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ON THE ARC AND CURVE COMPLEX OF A SURFACE

MUSTAFA KORKMAZ AND ATHANASE PAPADOPOULOS

ABSTRACT. We study the *arc and curve* complex $AC(S)$ of an oriented connected surface S of finite type with punctures. We show that if the surface is not a sphere with one, two or three punctures nor a torus with one puncture, then the simplicial automorphism group of $AC(S)$ coincides with the natural image of the extended mapping class group of S in that group. We also show that for any vertex of $AC(S)$, the combinatorial structure of the link of that vertex characterizes the type of a curve or of an arc in S that represents that vertex. We also give a proof of the fact if S is not a sphere with at most three punctures, then the natural embedding of the curve complex of S in $AC(S)$ is a quasi-isometry. The last result, at least under some slightly more restrictive conditions on S , was already known. As a corollary, $AC(S)$ is Gromov-hyperbolic.

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Keywords: curve complex ; arc complex ; arc and curve complex ; mapping class group.

1. INTRODUCTION

In this paper, $S = S_{g,n}$ is a connected orientable surface of genus $g \geq 0$ without boundary and with $n \geq 0$ punctures. The *mapping class group* of S , denoted by $\text{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of S . The *extended mapping class group* of S , $\text{Mod}^*(S)$, is the group of isotopy classes of all homeomorphisms of S .

We shall denote by \overline{S} the closed surface obtained by filling in the punctures of S , and by B the set of punctures when these points are considered as elements of \overline{S} . The elements of B are also called *distinguished points* of \overline{S} .

A simple closed curve on S or on \overline{S} is said to be *essential* if it is not homotopic to a point or to a puncture (or a distinguished point).

An *arc* in S or in \overline{S} is the homeomorphic image of a closed interval in \overline{S} whose interior is in $\overline{S} \setminus B$ and whose endpoints are in B . An arc is said to be *essential* if it is not homotopic (relative to B) to a point in \overline{S} .

When the setting is clear, we shall use the term “curve” instead of “essential simple closed curve” and the term “arc” instead of “essential arc”.

The *curve complex* of S , denoted by $C(S)$, is the abstract simplicial complex whose k -simplices, for each $k \geq 0$, are the sets of $k + 1$ distinct isotopy classes of curves in S that can be represented by pairwise disjoint curves on this surface. The curve complex has been introduced by Harvey [2], and

since then it has been investigated from various points of view (see in particular the papers [4], [5], [6] and [7]). We shall use below an alternative definition of $C(S_{g,n})$ in the case $(g, n) = (0, 4)$ or $(g, n) = (1, 1)$.

The *arc complex* of S , denoted by $A(S)$, is defined analogously, with curves replaced by arcs. The arc complex has also been studied by several authors, see for instance [1], [3], [4] and [5].

In this paper, we study the *arc and curve complex*, $AC(S)$, an abstract simplicial complex in which the curve complex and the arc complex naturally embed. The arc and curve complex is the abstract simplicial complex whose k -simplices, for each $k \geq 0$, are collections of $k + 1$ distinct isotopy classes of one-dimensional submanifolds which can be either essential simple closed curves or essential arcs in S , such that this collection can be represented by disjoint curves or arcs on the surface. The arc and curve complex was studied by Hatcher in [1], who proved that this complex is contractible.

Note that if $n = 0$, then there are no arcs on S , and in that case $AC(S_{g,0}) = C(S_{g,0})$. From now on, we assume that $n \geq 1$.

Each element of the extended mapping class group $\text{Mod}^*(S)$ acts naturally in a simplicial way on the complex $AC(S)$, and it is clear that the resulting map from $\text{Mod}^*(S)$ to the simplicial automorphism group $\text{Aut}(AC(S))$ of $AC(S)$ is a homomorphism.

There are natural simplicial maps from the curve complex $C(S)$ and from the arc complex $A(S)$ into the arc and curve complex, which extend the natural inclusions at the level of the vertices.

In this paper, we prove the following:

Theorem 1.1. *If the surface $S_{g,n}$ is not a sphere with one, two or three punctures nor a torus with one puncture, then the natural homomorphism $\text{Mod}^*(S_{g,n}) \rightarrow \text{Aut}(AC(S_{g,n}))$ is an isomorphism.*

The analogous result for the curve complex is due to Ivanov, Korkmaz and Luo ([4], [5], and [6]), Theorem 3.2 below. The analogous result for the arc complex has been recently obtained by Irmak and McCarthy ([3]), Theorem 3.1 below.

We shall give two different proofs of Theorem 1.1. The first one involves passing to the arc complex, and it uses the result by Irmak and McCarthy. The second proof involves passing to the curve complex, and it is based on the result of Ivanov, Korkmaz and Luo. We note that the proof of Irmak and McCarthy's result on the automorphisms of the arc complex does not use the result by Ivanov, Korkmaz and Luo.

We then show the following:

Theorem 1.2. *For any surface S , the combinatorics of the link at each vertex of $AC(S)$ is sufficient to characterize the type of curve or arc that represents such a vertex.*

We shall be more precise about the meaning of Theorem 1.2. Note that it is a local result, which does not immediately follow from Theorem 1.1.

If $(g, n) = (0, 4)$ or $(g, n) = (1, 1)$, the curve complex $C(S)$ is infinite discrete in the above definition. For Theorem 1.3 below, we change the definition of $C(S)$ in these two cases as follows: The complex $C(S)$ is the graph whose vertices are isotopy classes of simple closed curves and an edge

is placed between two vertices if two representatives of the vertices have minimal intersection number, which is 1 if $g = 1$ and 2 if $g = 0$.

Here and throughout the paper, we shall consider geometric realization of the simplicial complexes $AC(S)$ and $C(S)$, and not only these complexes as abstract simplicial complexes. In particular, we make these complexes into complete geodesic metric spaces in a natural way by making each simplex a regular Euclidean simplex of sidelength 1, as in [7]. We call these metrics on $AC(S)$ and $A(S)$ the *simplicial metrics* on these spaces. In the last section of the paper, we give a proof of the following result:

Theorem 1.3. *If the surface $S_{g,n}$ is not a sphere with at most three punctures, then the natural map $C(S_{g,n}) \rightarrow AC(S_{g,n})$ is a quasi-isometry. Furthermore, each point in $AC(S_{g,n})$ is at distance at most one from a point in the image of $C(S_{g,n})$.*

For the cases where $(g, n) = (0, 4)$ or $(1, 1)$, a special discussion is needed. The result given in Theorem 1.3, with slightly less general conditions on the topological type of the surface $S_{g,n}$, is not new; it is mentioned in Schleimer's notes [8]. Although the result is independent from the rest of the paper, we think it is useful to provide a complete proof of it.

The complex $AC(S)$ is finite if S is a sphere with at most three punctures. As a consequence of Theorem 1.3, and the fact that the curve complex is Gromov-hyperbolic (see [7]), we have the following corollary:

Corollary 1.4. *The arc and curve complex $AC(S_{g,n})$ of any surface $S_{g,n}$ is Gromov-hyperbolic.*

Note that the condition that $S_{g,n}$ is not a surface with at most three punctures is not excluded in this corollary, because finite metric spaces are always Gromov-hyperbolic.

2. MAXIMAL SIMPLICES IN $AC(S)$

The results of this section will be used in the proof of Theorem 1.1.

A simplex in $AC(S)$ is said to be *maximal* if it is maximal in the sense of inclusion.

Let Δ be a maximal simplex in $AC(S)$, and suppose that the vertices of Δ are isotopy classes of arcs. Then a set of disjoint representatives of Δ is an *ideal triangulation* of the closed surface \bar{S} , that is, a triangulation having all its vertices at the distinguished points. We shall also call Δ an *ideal triangulation* of S .

Let us take again a maximal simplex Δ in $AC(S)$ and let δ be a set of representatives of elements of Δ that are pairwise disjoint. Each component of the surface S_δ obtained by cutting S along δ is of one of the following three forms: (1) a triangle; (2) an annulus with one of its boundary components being a simple closed curve in δ and the other boundary component being an arc joining a puncture to itself; (3) a pair of pants all of whose boundary components are simple closed curves (see Figure 1).

We shall use the following:

Lemma 2.1. *A maximal simplex of Δ in $AC(S)$ that has maximal dimension consists of arcs, i.e. it is an ideal triangulation of S . The dimension of such a simplex is $6g + 3n - 7$.*

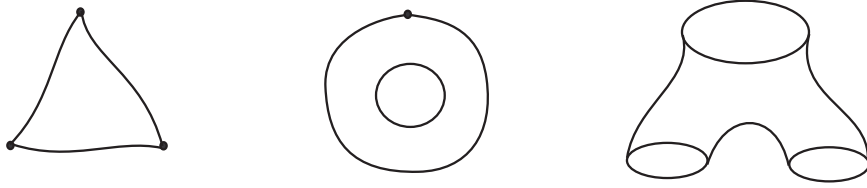


FIGURE 1. Possible components of the complement of a maximal simplex in the complex $AC(S)$.

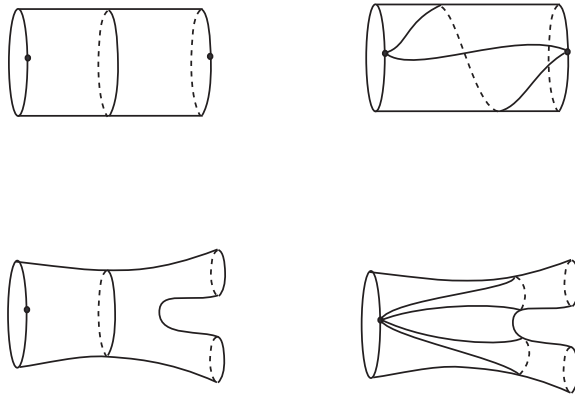


FIGURE 2. The dimension of the maximal simplex is increased in the presence of a curve.

Proof. The first statement is a consequence of the fact that if we start with a maximal simplex in $AC(S)$ having at least one vertex that is the isotopy class of a simple closed curve, then we can obtain a maximal simplex of one dimension higher by replacing one of these simple closed curves by two arcs (see Figure 2). Thus, a maximal simplex Δ having maximal dimension consists only of isotopy classes of arcs, and it is therefore the isotopy class of an ideal triangulation of S . The dimension statement follows then by a standard Euler characteristic argument, which gives the number of curves in an ideal triangulation of S . This number is $6g + 3n - 6 = -3\chi(S)$, which implies that the dimension of the maximal simplex Δ is $6g + 3n - 7$. \square

Lemma 2.2. *The dimension of a maximal simplex Δ in $AC(S)$ that has minimal dimension is $3g + 2n - 4$.*

Proof. The proof is analogous to the proof of Lemma 2.1. We use the fact that a maximal simplex that has minimal dimension has a maximal number of curves, which is $3g + n - 3$. If we cut the surface S along such a collection of curves, we get a collection of subsurfaces of Euler characteristic -1 . Each of these subsurfaces is either a disc with two punctures, or an annulus with one puncture or a pair of pants. On a disc with two punctures, there are systems of two disjoint nonisotopic arcs (and no systems of three nonisotopic arcs); on an annulus with one puncture there is only one essential arc; on a pair of

pants there are no arcs. Hence, the number of curves and arcs on S forming a maximal simplex of minimal dimension is $3g + n - 3 + n = 3g + 2n - 3$. The dimension of that simplex is therefore $3g + 2n - 4$. \square

From Lemmas 2.1 and 2.2, it follows that the dimension of maximal simplices in $AC(S)$ is bounded between $3g + 2n - 4$ and $6g + 3n - 7$. It can be seen, using the arguments of the proof of these lemmas that every integer between these two bounds is realized as the dimension of a maximal simplex.

3. PROOF OF THEOREM 1.1

We now give our first proof of Theorem 1.1. We may assume that the arc complex of the surface is not empty. Recall that we have the assumption $n \geq 1$. We use the following:

Theorem 3.1 (Irmak-McCarthy [3]). *Let S be a connected orientable surface of genus $g \geq 0$ with $n \geq 1$ punctures, which is not homeomorphic to a sphere with one, two or three punctures, or a torus with one puncture. Then the natural map $\text{Mod}^*(S) \rightarrow \text{Aut}(A(S))$ is an isomorphism. Furthermore, in the excluded cases, the natural map $\text{Mod}^*(S) \rightarrow \text{Aut}(A(S))$ is surjective and its kernel is the center of $\text{Mod}^*(S)$.* \square

For the proof of Theorem 1.1, let $\Phi : \text{Mod}^*(S) \rightarrow \text{Aut}(AC(S))$ denote the natural map. We define a natural map $\text{Aut}(AC(S)) \rightarrow \text{Aut}(A(S))$ as follows: Let φ be an automorphism of $AC(S)$. If a is an arc on S , then a is contained in a maximal simplex Δ of dimension $6g + 3n - 7$. (Here and in the sequel, we do not distinguish between an arc and its isotopy class. Likewise, we do not distinguish between a curve and its isotopy class.) Since $\varphi : AC(S) \rightarrow AC(S)$ is a simplicial automorphism, the image simplex $\varphi(\Delta)$ is also of dimension $6g + 3n - 7$ containing $\varphi(a)$. By Lemma 2.1, $\varphi(a)$ is an arc. It follows that φ maps arcs to arcs. Since the inverse of φ also maps arcs to arcs, the restriction of φ to arcs gives an automorphism $\tilde{\varphi} : A(S) \rightarrow A(S)$. It is easy to see that the resulting map $\varphi \mapsto \tilde{\varphi}$ is a homomorphism, which we denote by $\Psi : \text{Aut}(AC(S)) \rightarrow \text{Aut}(A(S))$.

Let $\Theta : \text{Mod}^*(S) \rightarrow \text{Aut}(A(S))$ be the natural map, which is an isomorphism by Theorem 3.1. Clearly, we have the equality $\Psi \circ \Phi = \Theta$, deducing that Φ is one-to-one and Ψ is onto.

$$\begin{array}{ccc} \text{Mod}^*(S) & \xrightarrow{\Theta} & \text{Aut}(A(S)) \\ \downarrow \Phi & & \uparrow \Psi \\ \text{Aut}(AC(S)) & & \end{array}$$

Let φ be an automorphism of $AC(S)$ such that φ fixes each arc. We claim that φ is the identity. This will show that Ψ is one-to-one, concluding the proof that the natural map Φ is onto.

Let b be a curve on S . It suffices to show that $\varphi(b) = b$. Suppose first that b is nonseparating. It is easy to see that there is a maximal simplex Δ in $AC(S)$ containing b such that all elements of $\Delta' = \Delta - \{b\}$ are arcs and that b is the only curve disjoint from all elements of Δ' . Since the automorphism φ fixes each element Δ' and $\varphi(b)$ is a curve disjoint from all elements of Δ' , we have $\varphi(b) = b$. If c is a separating curve on S , then it is easy to

find a maximal simplex Σ in $AC(S)$ containing c such that each element of $\Sigma' = \Sigma - \{c\}$ is either a nonseparating curve or an arc, and that c is the only curve disjoint from all elements of Σ' . Since φ fixes each element Σ' , it must fix c as well.

This completes the first proof of Theorem 1.1.

We now give our second proof of Theorem 1.1, using the curve complex instead of the arc complex. We use the following:

Theorem 3.2 (Ivanov-Korkmaz-Luo [4, 5, 6]). *Let S be a surface of genus g with n punctures. If $2g+n \geq 5$ then the natural map $\text{Mod}^*(S) \rightarrow \text{Aut}(C(S))$ is an isomorphism. \square*

Note that the hypothesis (and therefore the second proof of Theorem 1.1 that we give below) excludes the cases where the surface S is the sphere with four punctures or the torus with two punctures, but these cases were taken care in the first proof.

For the second proof of Theorem 1.1, let $\Phi : \text{Mod}^*(S) \rightarrow \text{Aut}(AC(S))$ be the natural map, as above. If φ is an automorphism of $AC(S)$, it maps arcs to arcs and curves to curves, as shown in the first proof above. Hence, φ induces an automorphism $\tilde{\varphi} : C(S) \rightarrow C(S)$, giving a homomorphism $\Psi : \text{Aut}(AC(S)) \rightarrow \text{Aut}(C(S))$ defined by $\varphi \mapsto \tilde{\varphi}$.

Let $\Theta : \text{Mod}^*(S) \rightarrow \text{Aut}(C(S))$ be the natural homomorphism, which is an isomorphism by Theorem 3.2. It is easy to see that $\Psi \circ \Phi = \Theta$, deducing that Φ is one-to-one and Ψ is onto.

$$\begin{array}{ccc} \text{Mod}^*(S) & \xrightarrow{\Theta} & \text{Aut}(C(S)) \\ \downarrow \Phi & & \uparrow \Psi \\ \text{Aut}(AC(S)) & & \end{array}$$

Let φ be an automorphism of $AC(S)$ such that φ fixes each curve. We claim that φ is the identity. This will show that Ψ is one-to-one, concluding the proof that the natural map Φ is onto.

Let a be an arc on S . It suffices to show that $\varphi(a) = a$.

Suppose first that a joins a puncture P to itself and that both boundary components of a regular neighborhood of $a \cup \{P\}$ are nontrivial curves. One can easily construct a (finite) set Σ of curves on S such that each element of Σ is disjoint from a , and that a is the only arc disjoint from all elements of Σ . Since φ fixes each element of Σ and $\varphi(a)$ is an arc disjoint from all elements of Σ , we conclude that $\varphi(a) = a$.

Suppose next that a joins two distinct punctures, say P and Q . One can construct a (finite) set Σ consisting of curves and arcs, as in the previous paragraph, on S such that each element of Σ is disjoint from a , and that a is the only arc disjoint from all elements of Σ . Since φ fixes each element of Σ and $\varphi(a)$ is an arc disjoint from all elements of Σ , we get that $\varphi(a) = a$.

Suppose finally that a joins a puncture P to itself, but one of the boundary components of a regular neighborhood of $a \cup \{P\}$ is trivial, i.e. bounds a disc with one puncture, say Q . Let D be the disk with two punctures P and Q such that a lies on D . Let a' be the arc joining P and Q on D , which is disjoint from a . Let Σ be a finite set of arcs and curves on S such that the

elements of Σ are fixed by φ , and that a and a' are the only arcs disjoint from the elements of Σ . Since φ fixes each element of $\Sigma \cup \{a'\}$, we conclude that $\varphi(a) = a$.

This completes the second proof of Theorem 1.1.

We conclude this section with a few words on the case of surfaces excluded by the hypothesis of Theorem 1.1.

In the case of a sphere with one puncture, $AC(S)$ is empty. In the case of a sphere with two punctures, $AC(S) = A(S)$ is a single point, and in the case of a sphere with three punctures, $AC(S) = A(S)$ is a finite complex (see Figure 3). In these two special cases, by Theorem 3.1 the natural homomorphism from $\text{Mod}^*(S_{g,n})$ to $\text{Aut}(AC(S_{g,n}))$ is surjective and its kernel is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, which is the center of $\text{Mod}^*(S_{g,n})$. Finally, in the case where S is a torus with one puncture, the arguments given in the first proof of Theorem 1.1 and the fact that $\text{Mod}^*(S_{g,n}) \rightarrow \text{Aut}(A(S_{g,n}))$ is surjective with kernel \mathbb{Z}_2 (see Theorem 3.1) show that the natural homomorphism $\text{Aut}(AC(S_{g,n})) \rightarrow \text{Aut}(A(S_{g,n}))$ is an isomorphism. This shows that we have an isomorphism $\text{Mod}^*(S_{g,n})/\mathbb{Z}_2 \simeq \text{Aut}(AC(S))$.

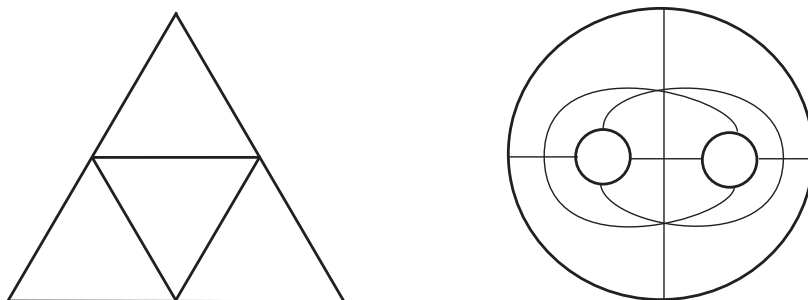


FIGURE 3. The figure on the left hand side represents the arc complex of the sphere with three punctures. It is a finite simplicial complex, having six vertices, nine edges and four two-cells. The vertices of this complex are the isotopy classes of the arcs represented in the right hand side figure, which represents a three-punctured sphere in which the punctures have been replaced by circles, for more visibility.

4. PROOF OF THEOREM 1.2

In this section, we consider the following classification of the vertices of $AC(S)$ into five types. A representative on S of a vertex is of one of the following types:

- (1) a separating closed curve;
- (2) an arc connecting a puncture to itself that is separating the surface;
- (3) a nonseparating closed curve;
- (4) an arc connecting a puncture to itself that is nonseparating the surface;
- (5) an arc connecting two distinct punctures.

The stars and the links of vertices will be used to distinguish combinatorially these types of vertices from each other. For the convenience of the reader, we recall the definition of the star and link of a vertex in a simplicial complex, which will be useful below. Given a vertex v in an abstract simplicial complex E , its *star*, $\text{St}(v)$, is the subcomplex of E whose simplices are the simplices of E that are of the following types: (1) the simplices of E that contain v , and (2) the faces of such simplices. The *link* of v , $\text{Lk}(v)$, is then the subcomplex of E whose simplices are those simplices of $\text{St}(v)$ whose intersection with v is empty.

We now prove Theorem 1.2.

In the above classification of vertices of $AC(S)$, Cases (1) and (2) can be distinguished from the three others by the fact that the link of a vertex in any one of these two cases is the join of two nonempty subcomplexes. Equivalently, the dual link, in each of these two cases, is the disjoint union of two nonempty sets. Here, by the *dual link* L^* of a link L , we mean an abstract graph whose vertices are the vertices of L , and two vertices of L^* are connected by an edge if and only if these vertices are not connected by an edge when they are considered as vertices in L . In cases (3)-(5), the dual link is connected.

Case (1) can be distinguished from Case (2) by the fact that the highest dimension of a maximal simplex of a vertex in Case (2) is bigger than the corresponding dimension for Case (1). (In fact, this dimension is $6g + 3n - 8$ in Case (1) and $6g + 3n - 7$ in Case (2).)

Thus, each of Cases (1) and (2) is completely distinguished from all the other cases.

Now, we deal with the remaining three cases. Case (3) is distinguished from the other two by looking at the highest dimension of maximal simplices containing such a vertex. The highest dimension of any maximal simplex containing a vertex in Case (3) is $6g + 3n - 8$, whereas in the other two cases this dimension is $6g + 3n - 7$.

It remains to distinguish Case (4) from Case (5). This is done by the following property. Consider on the closed surface \bar{S} a closed curve Z which is the boundary of a regular neighborhood of an arc joining the two distinguished points, representing a vertex in Case (5), which we call a . The vertex z represented by the curve Z is in $\text{Lk}(a)$ and it has the following property: z is represented by a separating curve and $\text{St}(z) \subset \text{St}(a)$. Note that we already distinguished the vertices in $AC(S)$ represented by arcs from those represented by closed curves by combinatorial information at the links of these vertices; likewise, we already distinguished combinatorially the vertices represented by separating closed curves from those represented by non-separating curves. Therefore, the property stated is a combinatorial property of the link of a . Now in the link of a vertex a of Type (4), there is no vertex x represented by a separating curve X satisfying $\text{St}(x) \subset \text{St}(a)$. Indeed, if X is any separating curve on S disjoint from an arc A representing the vertex a , then A is in one of the two connected components of $S \setminus X$. This component has positive genus (since A is also non-separating in that component), therefore we can find a closed curve Y in $S \setminus X$ that intersects A in exactly one point. Now the vertex y represented by Y is in $\text{St}(x)$, but

not in $\text{St}(a)$, showing that $\text{St}(x)$ is not a subset of $\text{St}(a)$. This completes the proof of the theorem.

Theorem 1.2 shows that not only a full automorphism of the complex $AC(S)$ recognizes the topological types of the vertices, as is implied by Theorem 1.1, but that also that local combinatorial data contained in the links of each vertex are sufficient to characterize these types.

5. PROOF OF THEOREM 1.3

We equip $C(S)$ and $AC(S)$ with the simplicial metrics recalled in the introduction. Let $C^0(S)$ be the set of vertices of $C(S)$. Since the dimension of simplices in the complex $AC(S)$ is uniformly bounded, the inclusion map $C^0(S) \rightarrow C(S)$ is a quasi-isometry. Thus, in order to show that the natural map $C(S) \rightarrow AC(S)$ is a quasi-isometry, it suffices to show that the natural map $C^0(S) \rightarrow AC(S)$ is a quasi-isometry. Thus, it suffices to reason on distances between vertices of $C(S)$.

We let $C^0(S) \rightarrow AC(S)$ be the natural inclusion map, and we let d_C (respectively d_{AC}) denote the distance function in $C(S)$ (respectively $AC(S)$). We identify $C^0(S)$ with its image in $AC(S)$.

To see that every point in $AC(S)$ is at distance at most one from a point in $C(S)$, we consider, for each essential arc A on S , one of the boundary components of a regular neighborhood of A . Since the surface S is not a sphere with at most three punctures, we can choose this boundary component to be essential. The isotopy class of such an essential simple closed curve is a vertex in $AC(S)$ which is at distance one from the vertex represented by the arc A .

For the proof of the first claim in Theorem 1.3, we prove that for every pair of vertices x and y in $C(S)$, we have

$$(1) \quad \frac{1}{2} d_C(x, y) \leq d_{AC}(x, y) \leq d_C(x, y)$$

if $2g + n \geq 5$,

$$(2) \quad d_C(x, y) \leq d_{AC}(x, y) \leq d_C(x, y) + 2$$

if $(g, n) = (1, 1)$, and

$$(3) \quad \frac{1}{2} d_C(x, y) \leq d_{AC}(x, y) \leq d_C(x, y) + 2$$

if $(g, n) = (0, 4)$.

This will complete the proof of Theorem 1.3.

If $x = y$ then clearly all inequalities hold. So we assume that x and y are distinct.

Suppose first that $2g + n \geq 5$. The second inequality in (1) is clear from the definitions. We prove the first one.

Suppose that $d_{AC}(x, y) = L$. Consider a simplicial path p of length L in $AC(S)$ joining x and y . We show that we can find a simplicial path p' of length $\leq 2L$ in $AC(S)$ joining these two vertices such that all the vertices of p' are in $C(S)$. The construction of the path p' is by induction. We describe the first step, and the induction step is similar.

There are three cases.

Case 1. The first edge of the simplicial path p joins the vertex x to a vertex z represented by a simple closed curve in S . In this case, we do not change this part of the path p , and we go to the next step, that is, we examine the part of the path p that starts at the second vertex, z .

Case 2. The first edge of the path p joins the vertex x to a vertex a represented by an arc A in S , and the second edge joins the vertex a to a vertex z represented by a curve. Note that x is not connected to z by an edge. Note also that a tubular neighborhood N of A is either a disk with two punctures or an annulus with one puncture. Since $2g + n \geq 5$, one of the boundary components, say V , of N is a simple closed curve not bounding a disk with at most one puncture. In this case, let v be the isotopy class of V . Note that v is a vertex of $C(S)$ which is at distance one from x and from z . We replace the path xaz by the path xvz . Such a step does not change the length of the path joining x to y .

Case 3. The first edge of the path p joins the vertex x to a vertex a represented by an arc A on S , and the second edge joins the vertex a to a vertex b that is also represented by an arc B . We may assume that A and B are disjoint. In this case, we distinguish nine subcases, depending on the relative position of A and B on the surface S . These cases are represented in Figure 4, (i) to (ix). In each case, we replace the path xab by either a path of the same length, xzb , or a path of length three, $xzvb$. Here, z is the isotopy class of a nontrivial boundary component of A , and v is the isotopy class of a nontrivial boundary component of $A \cup B$. The curves are represented in dashed lines in Figure 4, and one can check in each case that either a vertex z or a pair of adjacent vertices z and v with the appropriate requirements exists. Note that this also uses fact that the surface S is not a sphere with at most three punctures.

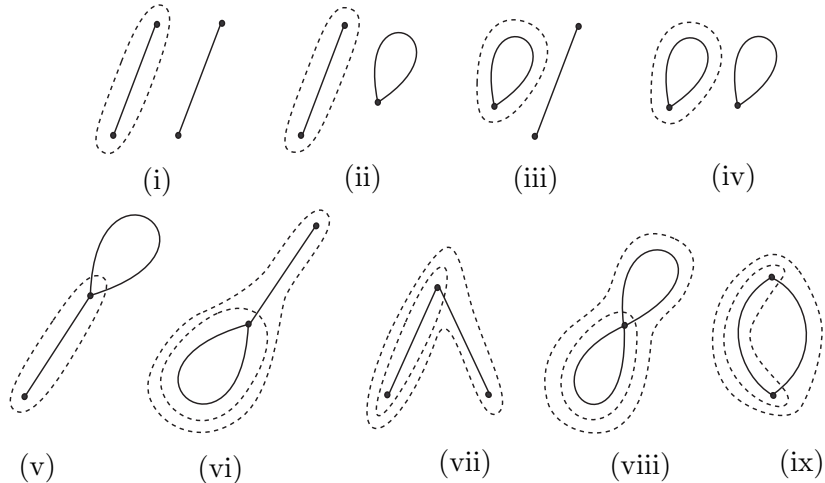


FIGURE 4. These nine cases, used in the proof of Theorem 1.3, are the various possibilities for an ordered pair of consecutive vertices of $AC(S)$ that represent by arcs on S . The dashed lines are closed curves used to replace the arcs.

Thus we get a simplicial path p' between x and y of length $\leq 2L$ such that all all vertices are in $C(S)$. This concludes the proof of (1).

Suppose now that $(g, n) = (1, 1)$. For each vertex z in $AC(S)$ represented by a curve, there is a unique vertex z' represented by an arc such that zz' is an edge in $AC(S)$. Conversely, for each vertex z' in $AC(S)$ represented by an arc there is a unique vertex z represented by a curve such that $z'z$ is an edge in $AC(S)$. From this, it is easy to see that for any two vertices z and v represented by curves

- (i) there are no simplicial paths of form zv , or zav , or azb in $AC(S)$, where a and b are represented by arcs, and
- (ii) if there is a simplicial path $zabv$ in $AC(S)$ then $d_C(z, v) = 1$.

Suppose that $d_C(x, y) = L$. Since x and y are distinct, we have $L \geq 3$ by (i). If $xx_1x_2 \cdots x_{L-1}y$ is a path of length L in $C(S)$, so that adjacent vertices have representatives intersecting only once, then $xx'x'_1x'_2 \cdots x'_{L-1}y'y$ is a path of length $L + 2$ in $AC(S)$, showing that $d_{AC}(x, y) \leq d_C(x, y) + 2$.

Suppose now that $d_{AC}(x, y) = L$. Let p be a simplicial path of length L in $AC(S)$ joining x and y . By (i), the vertices of p other than x and y are all represented by arcs. Thus the path p must be of the form $xx'x'_1 \cdots x'_{L-3}y'y$. Then by (ii), the path $xx_1x_2 \cdots x_{L-3}y$ is of length $L - 2$ in $C(S)$, showing that $d_C(x, y) \leq d_{AC}(x, y) - 2$. This finishes the proof of (2).

Suppose finally that $(g, n) = (0, 4)$. For each vertex a in $AC(S)$ represented by an arc, there is a unique vertex v_a represented by a curve such that av_a is an edge in $AC(S)$. For each vertex z in $AC(S)$ represented by a curve, there are two vertices a and b connected to z by an edge such that representatives of each of a and b joins different punctures. Let z' be any one of them. Note that if z and v are connected by an edge in $C(S)$, so that they have representatives intersecting twice, then $zz'v'v$ is a simplicial path in $AC(S)$ for any choice of z' and v' . It is now easy to see that for any two vertices z and v represented by curves

- (iii) there are no simplicial paths of the form zv , or zav in $AC(S)$, where a is represented by an arc, and
- (iv) if there is a simplicial path $zabv$ in $AC(S)$ then $d_C(z, v)$ is equal to 1 or 2.

Suppose that $d_C(x, y) = L$. Since x and y are distinct, we have $L \geq 3$. Let $xx_1x_2 \cdots x_{L-1}y$ be a path of length L in $C(S)$, so that adjacent vertices have representatives intersecting twice. Then $xx'x'_1x'_2 \cdots x'_{L-1}y'y$ is a path of length $L + 2$ in $AC(S)$, showing that $d_{AC}(x, y) \leq d_C(x, y) + 2$.

Suppose now that $d_{AC}(x, y) = L$. Let p is a simplicial path of length L in $AC(S)$ joining x to y . By (iii), we have $L \geq 3$. If there is a part of p of the form azb , then by changing z with z' for a suitable choice, we can assume that all vertices of p other than x and y are represented by arcs.

Thus the path p must be of the form $xa_1a_2 \cdots a_{L-1}y$, where each a_i is represented by an arc. If v_i denotes the vertex v_{a_i} , we get a sequence of vertices $x, v_2, v_3, \cdots, v_{L-2}, y$ in $C(S)$. Since the distance between adjacent vertices in this sequence is at most 2 by (iv), we get that $d_C(x, y) \leq 2(L-2)$, concluding that $\frac{1}{2} d_C(x, y) + 2 \leq d_{AC}(x, y)$.

This finishes the proof of (3), concluding the proof of Theorem 1.3.

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REFERENCES

- [1] A. Hatcher, On triangulations of surfaces, *Top. and its Appl.* 40 (1991), 189-194. A new version is available on the author's webpage.
- [2] W. J. Harvey, Geometric structure of surface mapping class groups, in: *Homological group theory* (Proc. Sympos., Durham, 1977). London Math. Soc. Lecture Note Ser., 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 255–269.
- [3] E. Irmak and J. D. McCarthy, Injective simplicial maps of the arc complex, to appear in *Turkish Mathematical Journal*.
- [4] N. V. Ivanov, Automorphisms of Teichmüller modular groups. *Lecture Notes in Math.*, No. 1346, Springer-Verlag, Berlin and New York, 1988, 199-270.
- [5] M. Korkmaz, Automorphisms of complexes of curves on punctured spheres and on punctured tori. *Topology and its Applications* 95 no. 2; 85-111 (1999).
- [6] F. Luo, Automorphisms of the complex of curves, *Topology* 39, no. 2, 283-298 (2000).
- [7] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1), 103–149 (1999).
- [8] S. Schleimer, Notes on the complex of curves, Notes from a mini-course given at Caltech in January 2005, available on the web.

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