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# BETA-EXPANSIONS, NATURAL EXTENSIONS AND MULTIPLE TILINGS

CHARLENE KALLE AND WOLFGANG STEINER

ABSTRACT. From the works of Rauzy and Thurston, we know how to construct (multiple) tilings of some Euclidean space using the conjugates of a Pisot unit  $\beta$  and the greedy  $\beta$ -transformation. In this paper, we consider different transformations generating expansions in base  $\beta$ . Under certain mild conditions, we show that they give multiple tilings. We also give a necessary and sufficient condition for the tiling property, generalizing the weak finiteness property (W) for greedy  $\beta$ -expansions. Remarkably, the symmetric  $\beta$ -transformation does not satisfy this condition when  $\beta$  is the smallest Pisot number or the Tribonacci number. This means that the Pisot conjecture on tilings cannot be extended to the symmetric  $\beta$ -transformation.

Closely related to these (multiple) tilings are natural extensions of the transformations, which have many nice properties: they are invariant under the Lebesgue measure; under certain conditions, they provide Markov partitions of the torus; they characterize the numbers with purely periodic expansion, and they allow determining any digit in an expansion without knowing the previous digits.

## 1. INTRODUCTION

Tilings generated by substitutions and the  $\beta$ -transformation are well-studied objects from various points of view. Tilings from substitutions were first introduced by Rauzy in the seminal paper [26] from 1982. For the  $\beta$ -transformation,  $T_\beta x = \beta x \pmod{1}$ , Thurston laid the ground work in [33] from 1989. The transformation  $T_\beta$  can be used to obtain the greedy  $\beta$ -expansion of every  $x \in [0, 1)$  by iteration. The expansions obtained in this way are expressions of the form

$$(1) \quad x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k},$$

where the digits  $b_k$  are all elements of the set  $\{0, 1, \dots, \lceil \beta \rceil - 1\}$ . Here,  $\lceil x \rceil$  denotes the smallest integer larger than or equal to  $x$ . The expansions that  $T_\beta$  produces are greedy in the sense that, for each  $n \geq 1$ ,  $b_n$  is the largest element of the set  $\{0, 1, \dots, \lceil \beta \rceil - 1\}$  such that  $\sum_{k=1}^n b_k \beta^{-k} \leq x$ . Thurston defined tiles in  $\mathbb{R}^{d-1}$  when  $\beta$  is a Pisot unit of degree  $d$ . In 1999, Akiyama ([1]) and Praggastis ([25]), independently of one another, showed that these tiles form a tiling of  $\mathbb{R}^{d-1}$  when  $\beta$  satisfies the finiteness property (F) defined in [17], which means that the set of numbers with finite greedy expansion is exactly  $\mathbb{Z}[\beta^{-1}] \cap [0, 1)$ . They also showed that the origin is an inner point of the central tile in this case.

The tiles can be constructed by using two-sided admissible sequences. These are sequences  $\dots w_{-1} w_0 w_1 w_2 \dots$  of elements from the digit set  $\{0, 1, \dots, \lceil \beta \rceil - 1\}$  such that each right-sided truncation  $w_k w_{k+1} \dots$  corresponds to an expansion generated by  $T_\beta$ . In some sense, this construction makes the non-invertible transformation  $T_\beta$  invertible at the level of sequences. In ergodic theory, a way to replace a non-invertible transformation by an invertible one, without losing its dynamics, is by constructing a version of the natural extension. A natural extension of a non-invertible dynamical system is an invertible dynamical system that contains the original dynamics as a subsystem and that is minimal in a measure theoretical sense. Much theory about natural extensions was developed by Rohlin ([27]). He gave a canonical way to construct a natural extension and showed that the natural extension is unique up to isomorphism. Many properties of the original dynamical system can be obtained through the natural extension, for

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example all mixing properties of the natural extension are inherited by the original system. The tilings described above and natural extensions of  $T_\beta$  are thus closely related concepts.

The question of whether or not Thurston's construction gives a tiling when conditions are relaxed, is equivalent to a number of questions in different fields in mathematics and computer science, like spectral theory (see Siegel [29]), the theory of quasicrystals (Arnoux et al. [6]), discrete geometry (Ito and Rao [20]) and automata ([29]). In [3], Akiyama defined a weak finiteness property (W) and proved that it is equivalent to the tiling property. He also stated there that it is likely that all Pisot units satisfy this condition (W). It is thus conjectured that we get a tiling of the appropriate Euclidean space for all Pisot units  $\beta$ . This is known as the Pisot conjecture and is discussed at length in the survey paper [7] by Berthé and Siegel. A class of Pisot numbers  $\beta$  satisfying (W) is given in [4].

The transformation  $T_\beta$  is not the only transformation that can be used to generate number expansions of the form (1) dynamically. In [11], Erdős et al. defined the lazy algorithm that also gives expansions with digits in the set  $\{0, 1, \dots, \lceil \beta \rceil - 1\}$ . In [10], Dajani and Kraaikamp gave a transformation, which they called the lazy transformation, that generates exactly these expansions in a dynamical way. This transformation is defined on the extended interval  $[0, \frac{\lceil \beta \rceil - 1}{\beta - 1}]$ . In 2005, Pedicini ([24]) introduced an algorithm that produces number expansions of the form (1), but with digits in an arbitrary finite set of real numbers  $A$ . He showed that if the difference between two consecutive elements in  $A$  is not too big, then every  $x$  in a certain interval has an expansion with digits in  $A$ . These expansions generalize the greedy expansions with digits in  $\{0, 1, \dots, \lceil \beta \rceil - 1\}$ , and are thus called greedy  $\beta$ -expansions with arbitrary digits. In [9], a lazy algorithm is given by Dajani and Kalle, that can be used to get lazy expansions with arbitrary digits. In the same article, both a greedy and lazy transformation are defined to generate expansions with arbitrary digits dynamically. Another type of transformations that generate expansions like (1), but with digits in  $\{-1, 0, 1\}$ , is given in [18] by Frougny and Steiner. It is shown there, among other things, that for specific  $\beta > 1$  and  $\alpha > 0$ , the transformation  $T : [-\beta\alpha, \beta\alpha] \rightarrow [-\beta\alpha, \beta\alpha]$  defined by  $Tx = \beta x - \lfloor \frac{x}{2\alpha} + \frac{1}{2} \rfloor$  provides  $\beta$ -expansions of minimal weight, i.e., expansions in which the number of non-zero digits is as small as possible. These expansions are interesting e.g. for applications to cryptography.

In this paper, we consider a class of piecewise linear transformations with constant slope, that contains all the transformations mentioned above. By putting some restrictions on  $\beta$  and on the digit set, we can mimic the construction of a tiling of a Euclidean space, as it is given in [3]. We establish some properties of the tiles we obtain by this construction and state conditions under which these tiles give a multiple tiling.

In Section 2, we define the class of transformations that we will be considering and give a complete characterization of the set of admissible sequences for these transformations. For the greedy  $\beta$ -transformation, this characterization was first given by Parry ([23]) and depends only on the expansion of 1. In our case, we have to consider the orbits of all the endpoints of the transformation. In this section, we impose only very mild restrictions on  $\beta$  and the digit set. In Section 3, we construct a version of the natural extension of the non-invertible transformation under consideration. We look in detail at the domain of the natural extension and give some examples of natural extensions. All this is done under the assumption that  $\beta$  is a Pisot unit and that the digit set is contained in  $\mathbb{Q}(\beta)$ . In Section 4, we define tiles in  $\mathbb{R}^{d-1}$  and show that almost every point is contained in the same finite number of tiles, i.e., the construction gives a multiple tiling. We give a necessary and sufficient condition for the multiple tiling to be a tiling, generalizing the (W) property. We show that the set of points with eventually periodic  $T$ -expansion is  $\mathbb{Q}(\beta)$ , and that the points with purely periodic  $T$ -expansion are characterized by the natural extension domain. The tiling property is also equivalent to the fact that the natural extension domain gives a tiling of the torus  $\mathbb{T}^d$ , which in turn allows to determine any digit in the  $T$ -expansion of a number without knowing the previous digits. For the examples of natural extensions defined in Section 3, we discuss whether they form tilings or multiple tilings. We also find quasi-periodic tilings which are not self-affine in the sense of [25] and [31].

Remarkable examples of double tilings come from the symmetric  $\beta$ -transformation defined by Akiyama and Scheicher ([5]), for two Pisot units  $\beta$ : the Tribonacci number and the smallest Pisot number. This means that the Pisot conjecture cannot be extended to the symmetric  $\beta$ -transformation. It is unclear why the Tribonacci number and the smallest Pisot number give double tilings while many other Pisot units give tilings, and it is possible that getting more insight into this question may lead to a proof or a disproof of the Pisot conjecture.

## 2. ADMISSIBLE EXPANSIONS

Throughout the paper, we consider transformations  $T : X \rightarrow X$  defined by  $Tx = \beta x - \delta_a$  for all  $x \in X_a$ ,  $a \in A$ , where  $A$  is a finite ordered set,  $X$  is the disjoint union of non-empty bounded sets  $X_a \subset \mathbb{R}$ ,  $\delta_a \in \mathbb{R}$  and  $\beta > 1$ . We are interested in the digital expansions generated by  $T$ , as defined in Definition 2.1. We denote by  $A^\omega$  the set of right infinite sequences with elements in  $A$ , and by  $\prec$  the lexicographical order on  $A^\omega$ .

**Definition 2.1.** Let  $T$  be as in the preceding paragraph. For  $x \in X$ , the sequence  $b(x) = b_1(x)b_2(x)\cdots \in A^\omega$  satisfying  $b_k(x) = a$  if  $T^{k-1}(x) \in X_a$ ,  $a \in A$ , is called the  $T$ -expansion of  $x$ . A sequence  $u \in A^\omega$  is called  $T$ -admissible if and only if  $u = b(x)$  for some  $x \in X$ .

Note that  $Tx = \beta x - \delta_{b_1(x)}$ , so  $x = (\delta_{b_1(x)} + Tx)/\beta$ , and inductively

$$(2) \quad x = \sum_{k=1}^n \frac{\delta_{b_k(x)}}{\beta^k} + \frac{T^n x}{\beta^n}$$

for all  $n \geq 1$ . Since  $X$  is bounded, we have  $\lim_{n \rightarrow \infty} (T^n x)/\beta^n = 0$  and thus  $x = \sum_{k=1}^{\infty} \delta_{b_k(x)}\beta^{-k}$ .

**Definition 2.2.** The value of a sequence  $v = (v_k)_{k \geq 1} \in A^\omega$  is  $\sum_{k \geq 1} \delta_{v_k}\beta^{-k}$  and is denoted by  $\cdot v$ .

In many cases, we can choose  $\delta_a = a$  for all  $a \in A$ . Then the definition of  $\cdot v$  corresponds to the usual definition of the ‘‘decimal point’’ in base  $\beta$ . Note that  $x = \cdot b(x)$ .

The aim of this section is to describe the  $T$ -admissible sequences, under some conditions on  $T$ .

**Lemma 2.3.** Let  $x, y \in X$  and assume that  $\sup X_a \leq \inf X_{a'}$  for all  $a < a'$ . Then we have

$$x < y \quad \text{if and only if} \quad b(x) \prec b(y).$$

*Proof.* Clearly,  $b(x) = b(y)$  is equivalent to  $x = y$ . So we can assume that there exists some  $k \geq 1$  such that  $b_1(x)\cdots b_{k-1}(x) = b_1(y)\cdots b_{k-1}(y)$  and  $b_k(x) \neq b_k(y)$ . Then  $x < y$  is equivalent to  $T^{k-1}x < T^{k-1}y$  by (2). Since we have  $T^{k-1}x \in X_{b_k(x)}$ ,  $T^{k-1}y \in X_{b_k(y)}$ , we obtain that  $T^{k-1}x < T^{k-1}y$ ,  $b_k(x) \neq b_k(y)$ , is equivalent to  $b_k(x) < b_k(y)$ . This proves the lemma.  $\square$

From now on, we assume that the sets  $X_a$  are half-open intervals.

**Definition 2.4.** Let  $\beta > 1$ , let  $A$  be a finite ordered set,  $X$  the union of non-empty intervals  $X_a = [\ell_a, r_a) \subset \mathbb{R}$ ,  $a \in A$ , with  $r_a \leq \ell_{a'}$  if  $a < a'$ , and  $\delta_a \in \mathbb{R}$  such that  $\beta X_a - \delta_a \subseteq X$ . Then we call the map  $T : X \rightarrow X$  defined by  $Tx = \beta x - \delta_a$  for all  $x \in X_a$ ,  $a \in A$ , a *right-continuous  $\beta$ -transformation*.

The corresponding *left-continuous  $\beta$ -transformation*  $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$  is defined by  $\tilde{T}x = \beta x - \delta_a$  for  $x \in \tilde{X}_a$ , where  $\tilde{X}$  is the (disjoint) union of the intervals  $\tilde{X}_a = (\ell_a, r_a]$ ,  $a \in A$ . For  $x \in \tilde{X}$ , the  $\tilde{T}$ -expansion of  $x$  is denoted by  $\tilde{b}(x)$ .

Notice that we have Lemma 2.3 also for  $\tilde{X}$  and the sequences  $\tilde{b}(x)$ . Then we obtain the following characterization of  $T$ -expansions and  $\tilde{T}$ -expansions. Note that  $b_1(\ell_a) = \tilde{b}_1(r_a) = a$  for all  $a \in A$ .

**Theorem 2.5.** Let  $T$  be a right-continuous  $\beta$ -transformation as in Definition 2.4 and  $\tilde{T}$  the corresponding left-continuous  $\beta$ -transformation. Then a sequence  $u = u_1u_2\cdots \in A^\omega$  is  $T$ -admissible if and only if

$$(3) \quad b(\ell_{u_k}) \preceq u_k u_{k+1} \cdots \prec \tilde{b}(r_{u_k}) \quad \text{for all } k \geq 1.$$

A sequence  $u = u_1u_2\cdots \in A^\omega$  is  $\tilde{T}$ -admissible if and only if

$$b(\ell_{u_k}) \prec u_k u_{k+1} \cdots \preceq \tilde{b}(r_{u_k}) \quad \text{for all } k \geq 1.$$

*Proof.* We will prove only the first statement, since the proof of the second one is very similar.

If  $u = b(x)$  for some  $x \in X$ , then  $T^{k-1}x \in [\ell_{u_k}, r_{u_k})$  and  $b(T^{k-1}x) = u_k u_{k+1} \cdots$  for all  $k \geq 1$ . By Lemma 2.3, we obtain immediately that  $b(\ell_{u_k}) \preceq b(T^{k-1}x)$ . We show that  $b(T^{k-1}x) \prec \tilde{b}(r_{u_k})$ . Since  $T^{k-1}x < r_{u_k}$ , there must be an index  $n$ , such that  $b_n(T^{k-1}x) \neq \tilde{b}_n(r_{u_k})$  and  $b_i(T^{k-1}x) = \tilde{b}_i(r_{u_k})$  for all  $i < n$ . Thus  $T^{n+k-2}x < \tilde{T}^{n-1}r_{u_k}$ , which implies that  $b_n(T^{k-1}x) < \tilde{b}_n(r_{u_k})$ . Hence,  $b(T^{k-1}x) \prec \tilde{b}(r_{u_k})$  and (3) holds.

For the other implication, suppose that  $u$  satisfies (3) and set  $x_k = \sum_{n=k+1}^{\infty} \delta_{u_n} \beta^{k-n}$  for  $k \geq 0$ . We will show that  $x_0 \in X$  and  $u = b(x_0)$ . First we show that  $x_k < r_{u_{k+1}}$  for all  $k \geq 0$ . Since  $u_{k+1} = \tilde{b}_1(r_{u_{k+1}})$ , there exists some  $s(k) > k$  such that  $u_{k+1} \cdots u_{s(k)} = \tilde{b}_1(r_{u_{k+1}}) \cdots \tilde{b}_{s(k)-k}(r_{u_{k+1}})$  and  $u_{s(k)+1} < \tilde{b}_{s(k)-k+1}(r_{u_{k+1}})$ . Then we have

$$r_{u_{k+1}} - x_k = \frac{\tilde{T}^{s(k)-k} r_{u_{k+1}}}{\beta^{s(k)-k}} - \sum_{n=s(k)+1}^{\infty} \frac{\delta_{u_n}}{\beta^{n-k}} > \frac{\ell_{\tilde{b}_{s(k)-k+1}(r_{u_{k+1}})} - x_{s(k)}}{\beta^{s(k)-k}} \geq \frac{r_{u_{s(k)+1}} - x_{s(k)}}{\beta^{s(k)-k}}$$

for all  $k \geq 0$ . By iterating  $s^n(k) = s(s^{n-1}(k)) > s^{n-1}(k)$ ,  $n \geq 1$ , we obtain

$$r_{u_{k+1}} - x_k > \lim_{n \rightarrow \infty} \frac{r_{u_{s^n(k)+1}} - x_{s^n(k)}}{\beta^{s^n(k)-k}} = 0,$$

where we have used that  $\{x_k : k \geq 0\}$  is bounded and that  $\lim_{n \rightarrow \infty} s^n(k) = \infty$ . Similarly, we can show that  $x_k \geq \ell_{u_{k+1}}$  for all  $k \geq 0$ . Therefore  $x_0 \in [\ell_{u_1}, r_{u_1}) \subset X$ , and we obtain inductively that  $b_k(x_0) = u_k$ ,  $T^k x_0 = x_k \in [\ell_{u_{k+1}}, r_{u_{k+1}})$  for all  $k \geq 0$ . This proves the theorem.  $\square$

*Remark 2.6.* From now on, we will consider only right-continuous  $\beta$ -transformations. By symmetry, all results can be easily adapted to the corresponding left-continuous  $\beta$ -transformations.

*Example 2.7.* Consider the classical greedy  $\beta$ -transformation,  $T_\beta x = \beta x \pmod{1}$ . This fits in the above framework if we take  $A = \{0, 1, \dots, \lceil \beta \rceil - 1\}$ ,  $X_a = [\frac{a}{\beta}, \frac{a+1}{\beta})$  for  $a \leq \lceil \beta \rceil - 2$ ,  $X_{\lceil \beta \rceil - 1} = [\frac{\lceil \beta \rceil - 1}{\beta}, 1)$ , and  $\delta_a = a$  for all  $a \in A$ . Parry gave a characterization of the  $T_\beta$ -admissible sequences in [23]. It only depends on the  $\tilde{T}$ -expansion of 1, since  $T\ell_a = 0$  for every  $a \in A$  and  $\tilde{T}r_a = 1$  for  $a < \lceil \beta \rceil - 1$ . The transformation  $\tilde{T}$  is sometimes called quasi-greedy  $\beta$ -transformation.

The lazy  $\beta$ -transformation with digits  $\delta_a = a$ ,  $a \in \{0, 1, \dots, \lceil \beta \rceil - 1\}$ , is given by  $\tilde{T}$ , where  $\ell_0 = 0$ ,  $r_a = \ell_{a+1} = \frac{1}{\beta} (\frac{\lceil \beta \rceil - 1}{\beta - 1} + a)$  for  $0 \leq a \leq \lceil \beta \rceil - 2$  and  $r_{\lceil \beta \rceil - 1} = \frac{\lceil \beta \rceil - 1}{\beta - 1}$ . For the lazy  $\beta$ -transformation, the characterization of the expansions depends only on the  $T$ -expansion of  $\frac{\lceil \beta \rceil - \beta}{\beta - 1}$ .

*Example 2.8.* Let  $\beta > 1$ ,  $A = \{0, 1, \dots, m\}$  and  $0 = \delta_0 < \delta_1 < \dots < \delta_m$  satisfying the condition

$$(4) \quad \max_{0 \leq a < m} (\delta_{a+1} - \delta_a) \leq \frac{\delta_m}{\beta - 1}.$$

This last condition was given by Pedicini in [24]. He gives an algorithm that produces expansions of the form (1) with digits in  $A$  and he proves that under condition (4) every  $x$  in the interval  $[0, \frac{\delta_m}{\beta - 1}]$  has such an expansion. The greedy  $\beta$ -transformation with digit set  $\{\delta_a : a \in A\}$ , as defined in [9], is obtained by setting  $X_a = [\frac{\delta_a}{\beta}, \frac{\delta_{a+1}}{\beta})$  for  $0 \leq a < m$  and  $X_m = [\frac{\delta_m}{\beta}, \frac{\delta_m}{\beta - 1})$ . It produces the same expansions as the algorithm given by Pedicini. In [24], Pedicini gives a characterization of all the sequences produced by the algorithm. From the above, we see that only the expansions  $\tilde{b}(r_a)$  play a role here.

*Example 2.9.* The *linear mod 1 transformations* are maps from the interval  $[0, 1)$  to itself, given by  $Tx = \beta x + \alpha \pmod{1}$  with  $\beta > 1$  and  $0 \leq \alpha < 1$ . They are well-studied, see for example [19] by Hofbauer and [13, 14, 15] by Flatto and Lagarias. Note that if we set  $\alpha = 0$ , then this is the classical greedy  $\beta$ -transformation on the interval  $[0, 1)$ .

We can obtain these transformations in the following way. Take  $A = \{0, 1, \dots, \lceil \beta + \alpha \rceil - 1\}$  and set  $\delta_a = a - \alpha$  for  $a \in A$ . Define  $X_0 = [0, \frac{1-\alpha}{\beta})$ ,  $X_a = [\frac{a-\alpha}{\beta}, \frac{a+1-\alpha}{\beta})$  for  $1 \leq a \leq \lceil \beta + \alpha \rceil - 2$ ,  $X_{\lceil \beta + \alpha \rceil - 1} = [\frac{\lceil \beta + \alpha \rceil - 1 - \alpha}{\beta}, 1)$ . Then  $Tx = \beta x + \alpha - a$  on  $X_a$ ,  $a \in A$ .

*Example 2.10.* In [18], some examples of specific transformations generating minimal weight expansions are given. These transformations are symmetric (up to the endpoints of the intervals) and depend on two parameters,  $\beta > 1$  and  $\alpha$ , which lies in an interval depending on  $\beta$ . They fit into the above framework by taking  $A = \{-1, 0, 1\}$ ,  $\delta_a = a$  and setting  $X_{-1} = [-\beta\alpha, -\alpha)$ ,  $X_0 = [-\alpha, \alpha)$  and  $X_1 = [\alpha, \beta\alpha)$ . Suppose an  $x \in X$  has a finite expansion, i.e., suppose that there is an  $N \geq 1$  such that  $b_n(x) = 0$  for all  $n > N$ . Then the absolute sum of digits of  $x$  is  $\sum_{k=1}^N |b_k(x)|$ . The transformations  $T$  from [18] generate expansions of minimal weight in the sense that if  $x \in \mathbb{Z}[\beta^{-1}]$ , then the absolute sum of digits of its  $T$ -expansion is less than or equal to that of all possible other expansions of  $x$  in base  $\beta$  with integer digits.

*Example 2.11.* In [5], Akiyama and Scheicher define *symmetric  $\beta$ -transformations*. For  $1 < \beta \leq 3$ , they can be obtained by taking  $A = \{-1, 0, 1\}$ ,  $\delta_a = a$ , and setting  $X_{-1} = [-\frac{1}{2}, -\frac{1}{2\beta})$ ,  $X_0 = [-\frac{1}{2\beta}, \frac{1}{2\beta})$  and  $X_1 = [\frac{1}{2\beta}, \frac{1}{2})$ . Then  $Tx = \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$ .

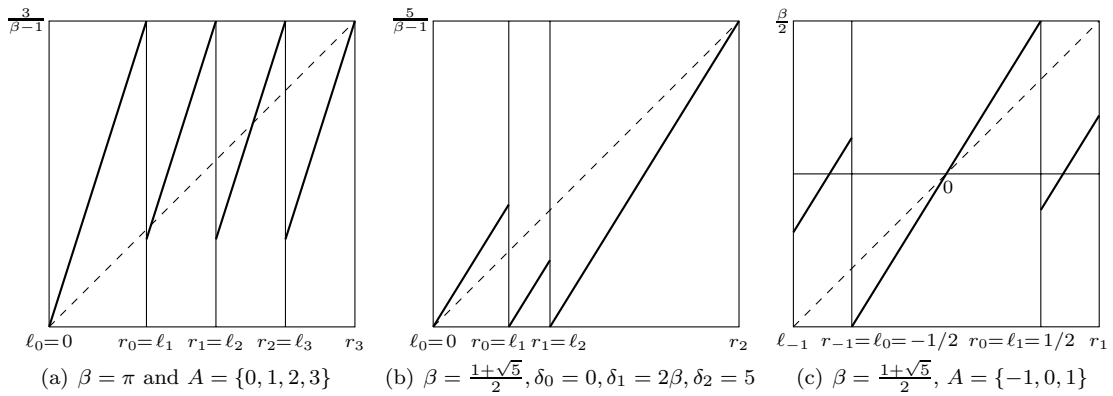


FIGURE 1. In (a), we see a lazy  $\beta$ -transformation, (b) shows a greedy transformation with arbitrary digits, and there is a minimal weight transformation in (c).

We will use the set of  $T$ -expansions to construct a natural extension and a multiple tiling for  $T$ . Therefore, we need to define the following sets. Similarly to the  $\beta$ -shift, we define the  $T$ -shift  $\bar{\mathcal{S}}$  as the set of two-sided infinite sequences  $u \in A^{\mathbb{Z}}$  such that every finite factor of  $u$  occurs in some  $T$ -expansion. Then we have

$$\begin{aligned} \bar{\mathcal{S}} &= \{(u_k)_{k \in \mathbb{Z}} \in A^{\mathbb{Z}} : b(\ell_{u_k}) \preceq u_k u_{k+1} \cdots \preceq \tilde{b}(r_{u_k}) \text{ for all } k \in \mathbb{Z}\} \\ &= \{(u_k)_{k \in \mathbb{Z}} : u_k u_{k+1} \cdots \in b(X) \cup \tilde{b}(\tilde{X}) \text{ for all } k \in \mathbb{Z}\}. \end{aligned}$$

The compact shift  $\bar{\mathcal{S}}$  is the closure of the non-compact shift

$$(5) \quad \begin{aligned} \mathcal{S} &= \{(u_k)_{k \in \mathbb{Z}} \in A^{\mathbb{Z}} : b(\ell_{u_k}) \preceq u_k u_{k+1} \cdots \prec \tilde{b}(r_{u_k}) \text{ for all } k \in \mathbb{Z}\} \\ &= \{(u_k)_{k \in \mathbb{Z}} : u_k u_{k+1} \cdots \in b(X) \text{ for all } k \in \mathbb{Z}\}. \end{aligned}$$

We will use the following notation for sequences. For a left-infinite sequence  $w = (w_k)_{k \leq 0} \in {}^{\omega}A$  and a right-infinite sequence  $v = (v_k)_{k \geq 1} \in A^{\omega}$ , let  $w \cdot v$  denote the sequence  $(u_k)_{k \in \mathbb{Z}}$  defined by  $u_k = w_k$  for all  $k \leq 0$  and  $u_k = v_k$  for all  $k \geq 1$ . Also,  $(b_1 \cdots b_n)^{\omega}$  denotes a block of digits repeated to the right, and  ${}^{\omega}(b_1 \cdots b_n)$  denotes a block of digits repeated to the left.

### 3. NATURAL EXTENSIONS

**3.1. Geometric realization of the natural extension.** Our goal in this section is to define a measure theoretical natural extension for the class of transformations defined in the previous section, under suitable assumptions on  $\beta$  and the digit set. This natural extension will allow us to define a multiple tiling of some Euclidean space. The set-up for this multiple tiling is similar to the one Thurston gave in [33] for the classical greedy  $\beta$ -transformation  $T_{\beta}x = \beta x \pmod{1}$ .

Let  $\beta > 1$  be a Pisot unit with minimal polynomial  $x^d - c_1x^{d-1} - \dots - c_d \in \mathbb{Z}[x]$  and  $\beta_2, \dots, \beta_d$  its Galois conjugates. Thus,  $|\beta_j| < 1$  and  $|c_d| = 1$ . Set  $\beta_1 = \beta$ . Let  $M_\beta$  be the companion matrix

$$M_\beta = \begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and let  $\mathbf{v}_j \in \mathbb{C}(\beta_j^{d-1}, \dots, \beta_j, 1)^t$ ,  $1 \leq j \leq d$ , be right eigenvectors of  $M_\beta$  such that  $\mathbf{v}_1 + \dots + \mathbf{v}_d = \mathbf{e}_1 = (1, 0, \dots, 0)^t$ . (If  $\beta_j \in \mathbb{R}$ , then  $\mathbf{v}_j \in \mathbb{R}^d$ , and if  $\beta_k$  is the complex conjugate of  $\beta_j$ , then the entries of  $\mathbf{v}_k$  are the complex conjugates of the entries of  $\mathbf{v}_j$ .) Since  $|\det M_\beta| = |c_d| = 1$ ,  $M_\beta^{-1}$  is a matrix with integer entries. Hence we have, for all  $k \in \mathbb{Z}$ ,

$$\beta^k \mathbf{v}_1 + \beta_2^k \mathbf{v}_2 + \dots + \beta_d^k \mathbf{v}_d = M_\beta^k (\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_d) = M_\beta^k \mathbf{e}_1 \in \mathbb{Z}^d.$$

Let  $H$  be the hyperplane of  $\mathbb{R}^d$  which is spanned by the real and imaginary parts of  $\mathbf{v}_2, \dots, \mathbf{v}_d$ . Then  $H \simeq \mathbb{R}^{d-1}$ .

Assume that  $\delta_a \in \mathbb{Q}(\beta)$  for every  $a \in A$ , and let  $\Gamma_j : \mathbb{Q}(\beta) \rightarrow \mathbb{Q}(\beta_j)$ ,  $1 \leq j \leq d$ , be the isomorphism defined by  $\Gamma_j(\beta) = \beta_j$ . Then we define

$$\psi(u) = \sum_{k \geq 1} \delta_{u_k} \beta^{-k} \mathbf{v}_1 - \sum_{k \leq 0} \sum_{j=2}^d \Gamma_j(\delta_{u_k}) \beta_j^{-k} \mathbf{v}_j \in \mathbb{R}^d$$

for two-sided infinite sequences  $u = (u_k)_{k \in \mathbb{Z}} \in A^{\mathbb{Z}}$ . The set  $\widehat{X} = \psi(\mathcal{S})$ , with  $\mathcal{S}$  defined in (5), will be our natural extension domain.

Define the maps  $\Phi : \mathbb{Q}(\beta) \rightarrow H$  and  $\Psi : \mathbb{Q}(\beta) \rightarrow \mathbb{Q}^d$  by

$$(6) \quad \Phi(x) = \sum_{j=2}^d \Gamma_j(x) \mathbf{v}_j, \quad \Psi(x) = \sum_{j=1}^d \Gamma_j(x) \mathbf{v}_j = x \mathbf{v}_1 + \Phi(x).$$

For left-infinite sequences  $w = (w_k)_{k \leq 0} \in {}^\omega A$ , we set

$$(7) \quad \varphi(w) = \sum_{k \leq 0} \Phi(\delta_{w_k} \beta^{-k}) = \sum_{k \leq 0} M_\beta^{-k} \Phi(\delta_{w_k}) \in H.$$

Then we have

$$\psi(\cdots u_{-1} u_0 u_1 u_2 \cdots) = (\cdot u_1 u_2 \cdots) \mathbf{v}_1 - \varphi(\cdots u_{-1} u_0).$$

The set  $\widehat{X} = \psi(\mathcal{S})$  is the disjoint union of the sets  $\widehat{X}_a = \psi(\{(u_k)_{k \in \mathbb{Z}} \in \mathcal{S} : u_1 = a\})$ ,  $a \in A$ . Therefore we can define a transformation  $\widehat{T} : \widehat{X} \rightarrow \widehat{X}$  by

$$\widehat{T}\mathbf{x} = M_\beta \mathbf{x} - \Psi(\delta_a) \quad \text{if } \mathbf{x} \in \widehat{X}_a.$$

We have indeed  $\widehat{T}\widehat{X} = \widehat{X}$  since, if we denote by  $\sigma$  the left-shift, then  $\sigma\mathcal{S} = \mathcal{S}$  and

$$\psi(\sigma u) = (\cdot u_2 u_3 \cdots) \mathbf{v}_1 - \varphi(\cdots u_0 u_1) = M_\beta (\cdot u_1 u_2 \cdots) \mathbf{v}_1 - \delta_{u_1} \mathbf{v}_1 - M_\beta \varphi(\cdots u_{-1} u_0) - \Phi(\delta_{u_1}) = \widehat{T}\psi(u)$$

for every  $u = (u_k)_{k \in \mathbb{Z}} \in \mathcal{S}$ . Note that  $\Psi(\delta_a) = \delta_a \mathbf{e}_1$  if  $\delta_a \in \mathbb{Q}$ .

Define the projection  $\pi : \widehat{X} \rightarrow X$  by  $\pi(\mathbf{x}) = x$  if  $\mathbf{x} = x \mathbf{v}_1 + \mathbf{y}$  for some  $\mathbf{y} \in H$ . Then we have

$$\pi(\widehat{T}\mathbf{x}) = \beta \pi(\mathbf{x}) - \delta_a = T\pi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \widehat{X}_a$$

and thus  $T \circ \pi = \pi \circ \widehat{T}$ .

Next we show that  $\widehat{X}$  is Lebesgue measurable. Since  $\mathcal{S}$  is not closed, we consider  $\widehat{Y} = \psi(\bar{\mathcal{S}})$ .

**Lemma 3.1.** *The set  $\widehat{Y}$  is compact and thus Lebesgue measurable.*

*Proof.* Note that  $\bar{\mathcal{S}}$  is a closed subset of a compact metric space  ${}^\omega A$ . Hence  $\bar{\mathcal{S}}$  is compact. Since  $\psi : \bar{\mathcal{S}} \rightarrow \mathbb{R}^d$  is a continuous function, we automatically have that  $\widehat{Y} = \psi(\bar{\mathcal{S}})$  is compact.  $\square$

A sequence  $(u_k)_{k \in \mathbb{Z}} \in \bar{\mathcal{S}}$  is not in  $\mathcal{S}$  if and only if  $u_k u_{k+1} \cdots = \bar{b}(r_a)$  for some  $a \in A$ ,  $k \in \mathbb{Z}$ . Therefore the projection of  $\hat{Y} \setminus \hat{X}$  on the line  $\mathbb{R}\mathbf{v}_1$  along  $H$  is a countable set and  $\lambda^d(\hat{Y} \setminus \hat{X}) = 0$ , where  $\lambda^d$  denotes the  $d$ -dimensional Lebesgue measure. This implies that  $\hat{X}$  is a Lebesgue measurable set with  $\lambda^d(\hat{X}) = \lambda^d(\hat{Y})$ .

Let  $\mathcal{B}$  be the Lebesgue  $\sigma$ -algebra on  $X$  and  $\hat{\mathcal{B}}$  the Lebesgue  $\sigma$ -algebra on  $\hat{X}$ . We want to prove that the system  $(\hat{X}, \hat{\mathcal{B}}, \lambda^d, \hat{T})$  is a version of the natural extension of the system  $(X, \mathcal{B}, \mu, T)$ , where the measure  $\mu$  on  $(X, \mathcal{B})$  is defined by  $\mu = \lambda^d \circ \pi^{-1}$ . In order to do this, we will show that there are sets  $\hat{N} \in \hat{\mathcal{B}}$  and  $M \in \mathcal{B}$ , such that all the following hold.

- (ne1)  $\lambda^d(\hat{N}) = \mu(M) = 0$ ,  $\hat{T}(\hat{X} \setminus \hat{N}) \subseteq \hat{X} \setminus \hat{N}$  and  $T(X \setminus M) \subseteq X \setminus M$ .
- (ne2) The projection map  $\pi : \hat{X} \setminus \hat{N} \rightarrow X \setminus M$  is measurable, measure preserving and surjective.
- (ne3)  $\pi(\hat{T}\mathbf{x}) = T\pi(\mathbf{x})$  for all  $\mathbf{x} \in \hat{X} \setminus \hat{N}$ .
- (ne4) The transformation  $\hat{T} : \hat{X} \setminus \hat{N} \rightarrow \hat{X} \setminus \hat{N}$  is invertible.
- (ne5)  $\bigvee_{k=0}^{\infty} \hat{T}^k \pi^{-1} \mathcal{B} = \hat{\mathcal{B}}$ , where  $\bigvee_{k=0}^{\infty} \hat{T}^k \pi^{-1} \mathcal{B}$  is the smallest  $\sigma$ -algebra containing the  $\sigma$ -algebras  $\hat{T}^k \pi^{-1} \mathcal{B}$  for all  $k \geq 0$ .

A map that satisfies (ne1)–(ne3) is called a *factor map*.

To identify the sets  $\hat{N}$  and  $M$  and to establish the invertibility of  $\hat{T}$  off of these sets, we need the following lemma.

**Lemma 3.2.** *For all  $a, a' \in A$  with  $a \neq a'$ , we have  $\lambda^d(\hat{T}\hat{X}_a \cap \hat{T}\hat{X}_{a'}) = 0$ .*

*Proof.* Recall that  $|\det M_\beta| = 1$  and that  $\hat{T}\hat{X} = \hat{X}$ , thus

$$\sum_{a \in A} \lambda^d(\hat{T}\hat{X}_a) = \sum_{a \in A} \lambda^d(M_\beta \hat{X}_a) = \sum_{a \in A} \lambda^d(\hat{X}_a) = \lambda^d(\hat{X}) = \lambda^d(\hat{T}\hat{X}) = \lambda^d\left(\bigcup_{a \in A} \hat{T}\hat{X}_a\right),$$

which proves the lemma.  $\square$

Let

$$\hat{N} = \bigcup_{n \in \mathbb{Z}} \hat{T}^n \left( \bigcup_{a, a' \in A, a \neq a'} \hat{T}\hat{X}_a \cap \hat{T}\hat{X}_{a'} \right).$$

Then, by Lemma 3.2,  $\lambda^d(\hat{N}) = 0$ . Note that  $\hat{T}$  is a bijection on  $\hat{X} \setminus \hat{N}$ . Hence  $\hat{T}$  is an a.e. invertible, measure preserving transformation on  $(\hat{X}, \hat{\mathcal{B}}, \lambda^d)$ , which proves (ne4). The measure  $\mu = \lambda^d \circ \pi^{-1}$ , defined on  $(X, \mathcal{B})$ , has its support contained in  $X$ . Hence, it is an invariant measure for  $T$ , that is absolutely continuous with respect to the Lebesgue measure, defined on the support of  $\mu$ . The projection map  $\pi : \hat{X} \rightarrow X$  is measurable and measure preserving and we have that  $T \circ \pi = \pi \circ \hat{T}$ . Let  $M = \{x \in X : \pi^{-1}\{x\} \subseteq \hat{N}\}$ . Then  $T(X \setminus M) \subseteq X \setminus M$  and  $\mu(M) = (\lambda^d \circ \pi^{-1})(M) \leq \lambda^d(\hat{N}) = 0$ . Since  $\pi$  is surjective from  $\hat{X} \setminus \hat{N}$  to  $X \setminus M$ ,  $\pi$  is a factor map from  $(\hat{X}, \hat{\mathcal{B}}, \lambda^d, \hat{T})$  to  $(X, \mathcal{B}, \mu, T)$ . This gives (ne1)–(ne3). In the next theorem, we prove (ne5).

**Theorem 3.3.** *Let  $T$  be a right-continuous  $\beta$ -transformation as in Definition 2.4 with a Pisot unit  $\beta$  and  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$ . Then the dynamical system  $(\hat{X}, \hat{\mathcal{B}}, \lambda^d, \hat{T})$  is a natural extension of the dynamical system  $(X, \mathcal{B}, \mu, T)$ .*

*Proof.* We have already shown (ne1)–(ne4). The only thing that remains in order to get the theorem is that

$$\bigvee_{k \geq 0} \hat{T}^k \pi^{-1}(\mathcal{B}) = \hat{\mathcal{B}}.$$

By the definition of  $\mathcal{S}$ , it is clear that  $\bigvee_{k \geq 0} \hat{T}^k \pi^{-1}(\mathcal{B}) \subseteq \hat{\mathcal{B}}$ . To show the other inclusion, take  $\mathbf{x}, \mathbf{x}' \in \hat{X}$ ,  $\mathbf{x} \neq \mathbf{x}'$ . Suppose first that  $\pi(\mathbf{x}) \neq \pi(\mathbf{x}')$ . Then there are two disjoint intervals  $B, B' \subset X$  with  $\pi(\mathbf{x}) \in B$  and  $\pi(\mathbf{x}') \in B'$ , thus  $\mathbf{x} \in \pi^{-1}(B)$  and  $\mathbf{x}' \in \pi^{-1}(B')$ . Now, suppose that  $\pi(\mathbf{x}) = \pi(\mathbf{x}') = x$ . There exist sequences  $w, w' \in {}^\omega A$  with  $w \cdot b(x), w' \cdot b(x) \in \mathcal{S}$  such that



$\mathbf{x} = x\mathbf{v}_1 - \varphi(w)$ ,  $\mathbf{x}' = x'\mathbf{v}_1 - \varphi(w')$ . Since  $\mathbf{x} \neq \mathbf{x}'$ , we have  $w \neq w'$ . Let  $n \geq 1$  be the first index such that  $w_{-n+1} \neq w'_{-n+1}$ , and set

$$x_n = \sum_{k=1}^n \frac{\delta_{w_{-n+k}}}{\beta^k} + \frac{x}{\beta^n}, \quad x'_n = \sum_{k=1}^n \frac{\delta_{w'_{-n+k}}}{\beta^k} + \frac{x}{\beta^n}.$$

Then  $x_n \neq x'_n$ , so there exist two disjoint intervals  $B, B' \subset X$ , such that  $x_n \in B$  and  $x'_n \in B'$ . Moreover,  $\mathbf{x} \in \widehat{T}^n \pi^{-1}(B)$  and  $\mathbf{x}' \in \widehat{T}^n \pi^{-1}(B')$ . By the invertibility of  $\widehat{T}$ , the sets  $\widehat{T}^n \pi^{-1}(B)$  and  $\widehat{T}^n \pi^{-1}(B')$  are disjoint a.e., hence for almost all points  $\mathbf{x}, \mathbf{x}' \in \widehat{X}$  we can find two disjoint elements of  $\widehat{T}^n \pi^{-1}(\mathcal{B})$ , such that one point is contained in one element and the other element contains the other point. This shows that  $\bigvee_{k \geq 0} \widehat{T}^k \pi^{-1}(\mathcal{B}) = \widehat{\mathcal{B}}$  and thus that  $(\widehat{X}, \widehat{\mathcal{B}}, \lambda^d, \widehat{T})$  is a natural extension of  $(X, \mathcal{B}, \mu, T)$ .  $\square$

We will prove in Section 4.4 that  $\widehat{Y}$  covers the torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  if  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ , and thus that  $\lambda^d(\widehat{X}) = \lambda^d(\widehat{Y}) > 0$ . If  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$ , then there exists some  $q \in \mathbb{Z}$  such that  $\{\delta_a : a \in A\} \subset q^{-1}\mathbb{Z}[\beta]$ . Then we can define a new right-continuous  $\beta$ -transformation with digits in  $\mathbb{Z}[\beta]$  by multiplying  $X_a$  and  $\delta_a$  with  $q$  for all  $a \in A$ . The new natural extension domain is  $q\widehat{X}$  and has positive Lebesgue measure by Lemma 4.23. Therefore, we always have  $\lambda^d(\widehat{X}) = \lambda^d(\widehat{Y}) > 0$ .

**3.2. Shape of the natural extension domain.** We can write

$$(8) \quad \widehat{X} = \bigcup_{x \in X} (x\mathbf{v}_1 - \mathcal{D}_x) \quad \text{with} \quad \mathcal{D}_x = \{\varphi(w) : w \cdot b(x) \in \mathcal{S}\},$$

where  $\varphi$  is as in (7) and  $\mathcal{S}$  as in (5). For the multiple tiling we will construct later on, the prototiles will be the sets  $\mathcal{D}_x$  for  $x \in \mathbb{Z}[\beta] \cap X$ . In this section, we show some properties of these sets.

**Lemma 3.4.** *Every set  $\mathcal{D}_x$ ,  $x \in X$ , is compact.*

*Proof.* Let  $x \in X$  and consider the subset  $\mathcal{W} = \{w \in {}^\omega A : w \cdot b(x) \in \mathcal{S}\}$  of the compact space  ${}^\omega A$ . We want to show that  $\mathcal{W}$  is closed and, hence, compact. Therefore, take some converging sequence  $(w^{(n)})_{n \geq 0} \subseteq \mathcal{W}$  and let  $\lim_{n \rightarrow \infty} w^{(n)} = w$ . For every  $k \geq 0$ , we can find some  $n_k \geq 0$  such that  $w_{-k}^{(n_k)} \cdots w_0^{(n_k)} = w_{-k} \cdots w_0$ . This implies that  $w_{-k} \cdots w_0 b(x)$  is  $T$ -admissible for every  $k \geq 0$ , thus  $w \cdot b(x) \in \mathcal{S}$ , and  $\mathcal{W}$  is closed. Since  $\mathcal{D}_x$  is the image of the compact set  $\mathcal{W}$  under the continuous map  $\varphi$ , it is compact as well.  $\square$

To distinguish different sets  $\mathcal{D}_x$ , we introduce the set  $\mathcal{V}$  below. We assume here that  $TX = X$ . Note that restricting  $T$  to  $X' = \bigcap_{n \geq 0} T^n X$  does not change  $\mathcal{S}$ , and that  $TX' = X'$ . We conjecture that  $X'$  is always a finite union of left-closed, right-open intervals, thus that the restriction of  $T$  to  $X'$  is a right-continuous  $\beta$ -transformation. In all our examples,  $T$  is surjective and thus  $X' = X$ .

For every  $a \in A$  with  $r_a \in X$ , let  $m_a$  be the minimal positive integer such that

$$\widetilde{T}^{m_a} r_a = T^{m_a} r_a,$$

with  $m_a = \infty$  if  $\widetilde{T}^k r_a \neq T^k r_a$  for all  $k \geq 1$ . Set

$$(9) \quad \mathcal{V} = \{\ell_a \notin \widetilde{X} : a \in A\} \cup \{r_a \notin X : a \in A\} \cup \bigcup_{1 \leq k < m_a, a \in A: r_a \in X} \{\widetilde{T}^k r_a, T^k r_a\}.$$

Then  $\mathcal{V}$  is a finite set if and only if, for every  $a \in A$  with  $r_a \in X$ ,  $\widetilde{b}(r_a)$  and  $b(r_a)$  are eventually periodic or  $m_a < \infty$ . Note that  $\{r_a \in X : a \in A\} = \{\ell_a \in \widetilde{X} : a \in A\}$ .

We will prove the following proposition.

**Proposition 3.5.** *Assume that  $TX = X$ . If  $x, y \in X$ ,  $x < y$ , and  $(x, y] \cap \mathcal{V} = \emptyset$ , then  $\mathcal{D}_x = \mathcal{D}_y$ .*

The main ingredient of the proof of Proposition 3.5 is the following simple lemma. We define  $\varphi$  on finite sequences  $v_1 \cdots v_n \in A^n$ ,  $n \geq 0$ , by

$$\varphi(v_1 \cdots v_n) = \sum_{k=1}^n \Phi(\delta_{v_k} \beta_j^{n-k}).$$

**Lemma 3.6.** *If  $x \in X \cap \tilde{X}$  and  $\tilde{T}^k x = T^k x$ ,  $k \geq 1$ , then  $\varphi(b_1(x) \cdots b_k(x)) = \varphi(\tilde{b}_1(x) \cdots \tilde{b}_k(x))$ .*

*Proof.* We have

$$\sum_{i=1}^k \delta_{\tilde{b}_i(x)} \beta^{k-i} = \beta^k x - \tilde{T}^k x = \beta^k x - T^k x = \sum_{i=1}^k \delta_{b_i(x)} \beta^{k-i}.$$

By applying  $\Phi$  to this equation, the lemma is proved.  $\square$

**Lemma 3.7.** *Assume that  $TX = X$ . Let  $x, y \in X$ ,  $x < y$ ,  $(x, y] \cap \mathcal{V} = \emptyset$ , and suppose that there is a finite sequence  $w_{-n} \cdots w_0$  such that  $w_{-n} \cdots w_0 b(x)$  is  $T$ -admissible but  $w_{-n} \cdots w_0 b(y)$  is not. Then there exists some sequence  $w'_{-n} \cdots w'_0$  such that  $w'_{-n} \cdots w'_0 b(y)$  is  $T$ -admissible and*

$$(10) \quad \varphi(w'_{-n} \cdots w'_0) = \varphi(w_{-n} \cdots w_0) + \mathcal{O}(\rho^n),$$

where  $\rho = \max_{2 \leq j \leq d} |\beta_j| < 1$  and the constant implied by the  $\mathcal{O}$ -symbol depends only on  $T$ .

*Proof.* Let  $k$ ,  $0 \leq k \leq n$ , be maximal such that

$$w_{-k} \cdots w_0 = \tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}).$$

The admissibility of  $w_{-n} \cdots w_0 b(x)$  implies  $b(\ell_{w_{-i}}) \preceq w_{-i} \cdots w_0 b(x) \prec w_{-i} \cdots w_0 b(y)$  for  $0 \leq i \leq n$ . For  $k < i \leq n$ , we have, by the maximality of  $k$ , that  $w_{-i} \cdots w_0 \neq \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$ , hence  $w_{-i} \cdots w_0 b(y) \prec \tilde{b}(r_{w_{-i}})$ . The non-admissibility of  $w_{-n} \cdots w_0 b(y)$  means therefore that  $w_{-i} \cdots w_0 b(y) \succeq \tilde{b}(r_{w_{-i}})$  for some  $0 \leq i \leq k$ . Together with the admissibility of  $w_{-n} \cdots w_0 b(x)$ , this gives  $w_{-i} \cdots w_0 = \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$ . Since  $w_{-i} \cdots w_0 b(x) < \tilde{T}^{k-i} r_{w_{-k}} \leq r_{w_{-i}}$ , we gain

$$x < \tilde{T}^{k+1} r_{w_{-k}} \leq \tilde{T}^{i+1} r_{w_{-i}} \leq y.$$

In order to prove the lemma, we want to replace the sequence  $w_{-k} \cdots w_0$  by a suitable sequence, depending on  $r_{w_{-k}}$ . Assume first  $r_{w_{-k}} \in X$ . Suppose that  $\tilde{T}^{k+1} r_{w_{-k}} \neq T^{k+1} r_{w_{-k}}$ , and let  $i$ ,  $0 \leq i \leq k$ , be maximal such that  $\tilde{T}^i r_{w_{-k}} = T^i r_{w_{-k}}$ . Since  $\tilde{T}^{i+1} r_{w_{-k}} \neq T^{i+1} r_{w_{-k}}$ , we have  $\tilde{T}^i r_{w_{-k}} = r_{w_{-k+i}} \in X$ . Furthermore, the maximality of  $i$  implies  $m_{w_{-k+i}} > k - i + 1$  and thus  $\tilde{T}^{k+1} r_{w_{-k}} = \tilde{T}^{k-i+1} r_{w_{-k+i}} \in \mathcal{V}$ , contradicting the assumption  $(x, y] \cap \mathcal{V} = \emptyset$ . Therefore we have  $\tilde{T}^{k+1} r_{w_{-k}} = T^{k+1} r_{w_{-k}}$ , and Lemma 3.6 gives

$$\varphi(w_{-k} \cdots w_0) = \varphi(b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}})).$$

Next we show that  $w_{-n} \cdots w_{-k-1} b(r_{w_{-k}})$  is  $T$ -admissible. We clearly have  $w_{-i} \cdots w_{-k-1} b(r_{w_{-k}}) \succ w_{-i} \cdots w_0 b(x) \succeq b(\ell_{w_{-i}})$  for  $k < i \leq n$ . Suppose that  $w_{-i} \cdots w_{-k-1} b(r_{w_{-k}}) \succeq \tilde{b}(r_{w_{-i}})$ . Since  $w_{-i} \cdots w_0 b(x) \prec \tilde{b}(r_{w_{-i}})$ , it follows that  $w_{-i} \cdots w_{-k-1} = \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i-k}(r_{w_{-i}})$ , hence

$$b(r_{w_{-k}}) \succeq \tilde{b}(\tilde{T}^{i-k} r_{w_{-i}}) \succ w_{-k} \cdots w_0 b(x) = \tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}) b(x).$$

Since  $b(r_{w_{-k}}) \succeq \tilde{b}(\tilde{T}^{i-k} r_{w_{-i}})$  implies  $r_{w_{-k}} \geq \tilde{T}^{i-k} r_{w_{-i}}$  and thus  $\tilde{b}(r_{w_{-k}}) \succeq \tilde{b}(\tilde{T}^{i-k} r_{w_{-i}})$ , we obtain  $\tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}) = \tilde{b}_{i-k+1}(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$ , hence  $w_{-i} \cdots w_0 = \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$ , contradicting the maximality of  $k$ . This shows that  $w_{-n} \cdots w_{-k-1} b(r_{w_{-k}})$  is  $T$ -admissible. If  $w_{-n} \cdots w_{-k-1} b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}}) b(y)$  is  $T$ -admissible as well, then we can take

$$w'_{-n} \cdots w'_0 = w_{-n} \cdots w_{-k-1} b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}}),$$

and (10) holds with vanishing error term.

Now assume that  $r_{w_{-k}} \notin X$ . If  $k < n$ , this implies  $\tilde{T} r_{w_{-k-1}} \leq r_{w_{-k}}$  and thus  $w_{-k-1} \cdots w_0 = \tilde{b}_1(r_{w_{-k-1}}) \tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}) = \tilde{b}_1(r_{w_{-k-1}}) \cdots \tilde{b}_{k+2}(r_{w_{-k-1}})$ , contradicting the maximality of  $k$ . Therefore, we must have  $k = n$ . Now, we show that there exists some  $h \geq 1$  and some  $a \in A$  such that  $r_a \in X$  and  $\tilde{T}^h r_a = r_{w_{-n}}$ . If such  $h$  and  $r_a$  did not exist, then every set  $\tilde{T}^{-h} \{r_{w_{-n}}\}$ ,  $h \geq 1$ , would consist only of numbers  $r_a \notin X$ . Since  $A$  is finite and  $\tilde{T}\tilde{X} = \tilde{X}$ ,  $\tilde{b}(r_{w_{-n}})$  would be purely periodic and  $\tilde{T}^{n+1} r_{w_{-n}} = r_{w_0} \notin X$ , contradicting the assumption  $(x, y] \cap \mathcal{V} = \emptyset$ . Therefore, we have  $\tilde{T}^h r_a = r_{w_{-n}}$  for some  $h \geq 1$  and  $r_a \in X$ . As above, we obtain  $\tilde{T}^{h+n+1} r_a = T^{h+n+1} r_a$ ,

$$\varphi(\tilde{b}_1(r_a) \cdots \tilde{b}_h(r_a) w_{-n} \cdots w_0) = \varphi(b_1(r_a) \cdots b_{n+h+1}(r_a)),$$

and thus

$$\varphi(w_{-n} \cdots w_0) = \varphi(b_{h+1}(r_a) \cdots b_{n+h+1}(r_a)) + \mathcal{O}(\rho^n).$$

If  $b_{h+1}(r_a) \cdots b_{n+h+1}(r_a)b(y)$  is  $T$ -admissible, then we are done.

If  $w_{-n} \cdots w_{-k-1}b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}})$  or  $b_{h+1}(r_a) \cdots b_{n+h+1}(r_a)$ , respectively, is not the desired sequence, then we iterate with this sequence as new  $w_{-n} \cdots w_0$  and  $\tilde{T}^{k+1}r_{w_{-k}}$  as new  $x$ . Since these numbers  $\tilde{T}^{k+1}r_{w_{-k}}$  increase and can take only finitely many values, the algorithm terminates. The number of instances of  $k = n$  is bounded by the size of  $A$ , thus the error term only depends on  $T$  and is  $\mathcal{O}(\rho^n)$ .  $\square$

*Remark 3.8.* We can prove Lemma 3.7 also in the other direction: Suppose that there is a finite sequence  $w_{-n} \cdots w_0$ , such that  $w_{-n} \cdots w_0b(y)$  is  $T$ -admissible, but  $w_{-n} \cdots w_0b(x)$  is not. Then, there is another sequence  $w'_{-n} \cdots w'_0$ , such that  $w'_{-n} \cdots w'_0b(x)$  is  $T$ -admissible and

$$\varphi(w'_{-n} \cdots w'_0) = \varphi(w_{-n} \cdots w_0) + \mathcal{O}(\rho^n).$$

The proof is done in exactly the same way.

*Proof of Proposition 3.5.* Take  $x, y \in X$  that satisfy the conditions of the proposition, and assume that there exists  $w = (w_k)_{k \leq 0}$  such that  $w \cdot b(x) \in \mathcal{S}$ , but  $w \cdot b(y) \in \mathcal{S}$ . For  $n \geq 0$ , let  $w_{-n}^{(n)} \cdots w_0^{(n)}$  be the sequence  $w'_{-n} \cdots w'_0$  given by Lemma 3.7. By the surjectivity of  $T$ , we can extend this sequence to a sequence  $w^{(n)} = (w_k^{(n)})_{k \leq 0}$  with  $w^{(n)} \cdot b(y) \in \mathcal{S}$ . Then we have  $\varphi(w^{(n)}) = \varphi(w) + \mathcal{O}(\rho^n)$ , hence  $\lim_{n \rightarrow \infty} \varphi(w^{(n)}) = \varphi(w)$ . Since  $\varphi(w^{(n)}) \in \mathcal{D}_y$  and  $\mathcal{D}_y$  is compact, we obtain that  $\varphi(w) \in \mathcal{D}_y$ . By Remark 3.8, we also obtain that  $\varphi(w) \in \mathcal{D}_x$  if  $\varphi(w) \in \mathcal{D}_y$ , and the proposition is proved.  $\square$

If  $\mathcal{V}$  is finite, then let the elements of  $\{\ell_a\} \cup (\mathcal{V} \cap X_a)$  be  $\ell_{a,0} < \ell_{a,1} < \cdots < \ell_{a,N_a}$  for every  $a \in A$ , the elements of  $\{r_a\} \cup (\mathcal{V} \cap \tilde{X}_a)$  be  $r_{a,0} < r_{a,1} < \cdots < r_{a,N_a}$ , i.e.,  $\ell_{a,0} = \ell_a$ ,  $r_{a,n} = \ell_{a,n+1}$  for  $0 \leq n < N_a$ ,  $r_{a,N_a} = r_a$ . Let  $X_{a,n} = [\ell_{a,n}, r_{a,n})$ . Then we have

$$(11) \quad \hat{X} = \bigcup_{a \in A} \bigcup_{0 \leq n \leq N_a} \hat{X}_{a,n} \quad \text{with} \quad \hat{X}_{a,n} = X_{a,n} \mathbf{v}_1 - \mathcal{D}_{\ell_{a,n}}.$$

The sets  $\mathcal{D}_x$  can be subdivided according to the following lemma.

**Lemma 3.9.** *Assume that  $TX = X$  and that  $\mathcal{V}$  is finite. For every  $x \in X$ , we have*

$$(12) \quad \mathcal{D}_x = \bigcup_{y \in T^{-1}\{x\}} \left( M_\beta \mathcal{D}_y + \Phi(\delta_{b_1(y)}) \right),$$

where the union is disjoint up to sets of measure zero.

*Proof.* Let  $x \in X$ . Then, for every  $a \in A$  such that  $ab(x)$  is  $T$ -admissible, there is a unique  $y \in T^{-1}\{x\}$  such that  $b(y) = ab(x)$ . Moreover, for each  $w \in {}^\omega A$ , we have  $wa \cdot b(x) \in \mathcal{S}$  if and only if  $w \cdot ab(x) \in \mathcal{S}$ . Since  $\varphi(wa) = M_\beta \varphi(w) + \Phi(\delta_a)$ , we obtain (12).

Let  $y \in T^{-1}\{x\}$  be in the interval  $X_{b_1(y),n}$  and  $y' \in T^{-1}\{x\}$ ,  $y' \neq y$ , be in  $X_{b_1(y'),n'}$ . Then

$$\hat{T} \hat{X}_{b_1(y),n} \cap \hat{T} \hat{X}_{b_1(y'),n'} = (TX_{b_1(y),n} \cap TX_{b_1(y'),n'}) \mathbf{v}_1 + (M_\beta \mathcal{D}_y + \Phi(\delta_{b_1(y)})) \cap (M_\beta \mathcal{D}_{y'} + \Phi(\delta_{b_1(y')})).$$

Since this set is contained in  $\hat{T} \hat{X}_{b_1(y)} \cap \hat{T} \hat{X}_{b_1(y')}$  and  $b_1(y) \neq b_1(y')$ , it has zero Lebesgue measure by Lemma 3.2. Since  $TX_{b_1(y),n} \cap TX_{b_1(y'),n'}$  is an interval of positive measure, this implies that  $\lambda^{d-1}((M_\beta \mathcal{D}_y + \Phi(\delta_{b_1(y)})) \cap (M_\beta \mathcal{D}_{y'} + \Phi(\delta_{b_1(y')}))) = 0$ .  $\square$

Using Lemma 3.9, we will show in Proposition 4.6 that the boundary of  $\mathcal{D}_x$  has zero measure.

Now, consider the measure  $\mu$ , defined by  $\mu(E) = (\lambda^d \circ \pi^{-1})(E)$  for all measurable sets  $E$ . If we take an interval  $[s, t] \subseteq X$ , then there exists some constant  $c > 0$  such that

$$\begin{aligned} \mu([s, t]) &= (\lambda^d \circ \pi^{-1}) \left( \bigcup_{a,n} X_{a,n} \cap [s, t] \right) = \sum_{a,n} c \lambda(X_{a,n} \cap [s, t]) \lambda^{d-1}(\mathcal{D}_{\ell_{a,n}}) \\ &= c \int_{[s,t]} \sum_{a,n} \lambda^{d-1}(\mathcal{D}_{\ell_{a,n}}) 1_{X_{a,n}} d\lambda. \end{aligned}$$

Hence, the support of  $\mu$  is the union of the intervals  $X_{a,n}$  with  $\lambda^{d-1}(\mathcal{D}_{\ell_{a,n}}) > 0$ . Since  $\lambda^d(\widehat{X}) > 0$ , the support of  $\mu$  has positive Lebesgue measure. Hence,  $\mu$  is absolutely continuous with respect to the Lebesgue measure on  $X$ . If the transformation  $T$  has a unique invariant measure that is absolutely continuous with respect to the Lebesgue measure, then  $\mu$  must be this measure. This is the case for the classical greedy and lazy  $\beta$ -transformations, and for the greedy and lazy  $\beta$ -transformations with arbitrary digits.

### 3.3. Examples of natural extensions.

*Example 3.10.* Let  $\beta$  be the golden mean, i.e., the positive solution of the equation  $x^2 - x - 1 = 0$ . The other solution of this equation is  $\beta_2 = -1/\beta$ . Then

$$M_\beta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_1 = \frac{1}{\beta^2 + 1} \begin{pmatrix} \beta^2 \\ \beta \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\beta^2 + 1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix}.$$

The greedy  $\beta$ -transformation is given by  $A = \{0, 1\}$ ,  $X_0 = [0, 1/\beta)$ ,  $X_1 = [1/\beta, 1)$ ,  $\delta_0 = 0$ ,  $\delta_1 = 1$ . We have  $T(\ell_0) = T(\ell_1) = \ell_0 = 0$ ,  $\tilde{T}(r_0) = r_1 = 1$ ,  $\tilde{T}(r_1) = r_0 = 1/\beta$ , thus  $\mathcal{V} = \{0, 1/\beta, 1\}$ . The transformation and its natural extension are depicted in Figure 2.

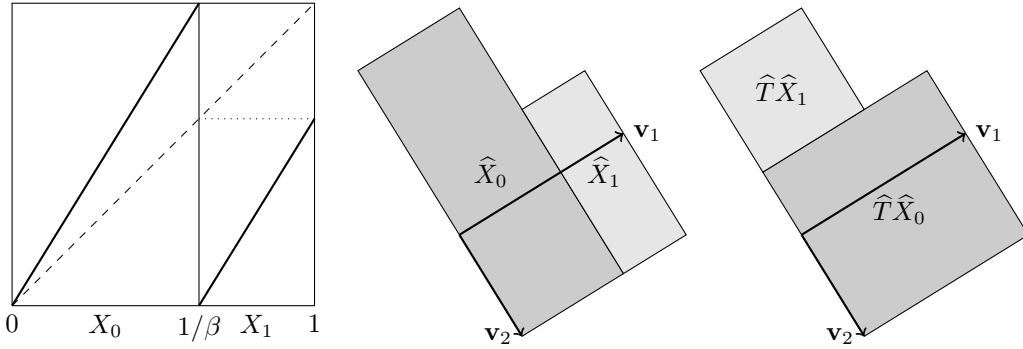


FIGURE 2. The (greedy)  $\beta$ -transformation,  $\beta = (1 + \sqrt{5})/2$ , and its natural extension.

*Example 3.11.* Let  $\beta$  be again the golden mean, but  $A = \{\bar{1}, 0, 1\}$ ,

$$X_{\bar{1}} = \left[ -\frac{\beta^2 + \beta^{-3}}{\beta^2 + 1}, -\frac{\beta + \beta^{-4}}{\beta^2 + 1} \right), \quad X_0 = \left[ -\frac{\beta + \beta^{-4}}{\beta^2 + 1}, \frac{\beta + \beta^{-4}}{\beta^2 + 1} \right), \quad X_1 = \left[ \frac{\beta + \beta^{-4}}{\beta^2 + 1}, \frac{\beta^2 + \beta^{-3}}{\beta^2 + 1} \right),$$

$\delta_{\bar{1}} = -1$ ,  $\delta_0 = 0$ ,  $\delta_1 = 1$ . The endpoints have the expansions

$$b(\ell_{\bar{1}}) = \bar{1}(00\bar{1}0)^\omega, \quad \tilde{b}(r_{\bar{1}}) = \bar{1}0010(000\bar{1})^\omega, \quad b(\ell_0) = b(r_{\bar{1}}) = 0\bar{1}(00\bar{1}0)^\omega, \\ \tilde{b}(r_0) = 01(0010)^\omega, \quad b(\ell_1) = b(r_0) = 100\bar{1}0(0001)^\omega, \quad \tilde{b}(r_1) = 1(0010)^\omega.$$

Since  $\tilde{T}^5 r_{\bar{1}} = T^5 r_{\bar{1}}$  and  $\tilde{T}^5 r_0 = T^5 r_0$ , we have  $m_{\bar{1}} = m_0 = 5$ , and thus

$$\mathcal{V} = \{ \cdot\bar{1}(00\bar{1}0)^\omega, \cdot(\bar{1}000)^\omega, \cdot\bar{1}0(0001)^\omega, \cdot(0\bar{1}00)^\omega, \cdot0\bar{1}0(0001)^\omega, \cdot(00\bar{1}0)^\omega, \cdot00\bar{1}0(0001)^\omega, \cdot0(000\bar{1})^\omega, \\ \cdot0(0001)^\omega, \cdot0010(000\bar{1})^\omega, \cdot(0010)^\omega, \cdot010(000\bar{1})^\omega, \cdot(0100)^\omega, \cdot10(000\bar{1})^\omega, \cdot(1000)^\omega, \cdot1(0010)^\omega \}.$$

The transformation and its natural extension are depicted in Figure 3. Note that, since  $\tilde{T}^5 r_0 = T^5 r_0 = \cdot(0001)^\omega \notin \mathcal{V}$ ,  $\mathcal{D}_x$  does not change at this point, but the shape of the sets  $\widehat{T}X_1$  and  $\widehat{T}X_0$  changes. This means that the decomposition of  $\mathcal{D}_x$  according to (12) changes. The same happens at  $\tilde{T}^5 r_{\bar{1}} = T^5 r_{\bar{1}} = \cdot(000\bar{1})^\omega$  with the sets  $\widehat{T}X_{\bar{1}}$  and  $\widehat{T}X_0$ .

*Example 3.12.* Let  $\beta$  be again the golden mean and  $A = \{\bar{1}, 0, 1\}$ , but  $X_{\bar{1}} = [-\frac{1}{2}, -\frac{1}{2\beta})$ ,  $X_0 = [-\frac{1}{2\beta}, \frac{1}{2\beta})$ ,  $X_1 = [\frac{1}{2\beta}, \frac{1}{2})$ ,  $\delta_{\bar{1}} = -1$ ,  $\delta_0 = 0$ ,  $\delta_1 = 1$ . This is an example of a symmetric  $\beta$ -transformation defined by Akiyama and Scheicher ([5]). The endpoints have the expansions

$$b(\ell_{\bar{1}}) = (\bar{1}01)^\omega, \quad \tilde{b}(r_{\bar{1}}) = (\bar{1}10)^\omega, \quad b(\ell_0) = b(r_{\bar{1}}) = 0(\bar{1}01)^\omega, \\ \tilde{b}(r_0) = 0(10\bar{1})^\omega, \quad b(\ell_1) = b(r_0) = (1\bar{1}0)^\omega, \quad \tilde{b}(r_1) = (10\bar{1})^\omega,$$

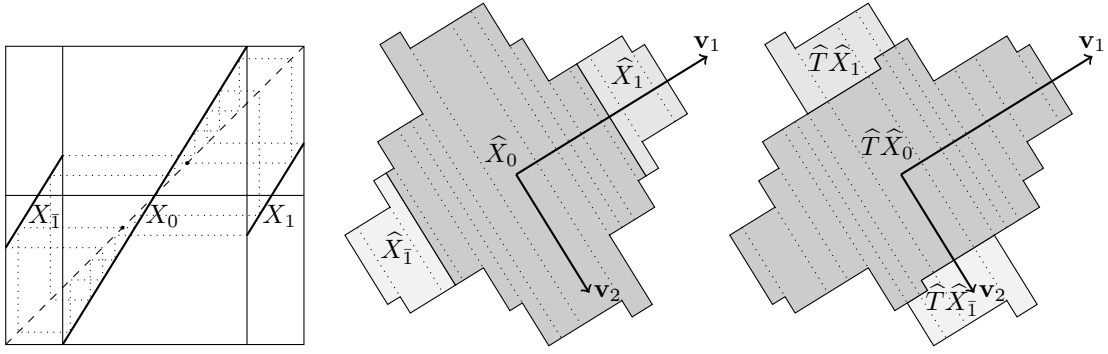


FIGURE 3. The transformation from Example 3.11 and its natural extension.

thus  $\mathcal{V} = \{-\frac{1}{2}, -\frac{1}{2\beta}, -\frac{1}{2\beta^2}, \frac{1}{2\beta^2}, \frac{1}{2\beta}, \frac{1}{2}\}$ . The transformation and its natural extension are depicted in Figure 4.

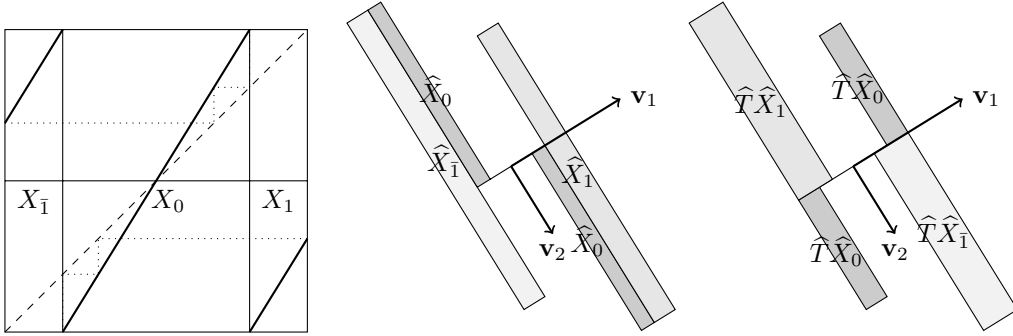


FIGURE 4. The transformation from Example 3.12 and its natural extension.

*Example 3.13.* Let  $\beta$  be again the golden mean, now  $A = \{\bar{1}, 1\}$ ,  $X_{\bar{1}} = [-1, 0]$ ,  $X_1 = [0, 1]$ ,  $\delta_{\bar{1}} = -1$ ,  $\delta_1 = 1$ . Then  $T^3(0) = T^2(-1) = T(-1/\beta) = 0$ ,  $\hat{T}^3(0) = \hat{T}^2(1) = \hat{T}(1/\beta) = 0$ , thus  $\mathcal{V} = \{-1, -1/\beta, 1/\beta, 1\}$ . The transformation and its natural extension are depicted in Figure 5.

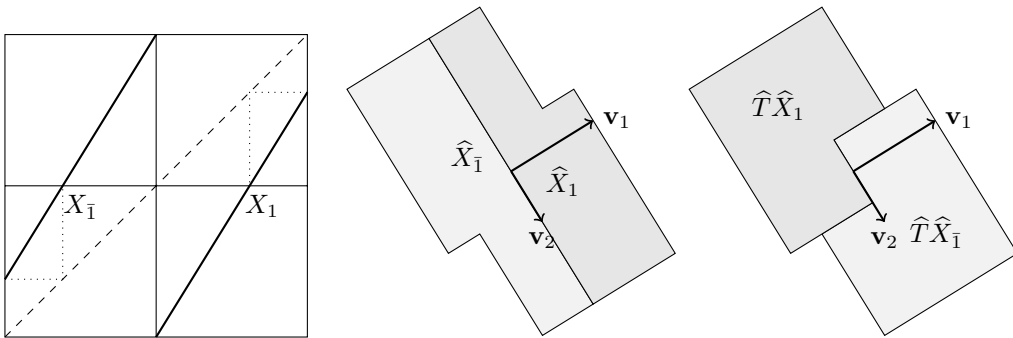


FIGURE 5. The transformation from Example 3.13 and its natural extension.

*Example 3.14.* Consider the Tribonacci number, i.e., let  $\beta$  be the real solution of the equation  $x^3 - x^2 - x - 1 = 0$ . Let  $A = \{\bar{1}, 0, 1\}$  and take  $X_{\bar{1}} = [\frac{-\beta}{\beta+1}, \frac{-1}{\beta+1}]$ ,  $X_0 = [\frac{-1}{\beta+1}, \frac{1}{\beta+1}]$ ,  $X_1 =$

$[\frac{1}{\beta+1}, \frac{\beta}{\beta+1})$ ,  $\delta_{\bar{1}} = -1$ ,  $\delta_0 = 0$ ,  $\delta_1 = 1$ . This is a minimal weight transformation as defined in [18]. The endpoints have the expansions

$$b(\ell_{\bar{1}}) = (\bar{1}00)^\omega, \tilde{b}(r_{\bar{1}}) = \bar{1}(010)^\omega, b(\ell_0) = (0\bar{1}0)^\omega, \tilde{b}(\ell_0) = (010)^\omega, b(\ell_1) = 1(0\bar{1}0)^\omega, \tilde{b}(r_1) = (100)^\omega,$$

and  $\mathcal{V} = \{\frac{-\beta}{\beta+1}, \frac{-1}{\beta+1}, \frac{-1/\beta}{\beta+1}, \frac{1/\beta}{\beta+1}, \frac{1}{\beta+1}, \frac{\beta}{\beta+1}\}$ . We see the transformation and  $\hat{X}$  in Figure 6.

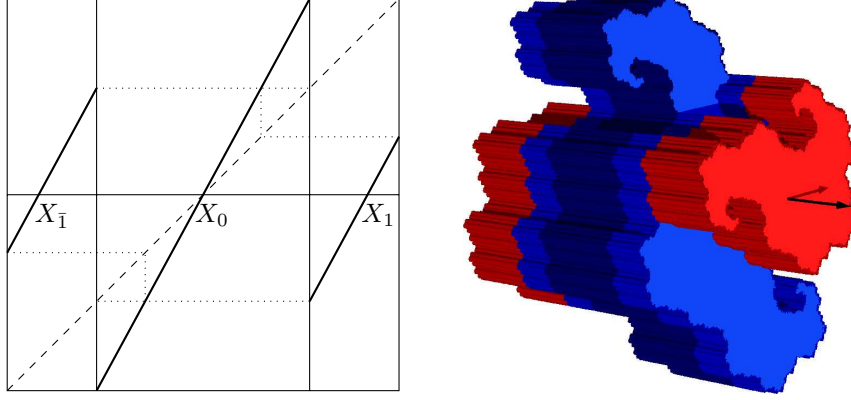


FIGURE 6. The transformation from Example 3.14 and its natural extension domain.

*Example 3.15.* Consider the smallest Pisot number, i.e., let  $\beta$  be the real solution of the equation  $x^3 - x - 1 = 0$ . Let  $A = \{\bar{1}, 0, 1\}$  and take  $X_{\bar{1}} = [\frac{-\beta^7}{\beta^8-1}, \frac{-\beta^6}{\beta^8-1})$ ,  $X_0 = [\frac{-\beta^6}{\beta^8-1}, \frac{\beta^6}{\beta^8-1})$ ,  $X_1 = [\frac{\beta^6}{\beta^8-1}, \frac{\beta^7}{\beta^8-1})$ . This is again a minimal weight transformation. The endpoints have the expansions

$$b(\ell_{\bar{1}}) = (\bar{1}0^7)^\omega, \tilde{b}(r_{\bar{1}}) = \bar{1}(0^610)^\omega, b(\ell_0) = (0\bar{1}0^6)^\omega, \tilde{b}(r_0) = (010^6)^\omega, b(\ell_1) = 1(0^6\bar{1}0)^\omega, \tilde{b}(r_1) = (10^7)^\omega,$$

and  $\mathcal{V} = \{\frac{\pm\beta^k}{\beta^8-1} : 0 \leq k \leq 7\}$ . We see the transformation and the domain  $\hat{X}$  in Figure 7.

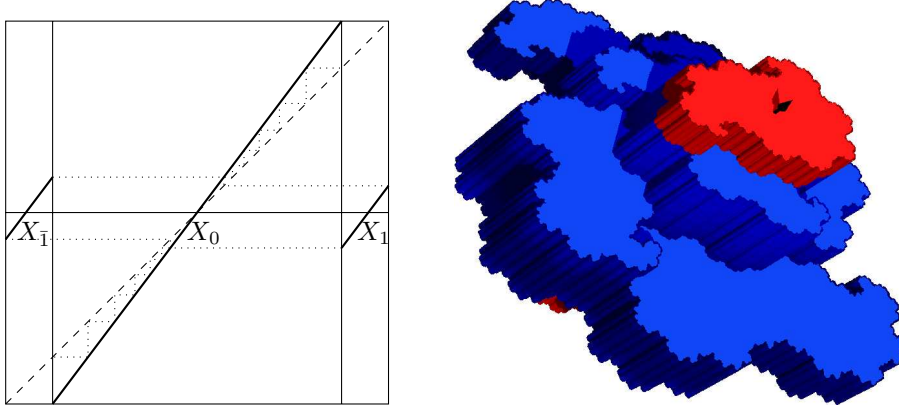


FIGURE 7. The transformation from Example 3.15 and its natural extension domain.

#### 4. TILINGS

In this section, we consider two types of (multiple) tilings which are closely related. The first one is an aperiodic (multiple) tiling of the hyperplane  $H$  by sets  $\mathcal{D}_x$  defined in (8). The second one is a periodic (multiple) tiling of  $\mathbb{R}^d$  by the closure  $\hat{Y}$  of the natural extension domain  $\hat{X}$ . We also show that the numbers with purely periodic  $T$ -expansion can be easily characterized by  $\hat{X}$ .

As for the natural extensions, we assume that  $T$  is a right-continuous  $\beta$ -transformation and  $\beta$  a Pisot unit. We furthermore assume that  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$  (except in Section 4.2). Note that all results can be adapted to  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$  by replacing  $\mathbb{Z}[\beta]$  with  $q^{-1}\mathbb{Z}[\beta]$  and  $\mathbb{Z}^d$  with  $q^{-1}\mathbb{Z}^d$  if  $\{\delta_a : a \in A\} \subset q^{-1}\mathbb{Z}[\beta]$ ,  $q \in \mathbb{Z}$ .

**4.1. Tiling of the contracting hyperplane.** We define tiles in the hyperplane  $H$  by

$$\mathcal{T}_x = \Phi(x) + \mathcal{D}_x \quad \text{for all } x \in \mathbb{Q}(\beta) \cap X,$$

with  $\Phi$  as in (6) and  $\mathcal{D}_x$  as in (8). We assume that  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ , and we will usually consider only  $x \in \mathbb{Z}[\beta] \cap X$ .

*Remark 4.1.* The tiles are often defined in  $\mathbb{R}^r \times \mathbb{C}^s$ , where  $r$  is the number of real conjugates and  $2s$  is the number of complex conjugates of  $\beta$ . We clearly have  $\mathbb{R}^r \times \mathbb{C}^s \simeq H$ . We choose to work in  $H$  because many statements are easier to formulate in  $H$  than in  $\mathbb{R}^r \times \mathbb{C}^s$ .

To show that the family  $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  is a *multiple tiling* of the space  $H$ , we have to show all of the following.

- (mt1) There are only finitely many different sets  $\mathcal{D}_x$ , and these sets are compact.
- (mt2) The family  $\mathcal{T}$  is *locally finite*, i.e., for every  $\mathbf{y} \in H$ , there is a positive  $r$  such that the set  $\{x \in \mathbb{Z}[\beta] \cap X : \mathcal{T}_x \cap B(\mathbf{y}, r) \neq \emptyset\}$  is finite.
- (mt3)  $\mathcal{T}$  gives a covering of  $H$ : for every  $\mathbf{y} \in H$ , there is a tile  $\mathcal{T}_x$ , such that  $\mathbf{y} \in \mathcal{T}_x$ .
- (mt4) Every set  $\mathcal{D}_x$ ,  $x \in \mathbb{Z}[\beta] \cap X$ , is the closure of its interior.
- (mt5) There is an integer  $m \geq 1$  such that almost all points of  $H$  are in exactly  $m$  different tiles. The number  $m$  is called the *covering degree* of the multiple tiling.

A *tiling* is a multiple tiling with covering degree 1.

If the set  $\mathcal{V}$  defined in (9) is finite (and  $TX = X$ ), then the tiles  $\mathcal{T}_x$ ,  $x \in \mathbb{Z}[\beta] \cap X$ , are translates of a finite collection of compact sets by Lemma 3.4 and Proposition 3.5. This proves (mt1). An important tool for showing (mt2)–(mt5) will be that the set of translation vectors,  $\Phi(\mathbb{Z}[\beta] \cap X)$ , is a *Delone set* in  $H$ , i.e., that it is uniformly discrete and relatively dense in  $H$ . This means the following.

- A set  $Z$  is *relatively dense* in  $H$  if there is an  $R > 0$ , such that, for every  $\mathbf{y} \in H$ ,  $B(\mathbf{y}, R) \cap Z \neq \emptyset$ .
- A set  $Z$  is *uniformly discrete* if there is an  $r > 0$  such that, for every  $\mathbf{y} \in Z$ , the set  $B(\mathbf{y}, r) \cap Z$  contains only one element.

In [22], Moody studied, among other things, Delone sets. He gives a detailed exposition of Meyer's theory, which was developed in [21]. According to Meyer, model sets for cut and project schemes are Delone. We will use this to prove Lemma 4.3. We also need the following lemma.

**Lemma 4.2.** *The map  $\Psi : \mathbb{Q}(\beta) \rightarrow \mathbb{Q}^d$  is bijective,  $\Psi(\mathbb{Z}[\beta]) = \mathbb{Z}^d$  and  $\Phi : \mathbb{Q}(\beta) \rightarrow H$  is injective.*

*Proof.* It can be easily seen from the structure of  $M_\beta$  that every  $\mathbf{x} \in \mathbb{Q}^d$  and every  $\mathbf{x} \in \mathbb{Z}^d$  can be written in a unique way as  $\mathbf{x} = \sum_{k=0}^{d-1} z_k M_\beta^k \mathbf{e}_1 = \Psi(\sum_{k=0}^{d-1} z_k \beta^k)$  with  $z_k \in \mathbb{Q}$  and  $z_k \in \mathbb{Z}$  respectively, which proves the first statements. Since  $\Phi(x) = \Psi(x) - x\mathbf{v}_1$  and the coefficients of  $\mathbf{v}_1$  are linearly independent over  $\mathbb{Q}$ ,  $\Phi$  is injective.  $\square$

**Lemma 4.3.** *Every set  $\Phi(q^{-1}\mathbb{Z}[\beta] \cap E)$ ,  $q \in \mathbb{Z} \setminus \{0\}$ , where  $E \subset \mathbb{R}$  is bounded and has non-empty interior, is uniformly discrete and relatively dense. In particular, this holds for  $\Phi(\mathbb{Z}[\beta] \cap X)$ .*

*Proof.* By Proposition 2.6 from [22], it suffices to show that  $\Phi(q^{-1}\mathbb{Z}[\beta] \cap E)$  is a model set.

Define two projections  $\pi_H : \mathbb{R}^d \rightarrow H$ ,  $\pi_1 : \mathbb{R}^d \rightarrow \mathbb{R}$ , by  $\pi_H(\mathbf{x}) + \pi_1(\mathbf{x})\mathbf{v}_1 = \mathbf{x}$  for  $\mathbf{x} \in \mathbb{R}^d$ , and set  $\iota(\mathbf{x}) = (\pi_H(\mathbf{x}), \pi_1(\mathbf{x}))$ . Then the pair  $(H \times \mathbb{R}, \iota(q^{-1}\mathbb{Z}^d))$  is a cut and project scheme, since  $\iota(q^{-1}\mathbb{Z}^d)$  is a lattice,  $\pi_H$  is injective on  $q^{-1}\mathbb{Z}^d$ ,  $\pi_1(q^{-1}\mathbb{Z}^d)$  is dense in  $\mathbb{R}$  and  $H \simeq \mathbb{R}^{d-1}$ . Indeed, the injectivity and denseness hold since  $\pi_H(\mathbf{x}) = \Phi \circ \Psi^{-1}(\mathbf{x})$  for  $\mathbf{x} \in \mathbb{Q}^d$  and  $\pi_1(q^{-1}\mathbb{Z}^d) = q^{-1}\mathbb{Z}[\beta]$ .

Now, let  $E \subset \mathbb{R}$  be a bounded set with non-empty interior. Then

$$\Phi(q^{-1}\mathbb{Z}[\beta] \cap E) = \{\pi_H(\mathbf{x}) : \mathbf{x} \in q^{-1}\mathbb{Z}^d, \pi_1(\mathbf{x}) \in E\}$$

is a model set, and therefore a Delone set by Proposition 2.6 from [22].  $\square$

**Corollary 4.4.** *The family  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  is locally finite.*

*Proof.* This follows immediately from the uniform discreteness of  $\Phi(\mathbb{Z}[\beta] \cap X)$ , the injectivity of  $\Phi$  and the fact that  $\|\varphi(w)\|$ ,  $w \in {}^\omega A$ , is bounded.  $\square$

Hence we have (mt2). The next lemma gives (mt3). Recall that  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ .

**Lemma 4.5.** *We have  $H = \bigcup_{x \in \mathbb{Z}[\beta] \cap X} \mathcal{T}_x$ .*

*Proof.* Let  $J = \bigcup_{x \in \mathbb{Z}[\beta] \cap X} \mathcal{T}_x$ . Every point  $\mathbf{y} \in J$  is of the form  $\mathbf{y} = \Phi(x) + \varphi(w)$ , with  $x \in \mathbb{Z}[\beta] \cap X$ ,  $w \in {}^\omega A$ ,  $w \cdot b(x) \in \mathcal{S}$ . We have  $M_\beta \mathbf{y} = \varphi(wb_1(x)) + \Phi(Tx)$ , with  $Tx \in \mathbb{Z}[\beta] \cap X$  since  $\delta_{b_1(x)} \in \mathbb{Z}[\beta]$ , and thus  $M_\beta \mathbf{y} \in J$ . Let  $X' = \bigcap_{k \geq 0} T^k X$ . Then every tile  $\mathcal{T}_x$ ,  $x \in \mathbb{Z}[\beta] \cap X'$ , is non-empty. The set  $X'$  has non-empty interior: Take an  $x \in X$  with  $T^n x = x$  for some  $n \geq 1$  (for the existence of such an  $x$ , see e.g. Section 4.2). Then there exists some  $\varepsilon > 0$  such that  $b_1(y) \cdots b_n(y) = b_1(x) \cdots b_n(x)$  for all  $y \in [x, x + \varepsilon]$ , which implies  $T^n[x, x + \varepsilon] = [x, x + \beta^n \varepsilon]$  and thus  $[x, x + \beta^n \varepsilon] \subseteq X'$ .

Since  $X'$  has non-empty interior,  $\Phi(\mathbb{Z}[\beta] \cap X')$  is relatively dense in  $H$ , thus  $J$  is also relatively dense. We have already shown that  $M_\beta J \subseteq J$ , hence  $J$  is dense in  $H$  since  $M_\beta$  is contracting. By the compactness of the tiles and the local finiteness of  $\mathcal{T}$ , we obtain  $J = H$ .  $\square$

Now we can prove that the boundary of every tile has zero measure, provided that  $\mathcal{V}$  is finite. This generalizes Theorem 3 of Akiyama ([1]).

**Proposition 4.6.** *If  $\mathcal{V}$  is finite (and  $TX = X$ ), then  $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$  for every  $x \in X$ .*

*Proof.* If  $\lambda^{d-1}(\mathcal{D}_x) = 0$ , then  $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$  since  $\mathcal{D}_x$  is compact.

Therefore we can assume  $\lambda^{d-1}(\mathcal{D}_x) > 0$ . We first show that  $\lambda^{d-1}(\partial \mathcal{D}_y) = 0$  for some  $y \in X$  with  $\lambda^{d-1}(\mathcal{D}_y) > 0$ , and then extend this property to arbitrary  $x \in X$ .

Since  $H = \bigcup_{z \in \mathbb{Z}[\beta] \cap X} \mathcal{T}_z$ , there exists, by Baire's theorem, some  $z \in \mathbb{Z}[\beta] \cap X$  such that  $\mathcal{T}_z$ , and thus  $\mathcal{D}_z$ , has an inner point (with respect to  $H$ ). By iterating (12), we obtain for all  $k \geq 1$  that  $\mathcal{D}_z$  is the (up to sets of measure zero) disjoint union of  $M_\beta^k \mathcal{D}_y + \varphi(b_1(y) \cdots b_k(y))$ ,  $y \in T^{-k}\{z\}$ . Since  $M_\beta^{-1}$  is expanding on  $H$ , there must be some  $y \in T^{-k}\{z\}$  for sufficiently large  $k$  such that  $M_\beta^k \mathcal{D}_y + \varphi(b_1(y) \cdots b_k(y))$  is contained in the interior of  $\mathcal{D}_z$ , and  $\lambda^{d-1}(\mathcal{D}_y) > 0$ . Then every point in  $\partial(M_\beta^k \mathcal{D}_y + \varphi(b_1(y) \cdots b_k(y)))$  lies also in  $M_\beta^k \mathcal{D}_{y'} + \varphi(b_1(y') \cdots b_k(y'))$  for some  $y' \in T^{-k}\{x\}$ ,  $y' \neq y$ . Since the intersection of these sets has zero measure, we obtain  $\lambda^{d-1}(\partial \mathcal{D}_y) = 0$ .

Now, consider a set  $M_\beta \mathcal{D}_z + \varphi(b_1(z))$ ,  $z \in T^{-1}\{y\}$ , in the subdivision of  $\mathcal{D}_y$ . Every point on the boundary of this set is either also on the boundary of another set from the subdivision, or not. If not, then the point is in  $\partial \mathcal{D}_y$ . Therefore, we have  $\lambda^{d-1}(\partial \mathcal{D}_z) = 0$  for every  $z \in T^{-1}\{y\}$ . It remains to show that, when iterating this argument, every  $\mathcal{D}_x$  with  $\lambda^{d-1}(\mathcal{D}_x) > 0$  occurs eventually in one of these subdivisions. Since  $\mathcal{D}_x$  does not change within  $X_{a,n}$  by Proposition 3.5 and (11), we set

$$\mathcal{N} = \{(a, n) : \lambda^{d-1}(\mathcal{D}_{\ell_{a,n}}) > 0\} = \{(a, n) : \lambda^d(\widehat{X}_{a,n}) > 0\}.$$

Let  $\mathcal{G}$  be the weighted oriented graph with set of nodes  $\mathcal{N}$  and an edge from  $(a, n)$  to  $(a', n')$  if and only if  $T^{-1}X_{a,n} \cap X_{a',n'} \neq \emptyset$ . Then we have  $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$  for every  $x \in X_{a',n'}$  if  $(a', n')$  can be reached from the node  $(a, n)$  satisfying  $y \in X_{a,n}$ .

Let the weight of an edge be  $\lambda^d(\widehat{T}^{-1}\widehat{X}_{a,n} \cap \widehat{X}_{a',n'}) > 0$ . Since  $\widehat{T}$  is bijective off of a set of Lebesgue measure zero, the sum of the weights of the outgoing edges as well as the sum of the weights of the ingoing edges must equal  $\lambda^d(\widehat{X}_{a,n})$  for every node  $(a, n) \in \mathcal{N}$ . This implies that every connected component of  $\mathcal{G}$  must be strongly connected, in the sense that if two nodes are in the same connected component, then there is a path from one node to the other and the other way around. By the definition of  $\mathcal{G}$ , the restriction of  $T$  to a strongly connected component is a right-continuous  $\beta$ -transformation, and this restriction changes the tiles in this component only by sets of measure zero. Therefore, the arguments of the preceding paragraph provide some  $y$



with  $\lambda^{d-1}(\partial\mathcal{D}_y) = 0$  in every connected component of  $\mathcal{G}$ , and we obtain  $\lambda^{d-1}(\partial\mathcal{D}_x) = 0$  for every  $x \in X$ .  $\square$

If we want that (mt4) holds, then we clearly have to exclude non-empty tiles of measure zero. If we also assume  $TX = X$ , this means that  $X$  has to be the support of the invariant measure  $\mu = \lambda^d \circ \pi^{-1}$ . Note that restricting  $T$  to the support of  $\mu$  changes the tiles only by sets of measure zero.

For a subset  $E \subseteq H$ , let  $\text{int}(E)$  denote the interior of  $E$ , and let  $\overline{E}$  be the closure of  $E$ .

**Lemma 4.7.** *Suppose that the support of the invariant measure  $\mu$  defined in Section 3 is  $X$  and that  $\mathcal{V}$  is finite. Then  $\mathcal{D}_x = \overline{\text{int}(\mathcal{D}_x)}$  for every  $x \in X$ .*

*Proof.* Since the tiles are compact, we have  $\overline{\text{int}(\mathcal{D}_x)} \subseteq \mathcal{D}_x$  for every  $x \in X$ . For the other inclusion, we show  $\mathcal{T}_x \subseteq \overline{\text{int}(\mathcal{T}_x)}$  for every  $x \in \mathbb{Z}[\beta] \cap X$ . If  $\mathbf{y} \in \mathcal{T}_x$ , then  $\mathbf{y} = \Phi(x) + \varphi(w)$  for some  $w = (w_k)_{k \leq 0} \in {}^\omega A$  with  $w \cdot b(x) \in \mathcal{S}$ . Set  $x_k = \cdot w_{-k+1} \cdots w_0 b(x)$  for  $k \geq 0$ . Then we have  $M_\beta^k(\Phi(x_k) + \mathcal{D}_{x_k}) \subseteq \mathcal{T}_x$ . Since  $\lambda^{d-1}(\mathcal{D}_{x_k}) > 0$  and  $\lambda^{d-1}(\partial\mathcal{D}_{x_k}) = 0$ , there exists a point  $\mathbf{y}_k \in M_\beta^k(\Phi(x_k) + \text{int}(\mathcal{D}_{x_k})) \subseteq \text{int}(\mathcal{T}_x)$ . Since  $\lim_{k \rightarrow \infty} \mathbf{y}_k = \mathbf{y}$ , we have  $\mathbf{y} \in \overline{\text{int}(\mathcal{T}_x)}$ .  $\square$

The family  $\mathcal{T}$  is *self-replicating*, i.e., for each element  $\mathcal{T}_x$  of  $\mathcal{T}$ , we can write  $M_\beta^{-1}\mathcal{T}_x$  as the (up to sets of measure zero) disjoint union of elements from  $\mathcal{T}$ . By Lemma 3.9, we have

$$(13) \quad M_\beta^{-1}\mathcal{T}_x = M_\beta^{-1}\left(\Phi(x) + \bigcup_{y \in T^{-1}\{x\}} (M_\beta\mathcal{D}_y + \Phi(\delta_{b_1(y)}))\right) = \bigcup_{y \in T^{-1}\{x\}} \mathcal{T}_y.$$

For  $\mathbf{y} \in H$  and  $r > 0$ , the *local arrangement* in  $B(\mathbf{y}, r)$  is the set

$$\mathcal{P}(B(\mathbf{y}, r)) = \{\mathcal{T}_x \in \mathcal{T} : \mathcal{T}_x \cap B(\mathbf{y}, r) \neq \emptyset\}.$$

As a next step in proving that the family  $\mathcal{T}$  is a multiple tiling, we will show that  $\mathcal{T}$  is *quasi-periodic*, i.e., for any  $r > 0$ , there is an  $R > 0$  such that, for any  $\mathbf{y}, \mathbf{y}' \in H$ , the local arrangement in  $B(\mathbf{y}, r)$  appears up to translation in the ball  $B(\mathbf{y}', R)$ .

**Proposition 4.8.** *If  $\mathcal{V}$  is finite (and  $TX = X$ ), then the family  $\mathcal{T}$  is quasi-periodic.*

*Proof.* Note first that, for each  $r > 0$ , there are only finitely many different local arrangements up to translation, since  $\Phi(\mathbb{Z}[\beta] \cap X)$  is uniformly discrete and there are only finitely many different sets  $\mathcal{D}_x$ .

Let  $r > 0$ . If two tiles  $\mathcal{T}_x$  and  $\mathcal{T}_{x+y}$  are in the same local arrangement, then

$$(14) \quad \|\Phi(y)\| < 2(r + \max_{w \in {}^\omega A} \|\varphi(w)\|) \quad \text{and} \quad y \in X - X.$$

By Lemma 4.3 and since  $\Phi$  is injective, the set of elements  $y \in \mathbb{Z}[\beta]$  satisfying (14) is finite. Call this set  $F$ , and note that  $0 \in F$ .

Now, take some local arrangement  $\mathcal{P}(B(\mathbf{y}, r))$  and  $x \in \mathbb{Z}[\beta] \cap X$  such that  $\mathcal{T}_x \in \mathcal{P}(B(\mathbf{y}, r))$ . For any  $y \in F$ , if  $x + y \in X_{a,n}$ , then there is an  $\varepsilon_y > 0$  such that  $[x + y, x + y + \varepsilon_y] \subseteq X_{a,n}$ , and if  $x + y \notin X$ , then there is an  $\varepsilon_y > 0$  such that  $[x + y, x + y + \varepsilon_y] \cap X = \emptyset$ . Let  $\varepsilon = \min_{y \in F} \varepsilon_y$ . Then, for all  $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and each  $y \in F$  such that  $\mathcal{T}_{x+y} \in \mathcal{P}(B(\mathbf{y}, r))$ , we have  $\mathcal{D}_{x+y+z} = \mathcal{D}_{x+y}$  and thus  $\mathcal{T}_{x+y+z} = \mathcal{T}_{x+y} + \Phi(z) \in \mathcal{P}(B(\mathbf{y} + \Phi(z), r))$ . Furthermore, every tile in  $\mathcal{P}(B(\mathbf{y} + \Phi(z), r))$  is of the form  $\mathcal{T}_{(x+z)+y} = \mathcal{T}_{x+y} + \Phi(z)$  by the choice of  $\varepsilon$ . Therefore,  $\mathcal{P}(B(\mathbf{y} + \Phi(z), r))$  is up to translation equal to  $\mathcal{P}(B(\mathbf{y}, r))$ .

By Lemma 4.3,  $\Phi(\mathbb{Z}[\beta] \cap [0, \varepsilon))$  is relatively dense in  $H$ . Thus, every local arrangement occurs relatively densely in  $H$ . Since the number of local arrangements is finite, the lemma is proved.  $\square$

Lemma 4.5 implies that every  $\mathbf{y} \in H$  lies in at least one tile. By the local finiteness of  $\mathcal{T}$ , there exists an  $m \geq 1$  such that every element of  $H$  is contained in at least  $m$  tiles in  $\mathcal{T}$  and there exist elements of  $H$  that are not contained in  $m + 1$  tiles. For this  $m$ , a point  $\mathbf{y} \in H$  lying in exactly  $m$  tiles is called an *m-exclusive point*. A point lying in exactly one tile is called an *exclusive point*. Similarly to Ito and Rao ([20]), we obtain the following proposition, which gives (mt5).

**Proposition 4.9.** *If  $\mathcal{V}$  is finite (and  $TX = X$ ), then there exists an  $m \geq 1$  such that almost every  $\mathbf{y} \in H$  is contained in exactly  $m$  tiles.*

*Proof.* We first show that the set of points that do not lie on the boundary of a tile is open and of full measure. Let  $C = \bigcup_{x \in \mathbb{Z}[\beta] \cap X} \partial \mathcal{T}_x$  denote the union of the boundaries of all the tiles in  $\mathcal{T}$ . This set is closed, since it is the countable union of closed sets that are locally finite. Hence,  $H \setminus C$  is open. By Proposition 4.6, we also have  $\lambda^{d-1}(C) = 0$ .

Let  $m$  be as in the paragraph preceding the proposition, and  $\mathbf{x} \in H$  be an  $m$ -exclusive point, lying in the tiles  $\mathcal{T}_{x_1}, \dots, \mathcal{T}_{x_m}$ . Since the tiles are closed, there is an  $\varepsilon > 0$  such that  $B(\mathbf{x}, \varepsilon) \cap \mathcal{T}_x = \emptyset$  for all  $x \in (\mathbb{Z}[\beta] \cap X) \setminus \{x_1, \dots, x_m\}$ . Since every point lies in  $m$  tiles, we have  $B(\mathbf{x}, \varepsilon) \subseteq \mathcal{T}_{x_k}$  for  $1 \leq k \leq m$ . By the self-replicating property,  $M_\beta^{-1} \mathcal{T}_{x_k}$  subdivides into tiles from  $\mathcal{T}$  with disjoint interior, hence almost every point in  $M_\beta^{-1} B(\mathbf{x}, \varepsilon)$  is also contained in exactly  $m$  tiles. The same holds for almost every point in  $M_\beta^{-n} B(\mathbf{x}, \varepsilon)$ , for all  $n \geq 1$ .

Now take a point  $\mathbf{y} \in H \setminus C$ . Since  $H \setminus C$  is open, there is an  $r > 0$  such that  $B(\mathbf{y}, r) \subseteq H \setminus C$ , i.e., every point in  $B(\mathbf{y}, r)$  lies in the same set of tiles. By the quasi-periodicity, translations of the local arrangement  $\mathcal{P}(B(\mathbf{y}, r))$  occur relatively densely in  $H$ . Since the matrix  $M_\beta^{-1}$  is expanding on  $H$ , there is therefore an  $n \geq 1$  such that  $M_\beta^{-n} B(\mathbf{x}, \varepsilon)$  contains a translation of  $\mathcal{P}(B(\mathbf{y}, r))$ , which implies that  $B(\mathbf{y}, r)$  and thus  $\mathbf{y}$  lie in exactly  $m$  tiles.  $\square$

We have now established all the properties of a multiple tiling.

**Theorem 4.10.** *Let  $T : X \rightarrow X$  be a right-continuous  $\beta$ -transformation as in Definition 2.4 with a Pisot unit  $\beta$  and  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ . Assume that the invariant measure  $\mu$ , given by the natural extension in Section 3.1, has support  $X$ , and that the set  $\mathcal{V}$ , defined by (9), is finite. Then the family  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  forms a multiple tiling of the hyperplane  $H$ .*

**Corollary 4.11.** *Let  $T$  satisfy the assumptions of Theorem 4.10. Then the family  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  forms a tiling of  $H$  if and only if there exists a point in  $H$  which lies in exactly one tile.*

*Remark 4.12.* If we replace (mt4) by the weaker condition that  $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$  for every  $x \in \mathbb{Z}[\beta] \cap X$ , then the condition that the support of  $\mu$  is equal to  $X$  can be replaced by the condition  $TX = X$  in Theorem 4.10 and Corollary 4.11.

We would like a way to find  $m$ -exclusive points. For that we need some results about the points with periodic  $T$ -expansion.

**4.2. Periodic expansions.** We first characterize eventually periodic expansions for a right-continuous  $\beta$ -transformation  $T$  with  $\beta$  a Pisot number (not necessarily a unit) and  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$ . This result was proved by Frank and Robinson ([16]) for a slightly smaller class of transformations, and generalizes the result by Bertrand ([8]) and Schmidt ([28]) for the classical  $\beta$ -transformation.

**Theorem 4.13.** *Let  $T$  be a right-continuous  $\beta$ -transformation as in Definition 2.4 with a Pisot number  $\beta$  and  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$ . Then the  $T$ -expansion of  $x \in X$  is eventually periodic if and only if  $x \in \mathbb{Q}(\beta)$ .*

*Proof.* If  $b(x) = b_1 \cdots b_m (b_{m+1} \cdots b_{m+n})^\omega$  is eventually periodic, then

$$x = \frac{\delta_{b_1}}{\beta} + \cdots + \frac{\delta_{b_m}}{\beta^m} + \frac{1}{\beta^n - 1} \left( \frac{\delta_{b_{m+1}}}{\beta^{m-n+1}} + \frac{\delta_{b_{m+2}}}{\beta^{m-n+2}} + \cdots + \frac{\delta_{b_{m+n}}}{\beta^m} \right),$$

which is clearly in  $\mathbb{Q}(\beta)$ .

For the other implication, let  $x \in \mathbb{Q}(\beta) \cap X$ . Since  $A$  is a finite set, there is a  $q \in \mathbb{Z}$ , such that  $x$  and  $\delta_a$  are in  $q^{-1} \mathbb{Z}[\beta]$  for all  $a \in A$ . Then  $T^k x \in q^{-1} \mathbb{Z}[\beta] \cap X$  for each  $k \geq 0$ . Furthermore, since  $\Phi(T^k x) = M_\beta^k \Phi(x) - \varphi(b_1(x) \cdots b_k(x))$  for all  $k \geq 0$  and  $M_\beta$  is contracting, we obtain that  $\Phi(T^k x)$  is bounded, uniformly in  $k$ . Therefore, by Lemma 4.3 and since  $\Phi$  is injective,  $\{T^k x : k \geq 0\}$  is a finite set, which implies that  $b(x)$  is eventually periodic.  $\square$

Note that the previous lemma implies the existence of points with purely periodic  $T$ -expansion. Now we look for all purely periodic expansions. Here,  $T$  satisfies the assumptions of Section 3.1, i.e.,  $\beta$  is a Pisot unit and  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$ . Denote by  $P$  the set of points with purely periodic  $T$ -expansion, i.e.,  $x \in P$  if and only if there exists some  $n \geq 1$  such that  $T^n x = x$ . Clearly,  $P \subset \mathbb{Q}(\beta)$ . We will consider also the sets  $P_q = P \cap q^{-1}\mathbb{Z}[\beta]$ ,  $q \in \mathbb{Z} \setminus \{0\}$ .

**Lemma 4.14.** *The origin  $\mathbf{0}$  belongs to  $\mathcal{T}_x$ ,  $x \in \mathbb{Q}(\beta) \cap X$ , if and only if  $x \in P$ .*

*Proof.* Suppose that  $x \in P$ , i.e.,  $b(x) = (b_1 \cdots b_n)^\omega$  for some  $n \geq 1$ . Then we clearly have  ${}^\omega(b_1 \cdots b_n) \cdot (b_1 \cdots b_n)^\omega \in \mathcal{S}$ . Furthermore,

$$\begin{aligned} \varphi({}^\omega(b_1 \cdots b_n)) &= \sum_{j=2}^d (\Gamma_j(\delta_{b_1})\beta_j^{n-1} + \cdots + \Gamma_j(\delta_{b_n})) (1 + \beta_j^n + \beta_j^{2n} + \cdots) \mathbf{v}_j \\ &= \sum_{j=2}^d \Gamma_j \left( \frac{\delta_{b_1}\beta_j^{n-1} + \cdots + \delta_{b_n}}{1 - \beta_j^n} \right) \mathbf{v}_j = - \sum_{j=2}^d \Gamma_j(x) \mathbf{v}_j = -\Phi(x). \end{aligned}$$

Hence,  $\mathbf{0} = \Phi(x) + \varphi({}^\omega(b_1 \cdots b_n))$  and thus  $\mathbf{0} \in \mathcal{T}_x$ .

Now, take an  $x \in \mathbb{Q}(\beta) \cap X$  with  $\mathbf{0} \in \mathcal{T}_x$ , i.e.,  $\mathbf{0} = \Phi(x) + \varphi(w)$  for some  $w \in {}^\omega A$ ,  $w \cdot b(x) \in \mathcal{S}$ . Let  $q \in \mathbb{Z}$  be such that  $x$  and  $\delta_a$  are in  $q^{-1}\mathbb{Z}[\beta]$  for all  $a \in A$ . For  $k \geq 0$ , set  $x_k = \cdot w_{-k+1} \cdots w_0 b(x)$  and  $w^{(k)} = \cdots w_{-k-1} w_{-k}$ . Then we have

$$\Phi(x_k) + \varphi(w^{(k)}) = M_\beta^{-k} (\Phi(x) + \varphi(w)) = \mathbf{0},$$

thus  $\mathbf{0} \in \mathcal{T}_{x_k}$  for all  $k \geq 0$ . Since  $x_k \in q^{-1}\mathbb{Z}[\beta] \cap X$  and the size of  $\mathcal{D}_{x_k}$  is bounded, we obtain by Lemma 4.3 that  $\{\Phi(x_k) : k \geq 0\}$  is finite. Since  $\Phi$  is injective,  $\{x_k : k \geq 0\}$  is finite as well, hence  $x_m = x_n$  for some  $m > n \geq 0$ . Since  $T^{m-n}x_m = x_n$  and  $T^m x_m = x$ , we obtain  $x \in P$ .  $\square$

Since  $\Phi(q^{-1}\mathbb{Z}[\beta] \cap X)$  is uniformly discrete, Lemma 4.14 implies that  $P_q$  is finite for each  $q \geq 1$ . The next theorem gives a simple characterization of all purely periodic points, cf. [20].

**Theorem 4.15.** *Let  $T$  be a right-continuous  $\beta$ -transformation as in Definition 2.4 with a Pisot unit  $\beta$  and  $\{\delta_a : a \in A\} \subset \mathbb{Q}(\beta)$ . Let  $\widehat{X}$  be the natural extension domain defined in Section 3.1,  $\Psi$  as in (6). Then the  $T$ -expansion of  $x \in X$  is purely periodic if and only if  $x \in \mathbb{Q}(\beta)$  and  $\Psi(x) \in \widehat{X}$ .*

*Proof.* If  $x \in P$ , then we clearly have  $x \in \mathbb{Q}(\beta) \cap X$ . By Lemma 4.14,  $x \in P$  is equivalent to  $\mathbf{0} \in \mathcal{T}_x$ , but this means that  $-\Phi(x) \in \mathcal{D}_x$ , i.e.,  $\Psi(x) = x\mathbf{v}_1 + \Phi(x) \in \widehat{X}$  by (8).  $\square$

Note that this theorem gives a nice characterization of rational numbers with purely periodic  $T$ -expansions, since we have  $\Gamma_j(x) = x$  for  $x \in \mathbb{Q}$  and thus  $\Psi(x) = x\mathbf{e}_1 = (x, 0, \dots, 0)^t$ .

**4.3. To find an  $m$ -exclusive point.** To determine the covering degree of the multiple tiling, we need to find an  $m$ -exclusive point. Recall that  $P_1$  is the set of purely periodic points in  $\mathbb{Z}[\beta]$ . Note that  $\mathbf{0}$  is an exclusive point if and only if  $P_1$  consists only of one element. This generalizes the (F) property, cf. [1]. If  $\mathbf{0}$  is contained in more than one tile, it is more difficult to determine the covering degree  $m$ . Corollary 4.18 provides an easy way to determine the number of tiles to which a point belongs. We restrict to points  $\Phi(z)$ ,  $z \in \mathbb{Z}[\beta] \cap [0, \infty)$ , because of the following lemma.

**Lemma 4.16** (Akiyama, [1]). *The set  $\Phi(\mathbb{Z}[\beta] \cap [0, \infty))$  is dense in  $H$ .*

*Proof.* By Lemma 4.3,  $\Phi(\mathbb{Z}[\beta] \cap [0, 1))$  is relatively dense in  $H$ . We have

$$\Phi(\mathbb{Z}[\beta] \cap [0, \infty)) = \bigcup_{n \geq 0} \Phi(\mathbb{Z}[\beta] \cap [0, \beta^n)) = \bigcup_{n \geq 0} M_\beta^n \Phi(\mathbb{Z}[\beta] \cap [0, 1)).$$

Since  $M_\beta$  is contracting, we obtain that  $\Phi(\mathbb{Z}[\beta] \cap [0, \infty))$  is dense in  $H$ .  $\square$

**Proposition 4.17.** *A point  $\Phi(z)$ ,  $z \in \mathbb{Z}[\beta] \cap [0, \infty)$ , lies in the tile  $\mathcal{T}_x$ ,  $x \in \mathbb{Z}[\beta] \cap X$ , if and only if there exists some  $p \in P_1$  and some  $\kappa \geq 0$  such that*

$$(15) \quad T^k(p_k + \beta^{-k}z) = x \quad \text{for all } k \geq \kappa,$$

where  $p_k \in P_1$  is defined by  $T^k p_k = p$ .

*Proof.* Let  $\Phi(z) \in \mathcal{T}_x$ , which means that  $\Phi(z) = \Phi(x) + \varphi(w)$  for some  $w \in {}^\omega A$  with  $w \cdot b(x) \in \mathcal{S}$ , and set  $x_k = .w_{-k+1} \cdots w_0 b(x)$ ,  $w^{(k)} = \cdots w_{-k-1} w_{-k}$ , for  $k \geq 0$ . Then

$$(16) \quad \Phi(x_k - \beta^{-k}z) = \Phi(x_k) - M_\beta^{-k}(\Phi(x) + \varphi(w)) = M_\beta^{-k}(\varphi(w_{-k+1} \cdots w_0) - \varphi(w)) = -\varphi(w^{(k)}),$$

therefore  $\Phi(x_k - \beta^{-k}z)$  is bounded, hence the set  $\{x_k - \beta^{-k}z : k \geq 0\}$  is finite by Lemma 4.3 and the injectivity of  $\Phi$ . This means that there exists some  $y$  such that  $x_k - \beta^{-k}z = y$  for infinitely many  $k \geq 0$ . Since  $x_k \in X$  and the intervals  $X_a$ ,  $a \in A$ , are left closed, we obtain  $y \in X$ .

Set  $K = \{k \geq 0 : x_k - \beta^{-k}z = y\}$ . Since the intervals  $X_{a,n}$  are right open, there is a  $\kappa_1$  such that  $\mathcal{D}_y = \mathcal{D}_{y+\beta^{-k}z}$  for all  $k \geq \kappa_1$ . Then  $\mathcal{D}_{x_k - \beta^{-k}z} = \mathcal{D}_{x_k}$  for all  $k \in K$ ,  $k \geq \kappa_1$ . Since  $\varphi(w^{(k)}) \in \mathcal{D}_{x_k}$ , we obtain  $\mathbf{0} \in \mathcal{T}_{x_k - \beta^{-k}z}$  by (16). By Lemma 4.14, there exists some  $p_k \in P_1$  such that  $x_k - \beta^{-k}z = p_k$ , which implies  $T^k(p_k + \beta^{-k}z) = T^k x_k = x$ .

By the finiteness of  $P_1$ , there is a  $\kappa_2$  such that, for each  $k > \kappa_2$ ,  $p' \in P_1$ ,

$$(17) \quad T(p' + \beta^{-k}z) = Tp' + \beta^{-k+1}z.$$

Let  $\kappa$  be the smallest  $k \in K$  with  $k \geq \max\{\kappa_1, \kappa_2\}$ . Let  $p = T^\kappa p_\kappa$  and, for each  $k \geq 1$ , let  $p_k$  be the unique element from  $P_1$  such that  $T^k p_k = p$ . Then, by (17), we have, for each  $k \geq \kappa$ ,

$$T^k(p_k + \beta^{-k}z) = T^\kappa(p_\kappa + \beta^{-\kappa}z) = T^\kappa x_\kappa = x,$$

which gives (15).

For the other direction, assume that (15) holds. Then we have sequences  $w^{(k)} \in {}^\omega A$ ,  $k \geq \kappa$ , such that  $w^{(k)} \cdot b(p_k + \beta^{-k}z) \in \mathcal{S}$ . Set

$$\mathbf{y}_k = M_\beta^k(\Phi(p_k + \beta^{-k}z) + \varphi(w^{(k)})) = \Phi(z) + M_\beta^k(\Phi(p_k) + \varphi(w^{(k)})).$$

Then we have  $\mathbf{y}_k \in \mathcal{T}_{T^k(p_k + \beta^{-k}z)} = \mathcal{T}_x$  and  $\lim_{k \rightarrow \infty} \mathbf{y}_k = \Phi(z)$ , thus  $\Phi(z) \in \mathcal{T}_x$ .  $\square$

**Corollary 4.18.** *Let  $z \in \mathbb{Z}[\beta] \cap [0, \infty)$  and  $\kappa \geq 0$  such that  $p + \beta^{-\kappa}z \in X$ ,  $b_1(p + \beta^{-\kappa-1}z) = b_1(p)$  for all  $p \in P_1$ . Then  $\Phi(z)$  lies exactly in the tiles  $\mathcal{T}_{T^\kappa(p + \beta^{-\kappa}z)}$ ,  $p \in P_1$ .*

*Proof.* Note that  $b_1(p + \beta^{-\kappa-1}z) = b_1(p)$  implies  $b_1(p + \beta^{-k}z) = b_1(p)$  for all  $k > \kappa$ , and thus  $T^{k-\kappa}(p + \beta^{-k}z) = T^{k-\kappa}p + \beta^{-\kappa}z$  for all  $p \in P_1$ . If we define  $p_k \in P_1$  by  $T^k p_k = p'$  for some fixed  $p' \in P_1$ , then we have therefore  $T^k(p_k + \beta^{-k}z) = T^\kappa(p_\kappa + \beta^{-\kappa}z)$  for all  $k \geq \kappa$ . Hence, by Proposition 4.17 we have  $\Phi(z) \in \mathcal{T}_{T^\kappa(p_\kappa + \beta^{-\kappa}z)}$ . For every  $p \in P_1$ , we have  $T^\kappa p = p'$  for some  $p' \in P_1$ , and thus  $\Phi(z) \in \mathcal{T}_{T^\kappa(p + \beta^{-\kappa}z)}$ . By Proposition 4.17,  $\Phi(z)$  cannot lie in other tiles.  $\square$

The main difficulty is to find some  $z \in \mathbb{Z}[\beta] \cap [0, \infty)$  such that the number of tiles to which  $\Phi(z)$  belongs is equal to the covering degree, i.e., such that  $\Phi(z)$  does not lie on the boundary of two tiles. By Corollary 4.18, it is relatively easy to find a point lying in exactly  $m$  tiles (if such a point exists). It is usually harder to show that  $m$  is indeed the covering degree of the multiple tiling, i.e., that there exists an open set lying in exactly  $m$  tiles. In case  $m = 1$ , however, the two problems are equivalent. The proof of the following proposition shows how to construct an exclusive point from points with weaker properties. This generalizes the (W) property and Proposition 2 in [3], where  $p_0 = 0$ , since  $T_\beta^\kappa(p + z) = 0$  means that the  $\beta$ -expansion of  $p + z$  is finite.

**Proposition 4.19.** *Let  $\varepsilon = \min_{p \in P_1} (r_{b_1(p)} - p)\beta$ . Then  $\mathcal{T}_{p_0}$ ,  $p_0 \in P_1$ , contains an exclusive point if and only if, for every  $p \in P_1$ , there exists some  $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and some  $\kappa \geq 0$  such that*

$$(18) \quad T^\kappa(p + z) = T^\kappa(p_0 + z) = p_0.$$

*Proof.* If  $\mathcal{T}_{p_0}$  contains an exclusive point, then it contains an exclusive point  $\Phi(z')$ ,  $z' \in \mathbb{Z}[\beta] \cap [0, \infty)$  by Lemma 4.16. For  $z'$ , there is a  $\kappa > 0$  such that the conditions of Corollary 4.18 are satisfied. Then, by Corollary 4.18, we have that  $T^\kappa(p + \beta^{-\kappa}z') = p_0$  for all  $p \in P$ , and  $b_1(p + \beta^{-\kappa-1}z') = b_1(p)$ . This implies  $\beta^{-\kappa-1}z' < r_{b_1(p)} - p$ , hence we can choose  $z = \beta^{-\kappa}z'$ .

For the converse, let  $P_1 = \{p_0, p_1, \dots, p_h\}$  and assume that, for every  $p \in P_1$ , there exists some  $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and some  $\kappa \geq 0$  such that (18) holds. It suffices to consider the case  $h \geq 2$ , since  $\mathbf{0}$  is an exclusive point of  $\mathcal{T}_{p_0}$  in case  $h = 0$  and, in case  $h = 1$ , Corollary 4.18 shows that  $\beta^\kappa z$  is an exclusive point of  $\mathcal{T}_{p_0}$  if  $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and  $\kappa$  are such that  $T^\kappa(p_1 + z) = T^\kappa(p_0 + z) = p_0$ . If

$h \geq 2$ , then we will recursively construct a point  $y_h$  that satisfies the conditions of Corollary 4.18 for some  $\kappa$  and for which  $T^\kappa(p + \beta^{-\kappa}y_h) = p_0$  for all  $p \in P_1$ .

First note that, if (18) holds for some  $\kappa, z$ , then it also holds for  $z' = \beta^{-n}z$  and  $\kappa' = \kappa + n$  if  $n$  is a multiple of the period lengths of  $p$  and  $p_0$ , since  $T^n(p + \beta^{-n}z) = T^n p + z = p + z$  and  $T^n(p_0 + \beta^{-n}z) = p_0 + z$ . Therefore, we can find arbitrarily small  $z$  such that (18) holds. For every such  $z$ , we can find arbitrarily large  $\kappa$  such that (18) holds since  $b(p_0)$  is purely periodic.

Choose, as a first step,  $z_1 \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and  $\kappa_1 \geq 1$  such that

- $T^{\kappa_1}(p_1 + z_1) = T^{\kappa_1}(p_0 + z_1) = p_0$ , which can be done by (18),
- $b_1(p + \beta^{-1}z_1) = b_1(p)$  for all  $p \in P_1$ , which can be done by choosing  $z_1$  sufficiently small,
- $T^{\kappa_1}(p_2 + z_1) \in P_1$ , which can be done by choosing  $\kappa_1$  sufficiently large since the  $T$ -expansion of  $p_2 + z_1$  is eventually periodic by Theorem 4.13.

Now, suppose that, for  $1 \leq n < h$ , we have  $\kappa_n \geq 0$  and  $z_n \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  that satisfy all of the following, where  $y_0 = 0$ ,  $y_n = \beta^{\kappa_n}(y_{n-1} + z_n)$  and  $s_n = \kappa_1 + \dots + \kappa_n$ .

- (i)  $T^{\kappa_n}(T^{s_{n-1}}(p_n + \beta^{-s_{n-1}}y_{n-1}) + z_n) = T^{\kappa_n}(p_0 + z_n) = p_0$ ,
- (ii)  $b_1(p + \beta^{-s_n}y_n) = b_1(p)$  for all  $p \in P_1$ ,
- (iii)  $T^{s_{n-1}}(p + \beta^{-s_n}y_n) = T^{s_{n-1}}(p + \beta^{-s_{n-1}}y_{n-1}) + z_n$  for all  $p \in P_1$ ,
- (iv)  $T^{s_n}(p_{n+1} + \beta^{-s_n}y_n) \in P_1$ .

Then, for  $n + 1$ , by (iv) and (18), we have arbitrarily small  $z_{n+1} \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and arbitrarily large  $\kappa_{n+1} \geq 0$  such that

$$T^{\kappa_{n+1}}(T^{s_n}(p_{n+1} + \beta^{-s_n}y_n) + z_{n+1}) = T^{\kappa_{n+1}}(p_0 + z_{n+1}) = p_0.$$

If we set  $y_{n+1} = \beta^{\kappa_{n+1}}(y_n + z_{n+1})$  and  $s_{n+1} = s_n + \kappa_{n+1}$ , then by choosing  $z_{n+1}$  small enough, and by (ii), we can get

$$b_1(p + \beta^{-s_{n+1}}y_{n+1}) = b_1(p + \beta^{-s_n}y_n + \beta^{-s_n}z_{n+1}) = b_1(p)$$

for all  $p \in P_1$ . By choosing  $z_{n+1}$  small enough, we also have

$$T^{s_n}(p + \beta^{-s_{n+1}}y_{n+1}) = T^{s_n}(p + \beta^{-s_n}y_n + \beta^{-s_n}z_{n+1}) = T^{s_n}(p + \beta^{-s_n}y_n) + z_{n+1}$$

for all  $p \in P_1$ . If  $n + 1 < h$ , then, by Theorem 4.13, we can choose  $\kappa_{n+1}$  large enough such that

$$T^{s_{n+1}}(p_{n+2} + \beta^{-s_{n+1}}y_{n+1}) = T^{\kappa_{n+1}}T^{s_n}(p_{n+2} + \beta^{-s_n}(y_n + z_{n+1})) \in P_1.$$

For  $n = 1$ , we have already chosen  $z_1 \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and  $\kappa_1 \geq 1$  with the properties (i)–(iv). Inductively, we obtain therefore some  $y_h \in \mathbb{Z}[\beta] \cap [0, \infty)$  and  $s_h \geq 1$  satisfying (i)–(iii). Note that, by (ii) and (iii),  $z = y_h$  satisfies all the conditions of Corollary 4.18 with  $\kappa = s_h$ . Furthermore, by (iii) and (i), we have, for  $1 \leq n \leq h$ ,

$$\begin{aligned} T^{s_h}(p_n + \beta^{-s_h}y_h) &= T^{\kappa_h}T^{s_{h-1}}(p_n + \beta^{-s_h}y_h) = T^{\kappa_h}(T^{s_{h-1}}(p_n + \beta^{-s_{h-1}}y_{h-1}) + z_h) \\ &= T^{\kappa_h}(\dots(T^{\kappa_{n+1}}(T^{\kappa_n}(T^{s_{n-1}}(p_n + \beta^{-s_{n-1}}y_{n-1}) + z_n) + z_{n+1}) \dots) + z_h) \\ &= T^{\kappa_h}(\dots(T^{\kappa_{n+1}}(p_0 + z_{n+1}) \dots) + z_h) = p_0. \end{aligned}$$

Similarly, we obtain

$$T^{s_h}(p_0 + \beta^{-s_h}y_h) = T^{\kappa_h}(\dots(T^{\kappa_2}(T^{\kappa_1}(p_0 + z_1) + z_2) \dots) + z_h) = p_0.$$

Hence, by Corollary 4.18,  $y_h$  is an exclusive point of  $\mathcal{T}_{p_0}$ .  $\square$

This leads us to define the following generalizations of the (F) and the (W) property, where  $P_1$  is the set of points in  $\mathbb{Z}[\beta] \cap X$  with purely periodic  $T$ -expansion and  $\varepsilon = \min_{p \in P_1} (r_{b_1(p)} - p)\beta$ .

(F) :  $P_1$  consists only of one element.

(W) :  $\exists p_0 \in P_1 : \forall p \in P_1 \exists z \in \mathbb{Z}[\beta] \cap [0, \varepsilon), \kappa \geq 0 : T^\kappa(p + z) = T^\kappa(p_0 + z) = p_0$ .

Clearly, (F) implies (W). We have the following corollary of Proposition 4.19.

**Corollary 4.20.** *Let  $T$  satisfy the assumptions of Theorem 4.10. Then the family  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  forms a tiling of  $H$  if and only (W) holds.*

*Remark 4.21.* The condition (W) given here is similar to the property (W') defined in [3] for the greedy  $\beta$ -transformation. Note that it is required in (W') that there exists  $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$  and  $\kappa \geq 0$  satisfying (18) for every  $\varepsilon > 0$ . As Corollary 4.20 shows, this requirement is not necessary, and  $\varepsilon = \min_{p \in P_1} (r_{b_1(p)} - p)\beta$  suffices.

*Remark 4.22.* It is not necessary that  $\bar{S}$  is a sofic shift to obtain a tiling of  $H$ . E.g., we can modify Example 3.11 by setting  $X_{\bar{1}} = [-\beta\alpha, -\alpha)$ ,  $X_0 = [-\alpha, \alpha)$ ,  $X_1 = [\alpha, \beta\alpha)$  for any  $\alpha \in (\frac{\beta}{\beta^2+1}, \frac{1}{2}]$ . Then we still have  $\tilde{b}_1(r_0) \cdots \tilde{b}_5(r_0) = 01001$ ,  $b_1(r_0) \cdots b_5(r_0) = 100\bar{1}0$ , and thus  $\tilde{T}^5 r_0 = T^5 r_0$ . By symmetry, we also have  $\tilde{T}^5 r_{\bar{1}} = T^5 r_{\bar{1}}$ , hence  $\mathcal{V}$  is finite. It was shown in [18] that these transformations give  $\beta$ -expansions of minimal weight, which means in particular that, for every  $x \in \mathbb{Z}[\beta] \cap X$ , there exists some  $n \geq 0$  such that  $T^n x = 0$ , hence  $P_1 = \{0\}$  and (F) holds. By Theorem 4.13,  $b(\ell_1) = b(\alpha)$  is aperiodic if  $\alpha \notin \mathbb{Q}(\beta)$ , which implies that the shift  $\bar{S}$  is not sofic.

If  $\bar{S}$  is not sofic, then  $\mathcal{T}$  is (usually) not  $M_\beta^{-1}$ -subdividing and therefore not a self-affine tiling in the sense of [25, 31], because different tiles  $\mathcal{T}_x$  of the same shape  $\mathcal{D}_x$  can subdivide in different ways according to (13). More precisely, iterating (13) gives rise to the same subdivisions for  $\mathcal{T}_x$  and  $\mathcal{T}_y$  only if no  $T^k(\ell_a)$  or  $\tilde{T}^k(r_a)$ ,  $a \in A$ ,  $k \geq 1$ , lies between  $x$  and  $y$ , and this gives a finite partition of  $X$  if and only if  $\bar{S}$  is sofic.

**4.4. Tiling of the torus.** Similarly to Ito and Rao ([20]), we can relate the tiling property of  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  to the tiling property of  $\{\mathbf{x} + \hat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$ , where  $\hat{Y}$  is the closure of the natural extension domain defined in Section 3.1. We show first the result announced at the end of Section 3.1.

**Lemma 4.23.** *If  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ , then  $\bigcup_{\mathbf{x} \in \mathbb{Z}^d} (\mathbf{x} + \hat{Y}) = \mathbb{R}^d$  and thus  $\lambda^d(\hat{X}) = \lambda^d(\hat{Y}) > 0$ .*

*Proof.* Set  $J = \bigcup_{\mathbf{x} \in \mathbb{Z}^d} (\mathbf{x} + \hat{Y})$ . The set  $\hat{Y}$  contains the sets  $x\mathbf{v}_1 - \mathcal{D}_x = \Psi(x) - \mathcal{T}_x$ ,  $x \in \mathbb{Z}[\beta] \cap X$ . Therefore,  $J$  contains the sets  $\Psi(y) - \mathcal{T}_x$ ,  $y \in \mathbb{Z}[\beta]$ ,  $x \in \mathbb{Z}[\beta] \cap X$ . By Lemma 4.5, we have  $\bigcup_{x \in \mathbb{Z}[\beta] \cap X} (\Psi(y) - \mathcal{T}_x) = y\mathbf{v}_1 + H$ . Since  $\mathbb{Z}[\beta]$  is dense in  $\mathbb{R}$ , it follows that  $J$  is dense in  $\mathbb{R}^d$ . By Lemma 3.1,  $\hat{Y}$  is compact and thus  $J = \mathbb{R}^d$ .

By Baire's theorem,  $\hat{Y}$  cannot be nowhere dense, thus  $\lambda^d(\hat{Y}) > 0$ .  $\square$

**Proposition 4.24.** *Let  $T$  satisfy the assumptions of Theorem 4.10. Then the family  $\{\mathbf{x} + \hat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$  forms a tiling of  $\mathbb{R}^d$  if and only if the family  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  forms a tiling of  $H$ .*

*Proof.* Assume first that  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  is not a tiling of  $H$ . Then  $\lambda^{d-1}(\mathcal{T}_x \cap \mathcal{T}_{x'}) > 0$  for some  $x, x' \in \mathbb{Z}[\beta] \cap X$ ,  $x' \neq x$ . Let  $x \in X_{a,n}$  and  $x' \in X_{a',n'}$ . Then

$$[0, r_{a,n} - x)\mathbf{v}_1 - \mathcal{T}_x = -\Psi(x) + [x, r_{a,n})\mathbf{v}_1 - \mathcal{D}_x \subseteq -\Psi(x) + \hat{Y}$$

and  $[0, r_{a',n'} - x')\mathbf{v}_1 - \mathcal{T}_{x'} \subseteq -\Psi(x') + \hat{Y}$ . From  $\lambda^{d-1}(\mathcal{T}_x \cap \mathcal{T}_{x'}) > 0$ , we see that

$$\lambda^d([0, r_{a,n} - x)\mathbf{v}_1 - \mathcal{T}_x) \cap ([0, r_{a',n'} - x')\mathbf{v}_1 - \mathcal{T}_{x'}) > 0$$

and thus  $\lambda^d((- \Psi(x) + \hat{Y}) \cap (- \Psi(x') + \hat{Y})) > 0$ , hence  $\{\mathbf{x} + \hat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$  is not a tiling.

Assume now that  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  forms a tiling of  $H$ . Consider a hyperplane  $y\mathbf{v}_1 + H$ ,  $y \in \mathbb{Z}[\beta]$ . We have  $(\mathbf{x} + \hat{X}) \cap (y\mathbf{v}_1 + H) \neq \emptyset$  for  $\mathbf{x} \in \mathbb{Z}^d$ ,  $\mathbf{x} = -x\mathbf{v}_1 - \Phi(x)$ , if and only if  $x + y \in X$ . Then

$$(\mathbf{x} + \hat{X}) \cap (y\mathbf{v}_1 + H) = y\mathbf{v}_1 - \Phi(x) - \mathcal{D}_{x+y} = y\mathbf{v}_1 + \Phi(y) - \mathcal{T}_{x+y} = \Psi(y) - \mathcal{T}_{x+y}.$$

Since  $\{\Psi(y) - \mathcal{T}_{x+y}\}_{x \in \mathbb{Z}[\beta] \cap (X-y)}$  is a tiling of  $y\mathbf{v}_1 + H$ , we have  $\lambda^{d-1}((\mathbf{x} + \hat{X}) \cap (\mathbf{x}' + \hat{X}) \cap (y\mathbf{v}_1 + H)) = 0$  for all  $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^d$ ,  $\mathbf{x} \neq \mathbf{x}'$ ,  $y \in \mathbb{Z}[\beta]$ . Since  $\mathbb{Z}[\beta]$  is dense in  $\mathbb{R}$  and  $\hat{X}$  has the shape given in (11), we obtain that  $\lambda^d((\mathbf{x} + \hat{X}) \cap (\mathbf{x}' + \hat{X})) = 0$ , which implies  $\lambda^d((\mathbf{x} + \hat{Y}) \cap (\mathbf{x}' + \hat{Y})) = 0$ . All other tiling conditions for  $\{\mathbf{x} + \hat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$  follow from the compactness of  $\hat{Y}$ , from Lemmas 4.23 and 4.7.  $\square$

*Remark 4.25.* If  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ , then  $\hat{T}$  is a toral automorphism since  $\hat{T}\mathbf{x} \equiv M_\beta \mathbf{x} \pmod{\mathbb{Z}^d}$ . If furthermore  $\bar{S}$  is a shift of finite type and  $\{\mathbf{x} + \hat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$  forms a tiling of  $\mathbb{R}^d$ , then  $\{\hat{X}_a\}_{a \in A}$  is a Markov partition of the torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ .

The next lemma shows that, when  $\{\mathbf{x} + \widehat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$  forms a tiling of  $\mathbb{R}^d$ , the partition  $\{\widehat{X}_a\}_{a \in A}$  can be used to determine the  $k$ -th digit  $b_k(x)$  in the  $T$ -expansion of  $x$  up to an error term of order  $\rho^k$ , with  $\rho = \max_{2 \leq j \leq d} |\beta_j| < 1$ . This can be used e.g. to obtain distribution properties of the  $k$ -th digit on polynomial sequences, see [32].

**Lemma 4.26.** *Assume that  $TX = X$  and  $\{\delta_a : a \in A\} \subset \mathbb{Z}[\beta]$ . Then we have*

$$\beta^{k-1}x\mathbf{v}_1 \in \widehat{X}_{b_k(x)} + M_\beta^{k-1}\mathcal{D}_x \pmod{\mathbb{Z}^d}$$

for every  $x \in X$ ,  $k \geq 1$ . If  $\mathbf{0} \in \mathcal{D}_x$ , then we have  $\beta^{k-1}x\mathbf{v}_1 \in \widehat{X}_{b_k(x)} \pmod{\mathbb{Z}^d}$ .

*Proof.* By definition, we have  $T^{k-1}x \in X_{b_k(x)}$ . Since  $TX = X$ , there exists some  $w \in {}^\omega A$  such that  $w \cdot b(x) \in \mathcal{S}$ , and thus  $\varphi(w) \in \mathcal{D}_x$ . Since  $\psi(\sigma u) = \widehat{T}\psi(u) \equiv M_\beta\psi(u) \pmod{\mathbb{Z}^d}$ , we obtain

$$\beta^{k-1}x\mathbf{v}_1 - M_\beta^{k-1}\varphi(w) \equiv M_\beta^{k-1}\psi(w \cdot b(x)) \equiv \psi(wb_1(x) \cdots b_{k-1}(x) \cdot b(T^{k-1}x)) \in \widehat{X}_{b_k(x)}$$

modulo  $\mathbb{Z}^d$ . If  $\mathbf{0} \in \mathcal{D}_x$ , then we can choose  $w$  such that  $\varphi(w) = \mathbf{0}$ .  $\square$

In certain cases, the partition  $\{\widehat{X}_a\}_{a \in A}$  determines exactly every digit  $b_k(x)$ .

**Corollary 4.27.** *Let  $T$  satisfy the assumptions of Theorem 4.10. Assume that (W) holds and that  $\mathbf{0}$  is an inner point of  $\mathcal{D}_x$  for every  $x \in X$ . Then we have, for every  $x \in X$ ,  $k \geq 1$ ,*

$$b_k(x) = a \quad \text{if and only if} \quad \beta^{k-1}x\mathbf{v}_1 \in \widehat{X}_a \pmod{\mathbb{Z}^d}.$$

*Proof.* By Lemma 4.26,  $b_k(x) = a$  implies  $\beta^{k-1}x\mathbf{v}_1 \in \widehat{X}_a \pmod{\mathbb{Z}^d}$ . Assume that  $\beta^{k-1}x\mathbf{v}_1 \in \widehat{X}_a \pmod{\mathbb{Z}^d}$  for some  $a \neq b_k(x)$ , i.e., that  $\beta^{k-1}x\mathbf{v}_1 \in \widehat{X}_a \cap \widehat{X}_{b_k(x)} \pmod{\mathbb{Z}^d}$ . From the proof of Lemma 4.26, we see that  $\beta^{k-1}x\mathbf{v}_1 \equiv (T^{k-1}x)\mathbf{v}_1 - \varphi(b_1(x) \cdots b_{k-1}(x)) \pmod{\mathbb{Z}^d}$ . Furthermore,  $\varphi(b_1(x) \cdots b_{k-1}(x))$  is an inner point of  $\mathcal{D}_{T^{k-1}x}$  since  $\mathcal{D}_{T^{k-1}x}$  contains  $\varphi(b_1(x) \cdots b_{k-1}(x)) + M_\beta^{k-1}\mathcal{D}_x$  and  $\mathbf{0}$  is an inner point of  $\mathcal{D}_x$ . Since every interval  $X_{a,n}$  is right open and by Lemma 4.7 we obtain that the intersection  $\widehat{X}_a \cap \widehat{X}_{b_k(x)} \pmod{\mathbb{Z}^d}$  has positive Lebesgue measure. Since  $\widehat{X}_a \cap \widehat{X}_{b_k(x)} = \emptyset$  in  $\mathbb{R}^d$ , this implies that  $\{\mathbf{x} + \widehat{Y}\}_{\mathbf{x} \in \mathbb{Z}^d}$  is not a tiling. By Proposition 4.24 and Corollary 4.20, this contradicts the (W) property.  $\square$

**4.5. Examples of (multiple) tilings.** Consider now the multiple tilings defined by the transformations in the examples of Section 3.3. In Figure 8, we see the natural extension domain  $\widehat{X}$  and its translation by the vectors  $(1, 0)^t$ ,  $(0, 1)^t$  and  $(1, 1)^t$  for Examples 3.10 and 3.11. The third picture in Figure 8 shows  $\widehat{X}$  and its translations by the vectors  $(2, 0)^t$ ,  $(0, 2)^t$  and  $(2, 2)^t$  for Example 3.13. It follows that  $\mathcal{T}$  is a tiling for Examples 3.10 and 3.11, while the covering degree is 4 for Example 3.13.

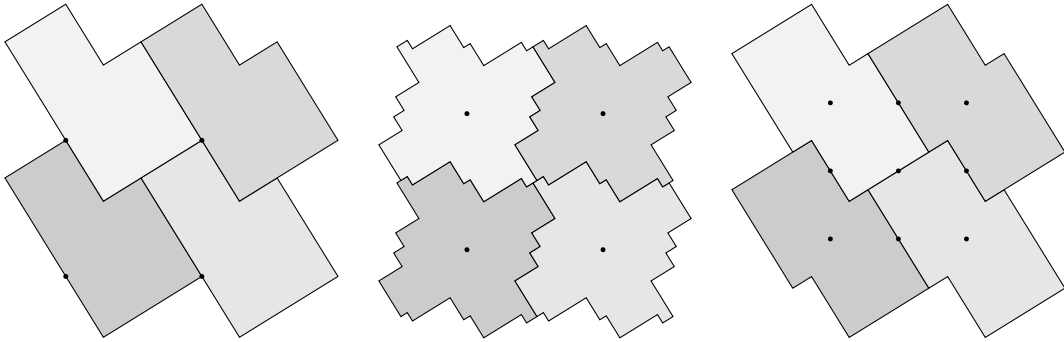


FIGURE 8. Translations of the natural extension domains of Examples 3.10 and 3.11 by integer vectors, and of Example 3.13 by vectors in  $2\mathbb{Z}^2$ .

To give a formal proof of these assertions, let  $\beta = (1 + \sqrt{5})/2$  and  $\{\delta_a : a \in A\} \subseteq \{-1, 0, 1\}$ . Then we have  $\varphi(w) \in [\frac{-1}{1-1/\beta}, \frac{1}{1-1/\beta}] \mathbf{v}_2$  since  $|\beta_2| = 1/\beta$ , thus  $|\Gamma_2(p)| \leq \frac{1}{1-1/\beta} = \beta^2$  for all

$p \in P$ . The only  $x \in \mathbb{Z}[\beta]$  with  $|x| < 1$ ,  $|\Gamma_2(x)| \leq \beta^2$ , are  $x = 0$ ,  $x = \pm(\beta - 1) = \pm 1/\beta$ ,  $x = \pm(\beta - 2) = \pm 1/\beta^2$ . We immediately obtain that  $P_1 = \{0\}$  in Examples 3.10 and 3.11, thus (F) holds. In Example 3.13, we have  $P_1 = \{-1, -1/\beta, -1/\beta^2, 0, 1/\beta^2\}$ . Using Corollary 4.18, it is easy to find a point lying in exactly 4 tiles, e.g.  $\Phi(1)$ . Hence the covering degree is at most 4. It is slightly harder to show that some set with non-empty interior is covered 4 times. Remember that  $\mathcal{V} = \{-1, -1/\beta, 1/\beta, 1\}$ . By (12), we have the decompositions

$$\mathcal{D}_{-1} = -\frac{\mathcal{D}_0}{\beta} + \mathbf{v}_2, \quad \mathcal{D}_{-1/\beta} = \left(-\frac{\mathcal{D}_{-1}}{\beta} - \mathbf{v}_2\right) + \left(-\frac{\mathcal{D}_{1/\beta^3}}{\beta} + \mathbf{v}_2\right), \quad \mathcal{D}_{1/\beta} = -\frac{\mathcal{D}_{-1/\beta^3}}{\beta} - \mathbf{v}_2.$$

Since  $\mathcal{D}_0 = \mathcal{D}_{-1/\beta^3} = \mathcal{D}_{1/\beta^3} = \mathcal{D}_{-1/\beta}$ , these decompositions provide a graph-directed IFS, which has a unique solution with non-empty compact sets (see e.g. [12]). This solution is

$$\mathcal{D}_{-1} = [-1/\beta, \beta^2]\mathbf{v}_2, \quad \mathcal{D}_{-1/\beta} = [-\beta^2, \beta^2]\mathbf{v}_2, \quad \mathcal{D}_{1/\beta} = [-\beta^2, 1/\beta]\mathbf{v}_2.$$

Therefore,  $[0, \beta]\mathbf{v}_2$  is covered 4 times by  $\mathcal{T}_0 = [-\beta^2, \beta^2]\mathbf{v}_2$ ,  $\mathcal{T}_{-1} = [-\beta, \beta]\mathbf{v}_2$ ,  $\mathcal{T}_{-1/\beta} = [-1, \beta^3]\mathbf{v}_2$  and  $\mathcal{T}_{1/\beta^2} = [0, 2\beta^2]\mathbf{v}_2$ . Note that an important difference between this transformation and all other examples we are considering is that we do not have  $\beta^{-1}x \in T^{-1}\{x\}$  here.

**4.6. Symmetric  $\beta$ -transformations.** The symmetric  $\beta$ -transformation is defined in [5] by  $Tx = \beta x - \lfloor \beta x + 1/2 \rfloor$  on  $X = [-1/2, 1/2)$ . If  $\beta \leq 3$ , then this means in our setting that  $A = \{\bar{1}, 0, 1\}$ ,  $X_{\bar{1}} = [-\frac{1}{2}, -\frac{1}{2\beta})$ ,  $X_0 = [-\frac{1}{2\beta}, \frac{1}{2\beta})$ ,  $X_1 = [\frac{1}{2\beta}, \frac{1}{2})$ ,  $\delta_{\bar{1}} = -1$ ,  $\delta_0 = 0$ ,  $\delta_1 = 1$ .

The case  $\beta < 2$  plays a special role. In this case, we have  $T^{-1}\{0\} = \{0\}$ , and the support  $X'$  of the invariant measure given by the natural extension in Section 3.1 is contained in  $[-\frac{1}{2}, \frac{\beta}{2} - 1) \cup [1 - \frac{\beta}{2}, \frac{1}{2})$ . Remember that it is natural to restrict  $T$  to  $X'$  when we consider multiple tilings. Then we have  $0 \notin P_1$ , thus there exists an  $x \in P_1 \setminus \{0\}$ , which implies that (F) cannot hold since we also have  $-x \in P_1$  by the symmetry of the transformation. (The fact that  $T$  is not symmetric on the endpoints of the intervals plays no role since the endpoints are not in  $\mathbb{Z}[\beta]$ .)

**4.6.1. Quadratic Pisot units.** If  $\beta$  is a quadratic Pisot unit, then Theorem 3.8 in [5] shows that (F) holds for the symmetric  $\beta$ -transformation if and only if  $\beta > 2$ .

The only quadratic Pisot unit with  $\beta < 2$  is the golden mean  $\beta = (1 + \sqrt{5})/2$ . The transformation and its natural extension is given in Example 3.12 and Figure 4, and the translations of  $\widehat{X}$  by integer vectors are depicted in Figure 9. By the considerations in Section 4.5, we obtain  $P_1 = \{\pm 1/\beta^2\}$ . Here, (W) holds since  $z = \frac{1}{\beta^5} < \varepsilon = \min\{(\frac{1}{2} - \frac{1}{\beta^2})\beta, (-\frac{1}{2\beta} + \frac{1}{\beta^2})\beta\} = \frac{1}{2\beta^3}$  gives

$$\begin{aligned} T^3(1/\beta^2 + z) &= T^3(2/\beta^3) = T^2(-1/\beta^3) = T(-1/\beta^2) = 1/\beta^2, \\ T^3(-1/\beta^2 + z) &= T^3(-2/\beta^4) = T^2(-2/\beta^3) = T(1/\beta^3) = 1/\beta^2. \end{aligned}$$

Hence, Corollary 4.20 implies that the symmetric  $\beta$ -transformation with a quadratic Pisot unit  $\beta$  always gives a tiling.

**4.6.2. Cubic Pisot units.** By Lemma 1 in [2],  $\beta > 1$  is a cubic Pisot unit if and only if it is the root of an irreducible polynomial  $x^3 - c_1x^2 - c_2x - c_3 \in \mathbb{Z}[x]$  with  $|c_3| = 1$  and  $|c_2 - 1| < c_1 + c_3$ . By Theorem 3.8 in [5], a cubic Pisot unit satisfies (F) with respect to the symmetric  $\beta$ -transformation if and only if

$$\beta > 2 \quad \text{and} \quad |\beta - c_1| < \frac{c_3}{\beta} + \frac{1}{2}.$$

In this paper, we consider the 4 cubic Pisots units  $\beta < 2$ . They are given by the polynomials  $x^3 - x - 1$  (smallest Pisot number),  $x^3 - x^2 - 1$ ,  $x^3 - 2x^2 + x - 1$  (square of the smallest Pisot number), and  $x^3 - x^2 - x - 1$  (Tribonacci number).

Let first  $\beta$  be the smallest Pisot number, i.e.  $\beta^3 = \beta + 1$ . We will show that  $\mathcal{T}$  is a double tiling. Here, the support of the invariant measure  $\mu$  is

$$X' = \left[-\frac{1}{2}, \frac{\beta^2}{2} - \beta\right) \cup \left[\beta^2 - \frac{\beta}{2} - \frac{3}{2}, \frac{\beta}{2} - 1\right) \cup \left[1 - \frac{\beta}{2}, \frac{3}{2} + \frac{\beta}{2} - \beta^2\right) \cup \left[\beta - \frac{\beta^2}{2}, \frac{1}{2}\right).$$



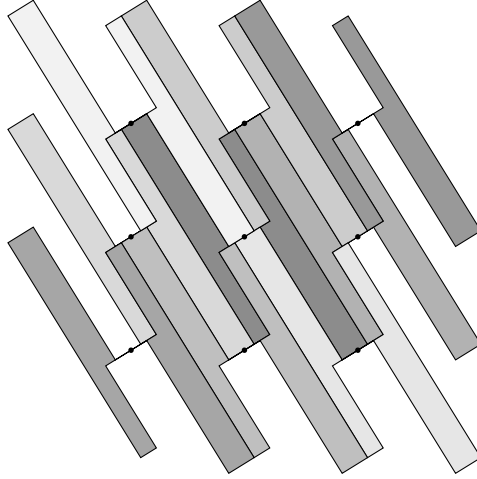


FIGURE 9. The natural extension domain and its translations by integer vectors for the symmetric  $\beta$ -transformation with  $\beta = (1 + \sqrt{5})/2$ .

The set  $P_1$  consists of the 8 points in the orbit of  $x = 3 - 2\beta$ , with  $b(x) = (01\bar{1}10\bar{1}1\bar{1})^\omega$ . Let  $T_Y$  be the induced transformation on

$$Y = \left[ \beta^2 - \frac{\beta}{2} - \frac{3}{2}, \frac{\beta}{2} - 1 \right) \cup \left[ 1 - \frac{\beta}{2}, \frac{3}{2} + \frac{\beta}{2} - \beta^2 \right).$$

We have  $T^2x \in Y$  for every  $x \in Y$  and thus  $T_Yx = T^2x = \beta^2x - \beta\delta_{b_1(x)} - \delta_{b_2(x)}$ . Therefore,  $T_Y$  is a right-continuous  $\beta^2$ -transformation with  $A_Y = \{\bar{1}1, 0\bar{1}, 01, 1\bar{1}\}$ ,  $Y_{\bar{1}1} = [\beta^2 - \frac{\beta}{2} - \frac{3}{2}, -\frac{1}{2\beta})$ ,  $Y_{0\bar{1}} = [-\frac{1}{2\beta}, \frac{\beta}{2} - 1)$ ,  $Y_{01} = [1 - \frac{\beta}{2}, \frac{1}{2\beta})$ ,  $Y_{1\bar{1}} = [\frac{1}{2\beta}, \frac{3}{2} + \frac{\beta}{2} - \beta^2)$ ,  $\delta_{\bar{1}1} = -\beta + 1 = -1/\beta^4$ ,  $\delta_{0\bar{1}} = -1$ ,  $\delta_{01} = 1$ ,  $\delta_{1\bar{1}} = 1/\beta^4$ . Moreover, we have  $T^{-1}Y \cap X' = X' \setminus Y$  and  $T^{-2}Y \cap X' = Y$ . Therefore, for every  $x \in \mathbb{Z}[\beta] \cap Y$ , the tile  $\mathcal{T}_x$  defined by  $T$  is equal to the tile  $\mathcal{T}_x^Y$  defined by  $T_Y$ . This implies

$$\bigcup_{x \in \mathbb{Z}[\beta] \cap Y} \mathcal{T}_x = \bigcup_{x \in \mathbb{Z}[\beta] \cap Y} \mathcal{T}_x^Y = H.$$

Since every  $x \in Y$  has a unique preimage  $T^{-1}x \in X' \setminus Y$ , we obtain  $\mathcal{T}_x = M_\beta \mathcal{T}_{T^{-1}x}$ . Hence, we also have  $\bigcup_{x \in \mathbb{Z}[\beta] \cap (X' \setminus Y)} \mathcal{T}_x = H$ , and  $\mathcal{T}$  is a multiple tiling of degree at least 2. By Corollary 4.18,  $\Phi(2)$  lies in  $\mathcal{T}_{3-2\beta}$  and  $\mathcal{T}_{2\beta^2-3\beta}$ . Thus,  $\mathcal{T}$  is a double tiling and  $\{\mathcal{T}_x^Y\}_{x \in \mathbb{Z}[\beta] \cap Y}$  a tiling, see Figure 10.

*Remark 4.28.* It is conjectured that the greedy  $\beta$ -transformation  $T_\beta x = \beta x - \lfloor \beta x \rfloor$  on  $X = [0, 1)$  produces a tiling for every Pisot unit  $\beta$ . This is a version of the *Pisot conjecture*, and it is known to be true for all cubic Pisot units ([4]). It is therefore quite surprising that this conjecture does not hold when we shift  $X$  from  $[0, 1)$  to  $[-1/2, 1/2)$  and set  $Tx = \beta x - \lfloor \beta x + 1/2 \rfloor$ .

If  $\beta$  is the square of the smallest Pisot number, i.e.  $\beta^3 = 2\beta^2 - \beta + 1$ , then the support of the invariant measure is  $[-\frac{1}{2}, \frac{\beta}{2} - 1) \cup [1 - \frac{\beta}{2}, \frac{1}{2})$  and  $P_1 = \{\pm(2 - \beta), \pm(2\beta - \beta^2)\}$ , with  $b(2 - \beta) = (010\bar{1})^\omega$ . By Corollary 4.18,  $\Phi(4)$  is an exclusive point of  $\mathcal{T}_{3+2\beta-2\beta^2}$ , see Figure 11. Note that  $M_\beta^{-3}\mathcal{T}$  seems to be equal to the tiling  $\{\mathcal{T}_x^Y\}_{x \in \mathbb{Z}[\beta] \cap Y}$  defined above, but we do not prove this relation in this paper.

The case  $\beta^3 = \beta^2 + 1$  is very similar to the preceding one. We have  $P_1 = \{\pm 1/\beta^2, \pm 1/\beta^3\}$ ,  $b(1/\beta^3) = (010\bar{1})^\omega$ , and  $\Phi(4)$  is an exclusive point of  $\mathcal{T}_{4-\beta-\beta^2}$ , see Figure 11.

Finally, let  $\beta$  be Tribonacci number, i.e.,  $\beta^3 = \beta^2 + \beta + 1$ . The support of the invariant measure is  $X' = [-\frac{1}{2}, -\frac{1}{2\beta^3}) \cup [\frac{1}{2\beta^3}, \frac{1}{2})$  and  $P_1 = \{\pm 1/\beta^3, \pm 1/\beta^2, \pm(1 - 1/\beta)\}$ , with  $b(1/\beta^3) = (01\bar{1})^\omega$ ,  $b(-1/\beta^3) = (0\bar{1}1)^\omega$ . The degree of the multiple tiling is at most 2 since  $\Phi(4)$  lies in the tiles  $\mathcal{T}_{3-\beta^2}$  and  $\mathcal{T}_{4-2\beta^2}$ , see Figure 12.

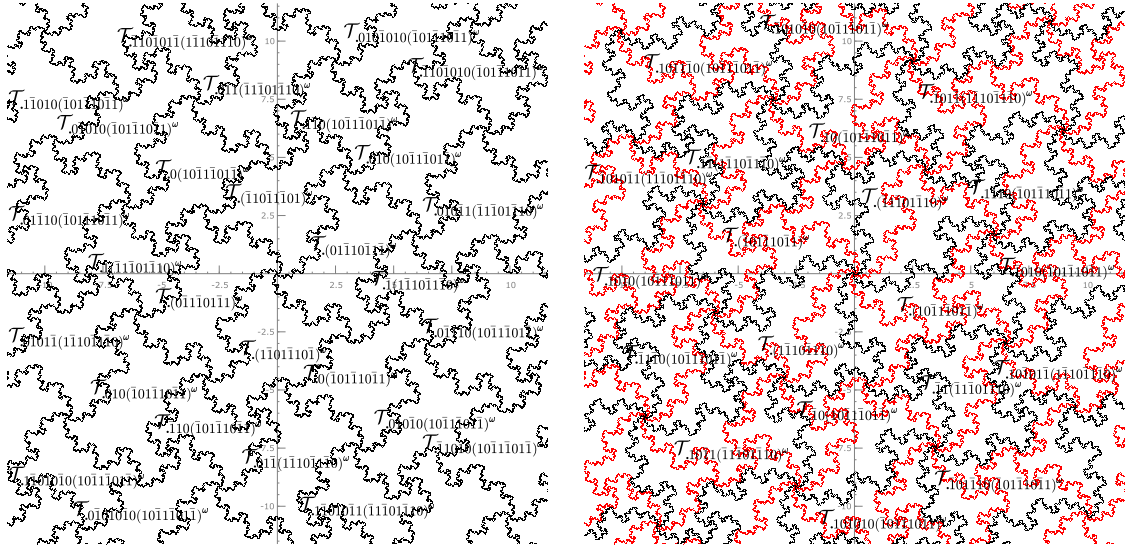


FIGURE 10. The tiling  $\{\mathcal{T}_x^Y\}_{x \in \mathbb{Z}[\beta] \cap Y} = \{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap Y}$  and the double tiling  $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$  from the symmetric  $\beta$ -transformation,  $\beta^3 = \beta + 1$ .

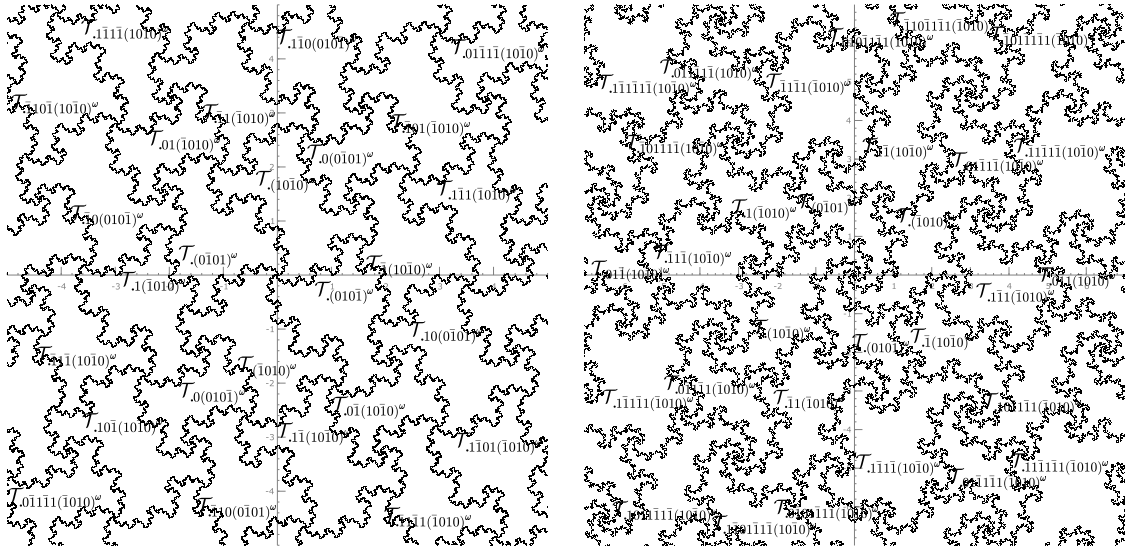


FIGURE 11. Tilings from the symmetric  $\beta$ -transformation,  $\beta^3 = 2\beta^2 - \beta + 1$  (left) and  $\beta^3 = \beta^2 + 1$  (right).

Proving that  $\mathcal{T}$  is not a tiling is more complicated than for the smallest Pisot number and for Example 3.13. We use the description of  $T$ -admissible sequences in Theorem 2.5 and the transducer in Figure 13 to show that the subtle  $M_\beta^5 \mathcal{T}_{\bar{1}0(10\bar{1})^\omega}$  of  $\mathcal{T}_{\bar{1}0(10\bar{1})^\omega}$  is also contained in  $\mathcal{T}_{(0\bar{1})^\omega}$ .

The endpoints of the intervals have the expansions

$$\begin{aligned} b(\ell_{\bar{1}}) &= (\bar{1}001)^\omega, & b(\ell_0) &= 0(\bar{1}001)^\omega, & b\left(\frac{1}{2\beta^3}\right) &= 000(100\bar{1})^\omega, & b(\ell_1) &= (1\bar{1}00)^\omega, \\ \tilde{b}(r_{\bar{1}}) &= (\bar{1}100)^\omega, & \tilde{b}\left(\frac{-1}{2\beta^3}\right) &= 000(\bar{1}001)^\omega, & \tilde{b}(r_0) &= 0(100\bar{1})^\omega, & \tilde{b}(r_1) &= (100\bar{1})^\omega. \end{aligned}$$

Therefore,  $\bar{\mathcal{S}}$  is a shift of finite type with forbidden sequences 11, 101, 1000, 1001,  $\bar{1}\bar{1}$ ,  $\bar{1}0\bar{1}$ ,  $\bar{1}000$ ,  $\bar{1}00\bar{1}$ . Restricting  $T$  to  $X'$  means that we also exclude 000. The set  $\mathcal{S}$  is obtained from  $\bar{\mathcal{S}}$  by excluding sequences ending with  $(100\bar{1})^\omega$ .

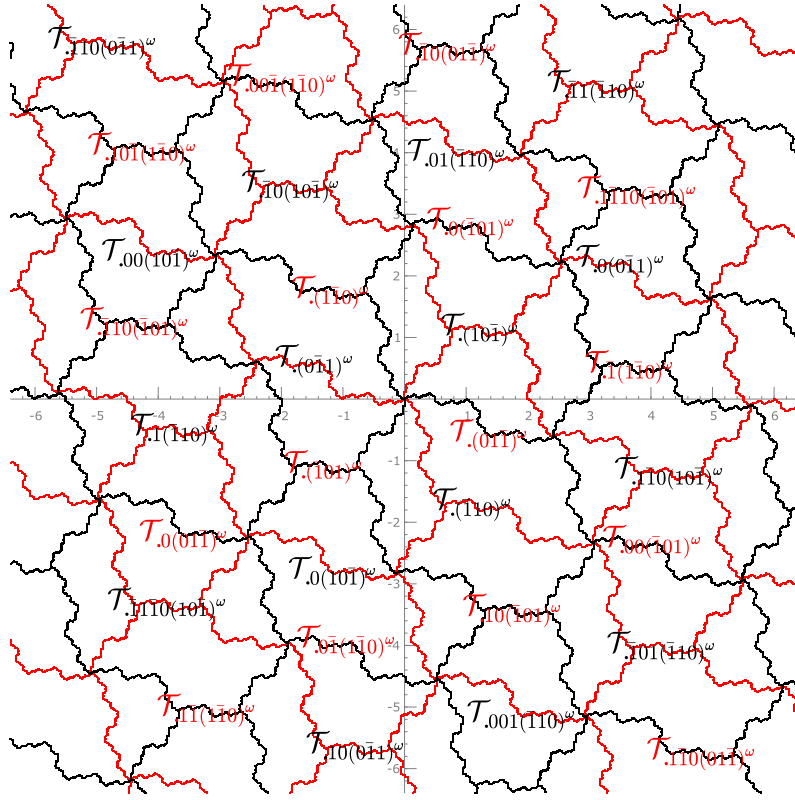


FIGURE 12. Double tiling from the symmetric  $\beta$ -transformation,  $\beta^3 = \beta^2 + \beta + 1$ .

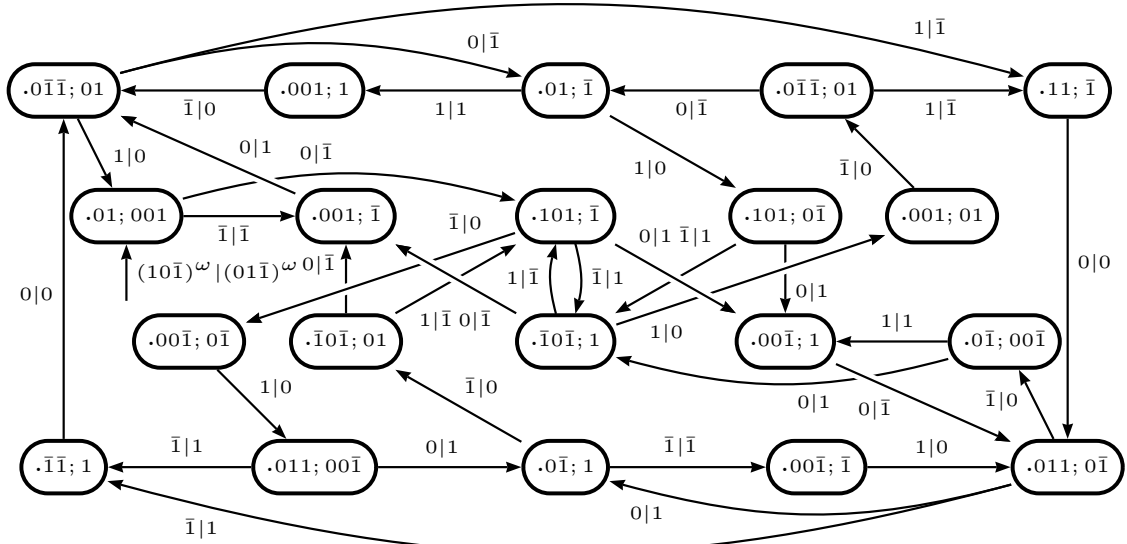


FIGURE 13. Transducer showing that  $\mathcal{T}$  is a double tiling,  $\beta^3 = \beta^2 + \beta + 1$ .

The nodes of the transducer in Figure 13 have labels  $(x; v)$ , where  $x = .u$  stands for the real number  $.u0^\omega$  and  $v$  is given by the outputs of the incoming paths. E.g., in  $(.01; 001)$  we have  $x = 1/\beta^2$ , and there exists an incoming path with edges  $\xleftarrow{1|0} \xleftarrow{0|0} \xleftarrow{\bar{1}|1}$ , which implies that the output of the outgoing transition can only be  $\bar{1}$ . (The incoming path  $\xleftarrow{1|0} \xleftarrow{0|1}$  imposes less restrictions on

the following output.) For every transition  $(x; v) \xrightarrow{a|a'} (x'; v')$ , we have  $x' = (x + \delta_a - \delta_{a'})/\beta$ . Consider a sequence of transitions starting in  $(.01; 001)$ ,

$$(.01; 001) = (x_0; v_0) \xrightarrow{w_0|w'_0} (x_1; v_1) \xrightarrow{w_{-1}|w'_{-1}} (x_2; v_2) \xrightarrow{w_{-2}|w'_{-2}} \dots$$

Since  $.(10\bar{1})^\omega = 1 - 1/\beta$  and  $.(01\bar{1})^\omega = 1/\beta^3$ , we have  $.(10\bar{1})^\omega - .(01\bar{1})^\omega = 1/\beta^2 = x_0$ . By the construction of the transducer, we obtain  $x_{k+1} = .w_{-k} \dots w_0 (10\bar{1})^\omega - .w'_{-k} \dots w'_0 (01\bar{1})^\omega$  for every  $k \geq 0$ . This means that

$$\varphi(w_{-k} \dots w_0) + \Phi(. (10\bar{1})^\omega) = \varphi(w'_{-k} \dots w'_0) + \Phi(. (01\bar{1})^\omega) + M_\beta^{k+1} \Phi(x_{k+1}).$$

It can be easily verified that  $\dots w'_{-1} w'_0 . (01\bar{1})^\omega \in \mathcal{S}$  since the forbidden sequences given above are avoided in the output of the transducer. Since  $x_k$  is bounded and  $M_\beta$  is contracting on  $H$ , we obtain therefore that  $\varphi(\dots w_{-1} w_0) + \Phi(. (10\bar{1})^\omega) = \varphi(\dots w'_{-1} w'_0) + \Phi(. (01\bar{1})^\omega) \in \mathcal{T}_{.(01\bar{1})^\omega}$ .

For proving  $M_\beta^5 \mathcal{T}_{.\bar{1}0\bar{1}(10\bar{1})^\omega} \subseteq \mathcal{T}_{.(01\bar{1})^\omega}$ , it remains to show that every sequence  $\bar{1}0\bar{1}0\bar{1}w_{-5}w_{-6} \dots$  satisfying  $\dots w_{-6}w_{-5}\bar{1}0\bar{1}0\bar{1} \cdot (10\bar{1})^\omega \in \mathcal{S}$  is the input of a path in the transducer. The paths with input  $\bar{1}0\bar{1}0\bar{1}$  starting in  $(.01; 001)$  lead to the set of states

$$\bar{Q}_1 = \{(. \bar{1}0\bar{1}; 1), (.00\bar{1}; 0\bar{1}), (.0\bar{1}; 0\bar{1}), (. \bar{1}\bar{1}; 1)\}.$$

We show that every path  $\bar{1}0\bar{1}0\bar{1}w_{-5} \dots w_{-k}$ ,  $k \geq 5$ , with  $w_{-k} = 1$  leads to one of the sets

$$\begin{aligned} Q_1 &= \{(.101; \bar{1}), (.001; 01), (.01; 001), (.11; \bar{1})\} \\ Q_2 &= \{(.101; \bar{1}), (.001; 01), (.00\bar{1}; 1), (.011; 00\bar{1})\}, \\ Q_3 &= \{(.01; 001), (.11; \bar{1}), (.101; 0\bar{1}), (.001; 1)\}, \\ Q_4 &= \{(.101; \bar{1}), (.001; 01), (.011; 00\bar{1}), (.11; \bar{1})\}, \\ Q_5 &= \{(.00\bar{1}; 1), (.101; \bar{1}), (.011; 0\bar{1})\}, \end{aligned}$$

and that every path  $\bar{1}0\bar{1}0\bar{1}w_{-5} \dots w_{-k}$ ,  $k \geq 5$ , with  $w_{-k} = \bar{1}$ , leads to one of the sets  $\bar{Q}_1, \bar{Q}_2, \bar{Q}_3, \bar{Q}_4, \bar{Q}_5$ , where  $\bar{Q}_i$  is defined by exchanging 1 and  $\bar{1}$  in  $Q_i$ . We have the transitions

$$\begin{aligned} \bar{Q}_1 &\xrightarrow{1} Q_2, \bar{Q}_1 \xrightarrow{01} Q_1, \bar{Q}_1 \xrightarrow{001} Q_3, \bar{Q}_2 \xrightarrow{1} Q_4, \bar{Q}_2 \xrightarrow{01} Q_3, \bar{Q}_2 \xrightarrow{001} Q_3, \bar{Q}_3 \xrightarrow{1} Q_5, \bar{Q}_3 \xrightarrow{01} Q_1, \\ \bar{Q}_3 &\xrightarrow{001} Q_3, \bar{Q}_4 \xrightarrow{1} Q_4, \bar{Q}_4 \xrightarrow{01} Q_3, \bar{Q}_4 \xrightarrow{001} Q_3, \bar{Q}_5 \xrightarrow{1} Q_1, \bar{Q}_5 \xrightarrow{01} Q_3, \bar{Q}_5 \xrightarrow{001} Q_3. \end{aligned}$$

Together with the symmetric transitions  $Q_i \xrightarrow{0^n \bar{1}} \bar{Q}_j$ , this shows inductively the assertion. Since  $\lambda^{d-1}(\mathcal{D}_{.\bar{1}0\bar{1}(10\bar{1})^\omega}) > 0$ , we have shown that  $\mathcal{T}$  is not a tiling, but a double tiling.

*Remark 4.29.* In the transducer in Figure 13, every output gives a  $T$ -admissible sequence. It is possible to construct a similar (but larger) transducer where both the input and the output give  $T$ -admissible sequences. This means that almost every point in  $\mathcal{T}_{.(10\bar{1})^\omega} \cap \mathcal{T}_{.(01\bar{1})^\omega}$  is represented by a unique path in the new transducer. Then this intersection has positive measure if and only if the largest eigenvalue of the new transducer is equal to  $\beta$ , cf. Corollary 5.3 in [29]. Since the set of differences  $y - x$  with  $\mathcal{T}_x \cap \mathcal{T}_y \neq \emptyset$  is finite by Proposition 4.8, this provides an effective method for deciding the tiling property whenever  $\bar{\mathcal{S}}$  is a sofic shift, cf. Theorem 4.1 in [30] and Theorem 5.4 in [29].

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