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# VAN DEN BERGH ISOMORPHISMS IN STRING TOPOLOGY

LUC MENICHI

ABSTRACT. Let  $M$  be a path-connected closed oriented  $d$ -dimensional smooth manifold and let  $\mathbb{k}$  be a principal ideal domain. By Chas and Sullivan, the shifted free loop space homology of  $M$ ,  $H_{*+d}(LM)$  is a Batalin-Vilkovisky algebra. Let  $G$  be a topological group such that  $M$  is a classifying space of  $G$ . Denote by  $S_*(G)$  the (normalized) singular chains on  $G$ . Suppose that  $G$  is discrete or path-connected. We show that there is a Van Den Bergh type isomorphism

$$HH^{-p}(S_*(G), S_*(G)) \cong HH_{p+d}(S_*(G), S_*(G)).$$

Therefore, the Gerstenhaber algebra  $HH^*(S_*(G), S_*(G))$  is a Batalin-Vilkovisky algebra and we have a linear isomorphism

$$HH^*(S_*(G), S_*(G)) \cong H_{*+d}(LM).$$

This linear isomorphism is expected to be an isomorphism of Batalin-Vilkovisky algebras. We also give a new characterization of Batalin-Vilkovisky algebra in term of derived bracket.

## 1. INTRODUCTION

We work over an arbitrary principal ideal domain  $\mathbb{k}$ . Let  $M$  be a compact oriented  $d$ -dimensional smooth manifold. Denote by  $LM := \text{map}(S^1, M)$  the free loop space on  $M$ . Chas and Sullivan [5] have shown that the shifted free loop homology  $H_{*+d}(LM)$  has a structure of Batalin-Vilkovisky algebra (Definition 21). In particular, they showed that  $H_{*+d}(LM)$  is a Gerstenhaber algebra (Definition 19). On the other hand, let  $A$  be a differential graded (unital associative) algebra. The Hochschild cohomology of  $A$  with coefficients in  $A$ ,  $HH^*(A, A)$ , is a Gerstenhaber algebra. These two Gerstenhaber algebras are expected to be related:

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*Key words and phrases.* String Topology, Batalin-Vilkovisky algebra, Hochschild cohomology, free loop space, derived bracket, Van den Bergh duality, Poincaré duality group, Calabi-Yau algebra.

**Conjecture 1.** *Let  $G$  be a topological group such that  $M$  is a classifying space of  $G$ . There is an isomorphism of Gerstenhaber algebras  $H_{*+d}(LM) \cong HH^*(S_*(G), S_*(G))$  between the free loop space homology and the Hochschild cohomology of the algebra of singular chains on  $G$ .*

Suppose that  $G$  is discrete or path-connected. In this paper, we define a Batalin-Vilkovisky algebra structure on  $HH^*(S_*(G), S_*(G))$  and an isomorphism of graded  $\mathbb{k}$ -modules

$$BFG^{-1} \circ \mathcal{D} : H_{*+d}(LM) \cong HH^*(S_*(G), S_*(G))$$

which is compatible with the two  $\Delta$  operators of the two Batalin-Vilkovisky algebras:  $BFG^{-1} \circ \mathcal{D} \circ \Delta = \Delta \circ BFG^{-1} \circ \mathcal{D}$ . Indeed, Burghilea, Fiedorowicz [4] and Goodwillie [16] gave an isomorphism of graded  $\mathbb{k}$ -modules

$$BFG : HH_*(S_*(G), S_*(G)) \xrightarrow{\cong} H_*(LM).$$

which interchanges Connes boundary map  $B$  and the  $\Delta$  operator on  $H_{*+d}(LM)$ :  $BFG \circ B = \Delta \circ BFG$ . And in this paper, our main result is:

**Theorem 2.** *(Theorems 42 and 40) Let  $G$  be a discrete or a path-connected topological group such that its classifying space  $BG$  is an oriented Poincaré duality space of formal dimension  $d$ . Then*

a) *there exists  $\mathbb{k}$ -linear isomorphisms*

$$\mathcal{D} : HH_{d-p}(S_*(G), S_*(G)) \xrightarrow{\cong} HH^p(S_*(G), S_*(G)).$$

b) *If  $B$  denotes Connes boundary map on  $HH_*(S_*(G), S_*(G))$  then  $\Delta := \mathcal{D} \circ B \circ \mathcal{D}^{-1}$  defines a structure of Batalin-Vilkovisky algebra on  $HH^*(S_*(G), S_*(G))$ , extending the canonical Gerstenhaber algebra structure.*

Note that when  $G$  is a discrete group, the algebra of normalized singular chains on  $G$ ,  $S_*(G)$  is just the group ring  $\mathbb{k}[G]$ .

To prove Conjecture 1 in the discrete or path-connected case, it suffices now to show that the composite  $BFG^{-1} \circ \mathcal{D}$  is a morphism of graded algebras. When  $\mathbb{k}$  is a field of characteristic 0 and  $G$  is discrete, this was proved by Vaintrob [32].

Suppose now that

(3)  $M$  is simply-connected and that  $\mathbb{k}$  is a field.

In this case, there is a more famous dual conjecture relating Hochschild cohomology and string topology.

**Conjecture 4.** *Under (3), there is an isomorphism of Gerstenhaber algebras  $H_{*+d}(LM) \cong HH^*(S^*(M), S^*(M))$  between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on  $M$ .*

And in fact, Theorem 2 is the Eckmann-Hilton or Koszul dual of the following theorem.

**Theorem 5.** ([10, Theorem 23] and [27, Theorem 22]) *Assume (3).*

a) *There exist isomorphism of graded  $\mathbb{k}$ -vector spaces*

$$FTV : HH^{p-d}(S^*(M), S^*(M)^\vee) \xrightarrow{\cong} HH^p(S^*(M), S^*(M)).$$

b) *The Connes coboundary  $B^\vee$  on  $HH^*(S^*(M), S^*(M)^\vee)$  defines via the isomorphism  $FTV$  a structure of Batalin-Vilkovisky algebra extending the Gerstenhaber algebra  $HH^*(S^*(M), S^*(M))$ .*

Jones [19] proved that there is an isomorphism

$$J : H_{p+d}(LM) \xrightarrow{\cong} HH^{-p-d}(S^*(M), S^*(M)^\vee)$$

such that the  $\Delta$  operator of the Batalin-Vilkovisky algebra  $H_{*+d}(LM)$  and Connes coboundary map  $B^\vee$  on  $HH^{*-d}(S^*(M), S^*(M)^\vee)$  satisfies  $J \circ \Delta = B^\vee \circ J$ . Therefore, as we explain in [27], to prove conjecture 4, it suffices to show that the composite  $FTV \circ J$  is a morphism of graded algebras.

In [9], together with Felix and Thomas, we prove that Hochschild cohomology satisfies some Eckmann-Hilton or Koszul duality.

**Theorem 6.** [9, Corollary 2] *Let  $\mathbb{k}$  be a field. Let  $G$  be a connected topological group. Denote by  $S^*(BG)$  the algebra of singular cochains on the classifying space of  $G$ . Then there exists an isomorphism of Gerstenhaber algebras*

$$Gerst : HH^*(S_*(G), S_*(G)) \xrightarrow{\cong} HH^*(S^*(BG), S^*(BG)).$$

Therefore under (3), conjectures 4 and 1 are equivalent and under (3), Theorem 2 as stated in this introduction follows from Theorem 5.

The problem is that the isomorphism  $Gerst$  in Theorem 6 does not admit a simple formula. On the contrary, as we explain in Theorems 42 and 40, in this paper, the isomorphism  $\mathcal{D}$  is very simple:  $\mathcal{D}^{-1}$  is given by the cap product with a fundamental class  $c \in HH_d(S_*(G), S_*(G))$ .

In [15, Theorem 3.4.3 i)], Ginzburg (See also [22, Proposition 1.4]) shows that for any Calabi-Yau algebra  $A$ , the Van den Bergh duality isomorphism  $\mathcal{D} : HH_{d-p}(A, A) \xrightarrow{\cong} HH^p(A, A)$  is  $HH^*(A, A)$ -linear:  $\mathcal{D}^{-1}$  is also given by the cap product with a fundamental class  $c \in HH_d(A, A)$ .

We now give the plan of the paper:

**Section 2:** We recall the definitions of the Bar construction, of the Hochschild (co)chain complex and of Hochschild (co)homology.

**Section 3:** We show that, for some augmented differential graded algebra  $A$  such that the dual of its reduced bar construction  $B(A)^\vee$  satisfies Poincaré duality, we have a Van den Bergh duality isomorphism  $HH_{d-p}(A, A) \cong HH^p(A, A)$  if  $A$  is connected (Proposition 12).

**Section 4:** There is a well known isomorphism between group (co)homology and Hochschild (co)homology. We show that, through this isomorphism, cap products in Hochschild (co)homology correspond to cap products in group (co)homology.

**Section 5:** We give a new characterization of Batalin-Vilkovisky algebras.

**Section 6:** Ginzburg proved that if Hochschild (co)homology satisfies a Van den Bergh duality isomorphism  $HH_{d-p}(A, A) \cong HH^p(A, A)$  then Hochschild cohomology has a Batalin-Vilkovisky algebra structure. We rewrite the proof of Ginzburg using our new characterization of Batalin-Vilkovisky algebras and insisting on signs.

**Section 7:** We show that a differential graded algebra quasi-isomorphic to an algebra satisfying Poincaré duality, also satisfies Poincaré duality (Proposition 38). Finally, we show our main theorem for path-connected topological group.

**Section 8:** We show our main theorem for discrete groups. Extending a result of Kontsevich [15, Corollary 6.1.4] and Lambre [22, Lemme 6.2], we also show that, over any commutative ring  $\mathbb{k}$ , the group ring  $\mathbb{k}[G]$  of an orientable Poincaré duality group is a Calabi-Yau algebra.

**Appendix:** We recall the notion of derived bracket following Kosmann-Schwarzbach [20]. We interpret our new characterization of Batalin-Vilkovisky algebra in term of derived bracket (Theorem 53). To any differential graded algebra  $A$ , we associate

- a new Lie bracket on  $A$  (Remark 50),
- a new Gerstenhaber algebra which is a sub algebra of the endomorphism algebra of  $HH_*(A, A)$  (Theorem 54).

We conjecture that Theorem 2 is true without assuming that  $G$  is discrete or path-connected. Note that the proof of the discrete case (Sections 4 and 8) is independent of the proof of the path-connected case (Sections 3 and 7).

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## 2. HOCHSCHILD HOMOLOGY AND COHOMOLOGY

We work over an arbitrary commutative ring  $\mathbb{k}$  except in sections 3 and 7, where  $\mathbb{k}$  is assumed to be a principal ideal domain. We use the graded differential algebra of [8, Chapter 3]. In particular, an element of lower degree  $i \in \mathbb{Z}$  is by the *classical convention* [8, p. 41-2] of upper degree  $-i$ . Differentials are of lower degree  $-1$ . All the algebras considered in this paper, are unital and associative. Let  $A$  be a differential graded algebra. Let  $M$  be a right  $A$ -module and  $N$  be a left  $A$ -module. Denote by  $sA$  the suspension of  $A$ ,  $(sA)_i = A_{i-1}$ . Let  $d_0$  be the differential on the tensor product of complexes  $M \otimes T(sA) \otimes N$ . We denote the tensor product of the elements  $m \in M$ ,  $sa_1 \in sA, \dots, sa_k \in sA$  and  $n \in N$  by  $m[a_1 | \dots | a_k]n$ . Let  $d_1$  be the differential on the graded vector space  $M \otimes T(sA) \otimes N$  defined by:

$$\begin{aligned} d_1 m[a_1 | \dots | a_k]n &= (-1)^{|m|} m a_1 [a_2 | \dots | a_k]n \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[a_1 | \dots | a_i a_{i+1} | \dots | a_k]n \\ &\quad - (-1)^{\varepsilon_{k-1}} m[a_1 | \dots | a_{k-1}] a_k n; \end{aligned}$$

Here  $\varepsilon_i = |m| + |a_1| + \dots + |a_i| + i$ .

The *bar construction of  $A$  with coefficients in  $M$  and in  $N$* , denoted  $B(M; A; N)$ , is the complex  $(M \otimes T(sA) \otimes N, d_0 + d_1)$ . The *bar resolution of  $A$* , denoted  $B(A; A; A)$ , is the differential graded  $(A, A)$ -bimodule  $(A \otimes T(sA) \otimes A, d_0 + d_1)$ . If  $A$  is augmented then the *reduced bar construction of  $A$* , denoted  $B(A)$ , is  $B(\mathbb{k}; A; \mathbb{k})$ .

Denote by  $A^{op}$  the opposite algebra of  $A$  and by  $A^e := A \otimes A^{op}$  the enveloping algebra of  $A$ . Let  $M$  be a differential graded  $(A, A)$ -bimodule. Recall that any  $(A, A)$ -bimodule can be considered as a left (or right)  $A^e$ -module. The *Hochschild chain complex* is the complex  $M \otimes_{A^e} B(A; A; A)$  denoted  $\mathcal{C}_*(A, M)$ . Explicitly  $\mathcal{C}_*(A, M)$  is the complex  $(M \otimes T(sA), d_0 + d_1)$  with  $d_0$  obtained by tensorization and [6, (10) p. 78]

$$\begin{aligned} d_1 m[a_1 | \dots | a_k] &= (-1)^{|m|} m a_1 [a_2 | \dots | a_k] \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[a_1 | \dots | a_i a_{i+1} | \dots | a_k] \\ &\quad - (-1)^{|sa_k| \varepsilon_{k-1}} a_k m[a_1 | \dots | a_{k-1}]. \end{aligned}$$

The *Hochschild homology of  $A$  with coefficients in  $M$*  is the homology  $H$  of the Hochschild chain complex:

$$HH_*(A, M) := H(\mathcal{C}_*(A, M)).$$

The *Hochschild cochain complex of  $A$  with coefficients in  $M$*  is the complex  $\text{Hom}_{A^e}(B(A; A; A), M)$  denoted  $\mathcal{C}^*(A, M)$ . Explicitly  $\mathcal{C}^*(A, M)$  is the complex

$$(\text{Hom}(T(sA), M), D_0 + D_1).$$

Here for  $f \in \text{Hom}(T(sA), M)$ ,  $D_0(f)([ ]) = d_M(f([ ])$ ,  $D_1(f)([ ]) = 0$ , and for  $k \geq 1$ , we have:

$$D_0(f)([a_1|a_2|\dots|a_k]) = d_M(f([a_1|a_2|\dots|a_k]) - \sum_{i=1}^k (-1)^{\bar{\epsilon}_i} f([a_1|\dots|d_A a_i|\dots|a_k])$$

and

$$\begin{aligned} D_1(f)([a_1|a_2|\dots|a_k]) &= -(-1)^{|sa_1|+|f|} a_1 f([a_2|\dots|a_k]) \\ &\quad - \sum_{i=2}^k (-1)^{\bar{\epsilon}_i} f([a_1|\dots|a_{i-1}a_i|\dots|a_k]) \\ &\quad + (-1)^{\bar{\epsilon}_k} f([a_1|a_2|\dots|a_{k-1}]) a_k, \end{aligned}$$

where  $\bar{\epsilon}_i = |f| + |sa_1| + |sa_2| + \dots + |sa_{i-1}|$ .

The *Hochschild cohomology of  $A$  with coefficients in  $M$*  is

$$HH^*(A, M) = H(\mathcal{C}^*(A, M)).$$

Suppose that  $A$  has an augmentation  $\varepsilon : A \rightarrow \mathbb{k}$ . Let  $\bar{A} := \text{Ker}\varepsilon$  be the augmentation ideal. We denote by  $\bar{B}(A) := (T s \bar{A}, d_0 + d_1)$  the normalized reduced Bar construction, by  $\bar{\mathcal{C}}_*(A, M) := (M \otimes T(s\bar{A}), d_0 + d_1)$  the normalized Hochschild chain complex and by  $\bar{\mathcal{C}}^*(A, M) := (\text{Hom}(T(s\bar{A}), M), D_0 + D_1)$  the normalized Hochschild cochain complex.

### 3. THE ISOMORPHISM BETWEEN HOCHSCHILD COHOMOLOGY AND HOCHSCHILD HOMOLOGY FOR DIFFERENTIAL GRADED ALGEBRAS

Let  $A$  be a differential graded algebra. Let  $P$  and  $Q$  be two  $A$ -bimodules.

The action of  $HH^*(A, Q)$  on  $HH_*(A, P)$  comes from a (right) action of the  $\mathcal{C}^*(A, Q)$  on  $\mathcal{C}_*(A, P)$  given by [6, (18) p. 82], [22]

$$\cap : \mathcal{C}_*(A, P) \otimes \mathcal{C}^*(A, Q) \rightarrow \mathcal{C}_*(A, P \otimes_A Q)$$

(7)

$$(m[a_1 | \dots | a_n], f) \mapsto (m[a_1 | \dots | a_n]) \cap f := \sum_{p=0}^n \pm (m \otimes_{A^*} f[a_1 | \dots | a_p])[a_{p+1} | \dots | a_n].$$

Here  $\pm$  is the Koszul sign  $(-1)^{|f|(|a_1| + \dots + |a_n| + n)}$  [27, Proof of Lemma 16].

Let  $f : A \rightarrow B$  be a morphism of differential graded algebras and let  $N$  be a  $B$ -bimodule. The linear map  $B \otimes_A N \rightarrow N$ ,  $b \otimes n \mapsto b.n$  is a morphism of  $B$ -bimodules. We call again cap product the composite

$$(8) \quad \mathcal{C}_*(A, B) \otimes \mathcal{C}^*(A, N) \xrightarrow{\cap} \mathcal{C}_*(A, B \otimes_A N) \rightarrow \mathcal{C}_*(A, N).$$

In this paper, our goal (statement 9) is to relate the cap product with  $B = A$  to the cap product with  $N = B = \mathbb{k}$ .

**Statement 9.** *Let  $A$  be an augmented differential graded algebra such that each  $A_i$  is  $\mathbb{k}$ -free,  $i \in \mathbb{Z}$ . Let  $N$  be an  $A$ -bimodule. Let  $c \in HH_d(A, A)$ . Denote by  $[m] \in Tor_d^A(\mathbb{k}, \mathbb{k})$  the image of  $c$  by the morphism*

$$HH_d(A, \varepsilon) : HH_d(A, A) \rightarrow HH_d(A, \mathbb{k}) = Tor_d^A(\mathbb{k}, \mathbb{k}).$$

Suppose that

- $Tor_i^A(\mathbb{k}, \mathbb{k}) = 0$  for  $i \leq -n$  or  $i \geq n$  for some positive integer  $n$ ,
- each  $Tor_i^A(\mathbb{k}, \mathbb{k})$  is of finite type,  $i \in \mathbb{Z}$ ,
- the morphism of right  $Ext_A^*(\mathbb{k}, \mathbb{k})$ -modules

$$Ext_A^p(\mathbb{k}, \mathbb{k}) \xrightarrow{\cong} Tor_{d-p}^A(\mathbb{k}, \mathbb{k}), a \mapsto [m] \cap a$$

is an isomorphism.

Then the morphism

$$\mathbb{D}^{-1} : HH^p(A, N) \xrightarrow{\cong} HH_{d-p}(A, N), a \mapsto c \cap a$$

is also an isomorphism.

This statement is the Eckmann-Hilton or Koszul dual of [27, Proposition 11]. In this section, we will prove this statement if  $A$  is connected. But we wonder if this statement is true in general or even for ungraded algebras.

*Property 10.* Let  $B$  and  $N$  be two complexes. Consider the natural morphism of complexes  $\Theta : B^\vee \otimes N \rightarrow \text{Hom}(B, N)$ , which sends  $\varphi \otimes n$  to the linear map  $f : B \rightarrow N$  defined by  $f(b) := \varphi(b)n$ . Suppose that each  $B_i$  is  $\mathbb{k}$ -free.

- 1) If  $B_i = 0$  for  $i \leq -n$  or  $i \geq n$  for some positive integer  $n$  and if each  $B_i$  is of finite type or
- 2) If  $H_i(B) = 0$  for  $i \leq -n$  or  $i \geq n$  for some positive integer  $n$  and if each  $H_i(B)$  is of finite type

then  $\Theta$  is a homotopy equivalence.

*Proof.* 1) Since  $B$  is bounded, the component of degree  $n$  of  $\text{Hom}(B, N)$  is the direct sum  $\bigoplus_{q \in \mathbb{Z}} \text{Hom}(B_{q-n}, N_q)$ . Since  $B_{q-n}$  is free of finite type,  $\text{Hom}(B_{q-n}, N_q)$  is isomorphic to  $B_{q-n}^\vee \otimes N_q$ . Therefore  $\Theta$  is an isomorphism.

2) Since  $\mathbb{k}$  is a principal ideal domain, the proof of [30, Lemma 5.5.9] shows that there exists a complex  $B'$  satisfying 1) homotopy equivalent to  $B$ . By naturality of  $\Theta$ ,  $\Theta$  is a homotopy equivalence of complexes.  $\square$

**Lemma 11.** *The statement holds whenever  $N$  is a trivial  $A$ -bimodule, i.e.  $a.n = \varepsilon(a)n = n.a$  for  $a \in A$  and  $n \in N$ .*

*Proof.* Since  $N$  is a trivial  $A$ -bimodule, the normalized Hochschild chain complex  $\overline{\mathcal{C}}_*(A, N)$  is just the tensor product of complexes  $\overline{\mathcal{C}}_*(A, \mathbb{k}) \otimes N = \overline{\mathcal{B}}(A) \otimes N$  (This is also true for the unnormalized Hochschild chain complex, but less obvious). And the normalized Hochschild cochain complex  $\overline{\mathcal{C}}^*(A, N)$  is just the Hom complex  $\text{Hom}(\overline{\mathcal{C}}_*(A, \mathbb{k}), N) = \text{Hom}(\overline{\mathcal{B}}(A), N)$ . Since the augmentation ideal of  $A$ ,  $\overline{A}$ , is  $\mathbb{k}$ -free,  $\overline{\mathcal{B}}(A)$  is also  $\mathbb{k}$ -free. Each  $H_i(\overline{\mathcal{B}}(A))$  is of finite type and  $H_i(\overline{\mathcal{B}}(A)) = \text{Tor}_i^A(\mathbb{k}, \mathbb{k})$  is null if  $i \leq -n$  or  $i \geq n$ . Therefore by part 2) of Property 10,  $\Theta : \overline{\mathcal{B}}(A)^\vee \otimes N \xrightarrow{\cong} \text{Hom}(\overline{\mathcal{B}}(A), N)$  is a quasi-isomorphism. A straightforward calculation shows that the following diagram commutes

$$\begin{array}{ccc} \overline{\mathcal{B}}(A)^\vee \otimes N & \xrightarrow[\cong]{\Theta} & \text{Hom}(\overline{\mathcal{B}}(A), N) = \overline{\mathcal{C}}^*(A, N) \\ & \searrow^{([m] \cap -) \otimes N} & \downarrow c \cap - \\ & & \overline{\mathcal{B}}(A) \otimes N = \overline{\mathcal{C}}_*(A, N) \end{array}$$

Since  $\overline{\mathcal{B}}(A)$  is  $\mathbb{k}$ -free and its dual  $\overline{\mathcal{B}}(A)^\vee$  is torsion free, by naturality of Kunnetth formula [30, Theorem 5.3.3],  $([m] \cap -) \otimes N$  is a quasi-isomorphism. Therefore  $c \cap -$  is also a quasi-isomorphism.  $\square$

**Proposition 12.** *The statement is true if  $A$  and  $N$  are non-negatively graded and  $H_0(\varepsilon) : H_0(A) \xrightarrow{\cong} \mathbb{k}$  is an isomorphism.*

*Proof.* For any  $k \geq 0$ , let  $F^k := \overline{A}^{\varepsilon^k} . N$ . The  $F^k$  form a decreasing filtration of sub- $A$ -bimodules and sub-complexes of  $N$ . Since  $F^k / F^{k+1}$  is a trivial  $A$ -bimodule, by Lemma 11, the morphism of complexes

$$\mathcal{C}^*(A, F^k / F^{k+1}) \xrightarrow{\cong} \mathcal{C}_*(A, F^k / F^{k+1}), a \mapsto c \cap a$$

is a quasi-isomorphism. By Noether theorem, we have the short exact sequences of  $A$ -bimodules

$$0 \rightarrow F^k/F^{k+1} \rightarrow N/F^{k+1} \rightarrow N/F^k \rightarrow 0.$$

Since  $Ts\bar{A}$  is  $\mathbb{k}$ -free, the functors  $\text{Hom}_{\mathbb{k}}(Ts\bar{A}, -)$  and  $- \otimes_{\mathbb{k}} Ts\bar{A}$  preserve short exact sequences. Therefore consider the morphism of short exact sequences of complexes induced by the cap product with  $c$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}^*(A, F^k/F^{k+1}) & \longrightarrow & \mathcal{C}^*(A, N/F^{k+1}) & \longrightarrow & \mathcal{C}^*(A, N/F^k) \longrightarrow 0 \\ & & \simeq \downarrow c \cap - & & \downarrow c \cap - & & \downarrow c \cap - \\ 0 & \longrightarrow & \mathcal{C}_*(A, F^k/F^{k+1}) & \longrightarrow & \mathcal{C}_*(A, N/F^{k+1}) & \longrightarrow & \mathcal{C}_*(A, N/F^k) \longrightarrow 0 \end{array}$$

Using the long exact sequences associated and the five lemma, by induction on  $k$ , we obtain that the morphism of complexes

$$\mathcal{C}^*(A, N/F^k) \xrightarrow{\cong} \mathcal{C}_*(A, N/F^k), a \mapsto c \cap a$$

is a quasi-isomorphism for all  $k \geq 0$ .

The two towers of complexes

$$\begin{array}{c} \cdots \rightarrow \mathcal{C}^*(A, N/F^{k+1}) \rightarrow \mathcal{C}^*(A, N/F^k) \rightarrow \cdots \\ \cdots \rightarrow \mathcal{C}_*(A, N/F^{k+1}) \rightarrow \mathcal{C}_*(A, N/F^k) \rightarrow \cdots \end{array}$$

satisfy the trivial Mittag-Leffler condition, since all the maps in the two towers are onto. Therefore by naturality of [33, Theorem 3.5.8], for each  $p \in \mathbb{Z}$ , we have morphism of short exact sequences induced by the cap product with  $c$

$$\begin{array}{ccccc} \lim_{\leftarrow}^1 HH^{p-1}(A, N/F^k) & \longrightarrow & H^p \lim_{\leftarrow} \mathcal{C}^*(A, N/F^k) & \longrightarrow & \lim_{\leftarrow} HH^p(A, N/F^k) \\ & & \downarrow \lim_{\leftarrow} c \cap - & & \downarrow \lim_{\leftarrow} c \cap - \\ \lim_{\leftarrow}^1 HH_{d+1-p}(A, N/F^k) & \longrightarrow & H_{d-p} \lim_{\leftarrow} \mathcal{C}_*(A, N/F^k) & \longrightarrow & \lim_{\leftarrow} HH_{d-p}(A, N/F^k) \end{array}$$

Using the five Lemma again, we obtain that the middle morphism

$$H(\lim_{\leftarrow} c \cap -) : H^p \lim_{\leftarrow} \mathcal{C}^*(A, N/F^k) \rightarrow H_{d-p} \lim_{\leftarrow} \mathcal{C}_*(A, N/F^k)$$

is an isomorphism.

Case 1: We now suppose that  $A$  and  $N$  are non-negatively graded and that  $\varepsilon : A_0 \xrightarrow{\cong} \mathbb{k}$  is an isomorphism. Then  $\overline{A}^{e^k}$  is concentrated in degrees  $\geq k$ . Therefore  $F^k$  and  $\mathcal{C}_*(A, F^k)$  are also concentrated in degrees  $\geq k$ . This means that for  $n < k$  their components of degree  $n$ ,  $(F^k)_n$  and  $[\mathcal{C}_*(A, N/F^k)]_n$  are trivial. Therefore the tower in degree  $n$

$$\cdots \rightarrow (N/F^{k+1})_n \rightarrow (N/F^k)_n \rightarrow \cdots$$

is constant and equal to  $N_n$  for  $k > n$ . This implies that  $N_n = \varprojlim (N/F^k)_n$ . Therefore as complexes and as  $A$ -bimodule,  $N = \varprojlim N/F^k$ .

Since  $\mathcal{C}_*(A, N/F^k)$  is the quotient  $\mathcal{C}_*(A, N)/\mathcal{C}_*(A, F^k)$ , we also have that as complexes,

$$\mathcal{C}_*(A, N) = \varprojlim \mathcal{C}_*(A, N/F^k)$$

The functor  $\mathcal{C}^*(A, -)$  from (differential)  $A$ -bimodules to complexes is a right adjoint. Therefore  $\mathcal{C}^*(A, -)$  preserves inverse limits. Since  $N = \varprojlim N/F^k$  in the category of (differential)  $A$ -bimodules, we obtain that as complex

$$\mathcal{C}^*(A, N) = \mathcal{C}^*(A, \varprojlim N/F^k) = \varprojlim \mathcal{C}^*(A, N/F^k).$$

Since for any  $k \geq 0$ , the following square commutes

$$\begin{array}{ccc} \mathcal{C}^*(A, N) & \longrightarrow & \mathcal{C}^*(A, N/F^k) , \\ \downarrow c \cap - & & \downarrow c \cap - \\ \mathcal{C}_*(A, N) & \longrightarrow & \mathcal{C}_*(A, N/F^k) \end{array}$$

the quasi-isomorphism

$$\varprojlim c \cap - : \varprojlim \mathcal{C}^*(A, N/F^k) \rightarrow \varprojlim \mathcal{C}_*(A, N/F^k)$$

coincides with  $c \cap - : \mathcal{C}^*(A, N) \rightarrow \mathcal{C}_*(A, N)$ .

Case 2: We now suppose that  $A$  and  $N$  are non-negatively graded and that  $H_0(\varepsilon) : H_0(A) \xrightarrow{\cong} \mathbb{k}$  is an isomorphism. Let  $\tilde{A}$  be the graded  $\mathbb{k}$ -module defined by  $\tilde{A}_0 = \mathbb{k}$ ,  $\tilde{A}_1 = \text{Ker } d : A_1 \rightarrow A_0$ ,  $\tilde{A}_n = A_n$  for  $n \geq 2$  (Compare with the cochain version in [8, p. 184]). Clearly  $\tilde{A}$  is a sub differential graded algebra of  $A$  and the inclusion  $j : \tilde{A} \xrightarrow{\cong} A$  is a quasi-isomorphism since  $\text{Im } d : A_1 \rightarrow A_0$  is equal to  $\overline{A}_0$ .

Since the augmentation ideals of  $A$  and  $\tilde{A}$ ,  $\overline{A}$  and  $\tilde{\tilde{A}}$ , are  $\mathbb{k}$ -free and non-negatively graded, by [23, 5.3.5] or [7, 4.3(iii)], the three morphisms  $HH_*(j, N) : HH_*(\tilde{A}, N) \xrightarrow{\cong} HH_*(A, N)$ ,  $HH^*(j, N) : HH^*(A, N) \xrightarrow{\cong} HH^*(\tilde{A}, N)$  and  $HH_*(j, j) : HH_*(\tilde{A}, \tilde{A}) \xrightarrow{\cong} HH_*(A, A)$  are all isomorphisms. Let  $\tilde{c} \in HH_d(\tilde{A}, \tilde{A})$  such that  $HH_d(j, j)(\tilde{c}) = c$ . Using the definition of the cap product, it is straightforward to check that the

following square commutes

$$\begin{array}{ccc}
 HH^*(A, N) & \xrightarrow[\cong]{HH^*(j, N)} & HH^*(\tilde{A}, N) \\
 \downarrow c \cap - & & \downarrow \tilde{c} \cap - \\
 HH_*(A, N) & \xleftarrow[\cong]{HH_*(j, N)} & HH_*(\tilde{A}, N)
 \end{array}$$

Let  $[\tilde{m}] \in \text{Tor}_d^{\tilde{A}}(\mathbb{k}, \mathbb{k})$  such that  $\text{Tor}_d^j(\mathbb{k}, \mathbb{k})([\tilde{m}]) = [m]$ . When  $N = \mathbb{k}$ , the previous square specializes to the following commutative square

$$\begin{array}{ccc}
 \text{Ext}_A^*(\mathbb{k}, \mathbb{k}) & \xrightarrow[\cong]{\text{Ext}_j^*(\mathbb{k}, \mathbb{k})} & \text{Ext}_{\tilde{A}}^*(\mathbb{k}, \mathbb{k}) \\
 [m] \cap - \downarrow \cong & & \downarrow [\tilde{m}] \cap - \\
 \text{Tor}_*^A(\mathbb{k}, \mathbb{k}) & \xleftarrow[\cong]{\text{Tor}_*^{\tilde{A}}(\mathbb{k}, \mathbb{k})} & \text{Tor}_*^{\tilde{A}}(\mathbb{k}, \mathbb{k})
 \end{array}$$

By hypothesis,  $[m] \cap -$  is an isomorphism. Therefore  $[\tilde{m}] \cap -$  is also an isomorphism. So since  $\tilde{A}_0 = \mathbb{k}$ , we have seen in the case 1, that

$$\tilde{c} \cap - : HH^*(\tilde{A}, N) \xrightarrow{\cong} HH_*(\tilde{A}, N)$$

is an isomorphism. Therefore

$$c \cap - : HH^*(A, N) \xrightarrow{\cong} HH_*(A, N)$$

is also an isomorphism. □

#### 4. COMPARISON OF THE CAP PRODUCTS IN HOCHSCHILD AND GROUP (CO)HOMOLOGY

Let  $G$  be a discrete group. Let  $M$  and  $N$  be two  $\mathbb{k}[G]$ -bimodules. Let  $\eta : \mathbb{k} \rightarrow \mathbb{k}[G]$  be the unit map. Let  $E : \mathbb{k}[G] \rightarrow \mathbb{k}[G \times G^{op}]$  be the morphism of algebras mapping  $g$  to  $(g, g^{-1})$ . Let

$$\tilde{\eta} : \mathbb{k}[G \times G^{op}] \otimes_{\mathbb{k}[G]} \mathbb{k} \rightarrow \mathbb{k}[G]$$

be the unique morphism of left  $\mathbb{k}[G \times G^{op}]$ -modules extending  $\eta$ . Since  $\mathbb{k}[G \times G^{op}]$  is flat as left  $\mathbb{k}[G]$ -module via  $E$  and since  $\tilde{\eta}$  is an isomorphism, by Eckmann-Schapiro [18, Chapt IV.Proposition 12.2], we obtain the well-known isomorphisms between Hochschild (co)homology and group (co)homology:

$$\text{Ext}_E^*(\eta, N) : HH^*(\mathbb{k}[G], N) = \text{Ext}_{\mathbb{k}[G \times G^{op}]}^*(\mathbb{k}[G], N) \xrightarrow{\cong} \text{Ext}_{\mathbb{k}[G]}^*(\mathbb{k}, \tilde{N}) = H^*(G, \tilde{N}).$$

and

$$\text{Tor}_*^E(M, \eta) : H_*(G, \tilde{M}) = \text{Tor}_*^{\mathbb{k}[G]}(\tilde{M}, \mathbb{k}) \xrightarrow{\cong} \text{Tor}_{\mathbb{k}[G \times G^{op}]}^*(M, \mathbb{k}[G]) = HH_*(\mathbb{k}[G], M).$$

Here  $\tilde{M}$  and  $\tilde{N}$  denote the  $\mathbb{k}[G]$ -modules obtained by restriction of scalar via  $E$ . Note that we regard any left  $\mathbb{k}[G]$ -module as a right  $\mathbb{k}[G]$ -module via  $g \mapsto g^{-1}$  [3, p. 55].

**Proposition 13.** *Remark that the canonical surjection*

$$q : \tilde{M} \otimes \tilde{N} \rightarrow \widetilde{M \otimes_{\mathbb{k}[G]} N}$$

is a morphism of  $\mathbb{k}[G]$ -modules.

i) *Cup product  $\cup$  in Hochschild cohomology versus cup product in group cohomology (slight extension of [29, Proposition 3.1]). The following diagram commutes*

$$\begin{array}{ccc} HH^*(\mathbb{k}[G], M) \otimes HH^*(\mathbb{k}[G], N) & \xrightarrow{\cup} & HH^*(\mathbb{k}[G], M \otimes_{\mathbb{k}[G]} N) \\ \downarrow \text{Ext}_E^*(\eta, M) \otimes \text{Ext}_E^*(\eta, N) & & \downarrow \text{Ext}_E^*(\eta, M \otimes_{\mathbb{k}[G]} N) \\ H^*(G, \tilde{M}) \otimes H^*(G, \tilde{N}) & \xrightarrow{\cup} & H^*(G, \tilde{M} \otimes \tilde{N}) \xrightarrow{H^*(G, q)} H^*(G, \widetilde{M \otimes_{\mathbb{k}[G]} N}) \end{array}$$

ii) *Cap products  $\cap$ . The following diagram commutes*

$$\begin{array}{ccc} HH_*(\mathbb{k}[G], M) \otimes HH^*(\mathbb{k}[G], N) & \xrightarrow{\cap} & HH_*(\mathbb{k}[G], M \otimes_{\mathbb{k}[G]} N) \\ \uparrow \text{Tor}_*^E(M, \eta) \otimes \text{Ext}_E^*(\eta, N)^{-1} & & \uparrow \text{Tor}_*^E(M \otimes_{\mathbb{k}[G]} N, \eta) \\ H_*(G, \tilde{M}) \otimes H^*(G, \tilde{N}) & \xrightarrow{\cap} & H_*(G, \tilde{M} \otimes \tilde{N}) \xrightarrow{H_*(G, q)} H_*(G, \widetilde{M \otimes_{\mathbb{k}[G]} N}) \end{array}$$

*Remark 14.* In the case  $M = N = \mathbb{k}[G]$ , the diagram i) in Proposition 13 means that

$$\text{Ext}_E^*(\eta, \mathbb{k}[G]) : HH^*(\mathbb{k}[G], \mathbb{k}[G]) \rightarrow H^*(G, \mathbb{k}[\tilde{G}])$$

is a morphism of graded algebras.

In the case  $N = \mathbb{k}[G]$ , the diagram ii) means that

$$\text{Tor}_*^E(M, \eta) : H_*(G, \tilde{M}) \rightarrow HH_*(\mathbb{k}[G], M)$$

is a morphism of right  $HH^*(\mathbb{k}[G], \mathbb{k}[G])$ -modules:

$$\text{Tor}_*^E(\eta, \mathbb{k}[G]) (\alpha \cap \text{Ext}_E^*(\eta, \mathbb{k}[G])(\varphi)) = \text{Tor}_*^E(\eta, \mathbb{k}[G])(\alpha) \cap \varphi$$

for any  $\alpha \in H_*(G, \tilde{M})$  and any  $\varphi \in HH^*(\mathbb{k}[G], \mathbb{k}[G])$ .

*Proof.* Siegel and Witherspoon [29, Proposition 3.1] proved i) using that for any projective resolution  $P$  of  $\mathbb{k}$  as left  $\mathbb{k}[G]$ -modules,

$$X := \mathbb{k}[G \times G^{op}] \otimes_{\mathbb{k}[G]} P$$

is a projective resolution of  $\mathbb{k}[G]$  as  $\mathbb{k}[G]$ -bimodules. Let  $\iota : P \hookrightarrow \tilde{X}$  the left  $\mathbb{k}[G]$ -linear map defined by  $\iota(x) = (1, 1) \otimes x$ . Using that

$$\text{Hom}_E(\iota, N) : \text{Hom}_{\mathbb{k}[G \times G^{op}]}(X, N) \xrightarrow{\cong} \text{Hom}_{\mathbb{k}[G]}(P, \tilde{N})$$

is an isomorphism of complexes inducing  $\text{Ext}_E^*(\eta, N)$  and that

$$M \otimes_E \iota : \tilde{M} \otimes_{\mathbb{k}[G]} P \xrightarrow{\cong} M \otimes_{\mathbb{k}[G \times G^{op}]} X$$

is an isomorphism of complexes inducing  $\text{Tor}_*^E(M, \eta)$ , Siegel and Witherspoon [29, Proposition 3.1] proved i). But one can also prove similarly ii).

We find more simple to give a proof of ii) using the Bar resolution.

Let  $\iota : B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}) \rightarrow B(\mathbb{k}[G]; \widetilde{\mathbb{k}[G]}; \mathbb{k}[G])$  be the linear map defined by

$$\iota(g_0[g_1 | \dots | g_n]) = g_0[g_1 | \dots | g_n]g_n^{-1} \dots g_0^{-1}.$$

Obviously  $\iota$  fits into the commutative diagram of left  $\mathbb{k}[G]$ -modules

$$\begin{array}{ccc} B(\mathbb{k}[G]; \widetilde{\mathbb{k}[G]}; \mathbb{k}[G]) & \longrightarrow & \mathbb{k}[G] \\ \uparrow \iota & & \uparrow \eta \\ B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}) & \longrightarrow & \mathbb{k} \end{array}$$

A straightforward computation shows that  $\iota$  is a morphism of complexes. Therefore  $\text{Hom}_E(\iota, N)$  is an morphism of complexes from  $\mathcal{C}^*(\mathbb{k}[G], N) \cong (\text{Hom}_{\mathbb{k}[G \times G^{op}]}(B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}[G]), N)$  to  $\text{Hom}_{\mathbb{k}[G]}(B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}), \tilde{N})$ . inducing  $\text{Ext}_E^*(\eta, N)$  and  $M \otimes_E \iota$  is an morphism of complexes from

$$B(\tilde{M}; \mathbb{k}[G]; \mathbb{k}) \cong \tilde{M} \otimes_{\mathbb{k}[G]} B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k})$$

to

$$M \otimes_{\mathbb{k}[G \times G^{op}]} B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}[G]) \cong \mathcal{C}_*(\mathbb{k}[G], M)$$

inducing  $\text{Tor}_*^E(M, \eta)$ . Explicitly  $M \otimes_E \iota$  is the morphism of complexes

$$B(\tilde{M}; \mathbb{k}[G]; \mathbb{k}) \rightarrow \mathcal{C}_*(\mathbb{k}[G], M)$$

defined by [11, (2.20)]

$$(15) \quad \xi(m[g_1 | \dots | g_n]) = g_n^{-1} \dots g_1^{-1} m[g_1 | \dots | g_n].$$

And  $\text{Hom}_E(\iota, N) : \mathcal{C}^*(\mathbb{k}[G], N) \rightarrow \text{Hom}(B(\mathbb{k}[G]), \tilde{N})$ ,  $d$  is the morphism of complexes  $\xi$  mapping  $\varphi \in \mathcal{C}^*(\mathbb{k}[G], N)$  to the linear map  $\xi(\varphi) : \mathbb{k}[G] \rightarrow \tilde{N}$  defined by

$$\xi(\varphi)([g_1 | \dots | g_n]) = \varphi([g_1 | \dots | g_n])g_n^{-1} \dots g_1^{-1}.$$

Both  $M \otimes_E \iota$  and  $\text{Hom}_E(\iota, N)$  are in fact isomorphisms of complexes. The inverse of  $M \otimes_E \iota$  is the morphism of complexes  $\Phi : \mathcal{C}_*(\mathbb{k}[G], M) \rightarrow B(\tilde{M}; \mathbb{k}[G]; \mathbb{k})$  defined by [23, 7.4.2.1]

$$\Phi(m[g_1 | \dots | g_n]) = g_1 \dots g_n m[g_1 | \dots | g_n].$$

Let  $F$  be any projective resolution of  $\mathbb{k}$  as left  $\mathbb{k}[G]$ -module. Let  $P$  and  $Q$  be two  $\mathbb{k}[G]$ -modules. The cap product in group cohomology is the composite [3, p. 113], denoted  $\cap$

$$\begin{array}{c} P \otimes_{\mathbb{k}[G]} F \otimes \text{Hom}_{\mathbb{k}[G]}(F, Q) \\ \downarrow \text{Id} \otimes_{\mathbb{k}[G]} \Delta \otimes_{\mathbb{k}[G]} \text{Id} \\ P \otimes_{\mathbb{k}[G]} (F \otimes F) \otimes \text{Hom}_{\mathbb{k}[G]}(F, Q) \\ \downarrow \gamma \\ (P \otimes Q) \otimes_{\mathbb{k}[G]} F \end{array}$$

where  $\gamma(a \otimes x \otimes y \otimes u) = (-1)^{|u||x|+|u||y|} a \otimes u(x) \otimes y$  and  $\Delta$  is a *diagonal approximation*. In the case,  $F$  is the Bar resolution  $B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k})$ , one can take  $\Delta$  to be the Alexander-Whitney map

$$AW : B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}) \rightarrow B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k}) \otimes B(\mathbb{k}[G]; \mathbb{k}[G]; \mathbb{k})$$

defined by [3, p. 108 (1.4)]:

$$AW(g_0|g_1|\dots|g_n) = \sum_{p=0}^n g_0|g_1|\dots|g_p \otimes g_0 \dots g_p|g_{p+1}|\dots|g_n.$$

Therefore the cap product

$$\cap : B(P; \mathbb{k}[G]; \mathbb{k}) \otimes \text{Hom}(B(\mathbb{k}[G]), Q), d \rightarrow B(P \otimes Q; \mathbb{k}[G]; \mathbb{k})$$

is the morphism of complexes mapping  $m|g_1|\dots|g_n \otimes u : G^p \rightarrow Q$  to  $m.g_1 \dots g_p \otimes u(g_1, \dots, g_p).g_1 \dots g_p|g_{p+1}|\dots|g_n$ . Using the explicit formula (7) for the cap product in Hochschild cohomology, it is easy to check that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}_*(\mathbb{k}[G], M) \otimes \mathcal{C}^*(\mathbb{k}[G], N) & \xrightarrow{\cap} & \mathcal{C}_*(\mathbb{k}[G], M \otimes_{\mathbb{k}[G]} N) \\ \downarrow \Phi \otimes \text{Hom}_E(\iota, N) & & \downarrow \Phi \\ B(\tilde{M}; \mathbb{k}[G]; \mathbb{k}) \otimes H^*(G, \tilde{N}) & \xrightarrow{\cap} & B(\tilde{M} \otimes \tilde{N}; \mathbb{k}[G]; \mathbb{k}) \xrightarrow{B(q; \mathbb{k}[G]; \mathbb{k})} B(\widetilde{M \otimes_{\mathbb{k}[G]} N}; \mathbb{k}[G]; \mathbb{k}) \end{array}$$

By applying homology, ii) is proved.  $\square$

**Definition 16.** [23, 7.4.5 when  $z=1$ ] Let  $\sigma : B(\mathbb{k}[G]) \hookrightarrow \mathcal{C}_*(\mathbb{k}[G], \mathbb{k}[G])$  be the linear map defined by

$$\sigma([g_1|\dots|g_n]) = g_n^{-1} \dots g_1^{-1}[g_1|\dots|g_n].$$

*Property 17.* i) [23, 7.4.5 when  $z=1$ ] The map  $\sigma$  is a morphism of cyclic modules.

ii) The morphism of complexes  $\sigma$  coincides with the composite

$$B(\mathbb{k}[G]) \xrightarrow{B(\eta, \mathbb{k}[G]; \mathbb{k})} B(\widetilde{\mathbb{k}[G]}; \mathbb{k}[G]; \mathbb{k}) \xrightarrow{\xi} \mathcal{C}_*(\mathbb{k}[G]; \mathbb{k}[G]).$$

Here  $\xi$  is the isomorphism of complexes defined by (15). Note that the unit map  $\eta : \mathbb{k} \rightarrow \widetilde{\mathbb{k}[G]}$  is a morphism of  $\mathbb{k}[G]$ -modules.

iii) In particular, in homology,  $\sigma$  coincides with

$$\mathrm{Tor}^E(\eta, \eta) : H_*(G; \mathbb{k}) \rightarrow HH_*(\mathbb{k}[G]; \mathbb{k}[G]).$$

iv) The map  $\sigma$  is a section of

$$\mathcal{C}_*(\mathbb{k}[G], \varepsilon) : \mathcal{C}_*(\mathbb{k}[G], \mathbb{k}[G]) \rightarrow \mathcal{C}_*(\mathbb{k}[G], \mathbb{k}) = B(\mathbb{k}[G]).$$

**Corollary 18.** *Let  $G$  be any discrete group. Let  $N$  be a  $\mathbb{k}[G]$ -bimodule. Let  $\sigma : H_*(G; \mathbb{k}) \rightarrow HH_*(\mathbb{k}[G]; \mathbb{k}[G])$  be the section of  $HH_*(\mathbb{k}[G], \varepsilon) : HH_*(\mathbb{k}[G], \mathbb{k}[G]) \rightarrow H_*(G, \mathbb{k})$  defined in Definition 16. Let  $z \in H_d(G, \mathbb{k})$  be any element in group homology. Then the following square commutes*

$$\begin{array}{ccc} H^p(G, \tilde{N}) & \xrightarrow{z \cap -} & H_{d-p}(G, \tilde{N}) \\ \mathrm{Ext}_E^*(\eta, N) \uparrow \cong & & \cong \downarrow \mathrm{Tor}_*^E(N, \eta) \\ HH^p(\mathbb{k}[G], N) & \xrightarrow{\sigma(z) \cap -} & HH_{d-p}(\mathbb{k}[G], N) \end{array}$$

*Proof.*

$$\begin{array}{ccccc} HH_*(\mathbb{k}[G], \mathbb{k}[G]) \otimes HH^*(\mathbb{k}[G], N) & \xrightarrow{\cap} & HH_*(\mathbb{k}[G], \mathbb{k}[G] \otimes_{\mathbb{k}[G]} N) \\ \mathrm{Tor}_*^E(\mathbb{k}[G], \eta) \otimes \mathrm{Ext}_E^*(\eta, N)^{-1} \uparrow & & \mathrm{Tor}_*^E(N, \eta) \uparrow \\ H_*(G, \widetilde{\mathbb{k}[G]}) \otimes H^*(G, \tilde{N}) & \xrightarrow{\cap} & H_*(G, \widetilde{\mathbb{k}[G]} \otimes \tilde{N}) \xrightarrow{H_*(G, q)} & H_*(G, \tilde{N}) \\ H_*(G, \eta) \otimes \mathrm{Id} \uparrow & & H_*(G, \eta \otimes \tilde{N}) \uparrow & \nearrow \cong \\ H_*(G, \mathbb{k}) \otimes H^*(G, \tilde{N}) & \xrightarrow{\cap} & H_*(G, \mathbb{k} \otimes \tilde{N}) \end{array}$$

The top rectangle commutes by ii) of Proposition 13 in the case  $M = \mathbb{k}[G]$ . The Bottom square commutes by naturality of the cap product in group (co)homology with respect to the morphism of  $\mathbb{k}[G]$ -modules  $\eta : \mathbb{k} \rightarrow \widetilde{\mathbb{k}[G]}$ . The Bottom triangle commutes by functoriality of  $H_*(G, -)$ . By ii) or iii) of Property 17, the vertical composite is

$$\sigma \otimes \mathrm{Ext}_E^*(\eta, N)^{-1} : H_*(G, \mathbb{k}) \otimes H^*(G, \tilde{N}) \rightarrow HH_*(\mathbb{k}[G], \mathbb{k}[G]) \otimes HH^*(\mathbb{k}[G], N).$$

□

## 5. A NEW DEFINITION OF BATALIN-VILKOVISKY ALGEBRAS

**Definition 19.** A *Gerstenhaber algebra* is a commutative graded algebra  $A$  equipped with a linear map  $\{-, -\} : A_i \otimes A_j \rightarrow A_{i+j+1}$  of degree 1 such that:

a) the bracket  $\{-, -\}$  gives  $A$  a structure of graded Lie algebra of degree 1. This means that for each  $a, b$  and  $c \in A$

$$(20) \quad \{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\} \text{ and}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)}\{b, \{a, c\}\}.$$

b) the product and the Lie bracket satisfy the following relation called the Poisson relation:

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|+1)|b|}b\{a, c\}.$$

**Definition 21.** A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra  $A$  equipped with a degree 1 linear map  $\Delta : A_i \rightarrow A_{i+1}$  such that  $\Delta \circ \Delta = 0$  and such that the bracket is given by

$$(22) \quad \{a, b\} = (-1)^{|a|} (\Delta(a \cup b) - (\Delta a) \cup b - (-1)^{|a|}a \cup (\Delta b))$$

for  $a$  and  $b \in A$ .

*Remark 23.* In (22), a sign (here the sign chosen is  $(-1)^{|a|}$ ) is needed (See [21, (1.6)] or [14, beginning of the Proof of Proposition 1.2]), since the Lie bracket of degree +1 is graded antisymmetric (equation (20)) while the associative product is graded commutative. Therefore the definition of Batalin-Vilkovisky algebra in [15, Theorem 3.4.3 (ii)] and [22, p. 1] has a sign problem.

The following characterization of Batalin-Vilkovisky algebras was proved by Koszul and rediscovered by Getzler and by Penkava and Schwarz.

**Proposition 24.** [21, p. 3] [14, Proposition 1.2] [28] *Let  $A$  be a commutative graded algebra  $A$  equipped with an operator  $\Delta : A_i \rightarrow A_{i+1}$  of degree 1 such that  $\Delta \circ \Delta = 0$ . Consider the bracket  $\{, \}$  of degree +1 defined by*

$$\{a, b\} = (-1)^{|a|} (\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b))$$

for any  $a, b \in A$ . Then  $A$  is a Batalin-Vilkovisky algebra if and only if  $\Delta$  is a differential operator of degree  $\leq 2$ , this means that for  $a, b$  and  $c \in A$ ,

$$(25) \quad \Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ - (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).$$

Remark that till now, in this section, it is not necessary that the algebras have an unit. Now if the algebras have an unit, we give a new characterization of Batalin-Vilkovisky algebra. One implication in this new characterization is inspired by Ginzburg proof of Proposition 30. As we will recall in the proof of Theorem 53, the converse in this characterization is due to [20, “the restriction of this derived bracket to  $A$  is the BV-bracket”, p. 1270].

**Proposition 26.** *Let  $A$  be a Gerstenhaber algebra  $A$  equipped with an operator  $\Delta : A \rightarrow A$  of degree 1 such that  $\Delta \circ \Delta = 0$ . For any  $a \in A$ , denote by  $l_a : A \rightarrow A$ , the left multiplication by  $a$ , explicitly  $l_a(b) = ab$ ,  $b \in A$ . Denote by  $[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$  the graded commutator of two endomorphisms  $f : A \rightarrow A$  and  $g : A \rightarrow A$ . Then  $A$  is a Batalin-Vilkovisky algebra if and only if for  $a, b \in A$ ,*

$$l_{\{a,b\}} = -[[l_a, \Delta], l_b] \quad \text{and} \quad \Delta(1) = 0.$$

*Proof.* For  $a$  and  $b \in A$ ,

$$\begin{aligned} [[l_a, \Delta], l_b] &= (l_a \circ \Delta - (-1)^{|a|} \Delta \circ l_a) \circ l_b \\ &\quad - (-1)^{|b|(|a|+1)} l_b \circ (l_a \circ \Delta - (-1)^{|a|} \Delta \circ l_a) \\ &= l_a \circ \Delta \circ l_b - (-1)^{|a|} \Delta \circ l_{ab} - (-1)^{|b|} l_{ab} \circ \Delta + (-1)^{|b|(|a|+1)+|a|} l_b \circ \Delta \circ l_a. \end{aligned}$$

Therefore by applying this equality of operators to  $c \in A$ , we have

$$(27) \quad -(-1)^{|a|} [[l_a, \Delta], l_b](c) = -(-1)^{|a|} a \Delta(bc) + \Delta(abc) \\ + (-1)^{|a|+|b|} ab \Delta(c) - (-1)^{|b|(|a|+1)} b \Delta(ac).$$

Suppose that  $A$  is a Batalin-Vilkovisky algebra. By Proposition 24, using (27), we obtain that

$$-(-1)^{|a|} [[l_a, \Delta], l_b](c) = \Delta(ab)c - (\Delta a)bc - (-1)^{|a|} a(\Delta b)c = (-1)^{|a|} \{a, b\}c.$$

Therefore  $-[[l_a, \Delta], l_b] = l_{\{a,b\}}$ . In the case  $a = b = c = 1$ , equation (25) gives  $\Delta(1) = 3\Delta(1) - 3\Delta(1) = 0$ .

Conversely, suppose that  $\Delta(1) = 0$  and that  $l_{\{a,b\}} = -[[l_a, \Delta], l_b]$ . Then using (27)

$$\{a, b\} = l_{\{a,b\}}(1) = (-1)^{|a|} (-(-1)^{|a|} a \Delta(b) + \Delta(ab) + 0 - (\Delta a)b).$$

Therefore, by Definition 21,  $A$  is a Batalin-Vilkovisky algebra.  $\square$

## 6. BATALIN-VILKOVISKY ALGEBRA STRUCTURES ON HOCHSCHILD COHOMOLOGY

Let  $A$  be a differential graded algebra. The cap product defined in Section 3,

$$HH_*(A, A) \otimes HH^*(A, A) \xrightarrow{\cap} HH_*(A, A), c \otimes a \mapsto c \cap a$$

is a right action.

Following Tsygan definition of a calculus, we want a left action. Therefore, we define as [22, Definition 1.2],

$$\mathcal{C}^*(A, A) \otimes \mathcal{C}_*(A, A) \rightarrow \mathcal{C}_*(A, A)$$

$$(28) \quad f \otimes c \mapsto i_f(c) = f \cdot c := (-1)^{|c||f|} c \cap f.$$

Explicitly

$$i_f(m[a_1 | \dots | a_n]) := \sum_{p=0}^n (-1)^{|m||f|} (m.f[a_1 | \dots | a_p])[a_{p+1} | \dots | a_n].$$

The sign in [6, (18) p. 82] is different. But with our choice of signs, we recover Proposition 2.6 in [6, p. 82]. Indeed for  $D, E \in \mathcal{C}^*(A, A)$  and  $c \in \mathcal{C}_*(A, A)$ ,

$$\begin{aligned} D \cdot (E \cdot c) &= (-1)^{|c||E|} D \cdot (c \cap E) = (-1)^{|c||E|+|D||c|+|D||E|} (c \cap E) \cap D \\ &= (-1)^{|c||E|+|D||c|+|D||E|} c \cap (E \cup D) = (-1)^{|D||E|} (E \cup D) \cdot c \end{aligned}$$

Since the cup product on  $HH^*(A, A)$  is graded commutative, for  $D, E \in HH^*(A, A)$  and  $c \in HH_*(A, A)$ , we have

$$(29) \quad D \cdot (E \cdot c) = (D \cup E) \cdot c,$$

i. e. a left action. Note that in [27], the author forgot to twist the right action by the sign  $(-1)^{|c||f|}$ , therefore has also a sign problem!

**Proposition 30.** [15, Theorem 3.4.3 (ii)] *Let  $c \in HH_d(A, A)$  such that the morphism of left  $HH^*(A, A)$ -modules*

$$HH^p(A, A) \xrightarrow{\cong} HH_{d-p}(A, A), a \mapsto a \cdot c$$

*is an isomorphism. If  $B(c) = 0$  then the Gerstenhaber algebra  $HH^*(A, A)$  equipped with  $-B$  is a Batalin-Vilkovisky algebra.*

*Proof.* Let us rewrite the proof of Victor Ginzburg using explicitly our Proposition 26 and our choice of signs. Denote by

$$\begin{aligned} HH^p(A, A) \otimes HH_j(A, A) &\rightarrow HH_{j-p+1}(A, A) \\ a \otimes x &\mapsto L_a(x) \end{aligned}$$

the action of the suspended graded Lie algebra  $HH^*(A, A)$  on  $HH_*(A, A)$ . Gelfand, Daletski and Tsygan [12] proved that the Gerstenhaber algebra  $HH^*(A, A)$  and Connes boundary map  $B$  on  $HH_*(A, A)$  form a calculus [6, p. 93]. In particular, we have the two relations

$$L_a = [B, i_a]$$

and [6, Proposition 2.9 p. 83]

$$(31) \quad i_{\{a,b\}} = (-1)^{|a|+1}[L_a, i_b].$$

Therefore

$$(32) \quad i_{\{a,b\}} = (-1)^{|a|+1}[[B, i_a], i_b] = [[i_a, B], i_b].$$

The operator  $\Delta$  on  $HH^*(A, A)$  is defined by

$$(\Delta a) \cdot c := -B(a \cdot c) \quad \text{for any } a \in HH^*(A, A).$$

Thus  $B(c) = 0$  implies  $\Delta(1) = 0$ . Since we have a left action (equation (29)),  $l_a(b) \cdot c = (a \cup b) \cdot c = a \cdot (b \cdot c) = i_a(b \cdot c)$  and so equation (32) is equivalent to

$$l_{\{a,b\}} = -[[l_a, \Delta], l_b].$$

Therefore, by Proposition 26,  $HH^*(A, A)$  is a Batalin-Vilkovisky algebra.  $\square$

*Remark 33.* (Signs)

i) In [6, Example 4.6 p. 93], Tsygan writes that it follows from [6, 2.9 p. 83], that  $i_{\{a,b\}} = [L_a, i_b]$ . We do not understand why he has no sign in this formula. We believe that from [6, 2.9 p. 83], the correct equation with the signs is equation (31) above.

ii) In a calculus, there is a third relation, that we do not use in this paper:

$$L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b.$$

Since  $ab = (-1)^{|a||b|} ba$ ,

$$L_{ab} = (-1)^{|a||b|} L_{ba} = (-1)^{|a||b|} L_b i_a + (-1)^{(|a|+1)|b|} i_b L_a$$

and therefore

$$(34) \quad [L_a, i_b] = (-1)^{|a||b|} [L_b, i_a]$$

Since  $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)} \{b, a\}$ ,

-if we suppose like Tsygan that  $i_{\{a,b\}} = [L_a, i_b]$ , we obtain that

$$(35) \quad [L_a, i_b] = -(-1)^{(|a|+1)(|b|+1)} [L_b, i_a].$$

The two equations (34) and (35) seem incoherent. Therefore we believe that the definition of calculus of Tsygan has some sign problem.

-on the contrary, if we suppose (31), we obtain again (34).

## 7. PROOF OF THE MAIN THEOREM FOR PATH-CONNECTED GROUPS

**Cap products associated to coalgebras.** Let  $C$  be a (differential graded) coalgebra. Then its dual  $C^\vee$  is a (differential graded) algebra. Let  $N$  be a left  $C$ -comodule. Denote by  $\Delta_N : N \rightarrow C \otimes N$  the structure map. Let  $\cap : N \rightarrow C^\vee \rightarrow N$  be the composite

$$N \otimes C^\vee \xrightarrow{\Delta_N \otimes C^\vee} C \otimes N \otimes C^\vee \xrightarrow{N \otimes \tau} C \otimes C^\vee \otimes N \xrightarrow{ev \otimes N} \mathbb{k} \otimes N \cong N.$$

Here  $\tau$  denotes the twist map given by  $n \otimes \varphi \mapsto (-1)^{|n||\varphi|} \varphi \otimes n$  and  $ev$  is the evaluation map defined by  $ev(c \otimes \varphi) = (-1)^{|\varphi||c|} \varphi(c)$ . Then  $N$  equipped with the cap product is a left  $C^\vee$ -module [31, Proposition 2.1.1]. In this paper, we are only interested in the case  $N = C$ .

*Example 36.* Let  $X$  be any topological space. The (normalized or unnormalized) singular chains of  $X$ ,  $S_*(X)$  forms a differential graded coalgebra [25, p. 244-5]. The cap product defined above associated to  $C = S_*(X)$ ,  $\cap : S_*(X) \otimes S^*(X) \rightarrow S_*(X)$  is the usual cap product.

*Example 37.* Let  $A$  be any augmented differential graded algebra. Then the reduced (normalized or not) Bar construction  $B(A) = \mathcal{C}_*(A, \mathbb{k})$  is a differential graded coalgebra. The diagonal  $\Delta : B(A) \rightarrow B(A) \otimes B(A)$  is given by

$$\Delta([a_1 | \dots | a_n]) = \sum_{p=0}^n [a_1 | \dots | a_p] \otimes [a_{p+1} | \dots | a_n].$$

The cap product defined above associated to  $C = B(A)$  is given by

$$\begin{aligned} \cap : B(A) \otimes B(A)^\vee &\rightarrow B(A) \\ [a_1 | \dots | a_n] \cap f &= \sum_{p=0}^n (-1)^{|f|(|a_1| + \dots + |a_n|)} f([a_1 | \dots | a_p]) [a_{p+1} | \dots | a_n]. \end{aligned}$$

Therefore this cap product coincides with the cap product on the Hochschild (co)chain complex  $\cap : \mathcal{C}_*(A, \mathbb{k}) \otimes \mathcal{C}^*(A, \mathbb{k}) \rightarrow \mathcal{C}_*(A, \mathbb{k})$  defined by (8) in the case  $N = B = \mathbb{k}$ .

**Proposition 38.** *Let  $f : C \xrightarrow{\cong} D$  be a quasi-isomorphism of coalgebras. Suppose that  $C$  and  $D$  are  $\mathbb{k}$ -free. Let  $\tilde{c} \in C$  and  $\tilde{d} \in D$  such that  $\tilde{d} = H_*(f)([\tilde{c}])$ . Consider the cap products defined above associated to the coalgebras  $C$  and  $D$ . Then*

*the morphism of right  $C^\vee$ -modules  $\tilde{c} \cap - : C^\vee \rightarrow C$  given by  $a \mapsto \tilde{c} \cap a$  is quasi-isomorphism if and only if*

*the morphism of right  $D^\vee$ -modules  $\tilde{d} \cap - : D^\vee \rightarrow D$  given by  $a \mapsto \tilde{d} \cap a$  is quasi-isomorphism.*

*Proof.* The transpose of  $f: f^\vee : D^\vee \rightarrow C^\vee$  is a morphism of differential graded algebras. Therefore  $f^\vee$  is a morphism of right  $D^\vee$ -modules. Dually, since  $f$  is a morphism of coalgebras,  $f$  is a morphism of left  $D$ -comodules and therefore is also a morphism of right  $D^\vee$ -modules, i. e.  $f(c \cap f^\vee(\varphi)) = c \cap \varphi$  for any  $c \in C$  and  $\varphi \in D^\vee$ . Note that if  $f$  is the coalgebra map  $S_*(\lambda) : S_*(X) \rightarrow S_*(Y)$  included by a continuous map  $\lambda : X \rightarrow Y$ , this formula is well known ([2, Chapter VI 5. Theorem (4)] or [17, p. 241]).

The composite of the morphism of right  $D^\vee$ -modules

$$D^\vee \rightarrow C^\vee \xrightarrow{\tilde{c} \cap -} C \xrightarrow{f} D$$

maps 1 to  $f(\tilde{c})$  and therefore coincides with the morphism of right  $D^\vee$ -modules  $D^\vee \rightarrow D$ ,  $a \mapsto f(\tilde{c}) \cap a$ . Since  $[\tilde{d}] = [f(\tilde{c})]$ , the two maps  $a \mapsto f(\tilde{c}) \cap a$  and  $a \mapsto \tilde{d} \cap a$  coincide after passing to homology. Therefore after passing to homology, the following square commutes

$$(39) \quad \begin{array}{ccc} D^\vee & \xrightarrow{f^\vee} & C^\vee \\ \tilde{d} \cap - \downarrow & & \downarrow \tilde{c} \cap - \\ D & \xrightarrow[\simeq]{f} & C \end{array}$$

Since both  $C$  and  $D$  are  $\mathbb{k}$ -free and  $\mathbb{k}$  is a principal ideal domain, by naturality of the universal coefficient theorem for cohomology,  $H_*(f^\vee)$  is an isomorphism since  $H_*(f)$  is an isomorphism. The lemma follows now from the square (39).  $\square$

**Theorem 40.** *Let  $M$  be a simply-connected oriented Poincaré duality space of formal dimension  $d$ . Let  $G$  be a topological group such that  $M$  is a classifying space for  $G$  or let  $G$  be  $\Omega M$  the (Moore) pointed loop space on  $M$ . Let  $[M] \in H_d(M)$  be its fundamental class. Let  $c$  the image of  $[M]$  through the composite*

$$H_*(M) \xrightarrow{H_*(s)} H_*(LM) \xrightarrow{BFG^{-1}} HH_*(S_*(G), S_*(G)).$$

Then

a) *The morphism of left  $HH^*(S_*(G), S_*(G))$ -modules*

$$\mathbb{D}^{-1} : HH^p(S_*(G), S_*(G)) \xrightarrow{\cong} HH_{d-p}(S_*(G), S_*(G)), \quad a \mapsto a.c,$$

*is an isomorphism.*

b) *The Gerstenhaber algebra  $HH^*(S_*(G), S_*(G))$  equipped with the operator  $\Delta := \mathbb{D} \circ B \circ \mathbb{D}^{-1}$  is a Batalin-Vilkovisky algebra.*

Here  $s$  denotes  $s : M \hookrightarrow LM$  the inclusion of the constant loops into  $LM$  and  $BFG$  is the isomorphism of graded  $\mathbb{k}$ -modules between the

free loop space homology of  $M$  and the Hochschild homology of  $S_*(G)$  introduced by Burghelea, Fiedorowicz [4] and Goodwillie [16]. Finally  $B$  denotes Connes boundary on  $HH_*(S_*(G), S_*(G))$ .

*Remark 41.* We expect that the above theorem can be extended to any path-connected topological monoid  $G$  instead of just the topological monoid of pointed Moore loop spaces  $\Omega M$  or instead of just any topological group.

*Proof.* By [7, Proposition 6.13 in the case  $F=\text{pt}$ ] when  $G$  is a topological group or by [7, Theorem 6.3] when  $G = \Omega M$ , there exists a differential graded coalgebra  $B(S_*(EG); S_*(G); \mathbb{k})$  and two quasi-isomorphisms of coalgebras

$$B(S_*(G)) \xleftarrow{\cong} B(S_*(EG); S_*(G); \mathbb{k}) \xrightarrow{\cong} S_*(M).$$

The induced isomorphism in homology is the well known isomorphism due to Moore [26, Corollary 7.29]

$$\theta : \text{Tor}^{S_*(G)}(\mathbb{k}, \mathbb{k}) = H_*(B(S_*(G))) \xrightarrow{\cong} H_*(M).$$

Let  $[m] \in H_*(B(S_*(G)))$  such that  $\theta([m]) = [M]$ . By Proposition 38 and Example 37, the cap product with  $[m]$ ,  $[m] \cap - : B(S_*(G))^\vee \xrightarrow{\cong} B(S_*(G))$ ,  $a \mapsto [m] \cap a$  is quasi-isomorphism.

Denote by  $ev : LM \rightarrow M$ ,  $l \mapsto l(0)$  the evaluation map. The isomorphism  $BFG$  of Goodwillie, Burghelea and Fiedorowicz fits into the commutative square.

$$\begin{array}{ccc} HH_*(S_*(G), S_*(G)) & \xrightarrow[\cong]{BFG} & H_*(LM) \\ HH_*(S_*(G), \varepsilon) \downarrow & & H_*(ev) \downarrow \\ HH_*(S_*(G), \mathbb{k}) & \xrightarrow[\cong]{\theta} & H_*(M) \end{array}$$

Here  $\varepsilon$  denote the augmentation of  $S_*(G)$ . Let  $c := BFG^{-1} \circ H_d(s)([M])$ . Since  $s$  is a section of the evaluation map  $ev$ ,  $HH_*(S_*(G), \varepsilon)(c) = [m]$ . So the hypothesis of statement 9 are satisfied for  $A = S_*(G)$ .

Let  $N$  be any non-negatively graded  $S_*(G)$ -bimodule. Since  $M$  is simply connected, by Proposition 12, we obtain that the morphism

$$\mathcal{D}^{-1} : HH^p(S_*(G), S_*(G)) \xrightarrow{\cong} HH_{d-p}(S_*(G), S_*(G)), \quad a \mapsto c \cap a$$

is an isomorphism. By taking  $N = S_*(G)$  and by passing from a right action to a left action by (28), we obtain a).

The isomorphism  $BFG$  of Goodwillie, Burghelea and Fiedorowicz satisfies  $\Delta \circ BFG = BFG \circ B$ . Consider  $M$  equipped with the trivial

$S^1$ -action. The section  $s : M \hookrightarrow LM$  is  $S^1$ -equivariant. Since

$$B(c) = B \circ BFG^{-1} \circ H_d(s)([M]) = BFG^{-1} \circ \Delta \circ H_d(s)([M]) = 0,$$

by Proposition 30, we obtain b).  $\square$

## 8. PROOF OF THE MAIN THEOREM FOR DISCRETE GROUPS

**Theorem 42.** *Let  $G$  be a discrete group such that its classifying space  $K(G, 1)$  is an oriented Poincaré duality space of formal dimension  $d$ . Let  $[M] \in H_d(G, \mathbb{k})$  be a fundamental class. Let  $c$  be the image of  $[M]$  by  $Tor_*^E(\eta, \eta) : H_*(G, \mathbb{k}) \rightarrow HH_*(\mathbb{k}[G], \mathbb{k}[G])$ . Then*

a) *The morphism of left  $HH^*(\mathbb{k}[G], \mathbb{k}[G])$ -modules*

$$\mathbb{D}^{-1} : HH^p(\mathbb{k}[G], \mathbb{k}[G]) \xrightarrow{\cong} HH_{d-p}(\mathbb{k}[G], \mathbb{k}[G]), a \mapsto a.c$$

*is an isomorphism.*

b) *The Gerstenhaber algebra  $HH^*(\mathbb{k}[G], \mathbb{k}[G])$  equipped with the operator  $\Delta := \mathbb{D} \circ B \circ \mathbb{D}^{-1}$  is a Batalin-Vilkovisky algebra.*

*Proof.* Let  $N$  be any  $\mathbb{k}[G]$ -bimodule. Since, by hypothesis,  $G$  is orientable Poincaré duality group, the cap product with  $[M]$  in group (co)homology gives an isomorphism ([3, 10.1 iv), Remark 1 and Example 1 p. 222],[13, Th 15.3.1])

$$[M] \cap - : H^p(G, \tilde{N}) \xrightarrow{\cong} H_{d-p}(G, \tilde{N}), a \mapsto [M] \cap a.$$

Therefore, by Corollary 18, the cap product with  $c = \sigma([M])$  in Hochschild (co)homology gives the isomorphism

$$c \cap - : HH^p(\mathbb{k}[G], N) \rightarrow HH_{d-p}(\mathbb{k}[G], N), a \mapsto c \cap a.$$

Taking  $N = \mathbb{k}[G]$  and passing from a right action to left action as in (28), we obtain a).

By i) of Property 17,  $\sigma : H_*(G; \mathbb{k}) \rightarrow HH_*(\mathbb{k}[G], \mathbb{k}[G])$  commute with Connes boundary map  $B$  on  $H_*(G; \mathbb{k})$  and on  $HH_*(\mathbb{k}[G], \mathbb{k}[G])$ . By a well known result of Karoubi (for example [23, E.7.4.8] or [33, Theorem 9.7.1]), Connes boundary map  $B$  is trivial on  $H_*(G; \mathbb{k})$ . Therefore  $B(c) = B \circ \sigma([M]) = \sigma \circ B([M]) = 0$ . By applying Proposition 30, we obtain b).  $\square$

*Property 43.* Let  $A$  and  $B$  be two algebras (differential graded if we want). Let  $N$  be an  $(A, A \otimes B)$ -bimodule. Let  $c \in HH_d(A, A)$ . Then

- i)  $HH^*(A, N)$  and  $HH_*(A, N)$  are two right  $B$ -modules and
- ii) the cap product

$$c \cap - : HH^p(A, N) \rightarrow HH_{d-p}(A, N), \quad a \mapsto c \cap a$$

is a morphism of right  $B$ -modules.

*Proof.* Since  $N$  is an  $(A^e, B)$ -bimodule,  $\mathcal{C}^*(A, N) \cong \text{Hom}_{A^e}(B(A; A; A), N)$  is a (differential graded) right  $B$ -module and its homology  $HH^*(A, N)$  is a right  $B$ -module. Similarly  $\mathcal{C}_*(A, N) \cong N \otimes_{A^e} B(A; A; A)$  and  $HH_*(A, N)$  are two right  $B$ -modules. Let  $c$  be  $a[a_1 | \dots | a_n] \in \mathcal{C}_n(A, A)$ . Let  $f \in \mathcal{C}^p(A, N)$ . By definition,  $c \cap f := \pm a f([a_1 | \dots | a_p])[a_{p+1} | \dots | a_n]$ . Therefore for any  $b \in B$ ,

$$\begin{aligned} (c \cap f) \cdot b &= \pm a f([a_1 | \dots | a_p]) b [a_{p+1} | \dots | a_n] = \\ &\pm a (f \cdot b) ([a_1 | \dots | a_p]) [a_{p+1} | \dots | a_n] = c \cap (f \cdot b). \end{aligned}$$

□

*Remark 44.* We will be only interested in the case  $N = A \otimes A$  and  $B = A^e$ . Here the  $A$ -bimodule structure on  $N$  is given by  $a \cdot (x \otimes y) \cdot b = ax \otimes yb$  and is called the *outer* structure [15, (1.5.1)]. And the right  $B$ -module on  $N$  is given by  $(x \otimes y) \cdot (a \otimes b) = xa \otimes by$ ,  $x \otimes y \in N$ ,  $a \otimes b \in B$  and is called the *inner* structure.

**Definition 45.** ([15, Definition 3.2.3, (3.2.5), Remark 3.2.8] or simply [1, Definition 2.1]) An ungraded algebra  $A$  is *Calabi-Yau* of dimension  $d$  if

i) viewed as an  $A$ -bimodule over itself,  $A$  admits a finite resolution by finite type projective  $A$ -bimodules, i. e. there exists an exact sequence of  $A^e$ -projective finite type module of the form

$$0 \rightarrow P_i \rightarrow P_{i-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0,$$

ii) for all  $k \neq d$ ,  $HH^k(A, A \otimes A) = 0$  and

iii) as  $(A, A)$ -bimodule,  $HH^d(A, A \otimes A)$  is isomorphic to  $A$  (Here the  $(A, A)$ -bimodule on  $HH^*(A, A \otimes A)$  is given by Property 43 and Remark 44).

**Proposition 46.** (Stated without proof in [15, Remark 3.4.2]) Let  $A$  be ungraded algebra. Let  $c \in HH_d(A, A)$ . Suppose that for every  $A$ -bimodule  $N$ ,  $c \cap - : HH^p(A, N) \xrightarrow{\cong} HH_{d-p}(A, N)$ ,  $a \mapsto c \cap a$ , is an isomorphism. Then  $A$  satisfies conditions ii) and iii) of Definition 45.

*Proof.* Let  $N$  be a free  $(A, A)$ -bimodule. Then  $HH_*(A, N) = 0$  if  $* \neq 0$ . Therefore  $HH^k(A, N) = 0$  if  $k \neq d$ . Suppose moreover that  $N$  is a  $(A, A \otimes B)$ -bimodule. The quasi-isomorphism of complexes  $\mathcal{C}_*(A, N) \cong N \otimes_{A^e} B(A; A; A) \xrightarrow{\cong} N \otimes_{A^e} A$  is a morphism of right  $B$ -modules. By Property 43,

$$c \cap - : HH^d(A, N) \rightarrow HH_0(A, N) \cong N \otimes_{A^e} A$$

is an isomorphism of right  $B$ -modules.

Let  $N$  be the  $(A, A)$ -bimodule  $A \otimes A$  with the outer structure and  $B = A^e$  (See Remark 44). Then  $N \otimes_{A^e} A = (A \otimes A) \otimes_{A^e} A \xrightarrow{\cong} A$ ,  $(x \otimes y) \otimes_{A^e} m \mapsto ymx$  is an isomorphism whose inverse is the map mapping  $a \mapsto (1 \otimes 1) \otimes_{A^e} a$ . A straightforward calculation shows that these isomorphisms are right  $A^e$ -linear. Therefore, we have proved that  $HH^d(A, A \otimes A)$  is isomorphic to  $A$  as right  $A^e$ -modules.  $\square$

**Theorem 47.** *Let  $\mathbb{k}$  be any commutative ring. Let  $G$  be a orientable Poincaré duality group of dimension  $d$ . Then its group ring  $\mathbb{k}[G]$  is a Calabi-Yau algebra of dimension  $d$ .*

When  $\mathbb{k}$  is a field of characteristic 0 this theorem was proved by Kontsevich [15, Corollary 6.1.4] and Lambre [22, Lemme 6.2].

*Proof.* By [3, Remark 2. p. 222], there exists a finite resolution  $P \xrightarrow{\cong} \mathbb{k}$  of  $\mathbb{k}$  by finite type projective  $\mathbb{k}[G]$ -left modules. Then  $X := \mathbb{k}[G \times G^{op}] \otimes_{\mathbb{k}[G]} P \xrightarrow{\cong} \mathbb{k}[G]$  is a finite type resolution of  $\mathbb{k}[G]$  by finite type projective  $\mathbb{k}[G]$ -bimodules.

In the proof of Theorem 42, we saw that for any  $\mathbb{k}[G]$ -bimodule  $N$ ,  $c \cap - : HH^p(\mathbb{k}[G], N) \xrightarrow{\cong} HH_{d-p}(\mathbb{k}[G], N)$ ,  $a \mapsto c \cap a$ , is an isomorphism. Therefore, by Proposition 46,  $\mathbb{k}[G]$  is a Calabi-Yau algebra of dimension  $d$ .  $\square$

## 9. APPENDIX

The key of the proof of Proposition 30 is the relation

$$i_{\{a,b\}} = (-1)^{|a|+1} [[B, i_a], i_b] = [[i_a, B], i_b].$$

In this appendix, we recall that  $[[i_a, B], i_b]$  is the *derived bracket* of  $i_a$  and  $i_b$  and we explain that this relation means that the morphism of graded algebras

$$HH^*(A, A) \rightarrow \text{End}(HH_*(A, A)), \quad a \mapsto i_a,$$

is a morphism of generalized Loday-Gerstenhaber algebras (Theorem 54)

**Definition 48.** [20, p. 1247] A *generalized Loday-Gerstenhaber algebra* is a (not necessarily commutative) graded algebra  $A$  equipped with a linear map  $\{-, -\} : A_i \otimes A_j \rightarrow A_{i+j+1}$  of degree 1 such that:

a) the bracket  $\{-, -\}$  gives  $A$  a structure of graded Leibniz algebra of degree 1. This means that for each  $a, b$  and  $c \in A$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)} \{b, \{a, c\}\}.$$

b) the product and the Leibniz bracket satisfy the following relation called the Poisson relation:

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|+1)|b|} b\{a, c\}.$$

**Proposition 49.** *Let  $A$  be a graded algebra equipped with an operator  $d : A_n \rightarrow A_{n+1}$  such that  $d \circ d = 0$  and such that  $d$  is a derivation. Then  $A$  equipped with the derived bracket defined by [20, (2.8)]*

$$[a, b]_d := (-1)^{|a|+1}[da, b]$$

*is a generalized Loday-Gerstenhaber algebra.*

*Proof.* Since  $A$  is an associative graded algebra, the bracket  $[-, -]$  defined by

$$[a, b] := ab - (-1)^{|a||b|}ba,$$

is a Lie bracket. Since  $d$  is a derivation for the associative product of  $A$ ,  $d$  is a derivation for the Lie bracket  $[-, -]$ . Therefore by [20, Proposition 2.1], the derived bracket  $[-, -]_d$  satisfies the graded Jacobi identity and  $d$  is a derivation for the derived bracket  $[-, -]_d$ . Since  $[-, -]_d$  does not satisfy in general anticommutativity,  $[-, -]_d$  is only a Leibniz bracket in the sense of Loday [24], and not a Lie bracket in general. The Lie bracket  $[-, -]$  satisfies the Poisson relation:

$$[a, bc] = [a, b]c + (-1)^{(|a|+1)|b|}b[a, c].$$

Therefore since  $[a, -]_d$  is the derivation  $(-1)^{|a|+1}[da, -]$ , the Leibniz bracket  $[-, -]_d$  also satisfies the Poisson relation [20, Proposition 2.2]:

$$[a, bc]_d = [a, b]_dc + (-1)^{(|a|+1)|b|}b[a, c]_d.$$

□

*Remark 50.* In Proposition 49, if instead, we define the bracket by

$$[a, b]_d := ad(b) - (-1)^{(|a|+1)(|b|+1)}bd(a)$$

then  $[-, -]_d$  satisfies anti-commutativity and Jacobi:  $[-, -]_d$  is a Lie bracket<sup>1</sup> of degree +1. But this time,  $[-, -]_d$  does not satisfy the Poisson relation. Note that again  $d$  is a derivation for  $[-, -]_d$ .

*Proof.* Let  $a \in A_{x-1}$ ,  $b \in B_{y-1}$  and  $c \in C_{z-1}$  be three elements of  $A$  of degrees  $x - 1$ ,  $y - 1$  and  $z - 1$ . Then

$$\begin{aligned} [a, [b, c]_d]_d &= ad(bdc) - (-1)^{zy}ad(cdb) \\ &\quad - (-1)^{xy+xz}b(dc)(da) + (-1)^{xy+xz+yz}c(db)(da), \end{aligned}$$

$$\begin{aligned} [[a, b]_d, c]_d &= a(db)(dc) - (-1)^{xy}b(da)(dc) \\ &\quad + (-1)^{zx+zy}cd(adb) + (-1)^{zx+zy+xy}cd(bda) \end{aligned}$$

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<sup>1</sup>We could not find this Lie bracket in the literature. So this Lie algebra structure might be new.

and

$$\begin{aligned} (-1)^{xy}[b, [a, c]_d]_d &= (-1)^{xy}bd(adc) - (-1)^{xy+xz}bd(cda) \\ &\quad - (-1)^{yz}a(dc)(db) + (-1)^{yz+xz}c(da)(db). \end{aligned}$$

Since  $d$  is a derivation and  $d^2 = 0$ ,  $d(adb) = (da)(db)$ . Therefore we have the Jacobi identity:

$$[a, [b, c]_d]_d = [[a, b]_d, c]_d + (-1)^{xy}[b, [a, c]_d]_d.$$

Since  $[da, b]_d = (da)(db)$  and  $[a, db]_d = -(-1)^{x(y+1)}(db)(da)$ ,

$$d([a, b]_d) = (da)(db) - (-1)^{xy}(db)(da) = [da, b]_d + (-1)^x[a, db]_d.$$

This means that  $d$  is a derivation for  $[-, -]_d$ .  $\square$

*Example 51.* (interior derivation) Let  $A$  be an associative graded algebra. Let  $\tau \in A_1$  such that  $\tau^2 = 0$ . Then  $d := [\tau, -]$  is a derivation of the associative product and  $d \circ d = 0$ . Therefore, we can apply the previous proposition. In this case, we denote the derived bracket  $[a, b]_d$  simply by  $[a, b]_\tau$  and [20, Example p. 1250]

$$[a, b]_\tau = (-1)^{|a|+1}[[\tau, a], b] = [[a, \tau], b].$$

**Corollary 52.** [20, Beginning of Section 2.4] *Let  $E$  be a graded  $\mathbb{k}$ -module equipped with an operator  $B : E_n \rightarrow E_{n+1}$  such that  $B \circ B = 0$ . Then  $\text{End}(E)$  equipped with the derived bracket  $[a, b]_B = [[a, B], b]$  is a generalized Loday-Gerstenhaber algebra.*

*Proof.* Apply Proposition 49 and Example 51, to  $\text{End}(E)$  equipped with the composition product.  $\square$

**Theorem 53.** (implicit in [20, p. 1269-70 pointed by Krasilshchik]) *Let  $A$  be a Batalin-Vilkovisky algebra. The morphism of graded algebras induced by left multiplication*

$$\Psi : A \rightarrow \text{End}(A), a \mapsto l_a$$

*is an injective morphism of generalized Loday-Gerstenhaber algebras.*

*Proof.* Since  $A$  is a graded module equipped with an operator  $\Delta : A_n \rightarrow A_{n+1}$  such that  $\Delta \circ \Delta = 0$ , by Corollary 52 applied to  $A$  and to  $B = -\Delta$ ,  $\text{End}(A)$  equipped with the derived bracket  $[f, g]_{-\Delta} = [[f, -\Delta], g]$  is a generalized Loday-Gerstenhaber algebra. By Proposition 26,

$$l_{\{a, b\}} = -[[l_a, \Delta], l_b] = [[l_a, B], l_b]$$

Therefore  $\Psi$  is a morphism of generalized Loday-Gerstenhaber algebra  $\square$

**Theorem 54.** *Let  $A$  be a differential graded algebra. Then*

1)  *$\text{End}HH_*(A, A)$  equipped with the derived bracket*

$$[a, b]_B = [a, B], b]$$

*is a generalized Loday-Gerstenhaber algebra.*

2) *The morphism of graded algebras induced by the action*

$$\Phi : HH^*(A, A) \rightarrow \text{End}HH_*(A, A), \quad a \mapsto i_a,$$

*is a morphism of generalized Loday-Gerstenhaber algebra. In particular, its image  $\Phi(HH^*(A, A))$  is a Gerstenhaber algebra.*

*Proof.* Since Connes boundary  $B : HH_*(A, A) \rightarrow HH_{*+1}(A, A)$  satisfies  $B \circ B = 0$ , by Corollary 52, we obtain 1).

Since  $i_{ab} = i_a \circ i_b$  (equation (29)) and  $i_{\{a,b\}} = [[i_a, B], i_b] = [i_a, i_b]_B$ ,  $\Psi$  is a morphism of generalized Gerstenhaber-Loday algebra.

Since  $HH^*(A, A)$  is a Gerstenhaber algebra,  $\Phi(HH^*(A, A))$  is also a Gerstenhaber algebra.  $\square$

*Remark 55.* If  $A$  is a differential graded algebra satisfying the hypothesis of Proposition 30, the morphism  $\Phi : HH^*(A, A) \hookrightarrow \text{End}HH_*(A, A)$  of Theorem 54 is injective and can be identified with the morphism  $\Psi$  of Theorem 53 for the Batalin-Vilkovisky algebra  $HH^*(A, A)$ .

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