Auxiliary functions in transcendence proofs

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Abstract

We discuss the role of auxiliary functions in the development of transcendental number theory.

Earlier auxiliary functions were completely explicit (§ 1). The earliest transcendence proof is due to Liouville (§ 1.1), who produced the first explicit examples of transcendental numbers at a time where their existence was not yet known; in his proof, the auxiliary function is just a polynomial in one variable. Hermite's proof of the transcendence of e (1873) is much more involved, the auxiliary function he builds (§ 1.2) is the first example of the Padé approximants (§ 1.3), which can be viewed as a far reaching generalization of continued fraction expansion [13, 14]. Hypergeometric functions (§ 1.4) are among the best candidates for using Padé approximations techniques.

Another tool which plays the role of auxiliary functions is produced by interpolation formulae (§ 2). They occurred in the theory after a question by Weierstraß (§ 2.1) on the so-called *exceptional set* S_f of a transcendental function f, which is the set of algebraic numbers α such that $f(\alpha)$ is algebraic. The answer to his question is that any set of algebraic numbers is the exceptional set of some transcendental function f; this shows that one should add further conditions in order to get transcendence criteria. One way is to replace *algebraic number* by *rational integer*: this gives rise to the study of integer–valued entire functions (§ 2.2) with the works of G. Pólya (1915), A.O. Gel'fond (1929) and many others. The connexion with transcendental number theory may not have been clear until the solution by A.O. Gel'fond in 1929 of the question of the transcendence of e^{π}, a special case of Hilbert's seventh problem (§ 2.3). Along these lines, recent developments are due to T. Rivoal, who renewed forgotten rational interpolation formulae (1935) of R. Lagrange (§ 2.4).

The simple (but powerful) construction by Liouville was extended to several variables by A. Thue (§ 3.1.1), who introduced the Dirichlet's box principle (pigeonhole principle) (§ 3) into the topic of Diophantine approximation in the early 1900's. In the 1920's, Siegel (§ 3.1) developed this idea and applied it in 1932 to transcendental number theory. This gave rise to the Gel'fond–Schneider method (§ 3.1.2) which produces the Schneider–Lang Criterion in one (§ 3.1.3) or several (§ 3.1.4) variables. Among many developments of this method are results on modular functions (§ 3.1.6). Variants of the auxiliary functions produced by Dirichlet's Box Principle are universal auxiliary functions, which have small Taylor coefficients at the origin (§ 3.2). Another approach, due to K. Mahler (§ 3.3), involves auxiliary functions whose existence is deduced from linear algebra instead of Thue–Siegel Lemma.

In 1991 M. Laurent introduced interpolation determinants (§ 4). Two years

later J.B. Bost used Arakhelov theory (§ 5) to prove slope inequalities, which dispenses of the choice of bases.

1 Explicit functions

1.1 Liouville

The first examples of transcendental numbers were produced by Liouville [48] in 1844. At that time, it was not yet known that transcendental numbers exist. The idea of Liouville is to show that all algebraic real numbers α are badly approximated by rational numbers. The simplest example is a rational number $\alpha = a/b$: for any rational number $p/q \neq a/b$, the inequality

$$\left|\frac{a}{b} - \frac{p}{q}\right| \geq \frac{1}{bq}$$

holds. For an irrational real number x, on the contrary, for any $\epsilon > 0$ there exists a rational number p/q such that

$$0 < \left|x - \frac{p}{q}\right| \leq \frac{\epsilon}{q}\cdot$$

This yields an irrationality criterion, which is the basic tool for proving the irrationality of specific numbers: *a real number x is irrational if and only if there exists a sequence $(p_n/q_n)_{n\geq 0}$ of distinct rational numbers with*

$$\lim_{n\to\infty} q_n \left|x - \frac{p_n}{q_n}\right| = 0.$$

This criterion is not too demanding: the quality of the approximation is not very strong. Indeed for an given an irrational number x, it is known that there exist much better rational approximations, since there exist infinitely many rational numbers p/q for which

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{2q^2}\cdot$$

It is a remarkable fact that on the one hand there exist such sharp approximations, and on the other hand we are usually not able to produce (or at least to show the existence of) much weaker rational approximations. Also in examples like $\zeta(3)$, there is a single known explicit sequence of rational approximations which arises from the irrationality proofs – of course, once the irrationality is established, the existence of other much better approximations follows, but so far no one is able to produce an explicit sequence of such approximations.

Liouville extended the irrationality criterion into a transcendence criterion. The proof by Liouville involves the irreducible polynomial $f \in \mathbb{Z}[X]$ of the given irrational algebraic number α. Since α is algebraic, there exists an irreducible polynomial $f \in \mathbb{Z}[X]$ such that $f(\alpha) = 0$. Let d be the degree of f. For $p/q \in \mathbb{Q}$ the number $q^d f(p/q)$ is a non–zero rational integer, hence

$$|f(p/q)| \geq \frac{1}{q^d}\cdot$$

On the other hand, it is easily seen that there exists a constant $c(\alpha) > 0$, depending only on α (and its irreducible polynomial f), such that

$$|f(p/q)| \leq c(\alpha) \left| \alpha - \frac{p}{q} \right|.$$

An explicit value for a suitable $c(\alpha)$ is given for instance in Exercice 3.6 of [104]. Therefore

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c'(\alpha)}{q^d}$$

with $c'(\alpha) = 1/c(\alpha)$.

Let ξ be a real number such that, for any $\kappa > 0$, there exists a rational number p/q with $q \geq 2$ satisfying

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^\kappa}\cdot$$

It follows from Liouville's inequality that ξ is transcendental. Real numbers satisfying this assumption are called *Liouville's numbers*. The first examples [48] produced by Liouville in 1844 involved properties of continued fractions, but already in the first part of his 1844 note he considered series

$$\sum_{m\geq 1} \frac{1}{\ell^{m!}}$$

for $\ell \in \mathbb{Z}_{\geq 2}$. Seven years later in [49] he refers to a letter from Goldbach to Euler for numbers

$$\sum_{m\geq 1} \frac{k_m}{10^{m!}}$$

where $(k_m)_{m\geq 1}$ is a sequence of integers in the range $\{0, \ldots, 9\}$. Next (p. 140 of [49]) he uses the same argument to prove the irrationality of

$$\sum_{m\geq 1} \frac{1}{\ell^{m^2}}$$

whose transcendence was proved only in 1996 by Yu. V. Nesterenko [61, 63].

We consider below (§ 3.1.1) extensions of Liouville's result by Thue, Siegel, Roth and Schmidt.

1.2 Hermite

During his course at the École Polytechnique in 1815 (see [84]), Fourier gave a simple proof for the irrationality of e, which can be found in many textbooks (for instance Th. 47 Chap. 4 § 7 of Hardy and Wright [36]). The idea is to truncate the Taylor expansion at the origin of the exponential function. In this proof, the auxiliary function is the tail of the Taylor expansion of the exponential function

$$e^z - \sum_{n=0}^{N} \frac{z^n}{n!}$$

which one specializes at $z = 1$. As noticed by F. Beukers, the proof becomes even shorter if one specializes at $z = -1$. This proof has been revisited by Liouville in 1840 [47] who succeeded to extend the argument and to prove that e^2 is not a quadratic number. This result is quoted by Hermite in his memoir [37]. Fourier's argument produces rational approximations to the number e, which are sharp enough to prove the irrationality of e but not the transcendence. The denominators of these approximations are $N!$. One idea of Hermite is to look for other rational approximations. Instead of the auxiliary functions $e^z - A(z)$ for $A \in \mathbb{Q}[z]$, Hermite introduces more general auxiliary functions $R(z) = B(z)e^z - A(z)$. He finds a polynomial B such that the Taylor expansion at the origin of $B(z)e^z$ has a large gap: he calls $A(z)$ the polynomial part of the expansion before the gap, so that the auxiliary function $R(z)$ has a zero of high multiplicity at the origin. Hermite gives explicit formulae for A, B and R, in particular the polynomials A and B have rational coefficients – the question is homogeneous, one may multiply by a denominator to get integer coefficients. Also he obtains upper bounds for these integer coefficients (they are not too large) and for the modulus of the remainder (which is small on a given disc). As an example, given $r \in \mathbb{Q} \setminus \{0\}$ and $\epsilon > 0$, one can use this construction to show the existence of A, B and R with $0 < |R(r)| < \epsilon$. Hence $e^r \notin \mathbb{Q}$. This gives another proof of Lambert's result on the irrationality of e^r for $r \in \mathbb{Q} \setminus \{0\}$, and this proof extends to the irrationality of π as well [12, 62].

Hermite [37] goes much further, since he obtains the transcendence of e. To achieve this goal, he considers *simultaneous rational approximations to the exponential function*, in analogy with Diophantine approximation. The idea is as follows. Let $B_0, B_1, \ldots, B_m$ be polynomials in $\mathbb{Z}[x]$. For $1 \le k \le m$, define

$$R_k(x) = B_0(x)e^{kx} - B_k(x).$$

Set $b_j = B_j(1)$, $0 \le j \le m$ and

$$R = a_1 R_1(1) + \cdots + a_m R_m(1).$$

The numbers a_j and b_j are rational integers, hence

$$a_0 b_0 + a_1 b_1 + \cdots + a_m b_m = b_0(a_0 + a_1 e + a_2 e^2 + \cdots + a_m e^m) - R$$

also. Therefore, if one can prove $0 < |R| < 1$, then one deduces

$$a_0 + a_1 e + \cdots + a_m e^m \neq 0.$$

Hermite's construction is more general: he produces rational approximations to the functions $1, e^{\alpha_1 x}, \ldots, e^{\alpha_m x}$, when $\alpha_1, \ldots, \alpha_m$ are pairwise distinct complex numbers. Let $n_0, \ldots, n_m$ be rational integers, all ≥ 0. Set $N = n_0 + \cdots + n_m$. Hermite constructs explicitly polynomials B_0, $B_1, \ldots$, B_m, with B_j of degree $N - n_j$, such that each of the functions

$$B_0(z)e^{\alpha_k z} - B_k(z), \quad (1 \leq k \leq m)$$

has a zero at the origin of multiplicity at least N.

Such functions are now known as *Padé approximations of the second kind* (or *of type II*).

1.3 Padé approximation

In his thesis in 1892, H.E. Padé studied systematically the approximation of complex analytic functions by rational functions. See Brezinski's papers [13, 14], where further references to previous works by Jacobi (1845), Faá de Bruno (1859), Sturm, Brioschi, Sylvester, Frobenius (1870), Darboux (1876), Kronecker (1881) are given.

There are two dual points of view, giving rise to the two types of *Padé Approximants* [24].

Let $f_0, \ldots, f_m$ be complex functions which are analytic near the origin and $n_0, \ldots, n_m$ be non–negative rational integers. Set $N = n_0 + \cdots + n_m$.

Padé approximants of type II are polynomials $B_0, \ldots, B_m$ with B_j having degree $\leq N - n_j$, such that each of the functions

$$B_i(z)f_j(z) - B_j(z)f_i(z) \quad (0 \leq i < j \leq m)$$

has a zero of multiplicity $> N$.

Padé approximants of type I are polynomials $P_1, \ldots, P_m$ with P_j of degree $\leq n_j$ such that the function

$$P_1(z)f_1(z) + \cdots + P_m(z)f_m(z)$$

has a zero at the origin of multiplicity at least $N + m - 1$.

For type I as well as type II, existence of Padé approximants follow from linear algebra: one compares the number of equations which are produced by the vanishing conditions on the one hand, with the number of coefficients of P, considered as unknowns, on the other. Unicity of the solution, up to a multiplicative constant (the linear system of equations is homogeneous) is true only in specific cases (*perfect systems*): it amounts to proving that the matrix of the system of equations is regular.

These approximants were also studied by Ch. Hermite for the exponentials functions in 1873 and 1893; later, in 1917, he gave further integral formulae for the remainder. For transcendence purposes, Padé approximants of type I have been used for the first time in 1932 by K. Mahler [51], who gave effective version of the transcendence theorems by Hermite, Lindemann and Weierstraß.

In the theory of Diophantine approximation, there are transference theorems, initially due to Khintchine (see for instance [16, 44]). Similar transference properties for Padé approximation have been considered by H. Jager [39] and J. Coates [18, 19].

1.4 Hypergeometric methods

Explicit Padé approximations are known only for restricted classes of functions; however, when they are available, they often produce very sharp Diophantine estimates. Among the best candidates for having explicit Padé Approximations are the hypergeometric functions. A. Thue [88] developed this idea in the early 20th Century and was able to solve explicitly several classes of Diophantine equations. There is a contrast between the measures of irrationality for instance which can be obtained by hypergeometric methods and those produced by other methods, like Baker's method (§ 3.1.2): typically, hypergeometric methods produce numerical constants with one or two digits (when the expected value is something like 2), where Baker's method produces constants with several hundreds digits. On the other hand, Baker's method works in much more general situations. Compared with the Thue–Siegel–Roth–Schmidt's method (§ 3.1.1), it has the great advantage of being explicit.

Among many contributors to this topic, we quote A. Thue, C.L. Siegel, A. Baker, G.V. Chudnovskii, M. Bennett, P. Voutier, G. Rhin, C. Viola, T. Rivoal... These works also involve sorts of auxiliary functions (integrals) depending on parameters which needs to be suitably selected in order to produce sharp estimates.

Chapter 2 of [24] deals with effective constructions in transcendental number theory and includes two sections (§ 6 and § 7) on generalized hypergeometric functions and series.

2 Interpolation methods

We discuss here another type of auxiliary function which occurred in works related with a question of Weierstraß on the exceptional set of a transcendental entire function. Recall that an *entire function* is a complex valued function which is analytic in $\mathbb{C}$. A function f is *algebraic* (over $\mathbb{C}(z)$) if f is a solution of a functional equation $P(z, f(z)) = 0$ for some non–zero polynomial $P \in \mathbb{C}[X, Y]$. An entire function is algebraic if and only if it is a polynomial. A function which is not algebraic is called *transcendental*[1].

[1] A polynomial whose coefficients are not all algebraic numbers is an algebraic function, namely is algebraic over $\mathbb{C}(z)$, but is a transcendental element over $\mathbb{Q}(z)$. However, as soon as a polynomial assumes algebraic values at infinitely many points (including derivatives), its coefficients are algebraic. Therefore for the questions we consider it makes no difference to consider algebraicity of functions over $\mathbb{C}$ or over $\mathbb{Q}$.

2.1 Weierstraß question

Weierstraß (see [53]) initiated the question of investigating the set of algebraic numbers where a given transcendental entire function f takes algebraic values.

Denote by $\overline{\mathbb{Q}}$ the *field of algebraic numbers* (algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). For an entire function f, we define the *exceptional set* S_f of f as the set of algebraic numbers α such that $f(\alpha)$ is algebraic:

$$S_f := \{\alpha \in \overline{\mathbb{Q}}\; ;\; f(\alpha) \in \overline{\mathbb{Q}}\}.$$

For instance Hermite–Lindemann's Theorem on the transcendence of $\log \alpha$ and e^β for α and β algebraic numbers is the fact that the exceptional set of the function e^z is $\{0\}$. Also the exceptional set of $e^z + e^{1+z}$ is empty, by the Theorem of Lindemann–Weierstrass. The exceptional set of functions like 2^z or $e^{i\pi z}$ is $\mathbb{Q}$, as shown by the Theorem of Gel'fond and Schneider.

The exceptional set of a polynomial is $\overline{\mathbb{Q}}$ if the polynomial has algebraic coefficients, otherwise it is finite. Also any finite set of algebraic numbers is the exceptional set of some entire function: for $s \geq 1$ the set $\{\alpha_1, \ldots, \alpha_s\}$ is the exceptional set of the polynomial $\pi(z-\alpha_1)\cdots(z-\alpha_s) \in \mathbb{C}[z]$ and also of the transcendental entire function $(z-\alpha_2)\cdots(z-\alpha_s)e^{z-\alpha_1}$. It does not seem easy (without Schanuel's conjecture) to produce explicit examples of functions having either $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}$ as exceptional set.

The study of exceptional sets started in 1886 with a letter of Weierstrass to Strauss. It was later developed by Strauss, Stäckel, Faber – see [53]. Further results are due to van der Poorten, Gramain, Surroca and others (see [35, 86]).

Among the results which were obtained, a typical one is the following: *if A is a countable subset of $\mathbb{C}$ and if E is a dense subset of $\mathbb{C}$, there exist transcendental entire functions f mapping A into E.*

Also van der Poorten noticed in [89] that there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbb{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.

The question of possible sets S_f has been solved in [38]: *any set of algebraic numbers is the exceptional set of some transcendental entire function.* Also multiplicities can be included, as follows: define the *exceptional set with multiplicity* of a transcendental entire function f as the subset of $(\alpha, t) \in \overline{\mathbb{Q}} \times \mathbb{Z}_{\geq 0}$ such that $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$. Here $f^{(t)}$ stands for the t-th derivative of f.

Then *any subset of $\overline{\mathbb{Q}} \times \mathbb{Z}_{\geq 0}$ is the exceptional set with multiplicities of some transcendental entire function f.* More generally, the main result of [38] is the following:

> *Let A be a countable subset of $\mathbb{C}$. For each pair (α, s) with $\alpha \in A$, and $s \in \mathbb{Z}_{\geq 0}$, let $E_{\alpha,s}$ be a dense subset of $\mathbb{C}$. Then there exists a transcendental entire function f such that*
>
> $$\left(\frac{d}{dz}\right)^s f(\alpha) \in E_{\alpha,s}$$
>
> *for all $(\alpha, s) \in A \times \mathbb{Z}_{\geq 0}$.*

One may replace $\mathbb{C}$ by $\mathbb{R}$: it means that one may take for the sets $E_{\alpha,s}$ dense subsets of $\mathbb{R}$, provided that one requires A to be a countable subset of $\mathbb{R}$.

The proof is a construction of an interpolation series (see § 2.2) on a sequence where each w occurs infinitely often. The coefficients of the interpolation series are selected recursively to be sufficiently small (and nonzero), so that the sum f of the series is a transcendental entire function.

This process yields uncountably many such functions. One may also require that they are algebraically independent over $\mathbb{C}(z)$ together with their derivatives. Further, at the same time, one may request each of these functions f to have a small growth, meaning that f satisfies $|f|_R \le |g|_R$ for all $R \ge 0$, where g is a given transcendental function with $g(0) \ne 0$.

A simple measure for the growth of an entire function f is the real valued function $R \mapsto |f|_R$, where

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

An entire function f has an *order of growth* $\le \varrho$ if for all $\epsilon > 0$ the inequality

$$|f|_R \le \exp\bigl(R^{\varrho+\epsilon}\bigr)$$

holds for sufficiently large R.

As a very special case (selecting A to be the set $\overline{\mathbb{Q}}$ of algebraic numbers and each $E_{\alpha,s}$ to be either $\overline{\mathbb{Q}}$ or its complement in $\mathbb{C}$), one deduces the existence of uncountably many algebraic independent transcendental entire functions f such that any Taylor coefficient at any algebraic point α takes a prescribed value, either algebraic or transcendental.

2.2 Integer–valued entire functions

In 1915, G. Pólya [68] initiated the study of *integer–valued entire functions*; he proved that *if f is a transcendental entire function such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}_{\ge 0}$, then*

$$\limsup_{R\to\infty} \frac{1}{R} \log |f|_R > 0. \tag{2.1}$$

An example is the function 2^z for which the lower bound is $\log 2$. A stronger version of the fact that 2^z *is the "smallest" entire transcendental function mapping the positive integers to rational integers* is the estimate

$$\limsup_{R\to\infty} 2^{-R}|f|_R \ge 1,$$

which is valid under the same assumptions as for (2.1).

The method involves interpolation series: given an entire function f and a sequence of complex numbers $(\alpha_n)_{n\ge1}$, define inductively a sequence $(f_n)_{n\ge0}$ of entire functions by $f_0 = f$ and, for $n \ge 0$,

$$f_n(z) = f_n(\alpha_{n+1}) + (z - \alpha_{n+1}) f_{n+1}(z).$$

Define, for $j \geq 0$,

$$P_j(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_j).$$

One gets an expansion

$$f(z) = A(z) + P_n(z) f_n(z),$$

where

$$A = a_0 + a_1 P_1 + \cdots + a_{n-1} P_{n-1} \in \mathbb{C}[z] \quad \text{and} \quad a_n = f_n(\alpha_{n+1}) \quad (n \geq 0).$$

Conditions for such an expansion to be convergent as $n \to \infty$ are known – see for instance [32].

Such interpolation series give formulae for functions with given values at the sequence of points α_n; when some of the α_n's are repeated, these formulae involve the successive derivatives of the function at the given point. For instance, for a constant sequence $\alpha_n = z_0$ for all $n \geq 1$, one obtains the Taylor series expansion of f at z_0. These formulae have been studied by I. Newton and J-L. Lagrange.

Analytic formulae for the coefficients a_n and for the remainder f_n follow from Cauchy's residue Theorem. Indeed, let $x,\ z,\ \alpha_1, \ldots, \alpha_n$ be complex numbers with $x \notin \{z,\ \alpha_1, \ldots, \alpha_n\}$. Starting from the easy relation

$$\frac{1}{x - z} = \frac{1}{x - \alpha_1} + \frac{z - \alpha_1}{x - \alpha_1} \cdot \frac{1}{x - z}, \tag{2.2}$$

one deduces by induction the next formula due to Hermite:

$$\begin{aligned} \frac{1}{x - z} = \sum_{j=0}^{n-1} & \frac{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_j)}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{j+1})} \\ & + \frac{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)} \cdot \frac{1}{x - z}. \end{aligned}$$

Let $\mathcal{D}$ be an open disc containing $\alpha_1, \ldots, \alpha_n$, let $\mathcal{C}$ denote the circumference of $\mathcal{D}$, let $\mathcal{D}'$ be an open disc containing the closure of $\mathcal{D}$. Assume f is analytic in $\mathcal{D}'$. Then

$$a_j = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(x)dx}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{j+1})} \qquad (0 \leq j \leq n - 1)$$

and

$$f_n(z) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(x)dx}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)(x - z)}.$$

Pólya applies these formulae to prove that if f is an entire function which does not grow too fast and satisfies $f(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 0}$, then the coefficients a_n in the expansion of f at the sequence $(\alpha_n)_{n \geq 1} = \{0, 1, 2, \ldots\}$ vanish for sufficiently large n, hence f is a polynomial.

Further works on this topic, using a variety of methods, are due to G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,... – and A.O. Gel'fond (see § 2.3).

2.3 Transcendence of e^π

Pólya's study of the growth of transcendental entire functions taking integral values at the positive rational integers was extended to the Gaussian integers by A.O. Gel'fond in 1929 [30]. By means of interpolation series at the points in $\mathbb{Z}[i]$, he proved that *if f is a transcendental entire function which satisfies $f(\alpha) \in \mathbb{Z}[i]$ for all $\alpha \in \mathbb{Z}[i]$, then*

$$\limsup_{R\to\infty} \frac{1}{R^2} \log |f|_R \geq \gamma. \tag{2.3}$$

The example of the Weierstraß sigma function attached to the lattice $\mathbb{Z}[i]$

$$\sigma(z) = z \prod_{\omega\in\mathbb{Z}[i]\setminus\{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}$$

(which is the Hadamard canonical product with a simple zero at any point of $\mathbb{Z}[i]$), shows that the constant γ cannot be larger than $\pi/2$. Often for such problems, dealing with a discrete subset of $\mathbb{C}$, replacing *integer values* by *zero values* gives some hint of what should be expected, at least for the order of growth (the exponent 2 of R^2 in the left hand side of formula (2.3)), if not for the value of the constant (the number γ in the right hand side of formula (2.3)). Other examples of Hadamard canonical products are

$$z \prod_{n\geq 1} \left(1 - \frac{z}{n}\right) e^{z/n} = -e^{\gamma z}\Gamma(-z)^{-1}$$

for the set $\mathbb{Z}_{>0} = \{1, 2, \dots\}$ of positive integers and

$$z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi^{-1}\sin(\pi z)$$

for the set $\mathbb{Z}$ of rational integers.

The initial admissible value computed by A.O. Gel'fond in 1929 for γ in (2.3) was pretty small, namely $\gamma = 10^{-45}$. It was improved by several mathematicians including Fukasawa, Gruman, Masser, until 1981 when F. Gramain [34] reached $\gamma = \pi/(2e)$, which is best possible, as shown by D.W. Masser [54] one year earlier. See [1] for recent results and extensions to number fields.

This work of Gel'fond's [30] turns out to have fundamental consequences on the development of transcendental number theory, due to its connexion with the number e^π. Indeed, the assertion that the number

$$e^\pi = 23{,}140\,692\,632\,779\,269\,005\,729\,086\,367\,\dots$$

is irrational is equivalent to saying that the function $e^{\pi z}$ cannot take all its values in $\mathbb{Q}(i)$ when the argument z ranges over $\mathbb{Z}[i]$. By expanding the function $e^{\pi z}$ into an interpolation series at the Gaussian integers, Gel'fond was able to prove the transcendence of e^π. More generally Gel'fond proved the transcendence of

α^β for α and β algebraic, $\alpha \neq 0$, $\alpha \neq 1$ and β imaginary quadratic. In 1930, Kuzmin extended the proof to the case where β is real quadratic, thus proving the transcendence of $2^{\sqrt{2}}$. The same year, Boehle proved that if β is algebraic of degree $d \geq 2$, then one at least of the $d-1$ numbers $\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$ is transcendental (see for instance [25]).

The next important step came from Siegel's introduction of further ideas in the theory (see § 3.1.2).

We conclude this subsection by noting that our knowledge of the Diophantine properties of the number e^π is far from being complete. It was proved recently by Yu.V. Nesterenko (see § 3.1.6) that the two numbers π and e^π are algebraically independent, but there is no proof so far that e^π is not a Liouville number.

2.4 Lagrange interpolation

Newton-Lagrange interpolation (§ 2.2) of a function yields a series of *polynomials*, namely linear combinations of products $(z-\alpha_1)\cdots(z-\alpha_n)$. Another type of interpolation has been devised in [40] by another Lagrange (René and not Joseph–Louis), in 1935, who introduced instead a series of *rational* fractions. Starting from the formula

$$\frac{1}{x-z} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)} + \frac{x-\beta}{x-\alpha}\cdot\frac{z-\alpha}{z-\beta}\cdot\frac{1}{x-z}$$

in place of (2.2), iterating and integrating as in § 2.2, one deduces an expansion

$$f(z) = \sum_{j=0}^{n-1} b_j \frac{(z-\alpha_1)\cdots(z-\alpha_j)}{(z-\beta_1)\cdots(z-\beta_j)} + R_n(z).$$

This approach has been developed in 2006 [69] by T. Rivoal, who applies it to the Hurwitz zeta function

$$\zeta(s,z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^s} \quad (s \in \mathbb{C},\ \Re e(s) > 1,\ z \in \mathbb{C}).$$

He expands $\zeta(2,z)$ as a Lagrange series in

$$\frac{z^2(z-1)^2\cdots(z-n+1)^2}{(z+1)^2\cdots(z+n)^2}\cdot$$

He shows that the coefficients of the expansion belong to $\mathbb{Q}+\mathbb{Q}\zeta(3)$. This enables him to produce a new proof of Apéry's Theorem on the irrationality of $\zeta(3)$.

Further, he gives a new proof of the irrationality of $\log 2$ by expanding

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+z}$$

into a Lagrange interpolation series. Furthermore, he gives a new proof of the irrationality of $\zeta(2)$ by expanding the function

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right)$$

as a Hermite–Lagrange series in

$$\frac{\bigl(z(z-1)\cdots(z-n+1)\bigr)^2}{(z+1)\cdots(z+n)}\cdot$$

It is striking that these constructions yield exactly the same sequences of rational approximations as the one produced by methods which look very much different [26].

Further developments of the interpolation methods should be possible. For instance Taylor series are the special case of Hermite's formula with a single point and multiplicities — they give rise to Padé approximants. Multiplicities could also be introduced in Lagrange–Rivoal interpolation.

3 Auxiliary functions arising from the Dirichlet's box principle

3.1 Thue–Siegel lemma

Here is a translation of a statement p. 213 of Siegel's paper [81]:

Lemma 3.1 (Thue–Siegel)**.** *Let*

$$\begin{aligned} y_1 &= a_{11}x_1+\cdots+a_{1n}x_n\\ &\vdots\\ y_m &= a_{m1}x_1+\cdots+a_{mn}x_n \end{aligned}$$

be m linear forms in n variables with rational integer coefficients. Assume $n > m$. Let $A \in \mathbb{Z}_{>0}$ be an upper bound for the absolute values of the mn coefficients a_{kl}. Then the system of homogeneous linear equations $y_1 = 0, \ldots, y_m = 0$ has a solution in rational integers $x_1, \ldots, x_n$, not all of which are 0, with absolute values less than $1 + (nA)^{m/(n-m)}$.

The fact that there is a non-trivial solution is a consequence of linear algebra, thanks to the assumption $n > m$. The point here is that Dirichlet's box principe shows the existence of a non-trivial solution satisfying an explicit upper bound. The estimate for that solution is essential in the proofs due to A. Thue, C.L. Siegel, and later A.O. Gel'fond, Th. Schneider, A. Baker, W.M. Schmidt and others (however, K. Mahler devised another method where such estimate is not required, linear algebra suffices - see § 3.3).

The initial proof by Thue and Siegel relied on the box principle. More sophisticated arguments have been introduced by K. Mahler, using geometry of numbers, and they yield to a number of developments which we do not survey here (see for instance [75]).

3.1.1 The origin of the Thue–Siegel Lemma

The first improvement of Liouville's inequality was reached by A. Thue in 1909 [87, 88]. Instead of evaluating the values at p/q of a polynomial in a single variable (viz. the irreducible polynomial of the given algebraic number α), he considers two approximations p_1/q_1 and p_2/q_2 of α and evaluates at the point $(p_1/q_1, p_2/q_2)$ a polynomial P in two variables. This polynomial $P \in \mathbb{Z}[X,Y]$ is constructed (or rather is shown to exist) by means of Dirichlet's box principle (Lemma 3.1). The required conditions are that P has zeroes of sufficiently large multiplicity at $(0,0)$ and at $(p_1/q_1, p_2/q_2)$. The multiplicity is weighted: this is what Thue called *index* of P at a point. The estimate for the coefficients of the solution of the system of linear equations in Lemma 3.1 plays an important role in the proof.

One of the main difficulties that Thue had to overcome was to produce a *zero estimate*, in order to find a non–zero value of some derivative of P.

A crucial feature of Thue's argument is that he needs to select a second approximation p_2/q_2 depending on a first one p_1/q_1. Hence a first very good approximation p_1/q_1 is required to produce an effective result from this method. In general, such arguments lead to sharp estimates for all p/q with at most one exception. This approach has been worked out by J.W.S. Cassels, H. Davenport and others to deduce *upper bounds for the number of solutions of certain Diophantine equations*. However, these results are not *effective*, meaning that they do not yield complete solutions of these equations. More recently, E. Bombieri has produced examples where a sufficiently good approximation exists for the method to work in an effective way. Later, he produced *effective refinements* to Liouville's inequality by extending the argument (see [24] Chap. 1 § 5.4).

Further improvement of Thue's method were obtained by C.L. Siegel in the 1920's: he developed Thue's method and succeeded in refining his irrationality measure for algebraic real numbers. In 1929, Siegel [81], thanks to a further sharpening of his previous estimate, derived his well known theorem on integer points on curves: *the set of integral points on a curve of genus* ≥ 1 *is finite*.

The introduction of the fundamental memoir [81] of C.L. Siegel in 1929 stresses the importance of Thue's idea involving the pigeonhole principle. In the second part of this fundamental paper he extends the Lindemann–Weierstraß Theorem (on the algebraic independence of $e^{\beta_1}, \dots, e^{\beta_n}$ when $\beta_1, \dots, \beta_n$ are $\mathbb{Q}$-linearly independent algebraic numbers) from the usual exponential function to a wide class of entire functions which he calls *E-functions*. He also introduces the class of G-functions which has been extensively studied since 1929. See also his monograph in 1949 [83], Shidlovskii's book [80] and the Encyclopaedia volume by Feldman and Nesterenko [24] for E–functions, André's book [3] and [24] Chap. 5 § 7 for G–functions. Among many developments related to G functions are the works of Th. Schneider, V.G. Sprindzuck and P. Dèbes related to algebraic functions (see [24] Chap. 5 § 7).

The work of Thue and Siegel on the rational approximation to algebraic numbers was extended by many a mathematician, including Th. Schneider, A.O. Gel'fond, F. Dyson, until K. F. Roth obtained in 1955 a result which is essentially optimal. For his proof he introduces polynomials in many variables.

A powerful higher dimensional generalization of Thue–Siegel–Roth's Theorem, the *Subspace Theorem*, was obtained in 1970 by W.M. Schmidt [75]; see also [7, 9]. Again, the proof involves a construction of an auxiliary polynomial in several variables, and one of the most difficult auxiliary results is a zero estimate (*index theorem*).

Schmidt's Subspace Theorem together with its variants (including effective estimates for the exceptional subspaces as well as results involving several valuations) have a large number of applications: to Diophantine approximation and Diophantine equations, to transcendence and algebraic independence, to the complexity of algebraic numbers – see [7].

We give here only a simplified statement of this fundamental result, which is already quite deep and very powerful.

Theorem 3.2 (Schmidt's Subspace Theorem - simplified form). *For $m \geq 2$ let $L_1, \ldots, L_m$ be independent linear forms in m variables with algebraic coefficients. Let $\epsilon > 0$. Then the set*

$$\{\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{Z}^m \; ; \; |L_1(\mathbf{x}) \cdots L_m(\mathbf{x})| \leq |\mathbf{x}|^{-\epsilon}\}$$

is contained in the union of finitely many proper subspaces of $\mathbb{Q}^m$.

3.1.2 Siegel, Gel'fond, Schneider

In 1932, C.L. Siegel [82] obtained the first results on the transcendence of elliptic integrals of the first kind (*a Weierstrass elliptic function cannot have simultaneously algebraic periods and algebraic invariants g_2, g_3*), by means of a very ingenious argument which involved an auxiliary function whose existence follows from the Dirichlet's box principle. This idea turned out to be crucial in the development of transcendental number theory.

The seventh of the 23 problems raised by D. Hilbert in 1900 is to prove the transcendence of the numbers α^β for α and β algebraic ($\alpha \neq 0$, $\alpha \neq 1$, $\beta \notin \mathbb{Q}$). In this statement, α^β stands for $\exp(\beta \log \alpha)$, where $\log \alpha$ is any[2] logarithm of α. The solution was achieved independently by A.O. Gel'fond [31] and Th. Schneider [76] in 1934. Consequences, already quoted by Hilbert, are the facts that $2^{\sqrt{2}}$ and e^π are transcendental. The question of the arithmetic nature of $2^{\sqrt{2}}$ was considered by L. Euler in 1748 [22].

The proofs by Gel'fond and Schneider are different, but both of them rest on some auxiliary function which arises from Dirichlet's box principle, following Siegel's contribution to the theory.

Let us argue by contradiction and assume that α, β and α^β are all algebraic, with $\alpha \neq 0$, $\alpha \neq 1$, $\beta \notin \mathbb{Q}$. Define $K = \mathbb{Q}(\alpha, \beta, \alpha^\beta)$. By assumption, K is a number field.

[2]The assumption $\alpha \neq 1$ can be replaced by the weaker assumption $\log \alpha \neq 0$. That means that one can take $\alpha = 1$, provided that we select for $\log \alpha$ a non–zero multiple of $2i\pi$ The result allowing $\alpha = 1$ is not more general: it amounts to the same to take $\alpha = -1$, provided that one replaces β by 2β.

A.O. Gel'fond's proof [31] rests on the fact that the two entire functions e^z and $e^{\beta z}$ are algebraically independent, they satisfy differential equations with algebraic coefficients and they take simultaneously values in K for infinitely many z, viz. $z \in \mathbb{Z} \log \alpha$.

Th. Schneider's proof [76] is different: he notices that the two entire functions z and $\alpha^z = e^{z \log \alpha}$ are algebraically independent, they take simultaneously values in K for infinitely many z, viz. $z \in \mathbb{Z} + \mathbb{Z}\beta$. He makes no use of differential equations, since the coefficient $\log \alpha$ which occurs by derivating the function α^z is not algebraic.

Schneider introduces a polynomial $A(X, Y) \in \mathbb{Z}[X, Y]$ in two variables and considers the auxiliary function

$$F(z) = A(z, \alpha^z)$$

at the points $m+n\beta$: these values γ_{mn} are in the number field $K = \mathbb{Q}(\alpha, \beta, \alpha^\beta)$.

Gel'fond also introduces also a polynomial $A(X, Y) \in \mathbb{Z}[X, Y]$ in two variables and considers the auxiliary function

$$F(z) = A(e^z, e^{\beta z});$$

the values γ_{mn} at the points $m \log \alpha$ of the derivatives $F^{(n)}(z)$ are again in the number field K.

With these notations, the proofs are similar: the first step is the existence of a non–zero polynomial A, of partial degrees bounded by L_1 and L_2, say, such that the associated numbers γ_{mn} vanish for certain values of m and n, say $0 \le m < M$, $0 \le n < N$. This amounts to showing that a system of linear homogeneous equations has a non–trivial solution; linear algebra suffices for the existence. In this system of equations, the coefficients are algebraic numbers in the number field K, the unknowns are the coefficients of the polynomial A, and we are looking for a solution in rational integers. There are several options at this stage: one may either require only coefficients in the ring of integers of K, in which case the assumption $L_1 L_2 > MN$ suffices. An alternative way is to require the coefficients to be in $\mathbb{Z}$, in which case one needs to assume $L_1 L_2 > MN[K : \mathbb{Q}]$.

This approach is not quite sufficient for the next steps: one will need estimates for the coefficients of this auxiliary polynomial A. This is where the Thue–Siegel Lemma occurs into the picture: by assuming that the number of unknowns, namely $L_1 L_2$, is slightly larger than the number of equations, say twice as large, this lemma produces a bound, for a non–trivial solution of the homogeneous linear system, which is sharp enough for the rest of the proof.

The second step is an induction: one proves that γ_{mn} vanishes for further values of (m, n). Since there are two parameters (m, n), there are several options for this extrapolation (increasing m, or n, or $m+n$ for instance), but anyway the idea is that if F has sufficiently many zeroes, then F takes rather small values on some disc (Schwarz Lemma), and so do its derivatives (Cauchy's inequalities). Further, an element of K which is sufficiently small should vanish (by a Liouville type inequality, or a so-called *size* inequality, or else the product formula – see for instance [41, 9, 24, 104] among many references on this topic).

For the last step, there are also several options: one may perform the induction with infinitely many steps and use an asymptotic zero estimate, or else stop after a finite number of steps and prove that some determinant does not vanish. The second method is more difficult and this is the one Schneider succeeded to complete, but his proof can be simplified by pursuing the induction forever.

There is a duality between the two methods. In Gel'fond 's proof, replace L_1 and L_2 by S_1 and S_2, and replace M and N by T_0 and T_1; hence the numbers γ_{mn} which arise are

$$\left(\frac{d}{dz}\right)^{t_0}\left(e^{(s_1+s_2\beta)z}\right)_{z=t_1\log\alpha}$$

while in Schneider's proof replacing L_1 and L_2 by T_1 and T_2, and replacing M and N by S_0 and S_1, then the numbers γ_{mn} which arise are

$$\left(z^{t_0}\alpha^{t_1 z}\right)_{z=s_1+s_2\beta}$$

It is easily seen that they are the same, namely

$$(s_1+s_2\beta)^{t_0}\alpha^{t_1 s_1}(\alpha^\beta)^{t_1 s_2}.$$

See [99] and § 13.7 of [104].

Gel'fond–Schneider Theorem was extended in 1966 by A. Baker [4], who proved the more general result that *if* $\log\alpha_1,\ldots,\log\alpha_n$ *are* $\mathbb{Q}$*–linearly independent logarithms of algebraic numbers, then the numbers* $1,\log\alpha_1,\ldots,\log\alpha_n$ *are linearly independent over* $\overline{\mathbb{Q}}$. The auxiliary function used by Baker may be considered as a function of several variables or as a function of a single complex variable, depending on the point of view (cf. [95]). The analytic estimate (Schwarz lemma) involves merely a single variable. The differential equations can be written with a single variable with transcendental coefficients. By introducing several variables, only algebraic coefficients occur. See also § 3.1.4.

Indeed, assume that $\alpha_1,\ldots,\alpha_n,\alpha_{n+1},\beta_0,\ldots,\beta_n$ are algebraic numbers which satisfy

$$\beta_0+\beta_1\log\alpha_1+\beta_n\log\alpha_n=\log\alpha_{n+1}$$

for some specified values of the logarithms of the α_j. Then the $n+2$ functions of n variables

$$z_0, e^{z_1},\ldots,e^{z_n}, e^{\beta_0 z_0+\beta_1 z_1+\cdots+\beta_n z_n}$$

satisfy differential equations with algebraic coefficients and take algebraic values at the integral multiples of the point

$$(1,\log\alpha_1,\ldots,\log\alpha_n)\in\mathbb{C}^n.$$

This situation is therefore an extension of the setup in Gel'fond's solution of Hilbert's seventh problem, and Baker's method can be viewed as an extension of Gel'fond's method. The fact that all points are on a complex line

$\mathbb{C}(1, \log\alpha_1, \ldots, \log\alpha_n) \subset \mathbb{C}^n$ means that Baker's method requires only tools from the theory of one complex variable.

On the other hand, the corresponding extension of Schneider's method requires several variables: under the same assumptions, consider the functions

$$z_0, z_1, \ldots, z_n, e^{z_0}\alpha_1^{z_1}\cdots\alpha_n^{z_n}$$

and the points in the subgroup of $\mathbb{C}^{n+1}$ generated by

$$\bigl(\{0\}\times\mathbb{Z}^n\bigr) + \mathbb{Z}(\beta_0, \beta_1, \ldots, \beta_n).$$

Since Baker's Theorem includes the transcendence of e, there is no hope to prove it without introducing the differential equation of the exponential function - in order to use the fact that the last function involves e and not another number in the factor e^{z_0}, we also take derivatives with respect to z_0. For this method, we refer to [104]. The duality between Baker's method and Schneider's method in several variables is explained below in § 3.2.

3.1.3 Schneider–Lang Criterion

In 1949, [78] Th. Schneider produced a very general statement on algebraic values of analytic functions, which can be used as *a principle for proofs of transcendence*. This statement includes a large number of previously known results like the Hermite–Lindemann and Gel'fond–Schneider Theorems. It also contains the so–called *Six exponentials Theorem* 3.8 (which was not explicitly in the literature then). To a certain extent, such statements provide partial answers to Weierstraß question (see § 2.1) that exceptional sets of a transcendental function are not too large; here one puts restrictions on the functions, while in Pólya's work concerning integer–valued entire functions, the assumptions were mainly on the points and the values (the mere condition on the functions were that they have a finite order of growth).

A few years later, in his book [79] on transcendental numbers, Schneider gave variants of this statement, which lose some generality but gained in simplicity.

Further simplifications were introduced by S. Lang in 1964 and the statement which is reproduced in his book on transcendental numbers [41] is the so-called *Criterion of Schneider–Lang* (see also the appendix of [42] as well as [95] Th. 3.3.1).

Theorem 3.3. *Let K be a number field and $f_1, \ldots, f_d$ be entire functions in $\mathbb{C}$. Assume that f_1 and f_2 are algebraically independent over K and have finite order of growth. Assume also that they satisfy differential equations: for $1 \le i \le d$, assume f_i' is a polynomial in $f_1, \ldots, f_d$ with coefficients in K. Then the set S of $w \in \mathbb{C}$ such that all $f_i(w)$ are in K is finite*[3]*.*

[3]For simplicity, we consider only *entire* functions; the results extends to meromorphic functions, and this is important for applications, for instance to elliptic functions. To deal with functions which are analytic in a disc only is also an interesting issue [90, 91, 92, 93, 94].

This statement includes the Hermite–Lindemann Theorem on the transcendence of e^{α}: take

$$K = \mathbb{Q}(\alpha, e^{\alpha}), \quad f_1(z) = z, \quad f_2(z) = e^z, \quad S = \{m\alpha \; ; \; m \in \mathbb{Z}\},$$

as well as the Gel'fond –Schneider Theorem on the transcendence of α^{β} following Gel'fond's method: take

$$K = \mathbb{Q}(\alpha, \beta, \alpha^{\beta}), \quad f_1(z) = e^z, \quad f_2(z) = e^{\beta z}, \quad S = \{m \log \alpha \; ; \; m \in \mathbb{Z}\}.$$

This criterion 3.3 does not include some of the results which are proved by means of Schneider's method (for instance it does not contain the *Six Exponentials Theorem* 3.8), but there are different criteria (not involving differential equations) for that purpose (see for instance [41, 95, 104]).

Here is the idea of the proof of the Schneider–Lang Criterion 3.3. We argue by contradiction: assume f_1 and f_2 take simultaneously their values in the number field K for different values $w_1, \ldots, w_S \in \mathbb{C}$. We want to show that there exists a non–zero polynomial $P \in \mathbb{Z}[X_1, X_2]$ such that the function $P(f_1, f_2)$ is the zero function: this will contradict the assumption that f_1 and f_2 are algebraically independent.

The first step is to show that there exists a non–zero polynomial $P \in \mathbb{Z}[X_1, X_2]$ such that $F = P(f_1, f_2)$ has a zero of high multiplicity, say $\geq T$, at each w_s, $(1 \leq s \leq S)$: we consider the system of ST homogeneous linear equations

$$\left(\frac{d}{dz}\right)^t F(w_s) = 0 \quad \text{for} \quad 1 \leq s \leq S \quad \text{and} \quad 0 \leq t < T \tag{3.4}$$

where the unknowns are the coefficients of P. If we require that the partial degrees of P are strictly less than L_1 and L_2 respectively, then the number of unknowns is $L_1 L_2$. Since we are looking for a polynomial P with rational integer coefficients, we need to introduce the degree $[K : \mathbb{Q}]$ of the number field K. As soon $L_1 L_2 > TS[K : \mathbb{Q}]$, there is a non–trivial solution. Further, the Thue–Siegel Lemma produces an upper bound for the coefficients of P.

The next step is an induction: the goal is to prove that $F = 0$. By induction on $T' \geq T$, one proves

$$\left(\frac{d}{dz}\right)^t F(w_s) = 0 \quad \text{for} \quad 1 \leq s \leq S \quad \text{and} \quad 0 \leq t < T'.$$

One already knows that it holds for $T' = T$ by (3.4).

The proof of this induction is the same as the ones by Gel'fond and Schneider of the transcendence of α^{β}, (see § 3.1.2), combining an analytic upper bound (Schwarz Lemma) and an arithmetic lower bound (Liouville's inequality). At the end of the induction, one deduces $F = 0$, which is the desired contradiction with the algebraic independence of f_1 and f_2.

As we have seen in § 3.1.2, such a scheme of proof is characteristic of the Gel'fond–Schneider's method.

The main analytic argument is Schwarz Lemma for functions of one variable, which produces an upper bound for the modulus of an analytic function having many zeroes. One also requires Cauchy's inequalities in order to bound the moduli of the derivatives of the auxiliary function.

In this context, a well known open problem raised by Th. Schneider (this is the second in the list of his 8 problems from his book [79]) is related with his proof of the transcendence of $j(\tau)$, where j is the modular function defined in the upper half plane $\Im m(z) > 0$ and τ is an algebraic point in this upper half plane which is not imaginary quadratic. Schneider himself proved the transcendence of $j(\tau)$, but his proof is not direct, it rests on the use of elliptic functions (one may apply the Schneider–Lang Criterion for meromorphic functions). Therefore, the question is to prove the same result by using modular functions. In spite of recent progress on transcendence and modular functions (see § 3.1.6), this problem is still open. The difficulty lies in the analytic estimate and the absence of a suitable Schwarz Lemma - the best results on this topic are due to I.Wakabayashi [90, 91, 92, 93, 94].

3.1.4 Higher dimension: several variables

In 1941, Th. Schneider [77] obtained an outstanding result on the values of Euler's Gamma and Beta functions: *for any rational numbers a and b such that none of a, b and $a+b$ is an integer, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

His proof involves a generalization of Gel'fond's method to several variables and yields a general transcendence criterion for functions satisfying differential equations with algebraic coefficients and taking algebraic values at the points of a large Cartesian product. He applies this criterion to the theta functions associated with the Jacobian of the Fermat curves. His transcendence results apply more generally to yield transcendence results on periods of abelian varieties.

After a suggestion of P. Cartier, S. Lang extended the classical results on the transcendence of the values of the classical exponential function to the exponential function of commutative algebraic groups. During this process, he generalized the one dimensional Schneider–Lang criterion 3.3 to several variables [41] Chap. IV. In this higher dimensional criterion, the conclusion is that the set of exceptional values in $\mathbb{C}^n$ cannot contain a large Cartesian product. This was the generalization to several variables of the fact that in the one dimensional case the exceptional set is finite (with a bound for the number of elements).

A simplified version of the Schneider–Lang Criterion in several variables for Cartesian products is the following ([104] Theorem 4.1).

Theorem 3.5 (Schneider–Lang Criterion in several variables)**.** *Let d and n be two integers with $d > n \geq 1$, K be a number field, and $f_1, \ldots, f_d$ be algebraically* *independent entire functions of finite order of growth. Assume, for $1 \le \nu \le n$ and $1 \le i \le d$, that the partial derivative $(\partial/\partial z_\nu)f_i$ of f_i belongs to the ring $K[f_1, \dots, f_d]$. Further, let $(\underline{y}_1, \dots, \underline{y}_n)$ be a basis of $\mathbb{C}^n$ over $\mathbb{C}$. Then the numbers*

$$f_i(s_1\underline{y}_1 + \cdots + s_n\underline{y}_n), \qquad \bigl(1 \le i \le d,\ (s_1, \dots, s_n) \in \mathbb{Z}^n\bigr)$$

do not all belong to K.

Besides the consequences already derived by Schneider in 1941, Lang gave further consequences of this result to commutative algebraic groups, especially abelian varieties [41] Chap. IV and [98] Chap. 5.

It is interesting from an historical point of view to notice that Bertrand and Masser [6] succeeded in 1980 to deduce Baker's Theorem form the Schneider–Lang Criterion 3.5 (see [104] § 4.2). They could also prove the elliptic analog of Baker's result and obtain the linear independence, over the field of algebraic numbers, of elliptic logarithms of algebraic points - at that time such a result was available only in the case of complex multiplication (by previous work of D.W. Masser).

According to [41], Historical Note of Chap. IV, M. Nagata suggested that in the higher dimensional version of the Schneider–Lang Criterion the conclusion could be that the exceptional set of points where all the functions take simultaneously values in a number field K is contained in an algebraic hypersurface, the bound for the number of points being replaced by a bound for the degree of the hypersurface. This program was fulfilled by E. Bombieri in 1970 [8].

Theorem 3.6 (Bombieri)**.** *Let d and n be two integers with $d > n \ge 1$, K be a number field, and $f_1, \dots, f_d$ be algebraically independent entire functions of finite order of growth. Assume, for $1 \le \nu \le n$ and $1 \le i \le d$, that the partial derivative $(\partial/\partial z_\nu)f_i$ of f_i belongs to the ring $K[f_1, \dots, f_d]$. Then the set of points $w \in \mathbb{C}^n$ where the d functions $f_1, \dots, f_d$ all take values in K is contained in an algebraic hypersurface.*

Bombieri produces an upper bound for the degree of such an hypersurface (see also [98] Th. 5.1.1). His proof [8] involves different tools, including L^2–estimates by L. Hörmander for functions of several variables. One main difficulty that Bombieri had to overcome was to generalize Schwarz Lemma to several variables, and his solution involves an earlier work by E. Bombieri and S. Lang [10], where they use Lelong's theory of the mass of zeroes of analytic functions in a ball. Chapter 7 of [98] is devoted to this question. The next statement (Proposition 3.7) follows from the results in Chapter 7 of [98]. Given a finite subset S of $\mathbb{C}^n$ and an integer $t \ge 1$, one defines $\omega_t(S)$ as the smallest degree of a polynomial having a zero at each point of S of multiplicity at least t. Then the sequence $\omega(t)/t$ has a limit $\Omega(S)$ which satisfies

$$\frac{1}{t+n-1}\omega_t(S) \le \Omega(S) \le \frac{1}{t}\omega_t(S) \le \omega_1(S)$$

for all $t \ge 1$.

Proposition 3.7 (Schwarz Lemma in several variables)**.** *Let S be a finite subset of $\mathbb{C}^n$ and ϵ be a positive real number. There exists a positive real number $r_0 = r_0(S, \epsilon)$ such that, for any positive integer t, any real numbers R and r with $R > r \geq r_0$ and any entire function having a zero of multiplicity $\geq t$ at each point of S,*
$$|f|_r \leq \left(\frac{4nr}{R}\right)^{t(\Omega(S)-\epsilon)} |f|_R.$$

More recent results on Schwarz Lemma in several variables are due to D. Roy [70, 71, 72, 73].

3.1.5 The six exponentials Theorem

Here is the six exponentials Theorem, due to Siegel, Lang and Ramachandra [41, 95, 104, 98].

Theorem 3.8 (Six exponentials Theorem)**.** *Let $x_1, \ldots, x_d$ be complex numbers which are linearly independent over $\mathbb{Q}$ and let $y_1, \ldots, y_\ell$ be also complex numbers which are linearly independent over $\mathbb{Q}$. Assume $d\ell > d + \ell$. Then one at least of the numbers*
$$e^{x_i y_j}, \quad (1 \leq i \leq d,\ 1 \leq j \leq \ell) \tag{3.9}$$
is transcendental.

The condition $d\ell > d + \ell$ in integers d and ℓ means that the relevant cases are $d = 2$ and $\ell = 3$ or $d = 3$ and $\ell = 2$ (there is a symmetry), hence the name (since $\ell d = 6$ in these cases). We refer to [41, 95] for more information on this topic.

The classical proof by Schneider's method involves an auxiliary function of the form
$$F(z) = P(e^{x_1 z}, \ldots, e^{x_d z}),$$
where the existence of the polynomial P follows from Dirichlet's box principle and the Thue–Siegel Lemma. The conditions which are required are that F vanishes at many points of the form $s_1 y_1 + \ldots + s_\ell y_\ell$ with varying integers $s_1, \ldots, s_\ell$, the induction shows that F vanishes at more points of this form, until one deduces that F vanishes at all such points, and the conclusion then easily follows. One could avoid an infinite induction by using a zero estimate.

A variant is to use an interpolation determinant (see § 4 and [104] Chap. 2). As an example of upper bound for an interpolation determinant, here is the statement of Lemma 2.8 in [104].

Lemma 3.10. *Let $\varphi_1, \ldots, \varphi_L$ be entire functions in $\mathbb{C}$, $\zeta_1, \ldots, \zeta_L$ be elements of $\mathbb{C}$, $\sigma_1, \ldots, \sigma_L$ nonnegative integers, and $0 < r \leq R$ be real numbers, with $|\zeta_\mu| \leq r$ $(1 \leq \mu \leq L)$. Then the absolute value of the determinant*
$$\Delta = \det\left(\left(\frac{d}{dz}\right)^{\sigma_\mu} \varphi_\lambda(\zeta_\mu)\right)_{1 \leq \lambda, \mu \leq L}$$

is bounded from above by

$$|\Delta| \leq \left(\frac{R}{r}\right)^{-L(L-1)/2+\sigma_1+\cdots+\sigma_L} L! \prod_{\lambda=1}^{L} \max_{1\leq\mu\leq L} \sup_{|z|=R} \left| \left(\frac{d}{dz}\right)^{\sigma_\mu} \varphi_\lambda(z) \right|.$$

Another method of proof of the six exponentials Theorem 3.8 is proposed in [101, 103]. The starting idea is to consider the function of two complex variables e^{zw}. If all numbers in (3.9) are algebraic, then this function takes algebraic values (in fact in a number field) at all points (z, w) where z is of the form $t_1x_1 + \cdots + t_dx_d$ and w of the form $s_1y_1 + \ldots + s_\ell y_\ell$ with integers $t_1, \ldots, t_d$, $s_1, \ldots, s_\ell$. The method does not work like this: a single function e^{zw} in two variables does not suffice, one needs several functions. For this reason one introduces *redundant variables.* Letting z_h and w_k be new variables, one investigates the values of the functions are $e^{z_h w_k}$ at the points of Cartesian products.

Such a device has been introduced in transcendental number theory in 1981 by P. Philippon [65] for giving a proof of an algebraic independence result announced by G.V. Chudnovskii - at that time the original proof was very complicated and the approach by Philippon, using a transcendence criterion from É. Reyssat, was a dramatic simplification. Now much sharper results are known. In 1981, introducing several variables was called *Landau's trick*, which is a homogeneity argument: letting the number of variables tend to infinity enables one to kill error terms.

The generalisation of the six exponentials Theorem to higher dimension, also with multiplicities, is one of the main topics of [104].

Further auxiliary functions occur in the works on algebraic independence by A.O. Gel'fond [33], G.V. Chudnovskii and others. A reference is [63].

3.1.6 Modular functions

The solution by the team at St Etienne [5] of the problems raised by Mahler and Manin on the transcendence of the values of the modular function $J(q)$ for algebraic values of q in the unit disc[4] involves an interesting auxiliary function: the general scheme of proof is the one of Gel'fond and Schneider. They construct their auxiliary function by means of the Thue–Siegel Lemma as a polynomial in z and $J(z)$ having a high multiplicity zero at the origin, say $\geq L$. They consider the exact order of multiplicity M, which by construction is $\geq L$, and they bound $|z^{-M}F(z)|$ in terms of $|z|$ (this amounts to using the easiest case of Schwarz Lemma in one variable with a single point). They apply this estimate to $z = q^S$ where S is the smallest integer for which $F(q^S) \neq 0$. Liouville's estimate gives the conclusion. See also [63], Chap. 2.

A similar construction was performed by Yu. V. Nesterenko in 1996 (see [61] and [63] Chap. 3), when he proved the algebraic independence of π, e^π and $\Gamma(1/4)$: his main result is that *for any q in the open set $0 < |q| < 1$, the*

[4]The connexion between $J(q)$ and $j(\tau)$ is $j(\tau) = J(e^{2i\pi\tau})$.

transcendence degree of the field

$$\mathbb{Q}\bigl(q, P(q), Q(q), R(q)\bigr)$$

is at least 3. Here, P, Q, R are the classical Ramanujan functions, which are sometimes denoted as E_2, E_4 and E_6 (Eisenstein series). The auxiliary function F is a polynomial in the four functions z, $E_2(z)$, $E_4(z)$, $E_6(z)$; like for the *théorème stéphanois* on the transcendence of $J(q)$, it is constructed by means of the Thue–Siegel Lemma so that it has a zero of large multiplicity, say M. In order to apply a criterion for algebraic independence, Nesterenko needs to establish an upper bound for M, and this is not an easy result. An alternative argument due to Philippon (see [63] Chap. 4) is to apply a measure of algebraic independence of Faisant and Philibert [23] on numbers of the form ω/π and η/π.

3.2 Universal auxiliary functions

3.2.1 A general existence theorem

In the Gel'fond–Schneider method, the auxiliary function is constructed by means of the Thue–Siegel Lemma, and the requirement is that it has many zeroes (multiplicity are there in Gel'fond's method, not in Schneider's method). There is an alternative construction which was initiated in a joint work with M. Mignotte in 1974 [60], in connexion with quantitative statements related with transcendence criteria like the Schneider–Lang criterion. This approach turned out to be specially efficient in another context, namely in extending Schneider's method to several variables [96]. The idea is to require that the auxiliary function F has small Taylor coefficients at the origin; it follows that its modulus on some discs will be small, hence its values (including derivatives, if one wishes) at points in such a disc will also be small. Combining Liouville's estimate with Cauchy's inequality for estimating the derivatives, one deduces that F has a lot of zeroes, more than would be reached by the Dirichlet's box principle. At this stage there are several options; the easy case is when a sharp zero estimate is known: we immediately reach the conclusion without any further extrapolation: in particular there is no need of Schwarz Lemmas in several variables. This is what happend in [96] for exponential functions in several variables, the zero estimate being due to D.W. Masser [55]. This result, dealing with products of multiplicative groups (tori), can be extended to commutative algebraic groups [97], thanks to the zero estimate of Masser and Wüstholz [57, 58].

A *zero estimate* is a statement which gives a lower bound for the degree of a polynomial which vanishes at given points (when multiplicities are introduced this is sometimes calles a *multiplicity estimate*). D.W. Masser developed the study of zero estimates, and P. Philippon, G. Wüstholz, Yu.V. Nesterenko were among those who contributed to the theory. Introductions to this subject have been written by D. Roy (Chapters 5 and 8 of [104] and Chapter 11 of [63]). Also Chapter 5 of [4] is devoted to multiplicity estimates.

A *Schwarz Lemma* is a statement which gives an upper bound for the maximum modulus of an analytic function which vanishes at given points - multiplicities may be there. We gave an example in Proposition 3.7.

When the function has only small values at these points, instead of zeoes, one speaks either of a *small value Lemma* or of an *approximate Schwarz Lemma* [66].

A Schwarz Lemma implies a zero estimate, and a small value Lemma implies a Schwarz Lemma. However, the assumptions for obtaining a small value Lemma for instance are usually stronger than for only a Schwarz Lemma: as an example, in one variable there is no need to introduce an assumption on the distance of the given points for a Schwarz Lemma, while for a small value Lemma it is requested.

This construction of universal auxiliary functions is developed in [99] (see Lemme 2.1) and [100].

Here is Proposition 4.10 of [104].

Proposition 3.11. *Let L and n be positive integers, N, U, V, R, r positive real numbers and $\varphi_1, \dots, \varphi_L$ entire functions in $\mathbb{C}^n$. Define $W = N + U + V$ and assume*

$$W \geq 12n^2, \quad e \leq \frac{R}{r} \leq e^{W/6}, \quad \sum_{\lambda=1}^{L} |\varphi_\lambda|_R \leq e^U$$

and

$$(2W)^{n+1} \leq LN\bigl(\log(R/r)\bigr)^n.$$

Then there exist rational integers $p_1, \dots, p_L$, with

$$0 < \max_{1 \leq \lambda \leq L} |p_\lambda| \leq e^N,$$

such that the function $F = p_1\varphi_1 + \cdots + p_L\varphi_L$ satisfies

$$|F|_r \leq e^{-V}.$$

For an application to algebraic independence, see [63] Prop. 3.3 Ch. 14.

3.2.2 Duality

In the papers [99] (see Lemme 2.1) and [100], a *dual* construction is performed, where auxiliary *analytic functionals* are constructed, and the duality between the methods of Schneider and Gel'fond can be explained by means of the Fourier–Borel transform (see [99], especially Lemmas 3.1 and 7.6, and [104], § 13.7). In the special case of exponential polynomials, it reduces to the relation

$$D^{\underline{\sigma}} \left(\underline{z}^{\underline{\tau}} e^{\underline{t}\underline{z}} \right) (\underline{s}) = D^{\underline{\tau}} \left(\underline{z}^{\underline{\sigma}} e^{\underline{s}\underline{z}} \right) (\underline{t}), \tag{3.12}$$

where $\underline{\sigma}$, $\underline{\tau}$, $\underline{s}$, $\underline{t}$, $\underline{z}$ stand for tuples

$$\begin{aligned} &\underline{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{Z}^n_{\geq 0}, \quad &&\underline{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{Z}^n_{\geq 0}, \\ &\underline{s} = (s_1, \dots, s_n) \in \mathbb{C}^n, &&\underline{t} = (t_1, \dots, t_n) \in \mathbb{C}^n, \\ &\underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \end{aligned}$$

and

$$D^{\underline{\sigma}} = \left(\frac{\partial}{\partial z_1}\right)^{\sigma_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\sigma_n}$$

(see [99] Lemma 3.1 and [104] Corollary 13.21), which is a generalization in several variables of the formula

$$\left(\frac{d}{dz}\right)^{\sigma} \left(z^{\tau} e^{tz}\right)(s) = \left(\frac{d}{dz}\right)^{\tau} \left(z^{\sigma} e^{sz}\right)(t)$$

for t, s in $\mathbb{C}$ and σ, τ non–negative integers, both sides being

$$\sum_{k=0}^{\min\{\tau,\sigma\}} \frac{\sigma!\tau!}{k!(\tau-k)!(\sigma-k)!} t^{\sigma-k} s^{\tau-k} e^{st}.$$

Another (less special) case of (3.12) is the duality between Schneider's method in several variables and Baker's method. In Baker's method one considers the values at $(t, t\log\alpha_1, \ldots, t\log\alpha_n)$ of

$$\left(\frac{\partial}{\partial z_0}\right)^{\tau_0} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\tau_n} \left(z_0^{\sigma} e^{s_1 z_1} \cdots e^{s_n z_n} e^{s_0(\beta_0 z_0 + \cdots + \beta_n z_n)}\right),$$

while Schneider's method in several variables deals with the values of

$$\left(\frac{\partial}{\partial z_0}\right)^{\sigma} \left(z_0^{\tau_0} z_1^{\tau_1} \cdots z_n^{\tau_n} (e^{z_0}\alpha_1^{z_1} \cdots \alpha_n^{z_n})^t\right)$$

at the points

$$(s_0\beta_0, s_1 + s_0\beta_1, \ldots, s_n + s_0\beta_n),$$

and these values are just the same, namely

$$\sum_{k=0}^{\min\{\tau_0,\sigma\}} \frac{\sigma!\tau_0!}{k!(\tau_0-k)!(\sigma-k)!} t^{\sigma-k} (s_0\beta_0)^{\tau_0-k} (s_1+s_0\beta_1)^{\tau_1} \cdots (s_n+s_0\beta_n)^{\tau_n} (\alpha_0^{s_0} \cdots \alpha_n^{s_n})^t$$

with

$$\alpha_0 = e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}.$$

This is a special case of (3.12) with

$$\underline{\tau} = (\tau_0, \ldots, \tau_n) \in \mathbb{Z}_{\geq 0}^{n+1}, \quad \underline{\sigma} = (\sigma_0, 0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^{n+1},$$

$$\underline{t} = (t, t\log\alpha_1, \ldots, t\log\alpha_n) \in \mathbb{C}^{n+1}, \quad \underline{s} = (s_0\beta_0, s_1+s_0\beta_1, \ldots, s_n+s_0\beta_n) \in \mathbb{C}^{n+1},$$

$$\underline{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}.$$

The construction of universal auxiliary functions is one of the tools in D. Roy's approach to Schanuel's Conjecture in [70, 71].

3.3 Mahler's Method

In 1929, K. Mahler [50] developed an original method to prove the transcendence of values of functions satisfying certain types of functional equations. This method was somehow forgotten, for instance it is not quoted among the 440 references of the survey paper [25] by N.I. Fel'dman, and A.B. Šidlovskiĭ. After the publication of the paper [52] by Mahler, several mathematicians (including K. Kubota, J.H. Loxton and A.J. van der Poorten) extended the method (see the Lecture Notes [64] by K. Nishioka for further references). The construction of the auxiliary function is similar to what is done in Gel'fond–Schneider's method, with a main difference: in place of the Thue–Siegel Lemma, only linear algebra is required. No estimate for the coefficients of the auxiliary polynomial is needed in Mahler's method.

The following example is taken from § 1.1 of [64]. Let $d \geq 2$ be a rational integer and α an algebraic number with $0 < |\alpha| < 1$. Let us prove that the number

$$\sum_{k=0}^{\infty} \alpha^{d^k}$$

is transcendental, a result due to K. Mahler [50]. The basic remark is that the function

$$f(z) = \sum_{k=0}^{\infty} z^{d^k}$$

satisfies the functional equation $f(z^d) = f(z) - z$. It is not difficult to check also that f is a transcendental function, which means that if P is a non–zero polynomial in $\mathbb{C}[X, Y]$, then the function $F(z) = P\bigl(z, f(z)\bigr)$ is not the zero function. From linear algebra, it follows that if L is a sufficiently large integer, there exist a non–zero polynomial P in $\mathbb{Z}[X, Y]$ of partial degrees $\leq L$ such that the associated function $F(z) = P\bigl(z, f(z)\bigr)$ has a zero at the origin of multiplicity $> L^2$. Indeed, the existence of P amounts to showing the existence of a non–trivial solution to a system of $L^2 + 1$ homogeneous linear equations with rational coefficients in $(L + 1)^2$ unknowns.

If T denotes the multiplicity of F at the origin, then the limit

$$\lim_{k \to \infty} F(\alpha^{d^k})\alpha^{-d^k T}$$

is a non–zero constant. This produces an upper bound for $|F(\alpha^{d^k})|$ when k is a sufficiently large positive integer, and this upper bound is not compatible with Liouville's inequality. Hence $f(\alpha)$ is transcendental.

Another instructive example of application of Mahler's method is given by D.W. Masser in the first of his Cetraro's lectures [56].

The numbers whose transcendence is proved by this method are not Liouville numbers, but they are quite well approximated by algebraic numbers. In the example we discussed, the number $f(\alpha)$ is very well approximated by the algebraic numbers

$$\sum_{k=0}^{K} \alpha^{d^k}, \qquad (K > 0).$$

4 Interpolation determinants

An interesting development of the saga of auxiliary functions took place in 1991, thanks to the introduction of interpolation determinants by M. Laurent. Its origin goes back to earlier works on a question raised by Lehmer which we first introduce.

4.1 Lehmer's Problem

Let θ be a non–zero algebraic integer of degree d. Mahler's *measure* of θ is

$$M(\theta) = \prod_{i=1}^{d} \max(1, |\theta_i|) = \exp\left(\int_0^1 \log|f(e^{2i\pi t}|dt\right),$$

where $\theta = \theta_1$ and $\theta_2, \cdots, \theta_d$ are the conjugates of θ and f the monic irreducible polynomial of θ in $\mathbb{Z}[X]$.

From the definition one deduces $M(\theta) \geq 1$. According to a well-known and easy result of Kronecker, $M(\theta) = 1$ *if and only if θ is a root of unity.*

D.H. Lehmer [46] asked whether *there is a constant $c > 1$ such that $M(\theta) < c$ implies that θ is a root of unity.*

Among many tools which have been introduced to answer this question, we only quote some of them which are relevant for our concern. In 1977, M. Mignotte [59] used ordinary Vandermonde determinants to study algebraic numbers whose conjugates are close to the unit circle. In 1978, C.L. Stewart [85] sharpend earlier results by Schinzel and Zassenhaus (1965) and Blanksby and Montgomery (1971) by introducing an auxiliary function (whose existence follows from the Thue–Siegel Lemma) and using an extrapolation similar to what is done in the Gel'fond–Schneider method.

Refined estimates were obtained by E. Dobrowolski in 1979 [21] using Stewart's approach together with congruences modulo p. He achieved the best unconditional result known so far in this direction (apart from some marginal improvements on the numerical value for c: *There is a constant c such that, for θ a non–zero algebraic integer of degree d,*

$$M(\theta) < 1 + c(\log\log d/\log d)^3$$

implies that θ is a root of unity.

In 1982, D. Cantor and E.G. Straus [15] revisited this method of Stewart and Dobrowolski by replacing the auxiliary function by a generalised Vandermonde determinant. The sketch of proof is the following: in Dobrowolski's proof, there is a zero lemma which can be translated into a statement that some matrix has a maximal rank; therefore some determinant is not zero. This determinant is bounded from above by means of Hadamard's inequality; the upper bound depends on $M(\theta)$. On the other hand, the absolute value of this determinant is shown to be big, because it has many factors of the form $\prod_{i,j} |\theta_i^p - \theta_j|^k$, for many primes p. The lower bound makes use of a Lemma due to Dobrowolski: *For θ not a root of unity,*

$$\prod_{i,j} |\theta_i^p - \theta_j| \geq p^d$$

for any prime p.

One may also prove the lower bound by means of a p–adic Schwarz Lemma: a function (here merely a polynomial) with many zeroes has a small (p–adic) absolute value. In this case the method is similar to the earlier one, with analytic estimates on one side and arithmetic ones (Liouville type, or product formula) on the other. See [104] § 3.6.5 and § 3.6.6.

Dobrowolski's result has been extended to several variables by F. Amoroso and S. David in [2] – the higher dimensional version is much more involved. We refer to S. David's survey [20] for further references on this topic. We only notice the role of the generalized Dirichlet exponents ω_t and Ω (see [98] § 1.3 and Chap. 7), which occurred in transcendence statements in connexion with multidimensional Schwarz Lemmas (cf. Proposition 3.7).

4.2 Laurent's interpolation determinants

In 1991, M. Laurent [43] discovered that one may get rid of the Dirichlet's box principle in Gel'fond–Schneider's method by means of his *interpolation determinants*. In the classical approach, there is a zero estimate (or vanishing estimate, also called multiplicity estimate when derivatives are there) which shows that some auxiliary function cannot have too many zeroes. This statement can be converted into the non–vanishing of some determinant. Laurent works directly with this determinant: a Liouville-type estimate produces a lower bound, the remarkable fact is that analytic estimates like Schwarz Lemma produce sharp upper bounds. Again, the analytic estimates depend only in one variable (even if the determinant is a value of a function in many variables, it suffices to restrict this function to a complex line – see [104] § 6.2). Therefore this approach is specially efficient when dealing with functions of several variables, where Schwarz Lemmas are lacking.

Baker's method is introduced in [104] § 10.2 with interpolation determinants and in § 10.3 with auxiliary functions.

Interpolation determinants are easy to use when a sharp zero estimate is available. If not, it is more tricky to prove the analytic estimate. However, it is possible to perform extrapolation in the transcendence methods involving interpolation determinants: an example is a proof of Pólya's theorem (2.1) by means of interpolation determinants [102].

Proving algebraic independence results by means of interpolation determinants was quoted as an open problem in [99] p. 257: at that time (1991) auxiliary functions (or auxiliary functionals) were required, together with a zero estimate (or an interpolation estimate). The first solution to this problem in 1997 [74] (see also [104] Corollary 15.10) makes a detour via measures of simultaneous approximations: such measures can be proved by means of interpolation determinants, and they suffice to produce algebraic independence statements (*small transcendence degree*: at least two numbers in certain given sets are algebraically independent). Conjecturally, this method should also produce *large transcendence degree* results - see [104] Conjecture 15.31. See also the work by P. Philippon [67].

Another approach, due to M. Laurent and D. Roy [45] (see also [63] Chap. 13) is based on the observation that in algebraic independence proofs, the determinants which occur produce sequences of polynomials having small values *together with their derivatives* at a given point. By means of a generalization of Gel'fond's transcendence criterion involving multiplicities, M. Laurent and D. Roy succeeded to get algebraic independence results. Further generalisations of Gel'fond's criterion involving not only multiplicities, but also several points (having some structure, either additive or multiplicative) are being investigated by D. Roy in connection with his original strategy towards a proof of Schanuel's Conjecture [70, 71].

5 Bost slope inequalities, Arakelov's Theory

Interpolation determinants require choices of bases. A further tool has been introduced by J-B. Bost in 1994 [11], where bases are no more required: the method is more intrinsic. His argument rests on Arakelov's Theory, which is used to produce *slope inequalities*. This new approach is specially interesting for results on abelian varieties obtained by transcendence methods, the examples developed by Bost being related with the work of D. Masser and G. Wüstholz on periods and isogenies of abelian varieties over number fields. Further estimates related to Baker's method and measures of linear independence of logarithms of algebraic points on abelian varieties have been achieved by E. Gaudron [27, 28, 29] using Bost's approach. An introduction to Bost method can be found in the Bourbaki lecture [17] by A. Chambert-Loir in 2002.

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