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► **To cite this version:**

Mustafa Korkmaz, Athanase Papadopoulos. On the arc and curve complex of a surface. 2009. hal-00405188v2

HAL Id: hal-00405188

<https://hal.science/hal-00405188v2>

Preprint submitted on 21 Jul 2009 (v2), last revised 11 Aug 2009 (v4)

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ON THE ARC AND CURVE COMPLEX OF A SURFACE

MUSTAFA KORKMAZ AND ATHANASE PAPADOPOULOS

ABSTRACT. We study the *arc and curve* complex $AC(S)$ of an oriented connected surface S of finite type with punctures. We show that except for some finitely many special surfaces, the simplicial automorphism group of $AC(S)$ coincides with the natural image of the extended mapping class group of S in that group. We also show that for any vertex of $AC(S)$, the combinatorial structure of the link of that vertex is sufficient to characterize the topological type of the curve or of the arc on S that represents this vertex. Finally, we show that the natural embedding of the curve complex in $AC(S)$ is a quasi-isometry.

AMS Mathematics Subject Classification: Primary 32G15; Secondary 20F38, 30F10.

Keywords: curve complex ; arc complex ; arc and curve complex ; mapping class group.

1. INTRODUCTION

In this paper, $S = S_{g,n}$ is a connected orientable surface of genus $g \geq 0$ without boundary and with $n \geq 0$ punctures. The *mapping class group* of S , denoted by $\text{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of S . The *extended mapping class group* of S , $\text{Mod}^*(S)$, is the group of isotopy classes of all homeomorphisms of S .

We shall denote by \overline{S} the closed surface S obtained by filling in the punctures of S , and by B the set of punctures when these points are considered as elements of \overline{S} . The elements of B in \overline{S} are also called *distinguished points* of \overline{S} .

A simple closed curve in S or in \overline{S} is said to be *essential* if it is not homotopic to a point or to a puncture (or a distinguished point).

An *arc* in S or in \overline{S} is the homeomorphic image of a closed interval in \overline{S} , whose interior is in $\overline{S} \setminus B$ and whose endpoints are on B . An arc is said to be *essential* if it is not homotopic (relative to B) to a point in \overline{S} .

When the setting is clear, we shall sometimes use the term “curve” instead of “essential simple closed curve” and the term “arc” instead of “essential arc”.

The *curve complex* of S , denoted by $C(S)$, is the abstract simplicial complex whose k -simplices, for each $k \geq 0$, are the sets of $k + 1$ distinct isotopy classes of curves in S that can be represented by pairwise disjoint curves on this surface. The curve complex has been introduced by Harvey [1], and since then it has been investigated from various points of view (see in particular the papers [4], [5], [6] and [7]).

The *arc complex* of S , denoted by $A(S)$, is defined analogously, with curves replaced by arcs. The arc complex has also been studied in several papers, see for instance [3], [4] and [5].

In this paper, we study the *arc and curve complex*, $AC(S)$, an abstract simplicial complex in which the curve complex and the arc complex naturally embed. The arc and curve complex is the abstract simplicial complex whose k -simplices, for each $k \geq 0$, are collections of $k + 1$ distinct isotopy classes of one-dimensional submanifolds which can be either essential simple closed curves on S or essential arcs in S , such that this collection can be represented by disjoint curves or arcs on the surface.

Note that if $n = 0$, then there are no arcs on S , and in that case $AC(S_{g,0}) = C(S_{g,0})$.

The arc and curve complex was introduced by Allen Hatcher in [2], who showed that this complex is contractible whenever it is not empty.

Each element of the extended mapping class group $\text{Mod}^*(S)$ acts naturally in a simplicial way on the complex $AC(S)$, and it is clear that the map that we obtain in this way from $\text{Mod}^*(S)$ to the simplicial automorphism group $\text{Aut}(AC(S))$ of $AC(S)$ is a homomorphism.

There are natural simplicial maps from the curve complex $C(S)$ and from the arc complex $A(S)$ into the arc and curve complex, which extend the natural inclusion map at the level of the vertices.

In this paper, we prove the following:

Theorem 1.1. *If the surface $S_{g,n}$ is not a sphere with one, two or three punctures nor a torus with one puncture, then the natural homomorphism $\text{Mod}^*(S_{g,n}) \rightarrow \text{Aut}(AC(S_{g,n}))$ is an isomorphism.*

The analogous result for the curve complex is due to Ivanov, Korkmaz and Luo ([4], [5], and [6]), Theorem 3.2 below. The analogous result for the arc complex has been recently obtained by Irmak and McCarthy ([3]), Theorem 3.1 below.

We shall give two different proofs of Theorem 1.1. The first one involves passing to the arc complex, and it uses the result by Irmak and McCarthy. The second proof involves passing to the curve complex, and it is based on the result of Ivanov, Korkmaz and Luo. We note that the proof of Irmak and McCarthy's result on the automorphisms of the arc complex does not use the result by Ivanov, Korkmaz and Luo.

We then show the following:

Theorem 1.2. *For any surface S , the combinatorics of the link at each vertex of $AC(S)$ is sufficient to characterize the type of curve or arc that such a vertex represents.*

We shall be more precise about the meaning of Theorem 1.2. Note that it gives a local result, which does not immediately follow from Theorem 1.1.

We also prove the following

Theorem 1.3. *If the surface $S_{g,n}$ is not a sphere with one, two or three punctures nor a torus with one puncture, then the natural map $C(S_{g,n}) \rightarrow AC(S_{g,n})$ is a quasi-isometry. Furthermore, each point in $AC(S_{g,n})$ is at distance at most one from a point in the image of $C(S_{g,n})$.*

As a consequence of Theorem 1.3, and since the curve complex is Gromov-hyperbolic (see [7]), we have the following

Corollary 1.4. *Under the hypothesis of Theorem 1.3, the arc and curve complex $AC(S_{g,n})$ is Gromov-hyperbolic.*

2. MAXIMAL SIMPLICES IN $AC(S)$

The results of this section will be used in the proof of Theorem 1.1.

A simplex in $AC(S)$ is said to be maximal if it is maximal in the sense of inclusion.

Let Δ be a maximal simplex in $AC(S)$, and suppose that the vertices of Δ are isotopy classes of arcs. Then a set of disjoint representatives of Δ is an ideal triangulation of the closed surface \bar{S} , that is, a triangulation having all its vertices at the distinguished points.

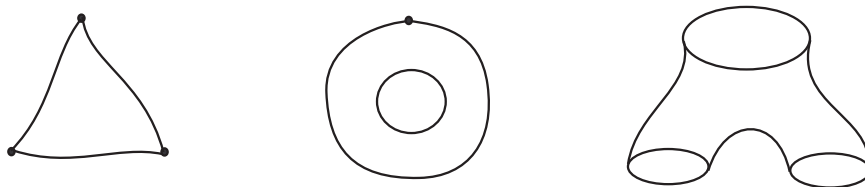


FIGURE 1. Components of the complement of a maximal simplex.

Suppose now that Δ is a maximal simplex in $AC(S)$ and let δ be a set of representatives of elements of Δ that are pairwise disjoint. Each component of the surface S_δ obtained by cutting S along δ is of one of the following three forms: (1) a triangle; (2) an annulus with one of its boundary components being a simple closed curve in δ , and the other boundary component an arc joining a puncture to itself; (3) a pair of pants all of whose boundary components are simple closed curves (see Figure 1).

We shall use the following

Lemma 2.1. *A maximal simplex Δ in $AC(S)$ that has maximal dimension consists of arcs, i.e. it is a triangulation of S . The dimension of such a simplex is $6g + 3n - 7$.*

Proof. The first statement is a consequence of the fact that if we start with a simplex in $AC(S)$ which has at least one vertex which is the isotopy class of a simple closed curve, then we can obtain a simplex of one dimension higher by replacing one of these simple closed curves by two arcs (see Figure 2). Thus, a maximal simplex Δ consists only of isotopy classes of arcs, and it is therefore the isotopy class of an ideal triangulation of S . The dimension statement follows then by a standard Euler characteristic argument, which gives the number of curves in an ideal triangulation of S . This number is equal to $6g + 3n - 6 = -3\chi(S)$, which implies that the dimension of the maximal simplex Δ is $6g + 3n - 7$. \square

Lemma 2.2. *The dimension of a maximal simplex Δ in $AC(S)$ that has minimal dimension is $3g + 2n - 4$.*

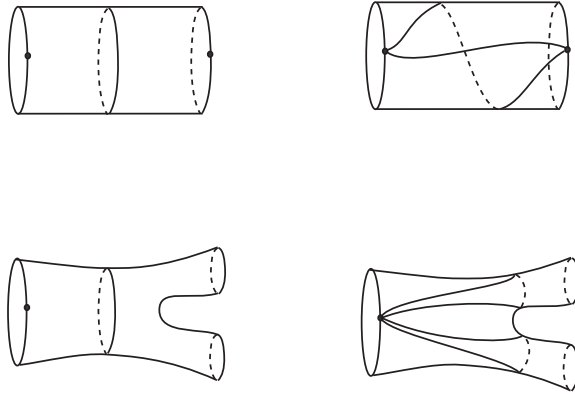


FIGURE 2. The dimension of the maximal simplex is increased in the presence of a curve.

Proof. The proof is analogous to the proof of Lemma 2.1. We use the fact that a maximal simplex that has minimal dimension has a maximal number of curves, which is $3g + 2n - 3$. If we cut the surface S along such curves, we get a collection of subsurfaces of Euler characteristic -1 . Each of these subsurfaces is either a disc with two punctures, or an annulus with one puncture or a pair of pants. On a disc with two punctures there are systems of two disjoint nonisotopic arcs (and no systems of three nonisotopic arcs), and on an annulus with one puncture there are essential arcs and no systems of two disjoint nonisotopic arcs. Hence, the number of curves and arcs on S forming a maximal simplex of minimal dimension is $3g + n - 3 + n = 3g + 2n - 3$. The dimension of that simplex is therefore $3g + 2n - 4$. \square

From Lemmas 2.1 and 2.2, it follows that the dimension of maximal simplices in $AC(S)$ is bounded between $3g + 2n - 4$ and $6g + 3n - 7$. It can be seen, using the arguments of the proof of these lemmas that every integer between these two bounds is realized as the dimension of a maximal simplex.

3. PROOF OF THEOREM 1.1

We now give a first proof of Theorem 1.1. We may assume that the arc complex of the surface is not empty, and we therefore assume that $n \geq 1$. We use the following:

Theorem 3.1 (Irmak-McCarthy [3]). *Let S be a surface of genus g with $n \geq 1$ punctures, which is not homeomorphic to a sphere with one, two or three punctures, or a torus with one puncture. Then the natural map $\text{Mod}^*(S) \rightarrow \text{Aut}(A(S))$ is an isomorphism. Furthermore, in the excluded cases, the map $\text{Mod}^*(S) \rightarrow \text{Aut}(A(S))$ is surjective and its kernel is the center of $\text{Mod}^*(S)$.* \square

To prove Theorem 1.1, let φ be an automorphism of $AC(S)$. If a is any arc on S , then it is contained in a maximal simplex Δ of dimension $6g + 3n - 7$.

(Here and in the sequel, we do not distinguish between an arc and its isotopy class.) Since φ is simplicial, the image simplex $\varphi(\Delta)$ is also of dimension $6g+3n-7$ and it contains $\varphi(a)$. By Lemma 2.1, $\varphi(a)$ is also an arc. It follows that φ maps arcs to arcs, inducing an automorphism $\tilde{\varphi} : A(S) \rightarrow A(S)$. It is easy to see that the resulting map $\varphi \mapsto \tilde{\varphi}$ is a homomorphism, which we denote by $\Psi : \text{Aut}(AC(S)) \rightarrow \text{Aut}(A(S))$. Using Theorem 3.1, we get from Ψ a natural isomorphism $\text{Aut}(A(S)) \rightarrow \text{Mod}^*(S)$. The homomorphism Ψ is injective, since if a homeomorphism f of the surface S that comes from an automorphism \bar{f} of $AC(S)$ fixes the homotopy classes of arcs, then it fixes each edge of some ideal triangulation of S , therefore it is isotopic to the identity. Thus, \bar{f} also fixes all the isotopy classes of curves on S , and the automorphism \bar{f} of $AC(S)$ is the identity.

Now we show that Ψ is onto. Let \bar{g} be an element of $\text{Aut}(A(S))$. Again, by Theorem 3.1, \bar{g} is induced by a homeomorphism $g : S \rightarrow S$. Now g also induces an automorphism of $AC(S)$ whose restriction to the set of isotopy classes of arcs is \bar{g} . Thus, \bar{g} is the image by Ψ of an automorphism of $AC(S)$, and Ψ is onto. This completes the first proof of Theorem 1.1.

We now give a second proof of Theorem 1.1. We use the following:

Theorem 3.2 (Ivanov-Korkmaz-Luo [4, 5, 6]). *Let S be a surface of genus g with n punctures. If $2g+n \geq 5$ then the natural map $\text{Mod}^*(S) \rightarrow \text{Aut}(C(S))$ is an isomorphism. \square*

Note that the hypothesis (and therefore the second proof below of Theorem 1.1) excludes the cases where the surface S is the sphere with four punctures or the torus with two punctures, but these cases were taken care in the first proof.

For the proof of Theorem 1.1, we first show that the natural map $\Phi : \text{Mod}^*(S) \rightarrow \text{Aut}(AC(S))$ is onto. Let φ be an automorphism of $AC(S)$. As in the first proof above, φ maps arcs to arcs. By considering the inverse of φ , we conclude that φ maps curves to curves. Hence, φ induces an automorphism $\tilde{\varphi} : C(S) \rightarrow C(S)$. By Theorem 3.2, there is a homeomorphism f of S such that $\tilde{\varphi} = f$. One can easily show that $\varphi = f$, which implies that Φ is onto. Thus, each simplicial automorphism is induced by a mapping class.

To prove the injectivity, consider a mapping class f that acts trivially on $AC(S)$. Then, f acts trivially on the maximal simplices of dimension $6g+3n-7$, which are triangulations of S . It is easy to see that if f is the identity on a triangulation, then it is the identity, showing that φ is injective.

This completes the second proof of Theorem 1.1.

We conclude this section with a few words on the case of surfaces excluded by the hypothesis of Theorem 1.1.

In the case of a sphere with one puncture, $AC(S)$ is empty. In the case of a sphere with two punctures, $AC(S) = A(S)$ is reduced to a single arc, and in the case of a sphere with three punctures, $AC(S) = A(S)$ is a finite complex (see Figure 3). Both special cases were analyzed in the paper [3], and in each of these cases the natural homomorphism from $\text{Mod}^*(S_{g,n})$ to $\text{Aut}(AC(S_{g,n}))$ is surjective and its kernel is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, which is the center of $\text{Mod}^*(S_{g,n})$. Finally, in the case where S is a torus with one hole, the

arguments given in the first proof of Theorem 1.1, together with the fact that the homomorphism from $\text{Mod}^*(S_{g,n})$ to $\text{Aut}(AC(S_{g,n}))$ is also surjective with kernel \mathbb{Z}_2 (see again the paper [3]), show that we have an isomorphism $\text{Mod}^*(S_{g,n})/\mathbb{Z}_2 \simeq \text{Aut}(AC(S))$.

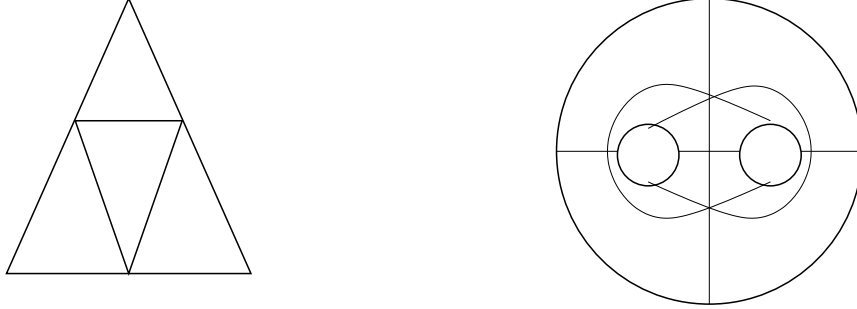


FIGURE 3. The figure on the left hand side represents the arc complex of the sphere with three punctures. It is a finite simplicial complex, having six vertices, nine edges and four two-cells. The vertices of this complex are the isotopy classes of the arcs on the right hand side figure, which represents a three-punctured sphere in which the punctures have been replaced by circles, for more lisibility.

4. PROOF OF THEOREM 1.2

In this section, we consider the following classification into five types of the vertices of $AC(S)$. A representative on S of such a vertex is of one of the following types:

- (1) a separating closed curve;
- (2) an arc separating the surface and connecting a puncture to itself;
- (3) a non-separating closed curve;
- (4) an arc that does not separate the surface and that connects a puncture to itself;
- (5) an arc connecting two distinct punctures.

The links of vertices will be used to distinguish combinatorially these kinds of vertices. For the convenience of the reader, we recall the definition of the link of a vertex in a simplicial complex. It is convenient to start with the definition of the star of a vertex. Given a vertex v in an abstract simplicial complex E , its star, $\text{St}(v)$, is the subcomplex of E whose simplices are (1) the simplices of E that contain v and (2) the faces of such simplices. The link of v , $\text{Lk}(v)$, is then the subcomplex of E whose simplices are the simplices of $\text{St}(v)$ whose intersection with v is empty.

Now we prove Theorem 1.2.

In the above classification of vertices of $AC(S)$, Cases (1) and (2) can be distinguished from the three others by the fact that the link of a vertex in these two cases is the join of two nonempty subcomplexes. Equivalently, the dual link, in each of these two cases, is the disjoint union of two nonempty sets. Here, the dual link of a link L is an abstract graph whose vertices are the vertices of L , and in which two vertices are connected by an edge

if and only if these vertices are not connected by an edge, where they are considered as vertices in L . In cases (3)-(5), the dual link is connected.

Case (1) can be distinguished from Case (2) by the fact that the highest dimension of the maximal simplex in the link of a vertex in Case (2) is bigger than the corresponding dimension for a link in Case (1). (In fact, this dimension is $6g + 3n - 9$ in Case (1) and $6g + 3n - 8$ in Case (2).)

Thus, each of Cases (1) and (2) is completely distinguished from all the other cases.

Now in the remaining three cases, Case (3) is distinguished from the other two by the fact that the highest dimension of the maximal simplex in its link is smaller than the highest dimension of the maximal simplex in the link of any of the two other cases. (In fact, the highest dimension of the maximal simplex in the link is $6g + 3n - 9$ in Case (3), whereas in the other two cases this dimension is $6g + 3n - 8$.)

Now it remains to distinguish Case (4) from Case (5). This is done by the following property. Consider on the closed surface \bar{S} a closed curve Z which is the boundary of a regular neighborhood of the arc joining the two distinguished points, representing a vertex in Case (5) which we call a . The vertex z in $\text{Lk}(a)$ has the following property: z is represented by a separating curve and $\text{Lk}(z) - \{a\} \subset \text{Lk}(a)$. (Note that we already distinguished by combinatorial information the vertices in $AC(S)$ represented by arcs from those represented by closed curves; likewise, we already distinguished combinatorially the vertices represented by separating closed curves from those represented by non-separating curves.) Now in the link of a vertex a of Type (4), there is no vertex x represented by a separating curve X satisfying $\text{Lk}(x) - \{a\} \subset \text{Lk}(a)$. Indeed, if X is any separating curve on S disjoint from an arc A representing the vertex a , then A is in one of the two connected components of $S \setminus X$. This component has positive genus (since A is also non-separating in that component). Therefore, we can find a closed curve Y in $S \setminus X$ that intersects A in exactly one point. Now the vertex y represented by Y is in $\text{Lk}(x) - \{a\}$, but not in $\text{Lk}(a)$, showing that $\text{Lk}(x) - \{a\}$ is not a subset of $\text{Lk}(a)$. This completes the proof of the theorem.

Theorem 1.2 shows that not only a full automorphism of the graph $AC(S)$ recognizes the topological types of the vertices, as is implied by Theorem 1.1, but that also that local combinatorial data contained in the links of each vertex are sufficient to characterize these types.

5. PROOF OF THEOREM 1.3

Let $C^0(S)$ be the set of vertices of $C(S)$. Since the dimension of simplices in $AC(S)$ is uniformly bounded, the inclusion map $C^0(S) \rightarrow C(S)$ is a quasi-isometry. Thus, to show that the natural map $C(S) \rightarrow AC(S)$ is a quasi-isometry, it suffices to show that the natural map $C^0(S) \rightarrow AC(S)$ is a quasi-isometry. Thus, it suffices to reason on distances between vertices of $C(S)$.

We let $\phi : C(S) \rightarrow AC(S)$ be the simplicial map extending the natural inclusion map at the level of the vertices, and we let d_C (respectively d_{AC}) denote the distance function in $C(S)$ (respectively $AC(S)$).

To see that every point in $AC(S)$ is at distance at most one from a point in the image of $C(S)$, we consider, for each essential arc a on S , one of the boundary components of a regular neighborhood of that arc. Since the surface S is not a sphere with at most three punctures, we can choose this boundary component such that it is essential. The isotopy class of this essential simple closed curve is a vertex in $AC(S)$ which is at distance one from the vertex representing the arc a .

We now prove that for every vertices x and y in $C(S)$, we have

$$d_{AC}(\phi(x), \phi(y)) \leq d_C(x, y) \leq 2d_{AC}(\phi(x), \phi(y)).$$

This will complete the proof of Theorem 1.3.

The first inequality is clear from the definitions. We prove the second one.

Suppose that $d_{AC}(\phi(x), \phi(y)) = L$. Consider a simplicial path p of length L in $AC(S)$ joining $\phi(x)$ and $\phi(y)$. We show that we can find a simplicial path p' of length $\leq 2L$ in $AC(S)$ joining these two vertices, and such that all the vertices of p' are in $\phi(C(S))$. The construction of the path p' is by induction. We describe the first step, and the induction step is similar.

There are three cases.

Case 1. The first edge of the simplicial path p joins the vertex $\phi(x)$ to a vertex z representing a simple closed curve in S , and the second edge joins a vertex z to a vertex v also representing a simple closed curve. In this case, we do not change this part of the path p , and we go to the next step, that is, we examine the part of the path p that starts at the second vertex, z .

Case 2. The first edge of the path p joins the vertex $\phi(x)$ to a vertex a represented an arc A in S , and the second edge joins the vertex a to a vertex z represented a closed curve Z . We may assume that $\phi(x)$ is not connected to z by an edge. Note that a tubular neighborhood N of A is either a disk with two punctures or an annulus with one puncture. Since $2g + n \geq 5$, one of the boundary components, say V , of that surface is a simple closed curve not bounding a disk with at most one puncture. In this case, let v be the isotopy class of V . Note that v is a vertex of $C(S)$ which is at distance one from $\phi(x)$ and from z . We replace the path $\phi(x)az$ by the path $\phi(x)vz$. Such a step does not change the length of the path joining $\phi(x)$ to $\phi(y)$.

Case 3. The first edge of the path p joins the vertex $\phi(x)$ to a vertex a represented by an arc A on S , and the second edge joins the vertex a to a vertex b that is also represented by an arc B . We may assume that A and B are disjoint. In this case, we distinguish nine subcases, depending on the relative position of A and B on the surface S . These cases are represented in Figure 4, (i) to (ix). In each case, we replace the path $\phi(x)ab$ by either a path of the same length, $\phi(x)zb$, or a path of length three, $\phi(x)zz'b$. Here, z is the isotopy class of a nontrivial boundary component of A , and z' is the isotopy class of a nontrivial boundary component of $A \cup B$. The simple closed curves are represented in dashed lines in Figure 4, and one can check in each case that either a vertex z or a pair of adjacent vertices z and z' with the appropriate requirements exist. This also uses fact that the surface S is not a sphere with ≤ 3 punctures.

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MUSTAFA KORKMAZ, DEPARTMENT OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, 06531 ANKARA, TURKEY.

E-mail address: korkmaz@metu.edu.tr

ATHANASE PAPADOPOULOS, MAX-PLANK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY, AND: INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG AND CNRS, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE.

E-mail address: papadopoulos@math.u-strasbg.fr

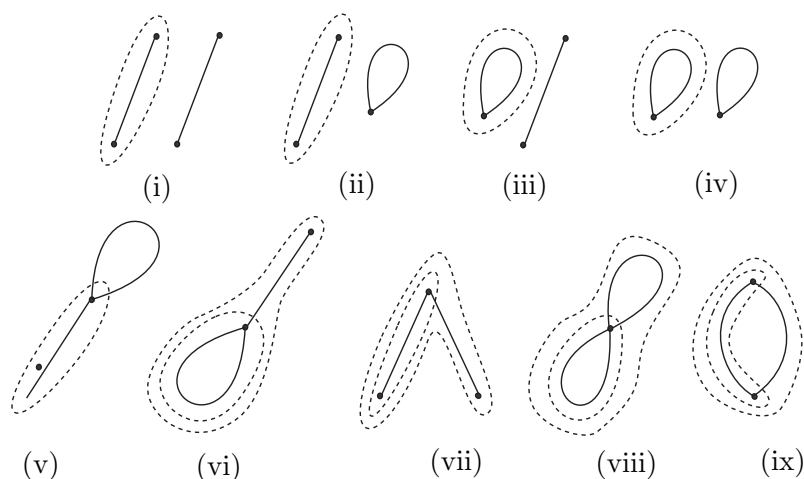


FIGURE 4. These nine cases, used in the proof of Theorem 1.3, are the various possibilities for an ordered pair of consecutive vertices of $AC(S)$ that represent by arcs on S . The dashed lines are closed curves used to replace the arcs.