D), whenever a statement concerns $k \leq l_n$ it is to be understood as $k = l_n$.

We have the following result.

THEOREM 3.2. *Let us assume that the prior is defined as in Definition 3.1 and for all $t \in \mathbb{R}$, $1 \leq k \leq l_n$ (or k_n^* in case (D)), assume that*

$$\frac{\pi_k(\theta)}{\pi_k(\theta - tB_{n,k})} = 1 + o(1), \quad \text{if} \quad \sum_{j=1}^{k}(\theta_j - \theta_{0j})^2 \leq \frac{(\log n)^2}{L(n)}\epsilon_n \tag{3.5}$$

uniformly over $\{\theta; \|\theta - \theta_0\|_2 \leq 3(\log n)^2\epsilon_n\}$. Assume also that

$$\sup_{k\leq l_n}(\|\sum_{j>k}\psi_{cj}\phi_j\|_\infty + \sqrt{k}\|\sum_{j>k}\psi_{cj}\phi_j\|_2) = o\left(\frac{(\log n)^{-2}}{\sqrt{n}\epsilon_n^2}\right). \tag{3.6}$$

Under assumptions of Theorem 3.1

- *for all* $z \in \mathbb{R}$

$$\mathbb{P}^{\pi}\left[\sqrt{n}(\Psi(f) - \Psi(P_n)) \leq z | X^n\right] = \sum_k p(k|X^n)\Phi_{V_{0k}}\left(z + \mu_{n,k}\right) + o_{\mathbb{P}_0}(1), \tag{3.7}$$

  where
  - $V_{0k} = F_0(\psi_c^2) - F_0(\Delta_\psi^2)$.
  - $\mu_{n,k} = -\sqrt{n}F_0[(\psi_c - \Pi_{f_0,k}\psi_c)\sum_{j\geq k+1}\theta_{0j}\phi_j] + G_n(\Delta_\psi)$

- *In the case (D), if* $\gamma \geq \beta$,

$$\mathbb{P}^{\pi}\left[\sqrt{n}(\Psi(f) - \Psi(P_n)) \leq z | X^n\right] = \Phi_{V_0}\left(z\right) + o_{\mathbb{P}_0}(1), \tag{3.8}$$

  where $V_0 = F_0(\psi_c^2)$.

The Bernstein-Von Mises property obtained in the case (D) is deduced by proving relation (3.6) in this case. Indeed if $\gamma \geq \beta$ there exists $a > 0$ such that $\sqrt{n}\epsilon_n^2 \leq n^{-a}$, besides

$$\begin{aligned} \|\sum_{j>k}\psi_{cj}\phi_j\|_\infty &\leq 1 + \|\sum_{j\leq k}\psi_{cj}\phi_j\|_\infty \\ &\leq C\log n \end{aligned}$$

and $\sum_{j\geq k+1}\psi_{cj}^2 = 0(1/k)$, so that relation (3.6) is satisfied. Apart from this argument the proof of Theorem 3.2 is given in Section 4.3. The first part of Theorem 3.2 shows that the posterior distribution of $\sqrt{n}(\Psi(f) - \Psi(P_n))$ is asymptotically a mixture of Gaussian distributions with variances $V_0 - F_0(\Delta_\psi^2)$ and mean values $\mu_{n,k}$ with weight $p(k|X^n)$. To obtain an asymptotic Gaussian distribution with mean zero and variance V_0 it is necessary for $\mu_{n,k}$ to be small whenever $p(k|X^n)$ is not. The conditions given in the second part of Theorem 3.2 ensure that this is the case, however they are not necessary conditions. Nevertheless, in Section 3.4, we give a counter-example for which the Bernstein-Von Mises property is not satisfied in the cases (PH) and (D) with $\gamma < \beta$.

We now discuss condition (3.5) in three different examples. Note first that $A_n \subset \{\theta; \|\theta - \theta_0\|_2 \leq 3(\log n)^2\epsilon_n\}$ with $\theta \in \Theta_k$, $k \leq l_n$.

- Gaussian: If g is Gaussian then for all $k \leq l_n$ (or k_n^* in the case of a

type (D) prior) and all $j \leq k$, $\theta_j \sim \mathcal{N}(0, \tau_0^2 j^{-2\beta})$ and for all $\theta \in A_n \cap \mathcal{F}_k$

$$\begin{aligned}
\frac{\sum_{j=1}^k \psi_{cj}^2 j^{2\beta}}{n} &\leq \frac{Ck^{2\beta}}{n} \leq O(n^{2\beta-1}\epsilon_n^{4\beta}) = o(1) \\
\frac{\sum_{j=1}^k \theta_j \psi_{cj} j^{2\beta}}{\sqrt{n}} &= \frac{\sum_{j=1}^k (\theta_j - \theta_{0j})\psi_{cj} j^{2\beta} + \sum_{j=1}^k \theta_{0j}\psi_{cj} j^{2\beta}}{\sqrt{n}} \\
&\leq \frac{C}{\sqrt{n}}\left[\|\theta - \theta_0\| k^{2\beta} + (k^{2\beta-\gamma}+1)\right] \\
&= o(1).
\end{aligned}$$

This implies that uniformly over A_n

$$\pi_k(\theta - B_{n,k}) = \pi_k(\theta)(1 + o(1))$$

- Laplace: If g is Laplace, $g(x) \propto e^{-|x|}$,

$$\left|\log\left(g\left(\frac{\theta_j - t\psi_{cj}/\sqrt{n}}{\sqrt{\tau_j}}\right)\right) - \log\left(g\left(\frac{\theta_j}{\sqrt{\tau_j}}\right)\right)\right| \leq C\frac{|\psi_{cj}|}{\sqrt{n}}$$

So that

$$\begin{aligned}
\left|\log\left[\frac{\pi_k(\theta - B_n)}{\pi_k(\theta)}\right]\right| &\leq C\frac{\sum_{j=1}^k j^\beta |\psi_{cj}|}{\sqrt{n}} \\
&\leq C\frac{k^\beta}{\sqrt{n}} = o(1),
\end{aligned}$$

for all $\gamma > 1/2$, $1 > \beta > 1/2$ in the cases (D) and (PH), and condition (3.5) is satisfied.

- Student: In the Student case for g we can use the calculations made in the Gaussian case since

$$\begin{aligned}
&\sum_{j=1}^k \log\left(1 + Cj^{2\beta}\theta_j^2\right) - \log\left(1 + Cj^{2\beta}(\theta_j - t\psi_{cj}/\sqrt{n})^2\right) \\
= \; & 0\left(\sum_{j=1}^k j^{2\beta}[(\theta_j - t\psi_{cj}/\sqrt{n})^2 - \theta_j^2]\right) \\
= \; & o(1)
\end{aligned}$$

Therefore in all these cases condition (3.5) is satisfied.

Interestingly Theorem 3.2 shows that parametric sieve models (increasing sequence of models) have a behaviour which is a mix between parametric

and nonparametric models. Indeed if the posterior distribution puts most of its mass on k's large enough the posterior distribution has a Bernstein Von Mises property centered on the empirical (nonparametric MLE) estimator with the correct variance whereas if it allows for k's that are not large enough (corresponding to $\sum_{j=1}^{k} j^{2\beta}[(\theta_j - t\psi_{cj}/\sqrt{n})^2 - \theta_j^2]$ or Δ_ψ not small enough) then the posterior distribution is not asymptotically Gaussian with the right centering, nor with the right variance. An extreme case corresponds to the situation where $F_0(\Delta_\psi^2) \neq o(1)$ under the posterior distribution, which is equivalent to

$$\exists k_0, \quad \text{s.t. } \forall \epsilon > 0 \quad \text{liminf}_{n\to\infty} \mathbb{P}_0^n \left[\mathbb{P}^\pi \left[k_0 | X^n\right] > \epsilon\right] > 0.$$

For each $k > 0$ fixed, if $\inf_{\theta\in\mathbb{R}^k} K(f_0, f_\theta) > 0$, since the model is regular, there exists $c > 0$ such that $\mathbb{P}_0^n \left[\mathbb{P}^\pi \left[k | X^n\right] > e^{-nc}\right] \to 1$. Therefore, $F_0(\Delta_\psi^2) \neq o(1)$ under the posterior distribution if there exists k_0 such that $\inf_{\theta\in\mathbb{R}^{k_0}} K(f_0, f_\theta) > 0$, i.e. if there exists $\theta_0 \in \mathbb{R}^{k_0}$ such that $f_0 = f_{\theta_0}$. In that case it can be proved that $\mathbb{P}^\pi[k_0|X^n] = 1 + o_P(1)$, see [4], and the Bernstein Von Mises theorem to be expected is the parametric one, under the model Θ_{k_0} which is regular. However, even if $\Delta_\psi = o_P(1)$, the posterior distribution might not satisfy the non parametric Bernstein Von Mises property with the correct centering. We illustrate in the following section this issue in the special case of the cumulative distribution function.

3.4. *An example: the cumulative distribution function.* As a special case, consider the functional on f to be the cumulative distribution function calculated at a given point x_0. As seen in Section 2, $\psi_c(x) = \mathbb{1}_{x\leq x_0} - F_0(x_0)$. We have $F_n(x_0) = P_n(\psi)$ and recall that the variance of $G_n(\psi)$ under P_0 is equal to $V_0 = F_0(x_0)(1 - F_0(x_0))$.

As an illustration, consider the case of the Fourier basis. The case of wavelet bases is dealt with in the same way. In other words for $\lambda \geq 1$, $\phi_{2\lambda-1}(x) = \sqrt{2}\sin(2\pi\lambda x)$, $\phi_{2\lambda}(x) = \sqrt{2}\cos(2\pi\lambda x)$ and $\phi_0(x) = 1$.

COROLLARY 3.1. *If the prior density g on the coefficients is Gaussian or Laplace then if $f_0 \in \mathcal{S}_\gamma$, with $\gamma \geq \beta$ and if the prior on k is the Dirac mass on k_n^* then the posterior distribution of $\sqrt{n}(F(x_0) - F_n(x_0))$ is asymptotically Gaussian with mean 0 and variance V_0.*

If the prior density g is Student and if $\gamma \geq \beta > 1$, then the same result remains valid.

This result is a direct application of Theorem 3.2.

Counter-example: In this remark we illustrate the fact that in the case of a random k, which leads to an adaptive minimax rate of convergence for the

posterior distribution we might not have a Bernstein - Von Mises theorem. Consider a density f_0 in the form

$$f_0 = \exp\left(\sum_{j\geq k_0} \theta_{0j}\phi_j(u)du - c(\theta_0)\right)$$

where k_0 is fixed but can be large and $\theta_{0,2j} = 0$ and

$$\theta_{0,2j-1} = \sin(2\pi jx)/[j^{\gamma+1/2}\sqrt{\log j}\log\log j].$$

Then for $J_1 > 3$

$$\begin{aligned}\sum_{j\geq J_1}\theta_{0j}^2 j^{2\gamma} &\leq \sum_{j\geq J_1}\frac{1}{j\log j\log\log j^2}\\ &\leq \int_{j_1}^{\infty}\frac{1}{x\log x(\log\log x)^2}dx\\ &= \frac{1}{\log\log J_1},\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{j\geq J_1}\theta_{0j}^2 &\leq \sum_{j\geq J_1}\frac{1}{j^{2\gamma+1}\log j\log\log j^2}\\ &\leq \int_{j_1}^{\infty}\frac{1}{x^{2\gamma+1}\log x(\log\log x)^2}dx\\ &= \left[-\frac{1}{2\gamma x^{2\gamma}\log x(\log\log x)^2}\right]_{J_1}^{\infty}(1+o(1))\\ &= \frac{1}{2\gamma J_1^{2\gamma}\log J_1(\log\log J_1)^2}(1+o(1)) \qquad (3.9)\end{aligned}$$

when $J_1 \to \infty$.

Consider a Poisson distribution on k with parameter $\nu > 0$ fixed then for such f_0, if $k_n = n^{1/(2\gamma+1)}(\log n)^{-2/(2\gamma+1)}(\log\log n)^{-2/(2\gamma+1)}$ and k_1 is large enough

$$\mathbb{P}^{\pi}[k \leq k_1 k_n | X^n] = 1 + o(1).$$

We now study the mean terms $\mu_{n,k}$ and we show that if $k \leq k_1 k_n$, $\mu_{n,k} \neq o(1)$ nor can $\mathbb{P}^{\pi}(k|X^n)$ be neglected.

First note that when $k \to \infty$ $G_n(\Delta_\psi) = o(1)$

$$\begin{aligned}
\mu_{n,k} &= \sqrt{n} F_0 \left[(\psi_c - \Pi_{f_0,k}\psi_c)(l_0 - \Pi_{f_0,k} l_0) \right] \\
&= \sqrt{n} F_0 \left[(\sum_{j=k+1}^{\infty} \psi_{cj}\phi_j)(l_0 - \Pi_{f_0,k} l_0) \right] \\
&= \sqrt{n} \int \left[(\psi_c - \Pi_{f_0,k}\psi_c)(l_0 - \Pi_{f_0,k} l_0) \right] \\
&\quad + \sqrt{n} \int (f_0 - 1) \left[(\psi_c - \Pi_{f_0,k}\psi_c)(l_0 - \Pi_{f_0,k} l_0) \right] \qquad (3.10)
\end{aligned}$$

We first consider the first term of the right hand side of (3.10).

$$\begin{aligned}
\mu_{n,k,1} &= \sqrt{n} \int \left[(\sum_{j=k+1}^{\infty} \psi_{cj}\phi_j)(l_0 - \Pi_{f_0,k} l_0) \right] \\
&= \sqrt{n} \sum_{j=k+1}^{\infty} \psi_{cj}\theta_{0j} \\
&= \sqrt{n} \sum_{l \geq k/2} \frac{\sin^2(2\pi x l)}{(2l+1)^{\gamma+3/2} \log(2l+1)^{1/2} \log\log(2l+1)}
\end{aligned}$$

and if $x = 1/4$ we have

$$\begin{aligned}
\mu_{n,k,1} &= \sqrt{n} \sum_{j \geq k/4 - 1/2} \frac{1}{(4j+3)^{\gamma+3/2} (\log 4j + 3)^{1/2} \log\log(4j+3)} \\
&\leq C\sqrt{n} \frac{k^{-\gamma-1/2}}{\sqrt{\log k} \log\log k} \\
\mu_{n,k,1} &\geq C'\sqrt{n} \frac{k^{-\gamma-1/2}}{\sqrt{\log k} \log\log k}.
\end{aligned}$$

Note that there exists $c > 0$ such that for all $k \leq k_n$

$$\mu_{n,k,1} \geq c\sqrt{\log n}.$$

We now consider the second term of (3.10). Let $M_{1,k}$ denote the projection on $(\phi_0, ..., \phi_k)$ with respect to the scalar product $< f, g >_2 = \int fg(u)du$ and note that

$$\Pi_{f_0,k} l_0 = M_{1,k} l_0 + \Pi_{f_0,k} [\sum_{j=k+1}^{\infty} \theta_{0j}\phi_j]$$

$$
\begin{aligned}
|\mu_{n,k,2}| &= \left|\sqrt{n}\int(f_0-1)\left[(\sum_{j=k+1}^{\infty}\psi_j\phi_j)(l_0-\Pi_{f_0,k}l_0)\right]\right| \\
&= \left|\sqrt{n}\int(f_0-1)\left[(\sum_{j=k+1}^{\infty}\psi_{cj}\phi_j)(l_0-M_{1,k}l_0)\right]\right| \\
&\quad + \left|\sqrt{n}\int(f_0-1)\left[(\sum_{j=k+1}^{\infty}\psi_j\phi_j)(M_{1,k}l_0-\Pi_{f_0,k}l_0)\right]\right| \\
&\leq 2|f_0-1|_\infty\left(\sum_{j=k+1}^{\infty}\psi_{cj}^2\right)^{1/2}\left(\sum_{j=k+1}^{\infty}\theta_{0,j}^2\right)^{1/2} \\
&\leq C\sqrt{n}|f_0-1|_\infty\frac{k^{-\gamma-1/2}}{\sqrt{\log k\log\log k}}
\end{aligned}
$$

By choosing k_0 large enough $|f_0-1|_\infty$ can be made as small as need be so that we finally obtain that there exists $c>0$ such that for all $k\leq k_n$

$$\mu_{n,k}\geq c\sqrt{\log n}.$$

Note that in case (D) with $\gamma<\beta$, the same calculations lead to

$$\mu_{n,k_n^*}\geq cn^{\frac{\beta-\gamma}{4\beta+2}}(\log n)^{-\frac{1}{2}}(\log\log n)^{-1}.$$

Thus in this case the posterior distribution is not asymptotically Gaussian with mean $F_n(x)$ and variance $F_0(x)(1-F_0(x))/n$. Whether it is asymptotically equivalent to a mixture of Gaussians is not clear. It would be a consequence of the way the posterior distribution of k concentrates as n goes to infinity. In the case (D), the posterior distribution is asymptotically Gaussian with mean $F_n(x)-\mu_{n,k_n^*}$.

4. Proofs. In this section we prove Theorems 2.1, 3.1 and 3.2. In the sequel, C denotes a generic positive constant whose value is of no importance.

4.1. *Proof of Theorem 2.1.* Let $Z_n=\sqrt{n}(\Psi(f)-\Psi(P_n))$. We have

$$\mathbb{P}^\pi\left\{A_n|X^n\right\}=1+o_{\mathbb{P}_0}(1). \tag{4.1}$$

So, it is enough to prove that conditionally on A_n and X^n, the distribution of Z_n converges to the distribution of a Gaussian variable whose variance is $F_0(\psi_c^2)$. This will be established if for any $t\in\mathbb{R}$,

$$\lim_{n\to+\infty}L_n(t)=\exp\left(\frac{t^2}{2}F_0\left[\psi_c^2\right]\right), \tag{4.2}$$

where $L_n(t)$ is the Laplace transform of Z_n conditionally on A_n and X^n:

$$\begin{aligned} L_n(t) &= \mathbb{E}^{\pi}\left[\exp(t\sqrt{n}(\Psi(f)-\Psi(P_n)))|A_n, X^n\right] \\ &= \frac{\mathbb{E}^{\pi}\left[\exp(t\sqrt{n}(\Psi(f)-\Psi(P_n)))\mathbb{1}_{A_n}(f)|X^n\right]}{\mathbb{P}^{\pi}\left\{A_n|X^n\right\}} \\ &= \frac{\int_{A_n}\exp\left(t\sqrt{n}(\Psi(f)-\Psi(P_n))+l_n(f)-l_n(f_0)\right)d\pi(f)}{\int_{A_n}\exp\left(l_n(f)-l_n(f_0)\right)d\pi(f)}. \end{aligned}$$

We set for any x,

$$B_{h,n}(x) = \int_0^1 (1-u)e^{uh(x)/\sqrt{n}}du. \tag{4.3}$$

so,

$$\exp\left(\frac{h(x)}{\sqrt{n}}\right) = 1 + \frac{h(x)}{\sqrt{n}} + \frac{h^2(x)}{n}B_{h,n}(x),$$

which implies that

$$f(x) - f_0(x) = f_0(x)\left(\frac{h(x)}{\sqrt{n}} + \frac{h^2(x)}{n}B_{h,n}(x)\right)$$

and

$$\begin{aligned} t\sqrt{n}(\Psi(f)-\Psi(P_n)) &= -tG_n(\psi_c) + t\sqrt{n}\left(\int \psi_c(x)(f(x)-f_0(x))dx\right) \\ &= -tG_n(\psi_c) + tF_0(h\psi_c) + \frac{t}{\sqrt{n}}F_0(h^2B_{h,n}\psi_c). \end{aligned}$$

Since

$$l_n(f) - l_n(f_0) = -\frac{F_0(h^2)}{2} + G_n(h) + R_n(h),$$

we have

$$\begin{aligned} L_n(t) &= \frac{\int_{A_n}\exp\left(G_n(h-t\psi_c) + tF_0(h\psi_c) + \frac{t}{\sqrt{n}}F_0(h^2B_{h,n}\psi_c) - \frac{F_0(h^2)}{2} + R_n(h)\right)d\pi(f)}{\int_{A_n}\exp\left(-\frac{F_0(h^2)}{2} + G_n(h) + R_n(h)\right)d\pi(f)} \\ &= \frac{\int_{A_n}\exp\left(-\frac{F_0((h-t\bar{\psi}_{t,n})^2)}{2} + G_n(h-t\bar{\psi}_{t,n}) + R_n(h-t\bar{\psi}_{t,n}) + U_{n,h}\right)d\pi(f)}{\int_{A_n}\exp\left(-\frac{F_0(h^2)}{2} + G_n(h) + R_n(h)\right)d\pi(f)}, \end{aligned}$$

where straightforward computations show that

$$\begin{aligned} U_{n,h} &= tF_0(h(\psi_c - \bar{\psi}_{t,n})) + \frac{t^2}{2}F_0(\bar{\psi}_{t,n}^2) + R_n(h) - R_n(h-t\bar{\psi}_{t,n}) + \frac{t}{\sqrt{n}}F_0(h^2B_{h,n}\psi_c) \\ &= tF_0(h\psi_c) + t\sqrt{n}F_0(\bar{\psi}_{t,n}) + \frac{t}{\sqrt{n}}F_0(h^2B_{h,n}\psi_c) \\ &= tF_0(h\psi_c) + n\log\left(F_0\left[\exp\left(\frac{h}{\sqrt{n}} - \frac{t\psi_c}{\sqrt{n}}\right)\right]\right) + \frac{t}{\sqrt{n}}F_0\left(h^2B_{h,n}\psi_c\right). \end{aligned}$$

Now, let us study each term of the last expression. We have

$$\begin{aligned} F_0\left[\exp\left(\frac{h}{\sqrt{n}}-\frac{t\psi_c}{\sqrt{n}}\right)\right] &= F_0\left[e^{\frac{h}{\sqrt{n}}}\left(1-\frac{t\psi_c}{\sqrt{n}}+\frac{t^2}{2n}\psi_c^2\right)\right]+0(n^{-\frac{3}{2}}) \\ &= 1-\frac{t}{\sqrt{n}}F_0\left[e^{\frac{h}{\sqrt{n}}}\psi_c\right]+\frac{t^2}{2n}F_0\left[e^{\frac{h}{\sqrt{n}}}\psi_c^2\right]+0(n^{-\frac{3}{2}}). \end{aligned}$$

So,

$$F_0\left[e^{\frac{h}{\sqrt{n}}}\psi_c\right]=\frac{F_0[h\psi_c]}{\sqrt{n}}+\frac{F_0[h^2B_{h,n}\psi_c]}{n};\quad F_0\left[e^{\frac{h}{\sqrt{n}}}\psi_c^2\right]=F_0\left[\psi_c^2\right]+\frac{F_0[h\psi_c^2]}{\sqrt{n}}+\frac{F_0[h^2B_{h,n}\psi_c^2]}{n}.$$

Note that, on A_n, we have $F_0(h^2)=0(nu_n^2)$ and $F_0\left(h^2B_{h,n}\right)=o(n)$. Therefore, uniformly on A_n,

$$\begin{aligned} F_0\left[\exp\left(\frac{h}{\sqrt{n}}-\frac{t\psi_c}{\sqrt{n}}\right)\right] &= 1-\frac{t}{\sqrt{n}}\left(\frac{F_0[h\psi_c]}{\sqrt{n}}+\frac{F_0[h^2B_{h,n}\psi_c]}{n}\right) \\ &\quad +\frac{t^2}{2n}\left(F_0\left[\psi_c^2\right]+\frac{F_0[h\psi_c^2]}{\sqrt{n}}+\frac{F_0[h^2B_{h,n}\psi_c^2]}{n}\right)+o\left(n^{-1}\right) \\ &= 1-\frac{t}{n}\left[F_0[h\psi_c]+\frac{F_0[h^2B_{h,n}\psi_c]}{\sqrt{n}}-\frac{tF_0(\psi_c^2)}{2}+o(1)\right] \\ &= 1+o\left(n^{-1/2}\right) \end{aligned}$$

and

$$n\log\left(F_0\left[\exp\left(\frac{h}{\sqrt{n}}-\frac{t\psi_c}{\sqrt{n}}\right)\right]\right) = -t\left[F_0(h\psi_c)+\frac{F_0[h^2B_{h,n}\psi_c]}{\sqrt{n}}-\frac{tF_0(\psi_c^2)}{2}\right]+o(1).$$

Finally,

$$U_{n,h}=\frac{t^2}{2}F_0\left[\psi_c^2\right]+o(1)$$

and up to a multiplicative factor equal to $1+o(1)$,

$$L_n(t)=\exp\left(\frac{t^2}{2}F_0\left[\psi_c^2\right]\right)\frac{\int_{A_n}\exp\left(-\frac{F_0((h-t\bar{\psi}_{t,n})^2)}{2}+G_n(h-t\bar{\psi}_{t,n})+R_n(h-t\bar{\psi}_{t,n})\right)d\pi(f)}{\int_{A_n}\exp\left(-\frac{F_0(h^2)}{2}+G_n(h)+R_n(h)\right)d\pi(f)}.$$

Finally (A3) implies (4.2) and the theorem is proved.

4.2. *Proof of Theorem 3.1.* We first give a preliminary lemma which will be used extensively in the sequel.

4.2.1. *Preliminary lemma.* Let us first state the following lemma.

LEMMA 4.1. *Set $K_n = \{1, 2, \dots, k_n\}$ with $k_n \in \mathbb{N}^*$. Assume either of the following two cases:*

- *$\gamma > 0$, $p = q = 2$ when Φ is the Fourier basis*
- *$0 < \gamma < r$, $2 \leq p \leq \infty$, $1 \leq q \leq \infty$ when Φ is the wavelet basis with r vanishing moments.*

Then the following results hold.

- *There exists a constant $c_{1,\Phi}$ depending only on Φ such that for any $\theta = (\theta_\lambda)_\lambda \in \mathbb{R}^{k_n}$,*

$$\left\| \sum_{\lambda \in K_n} \theta_\lambda \phi_\lambda \right\|_\infty \leq c_{1,\Phi} \sqrt{k_n} \|\theta\|_{\ell_2}. \tag{4.4}$$

- *If $\log(f_0) \in \mathcal{B}^\gamma_{p,q}(R)$, then there exists $c_{2,\gamma}$ depending on γ only such that*

$$\sum_{\lambda \notin K_n} \theta_{0\lambda}^2 \leq c_{2,\gamma}\, R^2 k_n^{-2\gamma}. \tag{4.5}$$

- *If $\log(f_0) \in \mathcal{B}^\gamma_{p,q}(R)$ with $\gamma > \frac{1}{2}$, then there exists $c_{3,\Phi,\gamma}$ depending on Φ and γ only such that:*

$$\left\| \sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda \right\|_\infty \leq c_{3,\Phi,\gamma}\, R\, k_n^{\frac{1}{2}-\gamma}. \tag{4.6}$$

PROOF. Let us first consider the Fourier basis. We have:

$$\begin{aligned} \left\| \sum_{\lambda \in K_n} \theta_\lambda \phi_\lambda \right\|_\infty &\leq \sum_{\lambda \in K_n} |\theta_\lambda| \times \|\phi_\lambda\|_\infty \\ &\leq \|\phi\|_\infty \sum_{\lambda \in K_n} |\theta_\lambda|, \end{aligned}$$

which proves (4.4). Inequality (4.5) follows from the definition of $\mathcal{B}^\gamma_{2,2} = W^\gamma$. To prove (4.6), we use the following inequality: for any x,

$$\begin{aligned} \left| \sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda(x) \right| &\leq \|\phi\|_\infty \sum_{\lambda \notin K_n} |\theta_{0\lambda}| \\ &\leq \|\phi\|_\infty \left(\sum_{\lambda \notin K_n} |\lambda|^{2\gamma} \theta_{0\lambda}^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda \notin K_n} |\lambda|^{-2\gamma} \right)^{\frac{1}{2}}. \end{aligned}$$

Now, we consider the wavelet basis. Without loss of generality, we assume that $\log_2(k_n+1) \in \mathbb{N}^*$. We have for any x,

$$\begin{aligned}\left|\sum_{\lambda\in K_n}\theta_\lambda\phi_\lambda(x)\right| &\leq \left(\sum_{\lambda\in K_n}\theta_\lambda^2\right)^{\frac{1}{2}}\left(\sum_{\lambda\in K_n}\phi_\lambda^2(x)\right)^{\frac{1}{2}} \\ &\leq \|\theta\|_{\ell_2}\left(\sum_{0\leq j\leq\log_2(k_n)}\sum_{k=0}^{2^j-1}\Psi_{jk}^2(x)\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\Psi(x)=0$ for $x\notin[-A,A]$,

$$\text{card}\left\{k\in\{0,\dots,2^j-1\}:\quad \Psi_{jk}(x)\neq 0\right\}\leq 3(2A+1).$$

(see [15], p. 282 or [16], p. 112). So, there exists c_Ψ depending only on Ψ such that

$$\left|\sum_{\lambda\in K_n}\theta_\lambda\phi_\lambda(x)\right| \leq \|\theta\|_{\ell_2}\left(\sum_{0\leq j\leq\log_2(k_n)}3(2A+1)2^jc_\Psi^2\right)^{\frac{1}{2}},$$

which proves (4.4). For the second point, we just use the inclusion $\mathcal{B}^\gamma_{p,q}(R)\subset\mathcal{B}^\gamma_{2,\infty}(R)$ and

$$\sum_{\lambda\notin K_n}\theta_{0\lambda}^2=\sum_{j>\log_2(k_n)}\sum_{k=0}^{2^j-1}\theta_{0jk}^2\leq R^2\sum_{j>\log_2(k_n)}2^{-2j\gamma}\leq\frac{R^2}{1-2^{-2\gamma}}k_n^{-2\gamma}.$$

Finally, for the last point, we have for any x:

$$\begin{aligned}\left|\sum_{\lambda\notin K_n}\theta_{0\lambda}\phi_\lambda(x)\right| &\leq \sum_{j>\log_2(k_n)}\left(\sum_{k=0}^{2^j-1}\theta_{0jk}^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{2^j-1}\Psi_{jk}^2(x)\right)^{\frac{1}{2}} \\ &\leq Ck_n^{\frac{1}{2}-\gamma},\end{aligned}$$

where $C\leq R(3(2A+1))^{\frac{1}{2}}c_\Psi(1-2^{\frac{1}{2}-\gamma})^{-1}$. □

4.2.2. *Proof of Theorem 3.1.* Denote for any n,

$$B_n(\epsilon_n)=\{f\in\mathcal{F}:\quad K(f_0,f)\leq\epsilon_n^2,\ V(f_0,f)\leq\epsilon_n^2\},$$

To prove Theorem 3.1, we use the following version of the theorem on posterior convergence rates. Its proof is not given, but it is a slight modification of Theorem 2.4 of [9].

THEOREM 4.1. *Let f_0 be the true density. We assume that there exists a constant c such that for any n, there exists $\mathcal{F}_n^* \subset \mathcal{F}$ and a prior π on $\mathcal{F}$ satisfying the following conditions:*

- *(A)*
$$\mathbb{P}^\pi\{\mathcal{F}_n^{*c}\} = o(e^{-(c+2)n\epsilon_n^2}).$$
- *(B) For any $j \in \mathbb{N}^*$, let*
$$S_{n,j} = \{f \in \mathcal{F}_n^* : \quad j\epsilon_n < h(f_0, f) \leq (j+1)\epsilon_n\},$$
and $H_{n,j}$ the Hellinger metric entropy of $S_{n,j}$. There exists $J_{0,n}$ (that may depend on n) such that for all $j \geq J_{0,n}$,
$$H_{n,j} \leq (K-1)nj^2\epsilon_n^2,$$
where K is an absolute constant.
- *(C) Let*
$$B_n(\epsilon_n) = \{f \in \mathcal{F} : \quad K(f_0, f) \leq \epsilon_n^2,\ V(f_0, f) \leq \epsilon_n^2\}.$$
Then,
$$\mathbb{P}^\pi\{B_n(\epsilon_n)\} \geq e^{-cn\epsilon_n^2}.$$

We have:
$$\mathbb{P}^\pi\{f : \quad h(f_0, f) \leq J_{0,n}\epsilon_n | X^n\} = 1 + o_P(1)$$

To prove Theorem 3.1 it is thus enough to prove that conditions (A), (B) and (C) of the previous result are satisfied. We consider $(\Lambda_n)_n$ the increasing sequence of subsets of $\mathbb{N}^*$ defined by $\Lambda_n = \{1, 2, \ldots, l_n\}$ with $l_n \in \mathbb{N}^*$. For any n, we set:

$$\mathcal{F}_n^* = \left\{ f_\theta \in \mathcal{F}_{l_n} : \quad f_\theta = \exp\left(\sum_{\lambda \in \Lambda_n} \theta_\lambda \phi_\lambda - c(\theta) \right),\ \|\theta\|_{\ell_2} \leq w_n \right\},$$

with

$$w_n = \exp(w_0 n^\rho (\log n)^q), \quad \rho > 0$$

Recall that

- $\epsilon_n = \epsilon_0 n^{-\frac{\gamma}{2\gamma+1}} (\log n)^{\frac{\gamma}{2\gamma+1}}$ in case (PH)
- $\epsilon_n = \epsilon_0 n^{-\frac{\beta}{2\beta+1}}$ in case (D).

Define l_n by

$$l_n = \frac{l_0 n\epsilon_n^2}{L(n)}, \tag{4.7}$$

where l_0 is some positive constant. When $\gamma, \beta > \frac{1}{2}$, we have

$$l_n\epsilon_n^2 \to 0. \tag{4.8}$$

Proof of condition (A):. We have, since $\sum_k \tau_k < \infty$

$$\begin{aligned}
\pi\left\{\mathcal{F}_n^{*c}\right\} &\leq \sum_{k>l_n} p(k) + \mathbb{P}^\pi\left\{\sum_{k\leq l_n}\theta_k^2 > w_n^2\right\} \\
&\leq C\exp\left(-l_n L(l_n)\right) + \sum_{k\leq l_n}\mathbb{P}^\pi\left\{\frac{\theta_k^2}{\tau_k} > w_n^2\right\} \\
&\leq C\exp\left(-l_0 n\epsilon_n^2\right) + \sum_{k\leq l_n}\mathbb{P}^\pi\left\{\exp\left(\frac{|\theta_k|^p}{2\tau_k^{p/2}}\right) > \exp\left(\frac{w_n^p}{2}\right)\right\} \\
&\leq C\exp\left(-l_0 n\epsilon_n^2\right) + Cl_n\exp\left(-\frac{w_n^p}{2}\right) \\
&\leq C\exp\left(-l_0 n\epsilon_n^2\right) + C\exp\left(-n^H\right)
\end{aligned}$$

for any positive $H > 0$. Hence,

$$\pi\left\{\mathcal{F}_n^{*c}\right\} \leq C\exp\left(-(l_0-1)n\epsilon_n^2\right)$$

and Condition (A) is proved.

Proof of condition (B):. We apply Lemma 4.1 with $K_n = \Lambda_n$ and $k_n = l_n$. For this purpose, we show that the Hellinger distance between two functions of $\mathcal{F}_n^*$ is related to the ℓ_2-distance of the associated coefficients. So, let us consider f_θ and $f_{\theta'}$ belonging to $\mathcal{F}_n^*$ with

$$f_\theta = \exp\left(\sum_{\lambda\in\Lambda_n}\theta_\lambda\phi_\lambda - c(\theta)\right), \quad f_{\theta'} = \exp\left(\sum_{\lambda\in\Lambda_n}\theta'_\lambda\phi_\lambda - c(\theta')\right).$$

Let us assume that $\|\theta'-\theta\|_{\ell_1} \leq c_1\epsilon_n l_n^{-1/2}$ with c_1 a positive constant, then using (4.4) and (4.8),

$$\left\|\sum_{\lambda\in\Lambda_n}(\theta'_\lambda-\theta_\lambda)\phi_\lambda\right\|_\infty \leq C\sqrt{l_n}\|\theta'-\theta\|_{\ell_2} \leq C\sqrt{l_n}\|\theta'-\theta\|_{\ell_1} \leq Cc_1\epsilon_n \to 0$$

and

$$\begin{aligned}
|c(\theta)-c(\theta')| &= \left|\log\left(\int_0^1 f_\theta(x)\exp\left(\sum_{\lambda\in\Lambda_n}(\theta'_\lambda-\theta_\lambda)\phi_\lambda(x)\right)\right)\right| \\
&\leq \left|\log\left(1+C\|\sum_{\lambda\in\Lambda_n}(\theta'_\lambda-\theta_\lambda)\phi_\lambda\|_\infty\right)\right| \\
&\leq C\|\sum_{\lambda\in\Lambda_n}(\theta'_\lambda-\theta_\lambda)\phi_\lambda\|_\infty.
\end{aligned}$$

Then,

$$\begin{aligned} h^2(f_\theta, f_{\theta'}) &= \int f_\theta(x) \left(\exp\left(\frac{1}{2} \sum_{\lambda \in \Lambda_n} (\theta'_\lambda - \theta_\lambda)\phi_\lambda(x) + \frac{1}{2}\left(c(\theta) - c(\theta')\right) \right) - 1 \right)^2 dx \\ &\leq \int_0^1 f_\theta(x) \left(\exp\left(C \| \sum_{\lambda \in \Lambda_n} (\theta'_\lambda - \theta_\lambda)\phi_\lambda \|_\infty \right) - 1 \right)^2 dx \\ &\leq C \| \sum_{\lambda \in \Lambda_n} (\theta_\lambda - \theta'_\lambda)\phi_\lambda \|_\infty^2 \\ (4.9) &\leq C l_n \|\theta - \theta'\|_{\ell_1}^2 \leq C l_n^2 \|\theta - \theta'\|_{\ell_2}^2 \end{aligned}$$

The next lemma establishes a converse inequality.

LEMMA 4.2. *There exists a constant $c \leq 1/2$ depending on γ, β,R and Φ such that if*

$$(j+1)^2 \epsilon_n^2 l_n \leq c \times \min\left(c_0, (1 - e^{-1})^2\right)$$

then for $f_\theta \in S_{n,j}$,

$$\|\theta_0 - \theta\|_{\ell_2}^2 \leq \frac{1}{c_0 c} (\log n)^2 h^2(f_0, f_\theta).$$

PROOF. Using Theorem 5 of [23], with $M_1 = \left(\int_0^1 \frac{f_0^2(x)}{f_\theta(x)} dx \right)^{\frac{1}{2}}$, if

$$h^2(f_0, f_\theta) \leq \frac{1}{2}(1 - e^{-1})^2,$$

we have

$$V(f_0, f_\theta) \leq 5 h^2(f_0, f_\theta) \left(|\log M_1| - \log(h(f_0, f_\theta)) \right)^2. \tag{4.10}$$

But

$$\begin{aligned} M_1 &= \int_0^1 f_0(x) \exp\left(\sum_{\lambda \in \Lambda_n} (\theta_{0\lambda} - \theta_\lambda)\phi_\lambda(x) + \sum_{\lambda \notin \Lambda_n} \theta_{0\lambda}\phi_\lambda(x) - c(\theta_0) + c(\theta) \right) dx \\ &\leq \int_0^1 f_0(x) \exp\left(C[\sqrt{l_n}\|\theta_0 - \theta\|_{\ell_2} + R \ell_n^{\frac{1}{2} - \gamma}] - c(\theta_0) + c(\theta) \right) dx, \end{aligned}$$

by using (4.4) and (4.6). Furthermore,

$$|c(\theta_0) - c(\theta)| \leq C[\sqrt{l_n}\|\theta_0 - \theta\|_{\ell_2} + R\, l_n^{\frac{1}{2} - \gamma}]. \tag{4.11}$$

So,

$$|\log M_1| \leq C[\sqrt{l_n}\|\theta_0 - \theta\|_{\ell_2} + R\, l_n^{\frac{1}{2}-\gamma}].$$

Finally, since $f_\theta \in S_{n,j}$ for $j \geq 1$,

$$\begin{aligned} V(f_0, f_\theta) &\leq 5h^2(f_0, f_\theta)\left(C[\sqrt{l_n}\|\theta_0 - \theta\|_{\ell_2} + R\, l_n^{\frac{1}{2}-\gamma}] - \log(\epsilon_n)\right)^2 \\ &\leq Ch^2(f_0, f_\theta)\left(l_n\|\theta_0 - \theta\|_{\ell_2}^2 + (\log n)^2\right). \end{aligned}$$

Since $f_0(x) \geq c_0$ for any x and $\int_0^1 \phi_\lambda(x)dx = 0$ for any $\lambda \in \Lambda$, we have

$$V(f_0, f_\theta) \geq c_0\|\theta_0 - \theta\|_{\ell_2}^2. \tag{4.12}$$

Combining (4.9) and (4.12), we conclude that

$$\|\theta_0 - \theta\|_{\ell_2}^2 \leq C(\log n)^2 h^2(f_0, f_\theta),$$

if $h^2(f_0, f_\theta)l_n \leq (j+1)^2\epsilon_n^2 l_n \leq 1/(2C)$. Lemma 4.2 is proved by taking $c = (\max(C, 1))^{-1}/2$. □

Now, under assumptions of Lemma 4.2, using (4.9), we obtain

$$H_{n,j} \leq \log\left((Cl_n(j+1)\log n)^{l_n}\right) \leq l_n \log\left(C\epsilon_n^{-1}\sqrt{l_n}\log n\right).$$

Then, since $l_n L(n) = l_0 n\epsilon_n^2$, we have

$$H_{n,j} \leq (K-1)nj^2\epsilon_n^2$$

as soon as

$$J_{0,n}^2 \geq \frac{j_0 \log n}{L(n)},$$

where j_0 is a constant and condition (B) is satisfied for such j's. Now, let j be such that

$$c(j+1)^2\epsilon_n^2 l_n > \min\left(\frac{c_0}{2}, \frac{1}{2}(1-e^{-1})^2\right). \tag{4.13}$$

In this case, since for $f_\theta \in \mathcal{F}_n^*$,

$$\|\theta\|_{\ell_1} \leq \sqrt{l_n}\|\theta\|_{\ell_2} \leq \sqrt{l_n}w_n,$$

for n large enough,

$$H_{n,j} \leq \log\left(\left(Cl_n w_n \epsilon_n^{-1}\right)^{l_n}\right) \leq 2l_n \log(w_n) \leq 2w_0 l_n n^\rho (\log n)^q.$$

Then, using (4.13), condition (B) is satisfied if w_0 and q are small enough and if

$$l_n^2(\log n)^q \leq n^{1-\rho},$$

which is true for n large enough, since $\gamma, \beta > \frac{1}{2}$, for ρ small enough.

Proof of condition (C). Let $k_n \in \mathbb{N}$, going to ∞ and $K_n = \{1, ..., k_n\}$, we assume that θ belongs to $A(u_n)$ where

(4.14)
$$A(u_n) = \left\{ \theta : \quad \theta_\lambda = 0 \text{ for every } \lambda \notin K_n \text{ and } \sum_{\lambda \in K_n} (\theta_{0\lambda} - \theta_\lambda)^2 \leq u_n^2 \right\},$$

where u_n goes to 0 such that

$$\sqrt{k_n} u_n \to 0. \tag{4.15}$$

We define for any $\lambda \in \Lambda$,

$$\beta_\lambda(f_0) = \int_0^1 \phi_\lambda(x) f_0(x) dx.$$

Let us introduce the following notations:

$$f_{0K_n} = \exp\left(\sum_{\lambda \in K_n} \theta_{0\lambda} \phi_\lambda(x) - c(\theta_{0K_n}) \right), \quad f_{0\bar{K_n}} = \exp\left(\sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda(x) - c(\theta_{0\bar{K_n}}) \right).$$

We have

$$\begin{aligned} K(f_0, f_{0K_n}) &= \sum_{\lambda \notin K_n} \theta_{0\lambda} \beta_\lambda(f_0) + c(\theta_{0K_n}) - c(\theta_0) \\ &= \sum_{\lambda \notin K_n} \theta_{0\lambda} \beta_\lambda(f_0) + \log\left(\int_0^1 f_0(x) e^{-\sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda(x)} dx \right). \end{aligned}$$

Using inequality (4.6) of Lemma 4.1 and a Taylor expansion of the function e^x we obtain

$$\begin{aligned} &\int_0^1 f_0(x) e^{-\sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda(x)} dx \\ = \; & 1 - \sum_{\lambda \notin K_n} \theta_{0\lambda} \beta_\lambda(f_0) + \frac{1}{2} \int_0^1 f_0(x) \left(\sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda(x) \right)^2 dx \times (1 + o(1)). \end{aligned}$$

We have

$$\left| \sum_{\lambda \notin K_n} \theta_{0\lambda} \beta_\lambda(f_0) \right| \leq \|f_0\|_2 \left(\sum_{\lambda \notin K_n} \theta_{0\lambda}^2 \right)^{\frac{1}{2}}$$

and

$$\int_0^1 f_0(x) \left(\sum_{\lambda \notin K_n} \theta_{0\lambda} \phi_\lambda(x) \right)^2 dx \leq \|f_0\|_\infty \sum_{\lambda \notin K_n} \theta_{0\lambda}^2$$

So,

$$\log\left(\int_0^1 f_0(x)e^{-\sum_{\lambda\notin K_n}\theta_{0\lambda}\phi_\lambda(x)}dx\right) = -\sum_{\lambda\notin K_n}\theta_{0\lambda}\beta_\lambda(f_0) - \frac{1}{2}\left(\sum_{\lambda\notin K_n}\theta_{0\lambda}\beta_\lambda(f_0)\right)^2 + \frac{1}{2}\int_0^1 f_0(x)\left(\sum_{\lambda\notin K_n}\theta_{0\lambda}\phi_\lambda(x)\right)^2 dx + o\left(\sum_{\lambda\notin K_n}\theta_{0\lambda}^2\right).$$

So, finally,

$$K(f_0, f_{0K_n}) = \frac{1}{2}\int_0^1 f_0(x)\left(\sum_{\lambda\notin K_n}\theta_{0\lambda}\phi_\lambda(x)\right)^2 dx - \frac{1}{2}\left(\sum_{\lambda\notin K_n}\theta_{0\lambda}\beta_\lambda(f_0)\right)^2 + o\left(\sum_{\lambda\notin K_n}\theta_{0\lambda}^2\right)$$

This implies that for n large enough,

$$K(f_0, f_{0K_n}) \le \|f_0\|_\infty \sum_{\lambda\notin K_n}\theta_{0\lambda}^2 \le Dk_n^{-2\gamma}.$$

Now, if $f_\theta \in \mathcal{F}_{k_n}$ with $f_\theta = \exp\left(\sum_{\lambda\in K_n}\theta_\lambda\phi_\lambda - c(\theta)\right)$, we have

$$\begin{aligned} K(f_0, f_\theta) &= K(f_0, f_{0K_n}) + \sum_{\lambda\in K_n}(\theta_{0\lambda} - \theta_\lambda)\beta_\lambda(f_0) - c(\theta_{0K_n}) + c(\theta) \\ &\le Dk_n^{-2\gamma} + \sum_{\lambda\in K_n}(\theta_{0\lambda} - \theta_\lambda)\beta_\lambda(f_0) - c(\theta_{0K_n}) + c(\theta). \end{aligned}$$

We set for any x,

$$T(x) = \sum_{\lambda\in K_n}(\theta_\lambda - \theta_{0\lambda})\phi_\lambda(x).$$

Using (4.4),

$$\|T\|_\infty \le C\sqrt{k_n}u_n \to 0.$$

So,

$$\int_0^1 f_{0K_n}(x)\exp(T(x))dx = 1 + \int_0^1 f_{0K_n}(x)T(x)dx + \int_0^1 f_{0K_n}(x)T^2(x)v(n,x)dx,$$

where v is a bounded function. Since $\log(1+u) \le u$ for any $u > -1$, for $\theta \in A(u_n)$ and n large enough,

$$\begin{aligned} |-c(\theta_{0K_n}) + c(\theta)| &= \left|\log\left(\int_0^1 f_{0K_n}(x)e^{T(x)}dx\right)\right| \\ &\le \int_0^1 f_{0K_n}(x)T(x)dx + \int_0^1 f_{0K_n}(x)T^2(x)v(n,x)dx \\ &\le \sum_{\lambda\in K_n}(\theta_\lambda - \theta_{0\lambda})\beta_\lambda(f_{0K_n}) + Dk_nu_n^2. \end{aligned}$$

So,

$$\begin{aligned} K(f_0, f_\theta) &\leq Dk_n^{-2\gamma} + \sum_{\lambda \in K_n} (\theta_{0\lambda} - \theta_\lambda)\left(\beta_\lambda(f_0) - \beta_\lambda(f_{0K_n})\right) \\ &\leq Dk_n^{-2\gamma} + u_n \|f_0 - f_{0K_n}\|_2 \end{aligned}$$

Using (4.6), we have

$$\|f_0 - f_{0K_n}\|_2^2 \leq \|f_0\|_\infty^2 \int_0^1 \left(1 - \exp\left(-\sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda(x) - c(\theta_{0K_n}) + c(\theta_0)\right)\right)^2 dx.$$

and

$$|c(\theta_{0K_n}) - c(\theta_0)| \leq \| \sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda \|_\infty.$$

Finally,

$$\|f_0 - f_{0K_n}\|_2 \leq D\| \sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda \|_\infty \leq Dk_n^{\frac{1}{2}-\gamma}.$$

and

$$K(f_0, f_\theta) \leq Dk_n^{-2\gamma} + Du_n k_n^{\frac{1}{2}-\gamma}. \tag{4.16}$$

We now bound $V(f_0, f_\theta)$. For this purpose, we refine the control of $|c(\theta_{0K_n}) - c(\theta_0)|$:

$$\begin{aligned} |c(\theta_{0K_n}) - c(\theta_0)| &= \left|\log\left(\int_0^1 f_0(x) \exp\left(-\sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda(x)\right) dx\right)\right| \\ &= \left|\log \int_0^1 f_0(x)\left(1 - \sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda(x) + w(n,x)\left(\sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda(x)\right)^2\right) dx\right|, \end{aligned}$$

where w is a bounded function. So,

$$\begin{aligned} |c(\theta_{0K_n}) - c(\theta_0)| &\leq D\left(\sum_{\lambda \notin K_n} |\theta_{0\lambda}\beta_\lambda(f_0)| + \int_0^1 \left(\sum_{\lambda \notin K_n} \theta_{0\lambda}\phi_\lambda(x)\right)^2 dx\right) \\ &\leq D\left(\sum_{\lambda \notin K_n} \theta_{0\lambda}^2\right)^{\frac{1}{2}} \leq Dk_n^{-\gamma}. \end{aligned}$$

In addition,

$$\begin{aligned} |c(\theta_{0K_n}) - c(\theta)| &\leq \sum_{\lambda \in K_n} |\theta_\lambda - \theta_{0\lambda}| \, |\beta_\lambda(f_{0K_n})| + Dk_n u_n^2 \\ &\leq u_n \left(\|f_0 - f_{0K_n}\|_2 + \|f_0\|_2 \right) + Dk_n u_n^2 \\ &\leq Du_n + Dk_n u_n^2 \end{aligned}$$

Finally,

$$V(f_0, f_\theta) \leq u_n^2 + Dk_n^{-2\gamma} + Dk_n u_n^2. \tag{4.17}$$

Now, let us consider the case (PH). We take k_n and u_n such that

$$k_n^{-2\gamma} \leq k_0 \epsilon_n^2 \quad \text{and } u_n = u_0 \epsilon_n k_n^{-\frac{1}{2}}, \tag{4.18}$$

where k_0 and u_0 are constants depending on $\|f_0\|_\infty$, γ, R and Φ. If k_0 and u_0 are small enough, then, by using (4.16) and (4.17),

$$K(f_0, f_\theta) \leq \epsilon_n^2 \quad \text{and} \quad V(f_0, f_\theta) \leq \epsilon_n^2.$$

So, Condition (C) is satisfied if

$$\mathbb{P}^\pi \left\{ A(u_n) \right\} \geq e^{-cn\epsilon_n^2},$$

where, $A(u_n)$ is defined in (4.14). We have:

$$\mathbb{P}^\pi \left\{ A(u_n) \right\} \geq \mathbb{P}^\pi \left\{ \theta : \sum_{\lambda \in K_n} (\theta_\lambda - \theta_{0\lambda})^2 \leq u_n^2 \right\} \times \exp\left(-c_1 k_n L(k_n) \right\}$$

The prior on θ implies that

$$\begin{aligned} P_1 &= \mathbb{P}^\pi \left\{ \theta : \sum_{\lambda \in K_n} (\theta_\lambda - \theta_{0\lambda})^2 \leq u_n^2 \right\} \\ &\geq \mathbb{P}^\pi \left\{ \theta : \sum_{\lambda \in K_n} \left| \sqrt{\tau_0} \lambda^{-\beta} G_\lambda - \theta_{0\lambda} \right| \leq u_n \right\} \\ &= \mathbb{P}^\pi \left\{ \theta : \sum_{\lambda \in K_n} \lambda^{-\beta} \left| G_\lambda - \tau_0^{-\frac{1}{2}} \lambda^\beta \theta_{0\lambda} \right| \leq \tau_0^{-\frac{1}{2}} u_n \right\} \\ &= \int \dots \int 1_{\left\{ \sum_{\lambda \in K_n} \lambda^{-\beta} \left| x_\lambda - \tau_0^{-\frac{1}{2}} \lambda^\beta \theta_{0\lambda} \right| \leq \tau_0^{-\frac{1}{2}} u_n \right\}} \prod_{\lambda \in K_n} g(x_\lambda) dx_\lambda \\ &\geq \int \dots \int 1_{\left\{ \sum_{\lambda \in K_n} \lambda^{-\beta} |y_\lambda| \leq \tau_0^{-\frac{1}{2}} u_n \right\}} \prod_{\lambda \in K_n} g\left(y_\lambda + \tau_0^{-\frac{1}{2}} \lambda^\beta \theta_{0\lambda} \right) dy_\lambda. \end{aligned}$$

Using (4.5), when $\gamma \geq \beta$, we have $\sup_{\lambda \in K_n} \left| \tau_0^{-\frac{1}{2}} \lambda^{\beta} \theta_{0\lambda} \right| < \infty$ and since

$$\sup_n \left\{ \tau_0^{-\frac{1}{2}} k_n^{\beta} u_n \right\} < \infty \tag{4.19}$$

using assumptions on the prior, there exists a constant D_3 such that

$$\begin{aligned} P_1 &\geq D_3^{k_n} \int ... \int 1_{\left\{ \sum_{\lambda \in K_n} \lambda^{-\beta} |y_\lambda| \leq \tau_0^{-\frac{1}{2}} u_n \right\}} \prod_{\lambda \in K_n} dy_\lambda \\ &\geq D_3^{k_n} \int ... \int 1_{\left\{ \sum_{\lambda \in K_n} |y_\lambda| \leq \tau_0^{-\frac{1}{2}} u_n \right\}} \prod_{\lambda \in K_n} dy_\lambda \\ &\geq \exp\left(-D_4 k_n \log n\right), \end{aligned} \tag{4.20}$$

where D_4 is a constant. When $\gamma < \beta$, since there exists $a, b > 0$ such that $\forall |y| \leq M$ for some positive M

$$g(y+u) \geq a \exp(-b|u|^{p^*})$$

using the above calculations we obtain if $p^* \leq 2$

$$\begin{aligned} P_1 &\geq D_3^{k_n} \exp\{-C \sum_{\lambda \in K_n} \lambda^{p^*\beta} |\theta_{0\lambda}|^{p^*}\} \sum \int ... \int 1_{\left\{ \sum_{\lambda \in K_n} \lambda^{-\beta} |y_\lambda| \leq \tau_0^{-\frac{1}{2}} u_n \right\}} \prod_{\lambda \in K_n} dy_\lambda \\ &\geq \exp\left[-C k_n^{1-p^*/2+\beta-\gamma}\right] \exp\left(-D_4 k_n \log n\right) \\ &\geq \exp\left(-(D_4+1) k_n \log n\right) \quad \text{if } \beta \leq 1/2 + p^*/2 \end{aligned}$$

and if $t > 2$

$$\begin{aligned} P_1 &\geq D_3^{k_n} \exp\{-C \sum_{\lambda \in K_n} \lambda^{p^*\beta} |\theta_{0\lambda}|^{p^*}\} \exp\left(-D_4 k_n \log n\right) \\ &\geq \exp\left(-(D_4+1) k_n \log n\right) \quad \text{if } \beta \leq 1/2 + 1/p^* \end{aligned}$$

So, Condition (C) is established as soon as $D_4 k_n \log n \leq c n \epsilon_n^2$. Using (4.18), this can be satisfied if and only if we take k_n such that

$$k_0^{-\frac{1}{2\gamma}} \epsilon_n^{-\frac{1}{\gamma}} \leq k_n \leq \frac{c n \epsilon_n^2}{D_4 \log n}, \tag{4.21}$$

which is possible if and only if ϵ_0 is large enough. In particular, this implies that

$$\sup_n \left\{ \epsilon_n \left(\frac{\log n}{n} \right)^{-\frac{\gamma}{2\gamma+1}} \right\} < \infty.$$

Note that when k_n satisfies (4.21), Conditions (4.15) and (4.19) are satisfied as well.

Similar computations show the result for the case (D).

4.3. *Proof of Theorem 3.2.* Our goal is to prove conditions (A1), (A2) and (A3) of Section 2.2 to apply Theorem 2.1. Let ϵ_n be the posterior concentration rate as obtained in Theorem 3.1.

Let us consider $f = f_\theta \in \mathcal{F}_k$ for $1 \leq k \leq l_n$, where $l_n = l_0 n\epsilon_n^2/L(n)$ in the case of type (PH) priors and $l_n = k_n^*$ in the case of type (D) priors. First, using the same upper bound as in the proof of Lemma 4.2 we have

$$V(f_0, f) \leq 2C(\log n)^2\epsilon_n^2, \tag{4.22}$$

as soon as $h(f_0, f) \leq \epsilon_n$. Thus, using (3.4), we have

$$\mathbb{P}^\pi \left\{ A_{u_n}^1 | X^n \right\} = 1 + o_{\mathbb{P}_0}(1)$$

with $u_n = u_0(\log n)^2\epsilon_n^2$, for a constant u_0 large enough. Note that we can restrict ourselves to $A_{u_n}^1 \cap (\cup_{k\leq l_n}\mathcal{F}_k)$, since $\mathbb{P}^\pi\left[(\cup_{k\leq l_n}\mathcal{F}_k)^c\right] \leq e^{-cn\epsilon_n^2}$ for any $c > 0$ by choosing l_0 large enough, see the proof of Theorem 3.1.

To establish (A2), we observe that

$$\begin{aligned}
\|\log f_\theta - \log f_0\|_\infty &\leq \|\sum_{\lambda\in\mathbb{N}^*}(\theta_{0\lambda} - \theta_\lambda)\phi_\lambda\|_\infty + |c(\theta) - c(\theta_0)| \\
&\leq C\left(\sqrt{l_n}\|\theta - \theta_0\|_{\ell_2} + l_n^{\frac{1}{2}-\gamma}\right) = 0(1),
\end{aligned}$$

by using Lemma 4.1 and (4.11). So, (A2) is implied by (A1). Now, let us establish (A3). Denote A_n the set defined in assumption (A2) and restricted to $(\cup_{k\leq l_n}\mathcal{F}_k)$. For any t, we study the term

$$\begin{aligned}
I_n &= \frac{\int_{A_n}\exp\left(-\frac{F_0((h_f - t\bar{\psi}_{t,n})^2)}{2} + G_n(h_f - t\bar{\psi}_{t,n}) + R_n(h_f - t\bar{\psi}_{t,n})\right)d\pi(f)}{\int_{A_n}\exp\left(-\frac{F_0(h_f^2)}{2} + G_n(h_f) + R_n(h_f)\right)d\pi(f)} \\
&= \frac{\sum_{1\leq k\leq l_n} p(k)\int_{A_n\cap\mathcal{F}_k}\exp\left(-\frac{F_0((h_f - t\bar{\psi}_{t,n})^2)}{2} + G_n(h_f - t\bar{\psi}_{t,n}) + R_n(h_f - t\bar{\psi}_{t,n})\right)d\pi_k(f)}{\sum_{1\leq k\leq l_n} p(k)\int_{A_n\cap\mathcal{F}_k}\exp\left(-\frac{F_0(h_f^2)}{2} + G_n(h_f) + R_n(h_f)\right)d\pi_k(f)}.
\end{aligned}$$

If we set

$$b_{n,k,t} = \frac{t\Pi_{f_0,k}\psi_c - t\psi_{\Pi,c,0}}{\sqrt{n}} = \frac{t}{\sqrt{n}}\sum_{\lambda=1}^{k}\psi_{\Pi,c,\lambda}\phi_\lambda,$$

we have using (4.4) and since $k \leq l_n$:

$$\begin{aligned}
\|b_{n,k,t}\|_\infty &\leq \frac{t\sqrt{k}}{\sqrt{c_0}\sqrt{n}}\|\Pi_{f_0,k}\psi_c - \psi_{\Pi,c,0}\|_{f_0} \\
&\leq \frac{2t\sqrt{l_n}}{\sqrt{c_0}\sqrt{n}}\|\psi_c\|_\infty = O(\epsilon_n).
\end{aligned}$$

for c_0 a constant. Recall that for $f_\theta \in \mathcal{F}_k$,

$$h_\theta = \sqrt{n}\left(\sum_{\lambda\in\mathbb{N}^*}(\theta_\lambda - \theta_{0\lambda})\phi_\lambda - c(\theta) + c(\theta_0)\right) \quad \text{and} \quad B_{n,k} = \frac{\psi_{\Pi,c,[k]}}{\sqrt{n}}$$

so, for $\theta' = \theta - tB_{n,k}$, with $H_n = (h_\theta - t\psi_c)/\sqrt{n}$ and $\Delta_\psi = \psi_c - \Pi_{f_0,k}\psi_c$

$$\begin{aligned} h_{\theta'} &= h_\theta - \sqrt{n}b_{n,k,t} + \sqrt{n}(c(\theta) - c(\theta - tB_{n,k})) \\ &= h_\theta - t\bar{\psi}_{t,n} + t(\psi_c - \Pi_{f_0,k}\psi_c) - \sqrt{n}\log\left[\frac{F_0(e^{H_n + t\Delta_\psi/\sqrt{n}})}{F_0(e^{H_n})}\right] \\ &= h_\theta - t\bar{\psi}_{t,n} + t\Delta_\psi - \Delta_n, \end{aligned}$$

with

$$\Delta_n = \sqrt{n}\log\left[\frac{F_0(e^{H_n + t\Delta_\psi/\sqrt{n}})}{F_0(e^{H_n})}\right].$$

Now, (4.11) implies $\|h_\theta\|_\infty/\sqrt{n} \leq \sqrt{k}\epsilon_n = o(1)$ and since $F(\Delta_\psi^2) = O(1)$, $\|\Delta_\psi\|_\infty = O(\sqrt{l_n}) = O(\sqrt{n}\epsilon_n)$,

$$\begin{aligned} F_0(e^{H_n + t\Delta_\psi/\sqrt{n}}) &= F_0\left(e^{H_n}\left(1 + \frac{t\Delta_\psi}{\sqrt{n}} + \frac{t^2\Delta_\psi^2}{2n}\right)\right) + 0\left(F(\Delta_\psi^2)\frac{\|\Delta_\psi\|_\infty}{n^{3/2}}\right) \\ &= F_0\left(e^{H_n}\left(1 + \frac{t\Delta_\psi}{\sqrt{n}} + \frac{t^2\Delta_\psi^2}{2n}\right)\right) + 0\left(\frac{\epsilon_n}{n}\right) \\ &= F_0\left(e^{H_n}\right) + \frac{t}{\sqrt{n}}F_0(e^{H_n}\Delta_\psi) + \frac{t^2}{2n}F_0(e^{H_n}\Delta_\psi^2) + o\left(\frac{1}{n}\right), \end{aligned}$$

Also, for any function v satisfying $F_0(|v|) < \infty$

$$F_0(e^{H_n}v) = F_0\left(ve^{h_\theta/\sqrt{n}}\right) - \frac{t}{\sqrt{n}}F_0\left(ve^{h_\theta/\sqrt{n}}\psi_c\right) + O\left(\frac{1}{n}\right). \tag{4.23}$$

Note that in the case $v = 1$ since $F_0(e^{h_\theta/\sqrt{n}}) = 1$ we can be more precise and obtain

$$\begin{aligned} F_0(e^{H_n}) &= 1 - \frac{t}{\sqrt{n}}F_0\left(e^{h_\theta/\sqrt{n}}\psi_c\right) + O(1/n) \\ &= 1 - \frac{tF_0(h_\theta\psi_c)}{n} + O\left(\frac{\epsilon_n^2}{\sqrt{n}} + \frac{1}{n}\right) = 1 + o\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{4.24}$$

Moreover

$$F_0\left(ve^{h_\theta/\sqrt{n}}\right) = F_0(v) + o(F_0(|v|)). \tag{4.25}$$

Therefore using (4.23) with $v = \Delta_\psi^2$ leads to

$$\frac{F_0(e^{H_n+t\Delta_\psi/\sqrt{n}})}{F_0(e^{H_n})} = 1 + \frac{t}{\sqrt{n}}\frac{F_0(e^{H_n}\Delta_\psi)}{F_0(e^{H_n})} + \frac{t^2}{2n}F_0(\Delta_\psi^2) + o\left(\frac{1}{n}\right),$$

and using (4.23) with $v = \Delta_\psi$ together with (4.24) and using (4.25)

$$\frac{t}{\sqrt{n}}F_0(e^{H_n}\Delta_\psi) = \frac{t}{\sqrt{n}}F_0\left(\Delta_\psi e^{h_\theta/\sqrt{n}}\right) - \frac{t^2}{n}F_0\left(\Delta_\psi\psi_c\right) + o\left(\frac{1}{n}\right).$$

Also

$$F_0\left(e^{h_\theta/\sqrt{n}}\Delta_\psi\right) = \frac{1}{\sqrt{n}}\left[F_0\left(h_\theta\Delta_\psi\right) + \frac{1}{\sqrt{n}}F_0\left(h_\theta^2 B_{h_\theta,n}\Delta_\psi\right)\right],$$

where $B_{h,n}$ is defined by (4.3). Since $F_0(e^{h_\theta/\sqrt{n}}\psi_c) = F_0(\psi_c) + o(1) = o(1)$, we thus obtain using the fact that $F_0(e^{H_n}) = 1 + o\left(\frac{1}{\sqrt{n}}\right)$ and $F_0(\psi_c\Delta_\psi) = F_0(\Delta_\psi^2)$

$$\frac{F_0(e^{H_n+t\Delta_\psi/\sqrt{n}})}{F_0(e^{H_n})} = 1 + \frac{t}{\sqrt{n}}F_0\left(e^{h_\theta/\sqrt{n}}\Delta_\psi\right) - \frac{t^2}{2n}F_0\left(\Delta_\psi^2\right) + o\left(\frac{1}{n}\right)$$

and finally,

$$\begin{aligned}
\Delta_n &= \sqrt{n}\log\left[\frac{F_0(e^{H_n+t\Delta_\psi/\sqrt{n}})}{F_0(e^{H_n})}\right] \\
&= tF_0\left(e^{h_\theta/\sqrt{n}}\Delta_\psi\right) - \frac{t^2}{2\sqrt{n}}F_0\left(\Delta_\psi^2\right) + o\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{t}{\sqrt{n}}\left[F_0\left(h_\theta\Delta_\psi\right) + \frac{F_0\left(h_\theta^2 B_{h_\theta,n}\Delta_\psi\right)}{\sqrt{n}} - \frac{t}{2}F_0(\Delta_\psi^2)\right] + o(n^{-1/2}). \qquad (4.26)
\end{aligned}$$

Moreover

$$F_0\left(h_\theta^2 B_{h_\theta,n}\Delta_\psi\right) = \frac{1}{2}F_0\left(h_\theta^2\Delta_\psi\right) + o\left(F_0\left(h_\theta^2|\Delta_\psi|\right)\right)$$

and by using (4.22),

$$\begin{aligned}
\frac{F_0\left(h_\theta^2|\Delta_\psi|\right)}{\sqrt{n}} &\leq \|\Delta_\psi\|_\infty\frac{F_0\left(h_\theta^2\right)}{\sqrt{n}} \\
&\leq C\|\Delta_\psi\|_\infty\sqrt{n}\left(\log n\right)^2\epsilon_n^2.
\end{aligned}$$

To bound $\|\Delta_\psi\|_\infty$, we write

$$\Delta_\psi = \psi_{+k} - \Pi_{f_0,k}(\psi_{+k}),$$

where ψ_{+k} is a linear function of the ϕ_j's for $j \geq k+1$. Then by using (4.4),

$$\begin{aligned}\|\Delta_\psi\|_\infty &\leq \|\psi_{+k}\|_\infty + \|\Pi_{f_0,k}\psi_{+k}\|_\infty \\ &\leq \|\psi_{+k}\|_\infty + C\sqrt{k}\|\Pi_{f_0,k}\psi_{+k}\|_{f_0} \\ &\leq \|\psi_{+k}\|_\infty + C\sqrt{k}\|\psi_{+k}\|_{f_0} \\ &\leq \|\psi_{+k}\|_\infty + \frac{C\sqrt{\|f_0\|_\infty}\sqrt{k}}{\sqrt{c_0}}\|\psi_{+k}\|_2.\end{aligned}$$

Under the assumption that

$$\sup_{k\leq l_n}(\|\psi_{+k}\|_\infty + \sqrt{k}\|\psi_{+k}\|_2) = o\left(\frac{1}{\sqrt{n}\epsilon_n^2\,(\log n)^2}\right),$$

we obtain that

$$\Delta_n = \frac{t}{\sqrt{n}}\left[F_0\left(h_\theta\Delta_\psi\right) - \frac{t}{2}F_0(\Delta_\psi^2)\right] + o(n^{-1/2}).$$

Note that $\Delta_n = o(1)$. Finally,

$$\begin{aligned}R_n(h_{\theta'}) &= \sqrt{n}F_0(h_{\theta'}) + \frac{F_0(h_{\theta'}^2)}{2} \\ &= R_n(h_\theta - t\bar{\psi}_{t,n}) - \sqrt{n}\Delta_n - \frac{t^2}{2}F_0(\Delta_\psi^2) + tF_0(h_\theta\Delta_\psi) - \Delta_n F_0(h_\theta) + o(1) \\ &= R_n(h_\theta - t\bar{\psi}_{t,n}) - \Delta_n F_0(h_\theta) + o(1)\end{aligned}$$

Recall that $h_{\theta'} = h_\theta - t\bar{\psi}_{t,n} + t\Delta_\psi - \Delta_n$, $\Delta_n = o(1)$ and $F_0(\Delta_\psi) = 0$. Note also that

$$\bar{\psi}_{t,n}(x) = \psi_c(x) + \frac{\sqrt{n}}{t}\log\left(F_0(e^{H_n})\right) = \psi_c(x) + o(1)$$

so that $F_0(\Delta_\psi\bar{\psi}_{t,n}) = F_0(\Delta_\psi^2) + o(1)$ and

$$\begin{aligned}-\frac{F_0(h_{\theta'}^2)}{2} &= -\frac{F_0((h_\theta - t\bar{\psi}_{t,n})^2)}{2} - \frac{F_0((t\Delta_\psi - \Delta_n)^2)}{2} - F_0((h_\theta - t\bar{\psi}_{t,n})(t\Delta_\psi - \Delta_n)) \\ &= -\frac{F_0((h_\theta - t\bar{\psi}_{t,n})^2)}{2} + \frac{t^2F_0(\Delta_\psi^2)}{2} - tF_0(h_\theta\Delta_\psi) + \Delta_n F_0(h_\theta) + o(1)\end{aligned}$$

Furthermore,

$$G_n(h_{\theta'}) = G_n(h_\theta - t\bar{\psi}_{t,n}) + tG_n(\Delta_\psi).$$

We set

$$\mu_{n,k} = -F_0(h_\theta\Delta_\psi) + G_n(\Delta_\psi)$$

and we finally obtain,

$$
\begin{aligned}
&-\frac{F_0((h_{\theta'})^2)}{2}+G_n(h_{\theta'})+R_n(h_{\theta'})\\
&\quad=-\frac{F_0((h_\theta-t\bar\psi_{t,n})^2)}{2}+R_n(h_\theta-t\bar\psi_{t,n})+G_n(h_\theta-t\bar\psi_{t,n})+t\mu_{n,k}\\
&\qquad+\frac{t^2F_0(\Delta_\psi^2)}{2}+o(1).
\end{aligned}
$$

Note that by orthogonality $F_0(h_\theta\Delta_\psi)=\sqrt{n}F_0[(\psi_c-\Pi_{f_0,k}\psi_c)\sum_{j\geq k+1}\theta_{0j}\phi_j]$ so that $\mu_{n,k}$ does not depend on θ and setting $T_k\theta=\theta-tB_{n,k}$ for all θ, we can write

$$
\begin{aligned}
J_k &:= \frac{\int_{A_n\cap\mathcal{F}_k}\exp\left(-\frac{F_0((h_f-t\bar\psi_{t,n})^2)}{2}+G_n(h_f-t\bar\psi_{t,n})+R_n(h_f-t\bar\psi_{t,n})\right)d\pi_k(f)}{\int_{A_n\cap\mathcal{F}_k}\exp\left(-\frac{F_0(h_f^2)}{2}+G_n(h_f)+R_n(h_f)\right)d\pi_k(f)}\\
&= e^{-\frac{t^2F_0(\Delta_\psi^2)}{2}}e^{-t\mu_{n,k}}\frac{\int_{\Theta_k\cap A_n'}e^{-\frac{F_0\left(h^2_{T_k\theta}\right)}{2}+G_n\left(h_{T_k\theta}\right)+R_n\left(h_{T_k\theta}\right)}d\pi_k(\theta)}{\int_{\Theta_k\cap A_n'}e^{-\frac{F_0(h_\theta^2)}{2}+G_n(h_\theta)+R_n(h_\theta)}d\pi_k(\theta)}(1+o_n(1)),
\end{aligned}
$$

where $A_n'=\{\theta:\ f_\theta\in A_n\}$. Moreover, for $k\leq l_n$, $\|B_{n,k}\|_2\leq C/\sqrt{n}$, where C depends on c_0 and $\|\psi_c\|_\infty$. So, if we set

$$
T_k(A_n')=\{\theta\in\Theta_k\cap A_n':\ \theta+tB_{n,k}\in A_n'\}
$$

for all $\theta\in T_k(A_n')$,

$$
\|\theta-\theta_0\|_{\ell_2}^2\leq 2(\log n)^4\epsilon_n^2+\frac{2c^2}{n}\leq 2\epsilon_n^2(\log n)^4(1+o_n(1))
$$

since $n\epsilon_n^2\to+\infty$. For all $\theta\in\Theta_k\cap A_n'$ such that $\|\theta-\theta_0\|_2\leq\frac{(\log n)^2\epsilon_n}{2}$

$$
\theta+tB_{n,k}\in A_n'\cap\Theta_k
$$

for n large enough and we can write

$$
A_{n,1}'=\left\{\theta\in A_n':\ \|\theta-\theta_0\|_{\ell_2}\leq\frac{(\log n)^2\epsilon_n}{2}\right\},\quad A_{n,2}'=\left\{\theta\in A_n':\ \|\theta-\theta_0\|_2\leq 3(\log n)^2\epsilon_n\right\}
$$

then

$$
\Theta_k\cap A_{n,1}'\subset T_k(A_n')\subset\Theta_k\cap A_{n,2}' \tag{4.27}
$$

and under assumption (3.5),

$$J_k \leq e^{-t^2\frac{F_0(\Delta_\psi^2)}{2}}e^{-t\mu_{n,k}}\frac{\int_{\Theta_k\cap A'_{n,2}} e^{-\frac{F_0(h_\theta^2)}{2}+G_n(h_\theta)+R_n(h_\theta)}d\pi_k(\theta)}{\int_{\Theta_k\cap A'_n} e^{-\frac{F_0(h_\theta^2)}{2}+G_n(h_\theta)+R_n(h_\theta)}d\pi_k(\theta)}(1+o_n(1)),$$

$$J_k \geq e^{-t^2\frac{F_0(\Delta_\psi^2)}{2}}e^{-t\mu_{n,k}}\frac{\int_{\Theta_k\cap A'_{n,1}} e^{-\frac{F_0(h_\theta^2)}{2}+G_n(h_\theta)+R_n(h_\theta)}d\pi_k(\theta)}{\int_{\Theta_k\cap A'_n} e^{-\frac{F_0(h_\theta^2)}{2}+G_n(h_\theta)+R_n(h_\theta)}d\pi_k(\theta)}(1+o_n(1)).$$

Therefore,

$$\begin{aligned}\zeta_n(t) &:= \mathbb{E}\left[\exp(t\sqrt{n}(\psi(f)-\psi(\mathbb{P}_n)))\mathbb{1}_{A_n}(f)|X^n\right]\\ &= e^{\frac{t^2F_0(\psi_c^2)}{2}}\left[\sum_{k=1}^{l_n}p(k|X^n)J_k\right](1+o_n(1))\\ &\leq \left[\sum_{k=1}^{l_n}p(k|X^n)\mathbb{1}_{\Theta_k\cap A'_n\neq\emptyset}e^{-t\mu_{n,k}}e^{t^2\frac{F_0(\psi_c{}^2)-F_0(\Delta_\psi^2)}{2}}\right](1+o_n(1))\end{aligned}$$

and

$$\zeta_n(t) \geq e^{t^2\frac{F_0(\psi_c{}^2)}{2}}\sum_{k=1}^{l_n}p(k|X^n)e^{-t\mu_{n,k}}e^{-t^2\frac{F_0(\Delta_\psi^2)}{2}}\pi\left[A'_{n,1}|X^n,k\right].$$

Besides under the above conditions on the prior, with probability converging to 1,

$$\pi\left[(A'_{n,1})^c|X^n\right]\leq e^{-nc\epsilon_n^2},$$

for some positive constant $c>0$. Then uniformly over k such that $\Theta_k\cap A'_{n,1}\neq\emptyset$

$$\pi\left[(A'_{n,1})^c|X^n,k\right]e^{-t\mu_{n,k}}=o(1)$$

and

$$\zeta_n(t) \geq e^{t^2\frac{F_0(\psi_c{}^2)}{2}}\sum_{k=1}^{l_n}p(k|X^n)\mathbb{1}_{\Theta_k\cap A_n\neq\emptyset}e^{-t\mu_{n,k}}e^{-t^2\frac{F_0(\Delta_\psi^2)}{2}}(1+o_n(1)).$$

This proves that the posterior distribution of $\sqrt{n}(\Psi(f)-\Psi(P_n))$ is asymptotically equal to a mixture of Gaussian distributions with variances $V_{0k}=F_0(\psi_c{}^2)-F_0(\Delta_\psi^2)$, means $-\mu_{n,k}$ and weights $p(k|X^n)$.

Now if $\|\Delta_\psi\| = o(1)$ $(k \to +\infty)$ $G_n(\Delta_\psi) = o_P(1)$ and with probability converging to 1,

$$|\mu_{n,k}| \leq \|f_0\|_\infty \sqrt{n} \left(\sum_{j=k+1}^{+\infty} \psi_{c,j}^2 \right)^{1/2} \left(\sum_{j=k+1}^{+\infty} \theta_{0j}^2 \right)^{1/2} + o_n(1).$$

Thus if $k = k_n^*$,

$$|\mu_{n,k}| = o\left(\sqrt{n}(k_n^*)^{-\gamma-1/2}\right) + o_n(1) = o_n(1)$$

and Equality (3.8) is proved.

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