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To cite this version:

Philippe Goutet. On the Zeta Function of a Family of Quintics. 2007. hal-00405347

HAL Id: hal-00405347

https://hal.science/hal-00405347v1

Preprint submitted on 20 Jul 2009

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On the zeta function of a family of quintics

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Abstract

In this article, we give a proof of the link between the zeta function of two families of hypergeometric curves and the zeta function of a family of quintics that was observed numerically by Candelas, de la Ossa, and Rodriguez Villegas. The method we use is based on formulas of Koblitz and various Gauss sums identities; it does not give any geometric information on the link.

Key words: quintic threefold, hypergeometric curves, zeta function factorization

1 Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic $p \neq 5$. In all this article, ψ will be an element of $\mathbb{F}_q$ and $\mathcal{M}_\psi$ the hypersurface of $\mathbb{P}^4_{\mathbb{F}_q}$ defined by

$$x_1^5 + \cdots + x_5^5 - 5\psi x_1 \ldots x_5 = 0.$$

It is non-singular if and only if $\psi^5 \neq 1$. We denote by $|\mathcal{M}_\psi(\mathbb{F}_{q^r})|$ the number of points of $\mathcal{M}_\psi$ over an extension $\mathbb{F}_{q^r}$ of degree r of $\mathbb{F}_q$. The zeta function of $\mathcal{M}_\psi$ is

$$Z_{\mathcal{M}_\psi/\mathbb{F}_q}(t) = \exp\left(\sum_{r=1}^{+\infty} |\mathcal{M}_\psi(\mathbb{F}_{q^r})| \frac{t^r}{r}\right).$$

Candelas, de la Ossa, and Rodriguez Villegas have given in [2,3] an explicit formula for $|\mathcal{M}_\psi(\mathbb{F}_{q^r})|$ in terms of Gauss sums. The formula takes the form

$$\begin{aligned}|\mathcal{M}_\psi(\mathbb{F}_{q^r})| = 1 + q^r + q^{2r} + q^{3r} \\ - N_w(q^r) - 10q^r N_a(q^r) - 15q^r N_b(q^r) + 24N_{\text{sing}}(q^r).\end{aligned}$$

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When $\mathcal{M}_\psi$ is non singular (that is, when $\psi^5 \neq 1$), the term $N_{\text{sing}}(q^r)$ is zero (see lemma 28 page 13 below). In this case, we have

$$Z_{\mathcal{M}_\psi/\mathbb{F}_q}(t) = \frac{P_w(t)P_a(qt)^{10}P_b(qt)^{15}}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}, \tag{1}$$

where P_w is the formal series $\exp\left(\sum_{r=1}^{+\infty} N_w(q^r)\frac{t^r}{r}\right)$, P_a and P_b being defined in a similar way. It is possible to associate to $\mathcal{M}_\psi$ a "mirror variety" $\mathcal{W}_\psi$ (obtained by quotient and then resolution of singularities, see [3, §10 p. 124]). When $\psi \neq 0$, Candelas, de la Ossa, and Rodriguez Villegas [3, §14 p. 149] show that

$$Z_{\mathcal{W}_\psi/\mathbb{F}_q}(t) = \frac{P_w(t)}{(1-t)(1-qt)^{101}(1-q^2t)^{101}(1-q^3t)}.$$

As the Betti numbers of $\mathcal{W}_\psi$ are (1,0,101,4,101,0,1), we know from the Weil conjectures that P_w is a polynomial of degree 4. When $\psi = 0$, the above formula for $Z_{\mathcal{W}_\psi/\mathbb{F}_q}(t)$ is still valid, but it is also possible to give a hypergeometric interpretation of $P_w(t)$; more precisely, we will show (see remark 25 page 12) that $P_w(t)$ is the numerator of the zeta function of the hypergeometric hypersurface

$$H_0 : y^5 = x_1x_2x_3(1-x_1-x_2-x_3).$$

Concerning P_a and P_b, define two affine curves by $\mathcal{A}_\psi : y^5 = x^2(1-x)^3(x-\psi^5)^2$ and $\mathcal{B}_\psi : y^5 = x^2(1-x)^4(x-\psi^5)$. We write the corresponding zeta functions as

$$Z_{\mathcal{A}_\psi/\mathbb{F}_q}(t) = \frac{P_{\mathcal{A}_\psi/\mathbb{F}_q}(t)}{1-qt} \quad \text{and} \quad Z_{\mathcal{B}_\psi/\mathbb{F}_q}(t) = \frac{P_{\mathcal{B}_\psi/\mathbb{F}_q}(t)}{1-qt},$$

where $P_{\mathcal{A}_\psi/\mathbb{F}_q}$ and $P_{\mathcal{B}_\psi/\mathbb{F}_q}$ are of degree 8 if $\psi \neq 0$ and of degree 4 if $\psi = 0$. On the basis of numerical observations (made in the case $q = p$ for $p \leq 101$), Candelas, de la Ossa, and Rodriguez Villegas conjecture that $P_a = P_{\mathcal{A}_\psi/\mathbb{F}_q}$ and $P_b = P_{\mathcal{B}_\psi/\mathbb{F}_q}$ if $\psi \neq 0$ and that $P_a = P^2_{\mathcal{A}_\psi/\mathbb{F}_q}$ and $P_b = P^2_{\mathcal{B}_\psi/\mathbb{F}_q}$ if $\psi = 0$.

In this article, we prove this conjecture by computing explicitly $|\mathcal{A}_\psi(\mathbb{F}_{q^r})|$ and $|\mathcal{B}_\psi(\mathbb{F}_{q^r})|$ in terms of Gauss sums using formulas given by Koblitz in [6, §5] and comparing them to those given for $|\mathcal{M}_\psi(\mathbb{F}_{q^r})|$ by Candelas, de la Ossa, and Rodriguez Villegas. More precisely, we show the following theorem (with analogous formulas for $\mathcal{B}_\psi$).

Theorem 1 *If $\mathcal{A}_\psi$ is the affine curve $y^5 = x^2(1-x)^3(x-\psi^5)^2$,*

$$|\mathcal{A}_\psi(\mathbb{F}_{q^r})| = q^r - N_a(q^r) \qquad \text{if } \psi \neq 0 \tag{2}$$

$$|\mathcal{A}_\psi(\mathbb{F}_{q^r})| = q^r - \frac{1}{2}N_a(q^r) \qquad \text{if } \psi = 0 \tag{3}$$

There are a number of remarks that can be made about the two factors $P_{\mathcal{A}_\psi}$ and $P_{\mathcal{B}_\psi}$.

Remark 2 *If $\rho \in \{1, 2, 4\}$ is the order of q in $(\mathbb{Z}/5\mathbb{Z})^\times$, then $P_{\mathcal{A}_\psi/\mathbb{F}_q}(t) = P_{\mathcal{A}_\psi/\mathbb{F}_{q^\rho}}(t^\rho)^{1/\rho}$ which implies that $P_{\mathcal{A}_\psi/\mathbb{F}_q}(t)$ is the ρ-th power of a polynomial. The same is true for $\mathcal{B}_\psi$.*

Remark 3 *If $\psi \neq 0$ and $\psi^5 \neq 1$, then $P_{\mathcal{A}_\psi/\mathbb{F}_q}$ and $P_{\mathcal{B}_\psi/\mathbb{F}_q}$ are squares.*

Remark 4 *If $\psi = 0$, $p \not\equiv 1 \mod 5$ and $q \equiv 1 \mod 5$, then $P_{\mathcal{A}_\psi/\mathbb{F}_q}(t) = P_{\mathcal{B}_\psi/\mathbb{F}_q}(t) = (1 - q't)^4$ where q' is the square root of q which is $\equiv 1 \mod 5$.*

By combining remark 2 and remark 4, one can easily prove the observations of [3, §12 p. 132] when $\psi = 0$ and $\rho = 2$ or $\rho = 4$. For a proof of remark 3, we refer to [3, §11.1 p. 129-130]; the argument is that it is possible to transform $\mathcal{A}_\psi$ and $\mathcal{B}_\psi$ into hyperelliptic curves and then use the existence of a special automorphism of these hyperelliptic curves to deduce that the Jacobian of these two curves is isogenous to a square.

The article is organized as follows. In section 2, we recall all the formulas on Gauss and Jacobi sums we will need. In section 3 we give the formulas for $|\mathcal{A}_\psi(\mathbb{F}_{q^r})|$, $|\mathcal{B}_\psi(\mathbb{F}_{q^r})|$ and prove remarks 2 and 4. In section 4 and section 5, we prove Theorem 1 when $\psi = 0$ and $\psi \neq 0$ respectively, and finally, in section 6, we mention what happens when the quintic is singular.

The method we use does not give any information on a geometric link between the two curves $\mathcal{A}_\psi$, $\mathcal{B}_\psi$ and the quintic $\mathcal{M}_\psi$, but can be generalized to give the explicit factorization of the zeta function of $x_1^n + \cdots + x_n^n - n\psi x_1 \ldots x_n = 0$, at least when n is a prime number. This will be the subject of a subsequent article.

2 Gauss and Jacobi sums formulas

Definition 5 (Gauss sums) *Let $\mathbf{\Omega}$ be an algebraically closed field of characteristic zero. Let $\varphi : \mathbb{F}_q \to \mathbf{\Omega}^*$ be a non trivial additive character of $\mathbb{F}_q$. For any character $\chi : \mathbb{F}_q^* \to \mathbf{\Omega}^*$ of $\mathbb{F}_q^*$, define the Gauss sum $G(\varphi, \chi)$ as being*

$$G(\varphi, \chi) = \sum_{x \in \mathbb{F}_q^*} \varphi(x)\chi(x).$$

If $\mathbf{1}$ is the trivial character of $\mathbb{F}_q^*$, we have $G(\varphi, \mathbf{1}) = -1$. Note that $G(\varphi, \chi^i)$ only depends on $i \mod q - 1$.

Proposition 6 (Reflection formula) *If χ is a non trivial character of $\mathbb{F}_q^*$,*

$$G(\varphi,\chi)G(\varphi,\chi^{-1}) = \chi(-1)q. \tag{4}$$

PROOF. See [1, Theorem 1.1.4 (a) p. 10]. □

Proposition 7 (Hasse-Davenport formula) *If χ is a character of $\mathbb{F}_q^*$,*

$$-G(\varphi\circ \mathrm{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}, \chi\circ N_{\mathbb{F}_{q^r}/\mathbb{F}_q}) = (-G(\varphi,\chi))^r. \tag{5}$$

PROOF. See [1, Theorem 11.5.2 p. 360]. □

Proposition 8 (Purity formula) *Let $d \geq 3$ be an integer and φ' a non trivial additive character of $\mathbb{F}_p$. Suppose there exists an integer s such that $p^s \equiv -1 \mod d$. If σ is the smallest integer such that $p^\sigma \equiv -1 \mod d$ and if we let $q = p^{2\sigma m}$ (so that $\sqrt{q} = p^{\sigma m}$ is an integer), then, for any multiplicative character χ of order d of $\mathbb{F}_q^*$,*

$$G(\varphi'\circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}, \chi) = \begin{cases} (-1)^{m-1}\sqrt{q} & \text{if } d \text{ is odd or if } \frac{p^\sigma+1}{d} \text{ is even} \\ -\sqrt{q} & \text{if } d \text{ is even and if } \frac{p^\sigma+1}{d} \text{ is odd} \end{cases} \tag{6}$$

PROOF. See [1, Theorem 11.6.3 p. 364]. □

Proposition 9 (Multiplication formula) *Let $d \geq 1$ be a divisor of $q-1$. If χ is a character of $\mathbb{F}_q^*$,*

$$\frac{G(\varphi,\chi^d)}{\prod_{\chi'^d=\mathbf{1}} G(\varphi,\chi\chi')} = \frac{\chi(d)^d}{\prod_{\substack{\chi'^d=\mathbf{1}\\ \chi'\neq\mathbf{1}}} G(\varphi,\chi')}. \tag{7}$$

PROOF. See [1, Theorem 11.3.5 p. 355]. □

Remark 10 *The product $\prod_{\substack{\chi'^d=\mathbf{1}\\ \chi'\neq\mathbf{1}}} G(\varphi,\chi')$ is given by*

$$\begin{cases} q^{\frac{d-1}{2}} & \text{if } d \text{ is odd} \\ (-1)^{\frac{(q-1)(d-2)}{8}} q^{\frac{d-2}{2}} G(\varphi,\varepsilon) & \text{if } d \text{ is even} \end{cases} \tag{8}$$

where ε is the character of order 2 of $\mathbb{F}_q^*$. This formula is a direct consequence of the reflection formula (4) by grouping Gauss sums by pair. When $\mathbf{\Omega} = \mathbb{C}$, it is possible to give an exact formula for $G(\varphi,\varepsilon)$ (see [1, Theorem 11.5.4 p. 362]), but we will not have any use of it as we will only be concerned with the case $d = 5$.

Definition 11 (Jacobi sums) *If χ and χ' are two characters of $\mathbb{F}_q^*$, define the corresponding Jacobi sum as being*

$$J(\chi,\chi') = \sum_{\substack{x,x'\in\mathbb{F}_q^* \\ x+x'=1}} \chi(x)\chi'(x') = \sum_{\substack{x\in\mathbb{F}_q \\ x\neq 0, x\neq 1}} \chi(x)\chi'(1-x).$$

Proposition 12 (Link with Gauss sums) *If χ and χ' are two characters of $\mathbb{F}_q^*$,*

$$J(\chi,\chi') = \begin{cases} \frac{G(\varphi,\chi)G(\varphi,\chi')}{G(\varphi,\chi\chi')} & \text{if } \chi\chi' \neq \mathbf{1} \\ \frac{1}{q}\frac{G(\varphi,\chi)G(\varphi,\chi')}{G(\varphi,\chi\chi')} & \text{if } \chi\chi' = \mathbf{1} \text{ and } \chi \neq \mathbf{1} \end{cases} \tag{9}$$

PROOF. For the first formula, see [1, Eq. (11.6.4) p. 365] (notice that there is nothing to prove if χ or χ' is trivial). For the second formula, use [1, Eq. (11.6.3) p. 365] and then the reflection formula (4) to see that $J(\chi,\chi^{-1}) = -\chi(-1) = \frac{1}{q}\frac{G(\varphi,\chi)G(\varphi,\chi^{-1})}{G(\varphi,\chi\chi^{-1})}$. $\square$

Proposition 13 (Jacobi sum of inverses) *If χ is a non trivial character of $\mathbb{F}_q^*$ such that $\chi(-1)=1$ and if η is an arbitrary character of $\mathbb{F}_q^*$,*

$$J(\chi^{-1},\eta^{-1}) = \begin{cases} \frac{1}{q}\frac{G(\varphi,\chi^{-1})G(\varphi,\chi\eta)}{G(\varphi,\eta)} & \text{if } \eta = \mathbf{1} \\ \frac{G(\varphi,\chi^{-1})G(\varphi,\chi\eta)}{G(\varphi,\eta)} & \text{otherwise} \end{cases} \tag{10}$$

PROOF. If χ and η are two characters of $\mathbb{F}_q^*$, we have

$$\begin{aligned} J(\chi^{-1},\chi\eta) &= \sum_{\substack{x,y\in\mathbb{F}_q^* \\ x+y=1}} \chi^{-1}(x)\chi\eta(y) = \sum_{\substack{x,y\in\mathbb{F}_q^* \\ x+y=1}} \chi^{-1}(\tfrac{x}{y})\eta^{-1}(\tfrac{1}{y}) \ ; \\ &= \sum_{\substack{x',y'\in\mathbb{F}_q^* \\ x'+y'=1}} \chi^{-1}(-x')\eta^{-1}(y') = \chi(-1)J(\chi^{-1},\eta^{-1}), \end{aligned}$$

where we made the change of variables $x' = -\frac{x}{y}$ et $y' = \frac{1}{y}$. Combining this formula with formula (9), we get at once formula (10) because χ is non trivial and $\chi(-1)=1$. $\square$

Proposition 14 (Fourier inversion formula) *If $f : \mathbb{F}_q^* \to \mathbf{\Omega}$ is a map,*

$$\forall \lambda \in \mathbb{F}_q^*, \quad f(\lambda) = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} \left(\sum_{\mu\in\mathbb{F}_q^*} f(\mu)\eta^{-1}(\mu) \right) \eta(\lambda). \tag{11}$$

PROOF. It is a direct consequence of the orthogonality relations for characters of the finite abelian group $\mathbb{F}_q^*$. $\square$

Remark 15 *From now on, we will take for additive character φ a character of the form $\varphi' \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ where φ' is a non trivial additive character of $\mathbb{F}_p$, so purity formula (6) will be valid.*

3 Number of points of the hypergeometric curves

In the introduction, we gave equations $y^5 = x^2(1-x)^3(x-\psi^5)^2$ and $y^5 = x^2(1-x)^4(x-\psi^5)$ for $\mathcal{A}_\psi$ and $\mathcal{B}_\psi$ respectively. When $\psi \neq 0$, we transform these equations into

$$y^5 = x^2(1-x)^3(1-\tfrac{1}{\psi^5}x)^2 \quad \text{and} \quad y^5 = x^2(1-x)^4(1-\tfrac{1}{\psi^5}x),$$

which does not change the number of points and is more convenient for our computations.

3.1 General remarks

To show Theorem 1, we only have to consider the case $r = 1$ as q is arbitrary. Moreover, the following lemma shows that we only need to consider the case $q \equiv 1 \mod 5$ to compute $|\mathcal{A}_\psi(\mathbb{F}_q)|$.

Lemma 16 *Let Q be a polynomial with coefficients in $\mathbb{F}_q$ and $\mathcal{C}$ the affine curve $y^5 = Q(x)$. If $q \not\equiv 1 \mod 5$, then $|\mathcal{C}(\mathbb{F}_q)| = q$.*

PROOF. Because $q \not\equiv 1 \mod 5$, the map $y \mapsto y^5$ is a bijection of $\mathbb{F}_q$ onto itself and so $\mathcal{C}$ has the same number of points than the curve $y = Q(x)$ i.e. q points. $\square$

Remark 17 *This lemma shows remark 2 of the introduction.*

3.2 The case $\psi = 0$

The curves $\mathcal{A}_0$ and $\mathcal{B}_0$ are isomorphic so have the same number of points. It is thus sufficient to give a formula for $|\mathcal{A}_0(\mathbb{F}_q)|$.

Theorem 18 *If $q \equiv 1 \mod 5$ and if $\mathcal{A}_0$ is the affine curve $y^5 = x^4(1-x)^3$,*

$$|\mathcal{A}_0(\mathbb{F}_q)| = q - \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \left(-\frac{1}{q}G(\varphi,\chi)^2 G(\varphi,\chi^3)\right).$$

PROOF. We recall the proof of this classical result (see for example [5, §1, p. 202-203] or [6, §5, p. 20]). It is a straightforward application of the method Weil used in [8].

The equation of $\mathcal{A}_0$ is of the type $y^5 = Q(x)$ where Q is a polynomial. The number of points of the affine curve $\mathcal{A}_0$ is given by

$$|\mathcal{A}_0(\mathbb{F}_q)| = |\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \mid y^5 = Q(x)\}|.$$

The fundamental remark is that, if $z \in \mathbb{F}_q$,

$$|\{y \in \mathbb{F}_q \mid y^5 = z\}| = \begin{cases} 1 & \text{if } z = 0 \\ 1 + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \chi(z) & \text{if } z \neq 0 \end{cases}$$

Thus,

$$\begin{aligned} |\mathcal{A}_0(\mathbb{F}_q)| &= |\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \mid y^5 = Q(x) \text{ and } Q(x) = 0\}| \\ &\quad + |\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \mid y^5 = Q(x) \text{ and } Q(x) \neq 0\}|; \\ &= \sum_{\substack{x\in\mathbb{F}_q \\ Q(x)=0}} 1 + \sum_{\substack{x\in\mathbb{F}_q \\ Q(x)\neq 0}} \left(1 + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \chi(Q(x))\right), \end{aligned}$$

and so

$$|\mathcal{A}_0(\mathbb{F}_q)| = q + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \sum_{\substack{x\in\mathbb{F}_q \\ Q(x)\neq 0}} \chi(Q(x)). \tag{12}$$

Replacing $Q(x)$ by $x^4(1-x)^3$, we obtain

$$|\mathcal{A}_0(\mathbb{F}_q)| = q + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \sum_{\substack{x\in\mathbb{F}_q \\ x\neq 0, x\neq 1}} \chi^4(x)\chi^3(1-x) = q + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} J(\chi^4, \chi^3).$$

We now use formula (9) to express the Jacobi sums in terms of Gauss sums

$$\begin{aligned} |\mathcal{A}_0(\mathbb{F}_q)| &= q + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \frac{G(\varphi,\chi^4)G(\varphi,\chi^3)}{G(\varphi,\chi^7)} = q + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \frac{G(\varphi,\chi^4)G(\varphi,\chi^3)}{G(\varphi,\chi^2)}; \\ &= q + \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \frac{1}{q} G(\varphi,\chi^4)G(\varphi,\chi^3)^2, \end{aligned}$$

by using the reflection formula (4) and the fact that $\chi(-1) = 1$ (because $\chi^5 = \mathbf{1}$). To obtain the announced formula, we just make the change of variable $\chi \mapsto \chi^2$. $\square$

Remark 19 *Assume as before that* $q \equiv 1 \mod 5$ *and replace* $\mathbb{F}_q$ *by* $\mathbb{F}_{q^r}$ *in the formula for* $|\mathcal{A}_\psi(\mathbb{F}_q)|$ *we have just obtained. As the characters of order* 5

of $\mathbb{F}_{q^r}^*$ *are exactly the* $\chi \circ \mathrm{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ *where* χ *is a character of order* 5 *of* $\mathbb{F}_q^*$*, the choice of additive character made in remark 15 and the Hasse-Davenport formula (5) show that*

$$|\mathcal{A}_0(\mathbb{F}_{q^r})| = q^r - \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^*} \\ \chi^5 = \mathbf{1}, \chi \neq \mathbf{1}}} \left(-\frac{1}{q} G(\varphi, \chi)^2 G(\varphi, \chi^3) \right)^r,$$

and so, when $q \equiv 1 \mod 5$*, the zeta function is given by*

$$Z_{\mathcal{A}_0/\mathbb{F}_q}(t) = \frac{\prod_{\substack{\chi^5 = \mathbf{1} \\ \chi \neq \mathbf{1}}} \left(1 + \frac{1}{q} G(\varphi, \chi)^2 G(\varphi, \chi^3) t\right)}{1 - qt}.$$

Remark 20 *Let us deduce remark 4 of the introduction. Because* $p \not\equiv 1 \mod 5$ *and* $q \equiv 1 \mod 5$*, we must have* $q = p^{2m}$ *if* $p \equiv -1 \mod 5$ *and* $q = p^{4m}$ *if* $p \equiv \pm 2 \mod 5$*. The purity formula (6) then shows that the numerator of the zeta function of* $\mathcal{A}_0$ *is*

$$(1 - (-1)^m \sqrt{q} t)^4 = (1 - q' t)^4,$$

where $q' = (-1)^m \sqrt{q}$ *is the square root of* q *which is* $\equiv 1 \mod 5$*.*

3.3 The case $\psi \neq 0$

When $\psi \neq 0$, the equations of both $\mathcal{A}_\psi$ and $\mathcal{B}_\psi$ are of the type $\mathcal{C}_\lambda : y^5 = x^a(1-x)^b(1-\lambda x)^{5-b}$ where $\lambda = \frac{1}{\psi^5}$ is a fifth power.

Theorem 21 *If* $q \equiv 1 \mod 5$ *and if* $\mathcal{C}_\lambda$ *is the affine curve* $y^5 = x^a(1-x)^b(1-\lambda x)^{5-b}$ *with* $\lambda \in \mathbb{F}_q^*$ *and* $1 \leq a, b \leq 4$*,*

$$|\mathcal{C}_\lambda(\mathbb{F}_q)| = q + \sum_{\substack{\chi^5 = \mathbf{1} \\ \chi \neq \mathbf{1}}} \left(\frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}_q^*}} q^{1-\nu} \frac{G(\varphi, \chi^a \eta) G(\varphi, \chi^b \eta)}{G(\varphi, \eta) G(\varphi, \chi^{a+b} \eta)} \eta(\lambda) \right),$$

where ν *is the number of trivial characters in the pair* $(\eta, \chi^{a+b}\eta)$*.*

PROOF. We follow the proof of Koblitz [6, Theorem 3 p. 18] and then derive the above formula.

Formula (12) is valid for $\mathcal{C}_\lambda$ and gives

$$\begin{aligned}|\mathcal{C}_\lambda(\mathbb{F}_q)| &= q + \sum_{\substack{\chi^5=\mathbf{1}\\ \chi\neq\mathbf{1}}} \sum_{\substack{x\in\mathbb{F}_q\\ x\neq 0, x\neq 1, x\neq 1/\lambda}} \chi^a(x)\chi^b(1-x)\chi^{-b}(1-\lambda x);\\ &= q + \sum_{\substack{\chi^5=\mathbf{1}\\ \chi\neq\mathbf{1}}} N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi},\end{aligned}$$

where, if χ is a non trivial character of order 5 of $\mathbb{F}_q^*$,

$$N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi} = \sum_{\substack{x\in\mathbb{F}_q\\ x\neq 0, x\neq 1, x\neq 1/\lambda}} \chi^a(x)\chi^b(1-x)\chi^{-b}(1-\lambda x).$$

Following Koblitz [6, p. 19], we make a Fourier transform. We choose a character η of $\mathbb{F}_q^*$, multiply $N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi}$ by $\eta^{-1}(\lambda)$, and sum on all $\lambda \neq 0$

$$\sum_{\lambda\in\mathbb{F}_q^*} N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi}\eta^{-1}(\lambda) = \sum_{\substack{x\in\mathbb{F}_q, \lambda\in\mathbb{F}_q^*\\ x\neq 0, x\neq 1, x\neq 1/\lambda}} \chi^a(x)\chi^b(1-x)\chi^{-b}(1-\lambda x)\eta^{-1}(\lambda).$$

We now make the change of variable $t = \lambda x$

$$\sum_{\lambda\in\mathbb{F}_q^*} N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi}\eta^{-1}(\lambda) = \sum_{\substack{x\in\mathbb{F}_q, t\in\mathbb{F}_q\\ x\neq 0, x\neq 1, t\neq 0, t\neq 1}} \chi^a(x)\chi^b(1-x)\chi^{-b}(1-t)\eta^{-1}(t)\eta(x).$$

Grouping the terms with only x together and only t together, we obtain two Jacobi sums

$$\begin{aligned}\sum_{\lambda\in\mathbb{F}_q^*} N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi}\eta^{-1}(\lambda) &= \left(\sum_{\substack{x\in\mathbb{F}_q\\ x\neq 0, x\neq 1}} (\chi^a\eta)(x)\chi^b(1-x)\right)\left(\sum_{\substack{t\in\mathbb{F}_q\\ t\neq 0, t\neq 1}} \chi^{-b}(1-t)\eta^{-1}(t)\right);\\ &= J(\chi^a\eta, \chi^b)J(\chi^{-b}, \eta^{-1}).\end{aligned}$$

We now use the Fourier inversion formula (11)

$$\begin{aligned}N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi} &= \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}}\left(\sum_{\mu\in\mathbb{F}_q^*} N_{\mathcal{C}_\mu/\mathbb{F}_q,\chi}\eta^{-1}(\mu)\right)\eta(\lambda);\\ &= \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} J(\chi^a\eta, \chi^b)J(\chi^{-b}, \eta^{-1})\eta(\lambda).\end{aligned}$$

Because $\chi^5 = \mathbf{1}$, we have $\chi(-1) = 1$ and so we can use formulas (9) and (10) to express these Jacobi sums in terms of Gauss sums and obtain

$$J(\chi^a\eta, \chi^b)J(\chi^{-b}, \eta^{-1}) = \frac{1}{q^\nu}\frac{G(\varphi, \chi^a\eta)G(\varphi, \chi^b)}{G(\varphi, \chi^{a+b}\eta)}\frac{G(\varphi, \chi^b\eta)G(\varphi, \chi^{-b})}{G(\varphi, \eta)}.$$

Finally, using the reflection formula (4),

$$N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi} = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} q^{1-\nu}\frac{G(\varphi,\chi^a\eta)G(\varphi,\chi^b\eta)}{G(\varphi,\eta)G(\varphi,\chi^{a+b}\eta)}\eta(\lambda), \tag{13}$$

which gives the announced formula. □

We now rewrite the formula for $|\mathcal{C}_\lambda(\mathbb{F}_q)|$ of the previous theorem in a form better suited for our use of it in §5.

Corollary 22 *Suppose that $q \equiv 1 \mod 5$ and let $\mathcal{C}_\lambda$ be the affine curve $y^5 = x^a(1-x)^b(1-\lambda x)^{5-b}$ with $\lambda \in \mathbb{F}_q^*$ a fifth power and $1 \le a, b \le 4$. If χ is a non trivial character of order 5 of $\mathbb{F}_q^*$, we can write*

$$|\mathcal{C}_\lambda(\mathbb{F}_q)| = q + 2N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi} + 2N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi^2},$$

where $N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi}$ is given by (13).

PROOF. It is a simple consequence of Theorem 21 and of the formula $N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi'} = N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi'^{-1}}$. To prove this last formula, write

$$N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi'^{-1}} = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} q^{1-\nu}\frac{G(\varphi,\chi'^{-a}\eta)G(\varphi,\chi'^{-b}\eta)}{G(\varphi,\eta)G(\varphi,\chi'^{-(a+b)}\eta)}\eta(\lambda),$$

and make the change of variable $\eta = \chi'^{a+b}\eta'$. It transforms $N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi'^{-1}}$ into $N_{\mathcal{C}_\lambda/\mathbb{F}_q,\chi'}$ because λ is a fifth power and because the number of trivial characters in the pair $(\eta, \chi'^{-(a+b)}\eta)$ in the same than in the pair $(\eta', \chi'^{a+b}\eta')$. □

4 Number of points of the quintic in the diagonal case

The aim of this section is to prove formula (3) of Theorem 1 of the introduction. As in the previous section, we restrict ourselves to the case $r = 1$ as q is arbitrary. We begin with the case $q \not\equiv 1 \mod 5$.

Lemma 23 *If $q \not\equiv 1 \mod 5$, the number of points of $\mathcal{M}_0 : x_1^5 + \cdots + x_5^5 = 0$ in $\mathbb{P}^4_{\mathbb{F}_q}$ is given by*

$$|\mathcal{M}_0(\mathbb{F}_q)| = 1 + q + q^2 + q^3.$$

PROOF. When $q \not\equiv 1 \mod 5$, the map $x \mapsto x^5$ is a bijection of $\mathbb{F}_q$ onto itself, so $\mathcal{M}_0$ has the same number of points as the hyperplane $x_1 + \cdots + x_5 = 0$. □

Comparing with Lemma 16, this proves formula (3) of Theorem 1 of the introduction when $q \not\equiv 1 \mod 5$. We now proceed to the case $q \equiv 1 \mod 5$.

Theorem 24 *If $q \equiv 1 \mod 5$, we can write*

$$|\mathcal{M}_0(\mathbb{F}_q)| = 1 + q + q^2 + q^3 - N_w(q) - 25qN_a(q),$$

where

$$N_w(q) = \sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \left(-\frac{1}{q}G(\varphi,\chi)^5\right);$$

$$N_a(q) = 2\sum_{\substack{\chi^5=\mathbf{1} \\ \chi\neq\mathbf{1}}} \left(-\frac{1}{q}G(\varphi,\chi)^2G(\varphi,\chi^3)\right).$$

PROOF. This is a direct consequence of Weil's formula [8, Eq. (3) p. 500]. More precisely, if we chose a non trivial character χ of order 5 of $\mathbb{F}_q^*$, we have

$$|\mathcal{M}_0(\mathbb{F}_q)| = 1 + q + q^2 + q^3 - \sum_{\substack{1\leq s_1,\ldots,s_5\leq 4 \\ s_1+\cdots+s_5\equiv 0 \mod 5}} \left(-\frac{1}{q}G(\varphi,\chi^{s_1})\ldots G(\varphi,\chi^{s_5})\right).$$

Up to permutation, there are only twelve 5-uples $(s_1,\ldots,s_5)$ that index this sum. Explicitly

$(s_1,\ldots,s_5)$	# PERMUTATIONS
$(1,1,1,1,1),(2,2,2,2,2),(3,3,3,3,3),(4,4,4,4,4)$	1
$(1,1,3,1,4),(2,2,1,2,3),(3,3,4,2,3),(4,4,2,1,4)$	20
$(1,1,3,2,3),(2,2,1,1,4),(3,3,4,1,4),(4,4,2,2,3)$	30

We are thus able to enumerate all the products of Gauss sums appearing in the previous formula for $|\mathcal{M}_0(\mathbb{F}_q)|$. Using the reflection formula (4) and grouping together the terms, we get the following table.

$(s_1,\ldots,s_5)$	$\frac{1}{q}G(\varphi,\chi^{s_1})\ldots G(\varphi,\chi^{s_5})$	MULTIPLICITY
(j,j,j,j,j)	$\frac{1}{q}G(\varphi,\chi^j)^5$	1
$(1,1,3,1,4)$ and $(1,1,3,2,3)$	$G(\varphi,\chi)^2G(\varphi,\chi^3)$	$20+30=50$
$(2,2,1,2,3)$ and $(2,2,1,1,4)$	$G(\varphi,\chi^2)^2G(\varphi,\chi)$	$20+30=50$
$(3,3,4,2,3)$ and $(3,3,4,1,4)$	$G(\varphi,\chi^3)^2G(\varphi,\chi^4)$	$20+30=50$
$(4,4,2,1,4)$ and $(4,4,2,2,3)$	$G(\varphi,\chi^4)^2G(\varphi,\chi^2)$	$20+30=50$

This gives the formula we want, namely

$$|\mathcal{M}_0(\mathbb{F}_q)| = 1 + q + q^2 + q^3 - \sum_{j=1}^{4} \left(-\frac{1}{q} G(\varphi, \chi^j)^5 \right) - 50 \sum_{j=1}^{4} \left(-G(\varphi, \chi^j)^2 G(\varphi, \chi^{3j}) \right). \qquad \square$$

By comparing with Theorem 18, we immediately obtain formula (3) of Theorem 1 of the introduction when $q \equiv 1 \mod 5$.

Remark 25 *When $\psi = 0$, the factor $N_w(q)$ comes from the hypergeometric hypersurface*

$$H_0 \ : \ y^5 = x_1 x_2 x_3 (1 - x_1 - x_2 - x_3).$$

More precisely, using the techniques from sections 3.1 and 3.2, it is straightforward to show that

$$|H_0(\mathbb{F}_q)| = q^3 - N_w(q).$$

This is in accordance with [4, Proposition 3.2].

5 Number of points of the quintic in the generic case

We now consider the case $\psi \neq 0$ and proceed to show formula (2) of Theorem 1 of the introduction. Let us first recall the result shown by Candelas, de la Ossa, and Rodriguez Villegas. Just like in the two previous sections, we restrict ourselves to the case $r = 1$ as q is arbitrary.

Theorem 26 (Candelas, de la Ossa, Rodriguez Villegas) *If $\psi \neq 0$, then*

$$|\mathcal{M}_\psi(\mathbb{F}_q)| = 1 + q + q^2 + q^3 - N_w(q) - 10qN_a(q) - 15qN_b(q) + 24N_{\text{sing}}(q),$$

where $N_a(q)$, $N_b(q)$ and $N_{\text{sing}}(q)$ vanish if $q \not\equiv 1 \mod 5$. Moreover,

$$|\mathcal{W}_\psi(\mathbb{F}_q)| = 1 + q + q^2 + q^3 - N_w(q) + 100(q + q^2),$$

where $\mathcal{W}_\psi$ is the "mirror" of $\mathcal{M}_\psi$. We also have the following formulas for $N_a(q)$, $N_b(q)$ and $N_{\text{sing}}(q)$ when $q \equiv 1 \mod 5$

$$\begin{aligned} -N_a(q) &= \tfrac{2}{q} N_{(0,0,0,1,4),\chi}(q) + \tfrac{2}{q} N_{(0,0,0,2,3),\chi}(q); \\ -N_b(q) &= \tfrac{2}{q} N_{(0,0,1,1,3),\chi}(q) + \tfrac{2}{q} N_{(0,0,1,2,2),\chi}(q); \\ N_{\text{sing}}(q) &= N_{(0,1,2,3,4),\chi}(q), \end{aligned}$$

where χ is a fixed character of order 5 of $\mathbb{F}_q^*$ and

$$N_{(s_1,s_2,s_3,s_4,s_5),\chi}(q) = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} q^{5-z-\delta}\frac{G(\varphi,\eta^5)}{\prod_{j=1}^5 G(\varphi,\chi^{s_j}\eta)}\eta(\tfrac{1}{(5\psi)^5}),$$

with $\delta\in\{0,1\}$ zero if and only if one of the product $\chi^{s_i}\eta$ is trivial and with z the number of trivial characters among the $\chi^{s_j}\eta$.

PROOF. As the computation is quite lengthly, we do not reproduce it here and refer to [2, §9] and [3, §14 p. 144-150]. The formulas we gave above differ from those given by Candelas, de la Ossa, and Rodriguez Villegas by a factor $\frac{1}{q-1}$ because they compute the number of points in the affine space instead of the projective space. Note also that, although they only showed the formulas for a specific choice of multiplicative and additive p-adic characters, it is straightforward to extend their method to arbitrary characters with values in any algebraic closed field of characteristic zero (we only need to replace the relations on the Dwork character by the orthogonality formulas). □

The first thing we do is modify the formulas of Candelas, de la Ossa, and Rodriguez Villegas thanks to the product formula (7).

Lemma 27 *With the notations of Theorem 26,*

$$N_{(s_1,s_2,s_3,s_4,s_5),\chi}(q) = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} q^{3-z-\delta}\frac{\prod_{j=1}^5 G(\varphi,\chi^j\eta)}{\prod_{j=1}^5 G(\varphi,\chi^{s_j}\eta)}\eta(\tfrac{1}{\psi^5}). \tag{14}$$

PROOF. In the formula for $N_{(s_1,s_2,s_3,s_4,s_5),\chi}(q)$ of Theorem 26, we write $\eta(\frac{1}{(5\psi)^5}) = \frac{1}{\eta(5)^5}\eta(\frac{1}{\psi^5})$ and replace $\eta(5)^5$ by $q^2\frac{G(\varphi,\eta^5)}{\prod_{j=1}^5 G(\varphi,\chi\eta)}$ as per formulas (7) and (8). □

The first consequence of this new formula is that $N_{\text{sing}}(q)$ is non zero only when $\psi^5=1$ i.e. only when the quintic is singular. In the case $q=p$, this was shown by Candelas, de la Ossa, and Rodriguez Villegas in [2, §9.3] by using the multiplication formula for the p-adic Gamma function. We have merely replaced this formula by the multiplication formula (7) for Gauss sums. It both simplifies the proof and allow to treat at the same time the case $q\neq p$.

Lemma 28 *When $q\equiv 1 \mod 5$,*

$$N_{(0,1,2,3,4),\chi}(q) = \begin{cases} 0 & \text{if } \psi^5\neq 1 \\ q^2 & \text{if } \psi^5 = 1 \end{cases}$$

and so the term $24N_{\mathrm{sing}}(q)$ *does not contribute to the zeta function* $Z_{\mathcal{M}_\psi/\mathbb{F}_q}(t)$ *when* $\psi^5 \neq 1$ *and gives rise to a factor* $\frac{1}{(1-q^2t)^{24}}$ *when* $\psi^5 = 1$.

PROOF. When $(s_1, \ldots, s_5) = (0, 1, 2, 3, 4)$, we have $z + \delta = 1$ for all values of η in formula (14) and the two products simplify completely. Thus,

$$N_{(0,1,2,3,4),\chi}(q) = \frac{q^2}{q-1} \sum_{\eta \in \widehat{\mathbb{F}_q^*}} \eta(\tfrac{1}{\psi^5}).$$

We now use the fact that when $\psi^5 \neq 1$, $\chi(\frac{1}{\psi^5})$ is a $\frac{q-1}{5}$-th root of unity $\neq 1$ and so

$$\sum_{\eta \in \widehat{\mathbb{F}_q^*}} \eta(\tfrac{1}{\psi^5}) = \begin{cases} 0 & \text{if } \psi^5 \neq 1 \\ q-1 & \text{if } \psi^5 = 1 \end{cases} \qquad \square$$

We now proceed to give the link between $N_a(q)$ and $|\mathcal{A}_\psi(\mathbb{F}_q)|$ and between $N_b(q)$ and $|\mathcal{B}_\psi(\mathbb{F}_q)|$.

Lemma 29 *Suppose* $q \equiv 1 \mod 5$. *With the notations of Corollary 22,*

$$N_{(0,0,0,1,4),\chi}(q) = qN_{\mathcal{A}_\psi/\mathbb{F}_q,\chi} \quad \textit{and} \quad N_{(0,0,0,2,3),\chi}(q) = qN_{\mathcal{A}_\psi/\mathbb{F}_q,\chi^2},$$

and so the term $-10qN_a(q)$ *gives rise to a factor* $P_{\mathcal{A}_\psi/\mathbb{F}_q}(qt)^{10}$ *in the zeta function* $Z_{\mathcal{M}_\psi/\mathbb{F}_q}(t)$.

PROOF. Let us for example do the computations for $N_{(0,0,0,1,4),\chi}(q)$. Using formula (14), we have, doing obvious simplifications,

$$N_{(0,0,0,1,4),\chi}(q) = \frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}_q^*}} q^{3-z-\delta} \frac{G(\varphi, \chi^2\eta)G(\varphi, \chi^3\eta)}{G(\varphi, \eta)^2} \eta(\tfrac{1}{\psi^5}).$$

If, as in Theorem 21, ν is the number of trivial characters in the pair (η, η) (here, $a = 2$ and $b = 3$, so $a + b = 5$), we easily see that $z + \delta = 1 + \nu$ and so $3 - z - \delta = 2 - \nu$. Thus, using formula (13) for $\mathcal{A}_\psi$,

$$N_{(0,0,0,1,4),\chi}(q) = \frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}_q^*}} q^{2-\nu} \frac{G(\varphi, \chi^2\eta)G(\varphi, \chi^3\eta)}{G(\varphi, \eta)^2} \eta(\tfrac{1}{\psi^5}) = qN_{\mathcal{A}_\psi/\mathbb{F}_q,\chi}.$$

The proof for $N_{(0,0,0,2,3),\chi}(q)$ is similar. $\square$

Lemma 30 *Suppose* $q \equiv 1 \mod 5$. *With the notations of Corollary 22,*

$$N_{(0,0,1,1,3),\chi}(q) = qN_{\mathcal{B}_\psi/\mathbb{F}_q,\chi} \quad \textit{and} \quad N_{(0,0,1,2,2),\chi}(q) = qN_{\mathcal{B}_\psi/\mathbb{F}_q,\chi^2},$$

and so the term $-15qN_b(q)$ *gives rise to a factor* $P_{\mathcal{B}_\psi/\mathbb{F}_q}(qt)^{15}$ *in the zeta function* $Z_{\mathcal{M}_\psi/\mathbb{F}_q}(t)$.

PROOF. The proof is the same than for the previous lemma. □

Lemma 29 gives formula (2) of Theorem 1 of the introduction. Lemma 30 gives the analogous formula for $\mathcal{B}_\psi$. We have thus proved Theorem 1 in all cases.

6 Remarks on the singular case

When $\psi^5 = 1$ and $q \equiv 1 \mod 5$, the only thing that changes by comparison with the non singular case is that $24N_{\text{sing}}(q) = 24q^2$ and so the term $24N_{\text{sing}}(q)$ gives rise to a factor $\frac{1}{(1-q^2t)^{24}}$ in $Z_{\mathcal{M}_\psi/\mathbb{F}_q}(t)$. Moreover, in that case, $N_a(q)$ and $N_b(q)$ have very simple expressions as the equations of $\mathcal{A}_\psi$ and $\mathcal{B}_\psi$ become $y^5 = x^2(1-x)^5$ and so, by using the same methods as in §3.1 and §3.2 for the case $\psi = 0$,

$$|\mathcal{A}_\psi(\mathbb{F}_q)| = |\mathcal{B}_\psi(\mathbb{F}_q)| = \begin{cases} q-4 & \text{if } q \equiv 1 \mod 5 \\ q & \text{otherwise} \end{cases}$$

The zeta function of $\mathcal{A}_\psi$ and $\mathcal{B}_\psi$ over $\mathbb{F}_q$ is thus $\frac{(1-t)^4}{1-qt}$ when $q \equiv 1 \mod 5$. This result was also shown in [3, §11.3 p. 131-132] by using a change of variable.

For the sake of completeness, let us mention that when $\psi^5 = 1$, the factor P_w has a modular origin. More precisely, using results of Schoen [7, p. 109-110], Candelas, de la Ossa, and Rodriguez Villegas [3, §12 p. 133-134] showed that, when $\psi^5 = 1$ and $q = p$,

$$P_w(t) = (1 - (\tfrac{5}{p})pt)(1 - a_p t + p^3t^2),$$

where $(\frac{\cdot}{p})$ is the quadratic character of $\mathbb{F}_p^*$ (Legendre symbol) and a_p the p-th Fourier coefficient of a weight 4 and level 25 modular form that can be computed explicitly thanks to the Dedekind η function (see [3, §12 p. 134])

$$\begin{aligned} f(q) &= \eta(q^5)^4\Big(\eta(q)^4 + 5\eta(q)^3\eta(q^{25}) + 20\eta(q)^2\eta(q^{25})^2 + 25\eta(q)\eta(q^{25})^3 + 25\eta(q^{25})^4\Big) \\ &= q + q^2 + 7q^3 - 7q^4 + 7q^6 + 6q^7 - 15q^8 + 22q^9 - 43q^{11} - 49q^{12} - 28q^{13} \\ &\quad + 6q^{14} + 41q^{16} + 91q^{17} + 22q^{18} - 35q^{19} + 42q^{21} - 43q^{22} + 162q^{23} - 105q^{24} \\ &\quad - 28q^{26} - 35q^{27} - 42q^{28} + 160q^{29} + 42q^{31} + 161q^{32} - 301q^{33} + 91q^{34} \\ &\quad - 154q^{36} - 314q^{37} - 35q^{38} - 196q^{39} - 203q^{41} + 42q^{42} + 92q^{43} + 301q^{44} \\ &\quad + 162q^{46} + 196q^{47} + 287q^{48} - 307q^{49} + 637q^{51} + 196q^{52} + 82q^{53} + \dots \end{aligned}$$

Acknowledgements

The author would like to thank his advisor, J. Oesterlé, for his comments on preliminary versions of this paper.

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