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Igor Chueshov, Annie Millet

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# STOCHASTIC 2D HYDRODYNAMICAL SYSTEMS: SUPPORT THEOREM

IGOR CHUESHOV AND ANNIE MILLET

ABSTRACT. We deal with a class of abstract nonlinear stochastic models with multiplicative noise, which covers many 2D hydrodynamical models including the 2D Navier-Stokes equation, 2D MHD models and 2D magnetic Bénard problems as well as some shell models of turbulence. Our main result describes the support of the distribution of solutions. Both inclusions are proved by means of a general result of convergence in probability for non linear stochastic PDEs driven by a Hilbert-valued Brownian motion and some adapted finite dimensional approximation of this process.

## 1. INTRODUCTION

Our goal in this paper is to continue the unified investigation of statistical properties of some stochastic 2D hydrodynamical models which started in our previous paper [6]. The model introduced there covers a wide class of mathematical coupled models from 2D fluid dynamics. This class includes the 2D Navier-Stokes equation, and also some other classes of two dimensional hydrodynamical models such as the magneto-hydrodynamic equation, the Boussinesq model for the Bénard convection and the 2D magnetic Bénard problem. We also cover the case of regular higher dimensional problems such as the 3D Leray  $\alpha$ -model for the Navier-Stokes equation and some shell models of turbulence. For details we refer to [6, Sect.2.1].

Our unified approach is based on an abstract stochastic evolution equation in some Hilbert space of the form

$$\partial_t u + Au + B(u, u) + R(u) = \Xi(t, u) \dot{W}, \quad u|_{t=0} = \xi, \quad (1.1)$$

where  $\dot{W}$  is a multiplicative noise white in time with spatial correlation. The hypotheses concerning the linear operator  $A$ , the bilinear mapping  $B$  and the operator  $R$  are stated below. These hypotheses guarantee unique solvability of problem (1.1).

For general abstract stochastic evolution equations in infinite dimensional spaces we refer to [8]. However the hypotheses in [8] do not cover our hydrodynamical type model. We also note that stochastic Navier-Stokes equations were studied by many authors (see, e.g., [4, 11, 17, 22] and the references therein). In [6] we prove existence, uniqueness and provide a priori estimates for a weak (variational) solution to the abstract problem of the form (1.1), where the forcing term may also include a stochastic control term with a multiplicative coefficient. In all the concrete hydrodynamical examples mentioned above, the diffusion coefficient may contain a small multiple of the gradient of the solution. This result contains the corresponding existence and uniqueness theorems and a priori bounds for 2D Navier-Stokes equation, the Boussinesq model of the Bénard convection, and also for the GOY and Sabra shell models of turbulence. Theorem 3.1 [6] generalizes the existence result for Boussinesq or MHD equations given in [10] or [3] to the case of multiplicative noise (see also [9]) and also covers new situations such as the 2D magnetic Bénard problem

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or the 3D Leray  $\alpha$ -model. Our main result in [6] is a Wentzell-Freidlin type large deviation principle (LDP) in an appropriate Polish space  $X$  for stochastic equations of the form (1.1) with  $\sigma^\varepsilon := \sqrt{\varepsilon}\sigma$  as  $\varepsilon \rightarrow 0$ , which describes the exponential rate of convergence of the solution  $u := u^\varepsilon$  to the deterministic solution  $u^0$ . One of the key arguments is a time increment control which provides the weak convergence needed in order to prove the large deviations principle. We refer to [6] for detailed discussion and references.

Another classical problem, where the lack of continuity of the solution as a function of the noise can be overcome by means of an appropriate drift correction, replacing the Itô integral by a Stratonovich one, is the Wong-Zakai characterization of the support of the distribution of the solution in the Polish space  $X$  where it lives. Our main result (see Theorem 3.1 below) provides this characterization for the abstract system (1.1) and thus covers a wide class of hydrodynamical models. The approach used here is similar to that introduced in [14] and [15]; see also [13] and [1] where related results were obtained for diffusion processes and [2], [5] and [16], where a support theorem for the one-dimensional heat or Burgers equation, and for general mild solutions to semi-linear abstract parabolic equations was proven along the same line. As in [6], the control equation is needed with some control defined in terms of both an element of the Reproducing Kernel Hilbert Space of the driving Brownian motion and an adapted linear finite-dimensional approximation of the Brownian. For this class of control equations we first establish a Wong-Zakai type approximation theorem (see Theorem 4.1), which, as we believe, has an independent interest. A key ingredient of the proof of this theorem is a time increment control with a better speed of convergence to zero than that needed in [6]. As in [15], since the linear interpolation is adapted and unlike in the original proof of [18], only one result of convergence in probability provides both inclusions needed to describe the support of the distribution.

To our best knowledge there is only one publication related to the support characterization of solutions to 2D hydrodynamical models. The short note [21], which states a characterization of the support for 2D Navier-Stokes equations with the Dirichlet boundary conditions, does not provide a detailed proof and refers to [20] concerning Wong-Zakai approximations. However, the argument used in [20] is incomplete and we were not able to fill the gaps.

The paper is organized as follows. In Section 2 we recall the mathematical model introduced in [6]. In this section we also formulate our abstract hypotheses. The description of the support is stated in section 3 under some additional integrability property on the solution. In Section 4 we formulate the Wong-Zakai type theorem, which is the main tool to characterize the support of the distribution of the solution to the stochastic hydrodynamical equations. We then show how the support characterization can be deduced. The proof of the convergence stated in Theorem 4.1 is given in section 6. It heavily depends on the time increment speed of convergence, which is proven in Section 5. In the appendix (see Section 7) we discuss with details the way our result can be applied to different classes of hydrodynamical models and give conditions which ensure that the solution fulfills the extra integrability assumption we have imposed.

## 2. DESCRIPTION OF THE MODEL

**2.1. Deterministic analog.** Let  $(H, |\cdot|)$  denote a separable Hilbert space,  $A$  be an (unbounded) self-adjoint positive linear operator on  $H$ . Set  $V = \text{Dom}(A^{\frac{1}{2}})$ . For  $v \in V$  set  $\|v\| = |A^{\frac{1}{2}}v|$ . Let  $V'$  denote the dual of  $V$  (with respect to the inner product  $(\cdot, \cdot)$  of  $H$ ). Thus we have the Gelfand triple  $V \subset H \subset V'$ . Let  $\langle u, v \rangle$  denote the duality between

$u \in V$  and  $v \in V'$  such that  $\langle u, v \rangle = (u, v)$  for  $u \in V$ ,  $v \in H$ , and let  $B : V \times V \rightarrow V'$  be a mapping satisfying the condition **(B)** given below.

The goal of this paper is to study stochastic perturbations of the following abstract model in  $H$

$$\partial_t u(t) + Au(t) + B(u(t), u(t)) + Ru(t) = f, \quad (2.1)$$

where  $R$  is a continuous operator in  $H$ . We assume that the mapping  $B : V \times V \rightarrow V'$  satisfies the following antisymmetry and bound conditions:

**Condition (B):**

- (1)  $B : V \times V \rightarrow V'$  is a bilinear continuous mapping.
- (2) For  $u_i \in V$ ,  $i = 1, 2, 3$ ,

$$\langle B(u_1, u_2), u_3 \rangle = -\langle B(u_1, u_3), u_2 \rangle. \quad (2.2)$$

- (3) There exists a Banach (interpolation) space  $\mathcal{H}$  possessing the properties
  - (i)  $V \subset \mathcal{H} \subset H$ ;
  - (ii) there exists a constant  $a_0 > 0$  such that

$$\|v\|_{\mathcal{H}}^2 \leq a_0 |v| \|v\| \quad \text{for any } v \in V; \quad (2.3)$$

- (iii) for every  $\eta > 0$  there exists  $C_\eta > 0$  such that

$$|\langle B(u_1, u_2), u_3 \rangle| \leq \eta \|u_3\|^2 + C_\eta \|u_1\|_{\mathcal{H}}^2 \|u_2\|_{\mathcal{H}}^2, \quad \text{for } u_i \in V, i = 1, 2, 3. \quad (2.4)$$

Note (see [6, Remark 2.1]) that the upper estimate in (2.4) can also be written in the following two equivalent forms:

- (iii-a) there exist positive constants  $C_1$  and  $C_2$  such that

$$|\langle B(u_1, u_2), u_3 \rangle| \leq C_1 \|u_3\|^2 + C_2 \|u_1\|_{\mathcal{H}}^2 \|u_2\|_{\mathcal{H}}^2, \quad \text{for } u_i \in V, i = 1, 2, 3; \quad (2.5)$$

- (iii-b) there exists a constant  $C > 0$  such that for  $u_i \in V$ ,  $i = 1, 2, 3$  we have:

$$|\langle B(u_1, u_2), u_3 \rangle| = |\langle B(u_1, u_3), u_2 \rangle| \leq C \|u_1\|_{\mathcal{H}} \|u_2\| \|u_3\|_{\mathcal{H}}. \quad (2.6)$$

For  $u \in V$  set  $B(u) := B(u, u)$ ; with this notation, relations (2.2), (2.3) and (2.6) yield for every  $\eta > 0$  the existence of  $C_\eta > 0$  such that for  $u_1, u_2 \in V$ ,

$$|\langle B(u_1), u_2 \rangle| \leq \eta \|u_1\|^2 + C_\eta |u_1|^2 \|u_2\|_{\mathcal{H}}^4. \quad (2.7)$$

Relations (2.2) and (2.7) imply that for any  $\eta > 0$  there exists  $C_\eta > 0$  such that

$$|\langle B(u_1) - B(u_2), u_1 - u_2 \rangle| = |\langle B(u_1 - u_2), u_2 \rangle| \leq \eta \|u_1 - u_2\|^2 + C_\eta |u_1 - u_2|^2 \|u_2\|_{\mathcal{H}}^4 \quad (2.8)$$

for all  $u_1, u_2 \in V$ . As it was explained in [6] the main motivation for condition **(B)** is that it covers a wide class of 2D hydrodynamical models including Navier-Stokes equations, magneto-hydrodynamic equations, Boussinesq model for the Bénard convection, magnetic Bénard problem, 3D Leray  $\alpha$ -model for Navier-Stokes equations, Shell models of turbulence (GOY, Sabra, and dyadic models).

**2.2. Noise.** We will consider a stochastic external random force  $f$  in equation (2.1), driven by a Wiener process  $W$  and whose intensity may depend on the solution  $u$ . More precisely, let  $Q$  be a linear positive operator in the Hilbert space  $H$  which belongs to the trace class, and hence is compact. Let  $H_0 = Q^{\frac{1}{2}}H$ . Then  $H_0$  is a Hilbert space with the scalar product

$$(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi), \quad \forall \phi, \psi \in H_0,$$

together with the induced norm  $|\cdot|_0 = \sqrt{(\cdot, \cdot)_0}$ . The embedding  $i : H_0 \rightarrow H$  is Hilbert-Schmidt and hence compact, and moreover,  $i i^* = Q$ . Let  $L_Q \equiv L_Q(H_0, H)$  denote the space of linear operators  $S : H_0 \mapsto H$  such that  $SQ^{\frac{1}{2}}$  is a Hilbert-Schmidt operator from

$H$  to  $H$ . The norm on the space  $L_Q$  is defined by  $|S|_{L_Q}^2 = \text{tr}(SQS^*)$ , where  $S^*$  is the adjoint operator of  $S$ . The  $L_Q$ -norm can be also written in the form:

$$|S|_{L_Q}^2 = \text{tr}([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k \geq 1} |SQ^{1/2}\psi_k|^2 = \sum_{k \geq 1} |[SQ^{1/2}]^*\psi_k|^2 \quad (2.9)$$

for any orthonormal basis  $\{\psi_k\}$  in  $H$ .

Let  $W(t)$  be a Wiener process defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , taking values in  $H$  and with covariance operator  $Q$ . This means that  $W$  is Gaussian, has independent time increments and that for  $s, t \geq 0$ ,  $f, g \in H$ ,

$$\mathbb{E}(W(s), f) = 0 \quad \text{and} \quad \mathbb{E}(W(s), f)(W(t), g) = (s \wedge t)(Qf, g).$$

We also have the representation

$$W(t) = \lim_{n \rightarrow \infty} W_n(t) \quad \text{in} \quad L^2(\Omega; H) \quad \text{with} \quad W_n(t) = \sum_{1 \leq j \leq n} q_j^{1/2} \beta_j(t) e_j, \quad (2.10)$$

where  $\beta_j$  are standard (scalar) mutually independent Wiener processes,  $\{e_j\}$  is an orthonormal basis in  $H$  consisting of eigen-elements of  $Q$ , with  $Qe_j = q_j e_j$ . For details concerning this Wiener process we refer to [8], for instance. Let  $(\mathcal{F}_t, t \geq 0)$  denote the Brownian filtration, that is the smallest right-continuous complete filtration with respect to which  $(W(t), t \geq 0)$  is adapted.

We now define some adapted approximations of the processes  $\beta_j$  and  $W$ . For all integers  $n \geq 1$  and  $k = 0, 1, \dots, 2^n$ , set  $t_k = kT2^{-n}$  and define step functions  $\underline{s}_n, s_n, \bar{s}_n : [0, T] \rightarrow [0, T]$  by the formulas

$$\underline{s}_n = t_k, \quad s_n = t_{k-1} \vee 0, \quad \bar{s}_n = t_{k+1} \quad \text{for} \quad s \in [t_k, t_{k+1}[. \quad (2.11)$$

It is clear that  $s_n < \underline{s}_n < \bar{s}_n$ . Now we set  $\dot{\beta}_j^n(s) = T^{-1}2^n (\beta_j(\underline{s}_n) - \beta_j(s_n))$ , for every  $s \in [0, T]$ , and thus we obtain an adapted approximation for  $\dot{\beta}_j(s)$  given by the formula

$$\dot{\beta}_j^n(s) = T^{-1}2^n [\beta_j(t_k) - \beta_j(t_{k-1} \vee 0)], \quad \text{for} \quad s \in [t_k, t_{k+1}[. \quad (2.12)$$

Clearly  $\dot{\beta}_j^n(s) \equiv 0$  for  $s \in [0, t_1[$  and  $\dot{\beta}_j^n(s) = T^{-1}2^n \beta_j(t_1)$  for  $s \in [t_1, t_2[$ . We also let

$$\tilde{W}^n(s) = \sum_{1 \leq j \leq n} \dot{\beta}_j^n(s) q_j^{1/2} e_j \quad (2.13)$$

denote the corresponding finite-dimensional adapted approximation of  $\dot{W}$ .

**Lemma 2.1.** *There exists an absolute constant  $\alpha_0 > 0$  such that for every  $\alpha > \alpha_0/\sqrt{T}$  and  $t \in [0, T]$  we have as  $n \rightarrow \infty$*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{1 \leq j \leq n} \sup_{s \leq t} |\dot{\beta}_j^n(s)| \geq \alpha n^{1/2} 2^{n/2} \right) = \lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{s \leq t} |\tilde{W}^n(s)|_{H_0} \geq \alpha n 2^{\frac{n}{2}} \right) = 0.$$

*Proof.* One can see that

$$\tilde{\Omega}_n(t) = \left\{ \sup_{1 \leq j \leq n} \sup_{s \leq t} |\dot{\beta}_j^n(s)| \geq \alpha n^{1/2} 2^{n/2} \right\} \subset \bigcup_{1 \leq j \leq n} \bigcup_{0 \leq k < 2^n} \left\{ |\gamma_j^k| \geq \alpha T^{1/2} n^{1/2} \right\},$$

where  $\gamma_j^k = T^{-1/2} 2^{n/2} [\beta_j(t_{k+1}) - \beta_j(t_k)]$  are independent standard normal Gaussian random variables. Therefore, if  $\alpha > \sqrt{T^{-1} 2 \ln 2}$  we have

$$\mathbb{P}(\tilde{\Omega}_n(t)) \leq n 2^n P(|\gamma_1^0| \geq \alpha \sqrt{Tn}) = \frac{n 2^{n+1}}{\sqrt{2\pi}} \int_{\alpha \sqrt{Tn}}^{\infty} e^{-z^2/2} dz$$

$$\leq \frac{n^{1/2}2^{n+1}}{\alpha\sqrt{2\pi T}} \int_{\alpha\sqrt{Tn}}^{\infty} ze^{-z^2/2} dz = \frac{2n^{1/2}}{\alpha\sqrt{2\pi T}} \exp \left[ n \left( -\alpha^2 T/2 + \ln 2 \right) \right].$$

This proves the first convergence result. The second one follows immediately from the estimate

$$\sup_{s \leq t} \left| \dot{\widehat{W}}^n(s) \right|_{H_0}^2 = \sup_{s \leq t} \sum_{1 \leq j \leq n} \left| \dot{\beta}_j^n(s) \right|^2 \leq n \sup_{1 \leq j \leq n} \sup_{s \leq t} \left| \dot{\beta}_j^n(s) \right|^2.$$

□

In the sequel, we will localize the processes using the following set:

$$\Omega_n(t) = \left\{ \sup_{j \leq n} \sup_{s \leq t} \left| \dot{\beta}_j^n(s) \right| \leq \alpha n^{1/2} 2^{n/2} \right\} \cap \left\{ \sup_{s \leq t} \left| \dot{\widehat{W}}^n(s) \right|_{H_0} \leq \alpha n 2^{n/2} \right\}. \quad (2.14)$$

It is clear that  $\Omega_n(t) \subset \Omega_n(s)$  for  $t > s$  and  $\Omega_n(t) \in \mathcal{F}_t$ . Furthermore, Lemma 2.1 implies that  $\mathbb{P}(\Omega_n(T)^c) \rightarrow 0$  as  $n \rightarrow \infty$ .

For any  $n \geq 1$ , we introduce the following localized processes  $\dot{\beta}_j^n$  and  $\dot{\widehat{W}}^n$ :

$$\dot{\beta}_j^n(t) = \dot{\beta}_j^n(t) 1_{\Omega_n(t)}, \quad j \leq n, \quad \dot{W}^n(t) = \dot{\widehat{W}}^n(t) 1_{\Omega_n(t)}. \quad (2.15)$$

For all integers  $n$  and  $j = 1, \dots, n$ ,  $(\dot{\beta}_j^n(t), 0 \leq t \leq T)$  (resp.  $(\dot{W}^n(t), 0 \leq t \leq T)$ ) are  $(\mathcal{F}_t)$ -adapted  $\mathbb{R}$  (resp.  $H_0$ ) valued processes.

**2.3. Diffusion coefficients.** We need below two diffusion coefficients  $\sigma$  and  $\tilde{\sigma}$  which map  $H$  into  $L_Q(H_0, H)$ . They are assumed to satisfy the following growth and Lipschitz conditions:

**Condition (S):** *The maps  $\sigma, \tilde{\sigma}$  belong to  $\mathcal{C}(H; L_Q(H_0, H))$  and satisfy:*

(1) *There exist non-negative constants  $K_i$  and  $L$  such that for every  $u, v \in H$ :*

$$|\sigma(u)|_{L_Q}^2 + |\tilde{\sigma}(u)|_{L_Q}^2 \leq K_0 + K_1 |u|^2, \quad (2.16)$$

$$|\sigma(u) - \sigma(v)|_{L_Q}^2 + |\tilde{\sigma}(u) - \tilde{\sigma}(v)|_{L_Q}^2 \leq L |u - v|^2. \quad (2.17)$$

(2) *Moreover, for every  $N > 0$ ,*

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq N} |\tilde{\sigma}(u) - \tilde{\sigma}(u) \circ \Pi_n|_{L_Q} = 0, \quad (2.18)$$

where  $\Pi_n : H_0 \rightarrow H_0$  denote the projector defined by  $\Pi_n u = \sum_{k=1}^n (u, e_k) e_k$ , where  $\{e_k, k \geq 1\}$  is the orthonormal basis of  $H$  made by eigen-elements of the covariance operator  $Q$  and used in (2.10).

**Condition (DS):** *For every integer  $j \geq 1$  let  $\sigma_j, \tilde{\sigma}_j : H \mapsto H$  be defined by*

$$\sigma_j(u) = q_j^{1/2} \sigma(u) e_j, \quad \tilde{\sigma}_j(u) = q_j^{1/2} \tilde{\sigma}(u) e_j, \quad \forall u \in H. \quad (2.19)$$

We assume that the maps  $\tilde{\sigma}_j$  are twice Fréchet differentiable and satisfy

(1) *For every integer  $N \geq 1$  there exist positive constants  $C_i(N)$ ,  $i = 1, 2, 3$  such that:*

$$C_1(N) := \sup_j \sup_{|u| \leq N} |D\tilde{\sigma}_j(u)|_{L(H, H)} < \infty, \quad (2.20)$$

$$C_2(N) := \sup_j \sup_{|u| \leq N} |D^2\tilde{\sigma}_j(u)|_{L(H \times H, H)} < \infty, \quad (2.21)$$

$$\sup_j \sup_{|u| \leq N} \|[D\tilde{\sigma}_j(u)]^* v\| \leq C_3(N) \|v\| \quad \text{for every } v \in V. \quad (2.22)$$

(2) For every integer  $n \geq 1$ , let the functions  $\varrho_n, \tilde{\varrho}_n : H \mapsto H$  be defined by

$$\varrho_n(u) = \sum_{1 \leq j \leq n} D\tilde{\sigma}_j(u) \sigma_j(u), \quad \tilde{\varrho}_n(u) = \sum_{1 \leq j \leq n} D\tilde{\sigma}_j(u) \tilde{\sigma}_j(u), \quad \forall u \in H, \quad (2.23)$$

where  $\sigma_j$  and  $\tilde{\sigma}_j$  are given by (2.19). For every integer  $N \geq 1$  there exist positive constants  $\bar{K}_N, \bar{C}_N$  such that:

$$\sup_{|u| \leq N} \sup_n \{|\varrho_n(u)| + |\tilde{\varrho}_n(u)|\} \leq \bar{K}_N, \quad (2.24)$$

$$\sup_{|u|, |v| \leq N} \sup_n \{|\varrho_n(u) - \varrho_n(v)| + |\tilde{\varrho}_n(u) - \tilde{\varrho}_n(v)|\} \leq \bar{C}_N |u - v|. \quad (2.25)$$

(3) Furthermore, there exist mappings  $\varrho, \tilde{\varrho} : H \mapsto H$  such that every integer  $N \geq 1$

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq N} \{|\varrho_n(u) - \varrho(u)| + |\tilde{\varrho}_n(u) - \tilde{\varrho}(u)|\} = 0. \quad (2.26)$$

**Remark 2.2.** As a simple (non-trivial) example of diffusion coefficient  $\sigma$  and  $\tilde{\sigma}$  satisfying Conditions **(S)** and **(DS)**, we can consider the case when  $\tilde{\sigma}(u)$  is proportional to  $\sigma(u)$ , i.e.  $\tilde{\sigma}(u) = c_0 \sigma(u)$  for some constant  $c_0$  and  $\sigma(u)$  is an affine function of  $u$  of the form:

$$\sigma(u)f = \sum_{j \geq 1} f_j \sigma_j(u) \quad \text{for} \quad f = \sum_{j \geq 1} f_j \sqrt{q_j} e_j \in H_0,$$

where  $\sigma_j(u) = g_j + S_j u$ ,  $j = 1, 2, \dots$ . Here  $g_j \in H$  satisfy  $\sum_{j \geq 1} |g_j|^2 < \infty$  and  $S_j : H \mapsto H$  are linear operators such that  $S_j^* : V \mapsto V$  and  $\sum_{j \geq 1} |S_j|_{L(H,H)}^2 + \sup_{j \geq 1} |S_j^*|_{L(V,V)}^2 < \infty$ . For instance, in the case  $H = L_2(D)$  and  $V = H^1(D)$ , where  $D$  is a bounded domain in  $\mathbb{R}^d$ , our framework includes diffusion terms of the form

$$\sigma(u)dW(t) = \sum_{1 \leq j \leq N} (g_j(x) + \phi_j(x)u(x))d\beta_j(t),$$

where  $g_j \in L_2(D)$  and  $\phi_j \in C^1(\bar{D})$ ,  $j = 1, 2, \dots, N$  are arbitrary functions. In this situation, another possibility to satisfy Conditions **(S)** and **(DS)** is

$$\sigma(u)dW(t) = \sum_{1 \leq j \leq N} s_j([\mathcal{R}_j u](x))d\beta_j(t),$$

where  $s_j : \mathbb{R} \mapsto \mathbb{R}$  are  $\mathcal{C}^2$ -functions such that  $s_j'$  and  $s_j''$  are bounded, and  $[\mathcal{R}_j u](x) = \int_D r_j(x, y)u(y)dy$  with sufficiently smooth kernels  $r_j(x, y)$ .

In order to define the sequence of processes  $u^n$  converging to  $u$  in the Wong-Zakai approximation, we need a control term, that is a coefficient  $G$  of the process acting on an element of the RKHS of  $W$ . We impose that  $G$  and  $R$  satisfy the following:

**Condition (GR):** Let  $G : H \mapsto L(H_0, H)$  and  $R : H \mapsto H$  satisfy the following growth and Lipschitz conditions:

$$|G(u)|_{L(H_0, H)}^2 \leq K_0 + K_1 |u|^2, \quad |G(u) - G(v)|_{L(H_0, H)}^2 \leq L |u - v|^2, \quad (2.27)$$

$$|R(u)| \leq R_0(1 + |u|), \quad |R(u) - R(v)| \leq R_1 |u - v|, \quad (2.28)$$

for some nonnegative constants  $K_i, R_i$ ,  $i = 0, 1, L$  and for every  $u, v \in H$  (we can assume that  $K_i$  and  $L$  are the same constants as in (2.16) and (2.17)).

**2.4. Basic problem.** Let  $X := \mathcal{C}([0, T]; H) \cap L^2(0, T; V)$  denote the Banach space endowed with the norm defined by

$$\|u\|_X = \left\{ \sup_{0 \leq s \leq T} |u(s)|^2 + \int_0^T \|u(s)\|^2 ds \right\}^{\frac{1}{2}}. \quad (2.29)$$

The class  $\mathcal{A}$  of admissible random shifts is the set of  $H_0$ -valued  $(\mathcal{F}_t)$ -predictable stochastic processes  $h$  such that  $\int_0^T |h(s)|_0^2 ds < \infty$ , a.s. For any  $M > 0$ , let

$$S_M = \left\{ h \in L^2(0, T; H_0) : \int_0^T |h(s)|_0^2 ds \leq M \right\}, \quad \mathcal{A}_M = \{h \in \mathcal{A} : h(\omega) \in S_M, a.s.\}. \quad (2.30)$$

Assume that  $h \in \mathcal{A}_M$  and  $\xi \in H$  is  $\mathcal{F}_0$ -measurable random element such that  $\mathbb{E}|\xi|^4 < \infty$ . Then under the conditions **(B)**, **(GR)**, (2.16) and (2.17) in **(S)**, Theorem 3.1 [6] implies that there exists a unique  $(\mathcal{F}_t)$ -predictable solution  $u \in X$  to the stochastic problem:

$$\begin{aligned} u(t) &= \xi - \int_0^t [Au(s) + B(u(s)) + R(u(s))] ds \\ &\quad + \int_0^t (\sigma + \tilde{\sigma})(u(s)) dW(s) + \int_0^t G(u(s))h(s) ds, \quad \text{a.s. for all } t \in [0, T]. \end{aligned} \quad (2.31)$$

This solution is weak in the PDE sense and strong in the probabilistic meaning. Moreover, for this solution there exists a constant  $C := C(K_i, L, R_i, T, M)$  such that for  $h \in \mathcal{A}_M$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |u(t)|^4 + \int_0^T \|u(t)\|^2 dt + \int_0^T \|u(t)\|_{\mathcal{H}}^4 dt \right) \leq C (1 + E|\xi|^4). \quad (2.32)$$

**2.5. Approximate problem.** We also consider the evolution equation on the time interval  $[0, T]$ :

$$\begin{aligned} u^n(t) &= \xi - \int_0^t [Au^n(s) + B(u^n(s)) + R(u^n(s))] ds + \int_0^t \sigma((u^n(s))) dW(s) \\ &\quad + \int_0^t \tilde{\sigma}(u^n(s)) \tilde{W}^n(s) ds - \int_0^t \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)) ds + \int_0^t G(u^n(s))h(s) ds, \quad \text{a.s.}, \end{aligned} \quad (2.33)$$

where  $\tilde{W}^n(t)$  is defined in (2.13). Let again  $h \in \mathcal{A}_M$ ,  $\xi$  be  $\mathcal{F}_0$  measurable such that  $\mathbb{E}|\xi|^4 < +\infty$ . Then Theorem 3.1 [6] shows that problem (2.33) has unique  $(\mathcal{F}_t)$ -predictable solution  $u^n \in X$ . This solution also satisfies (2.32), but the constant  $C$  in the right hand side depends on the constants  $K_i, L, R_i, T, M$ , and in addition, it may also depend on  $n$ .

### 3. SUPPORT THEOREM

We consider the following stochastic perturbation of the evolution equation (2.1):

$$dU(t) + [AU(t) + B(U(t)) + R(U(t))] dt = \Xi(U(t)) dW(t), \quad U(0) = \xi \in H, \quad (3.1)$$

where  $\Xi \in \mathcal{C}(H; L_Q(H_0, H))$  is such that conditions **(S)** and **(DS)** hold with  $\tilde{\sigma} \equiv \sigma \equiv \Xi$ . Thus problem (3.1) is a special case of problem (2.31).

Let  $\phi \in L^2(0, T; H_0)$  and  $\varrho \Xi \equiv \varrho$  be defined by (2.23) and (2.26) with  $\tilde{\sigma} = \sigma = \Xi$ . We also consider the following (deterministic) nonlinear PDE

$$\partial_t v_\phi(t) + Av_\phi(t) + B(v_\phi(t)) + R(v_\phi(t)) + \frac{1}{2} \varrho \Xi(v_\phi(t)) = \Xi(v_\phi(t)) \phi(t), \quad v_\phi(0) = \xi \in H. \quad (3.2)$$

If  $B(u)$  satisfies condition **(B)** and  $R : H \mapsto H$  possesses property (2.28) we can use Theorem 3.1 in [6] to obtain the existence (and uniqueness) of the solution  $v_\phi$  to (3.2) in the space  $X = \mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ . Let

$$\mathcal{L} = \{v_\phi : \phi \in L^2(0, T; H_0)\} \subset X.$$

Our main result is the following:

**Theorem 3.1.** *Let conditions (B) and (S), (DS) with  $\tilde{\sigma} \equiv \sigma \equiv \Xi$  be in force. Assume that  $R : H \mapsto H$  satisfies (2.28). Let  $U$  denote the solution to the stochastic evolution equation (3.1) with deterministic initial data  $\xi \in H$ . Suppose that:*

- (i)  $t \mapsto \|U(t)\|_{\mathcal{H}}$  is continuous on  $[0, T]$  almost surely,
- (ii) there exists  $q > 0$  such that for any constant  $C > 0$  we have

$$\mathbb{E} \left( \sup_{[0, \tau_C]} \|U(t)\|_{\mathcal{H}}^q \right) < \infty, \quad (3.3)$$

where  $\tau_C := \inf \{ t : \sup_{s \leq t} |U(s)|^2 + \int_0^t \|U(s)\|^2 ds \geq C \} \wedge T$  is a stopping time.

Then  $\text{supp } U(\cdot) = \bar{\mathcal{L}}$ , where  $\bar{\mathcal{L}}$  is the closure of  $\mathcal{L}$  in  $X$  and  $\text{supp } U(\cdot)$  denotes the support of the distribution  $\mathbb{P} \circ U^{-1}$ , i.e., the support of the Borel measure on  $X$  defined by  $\mu(\mathcal{B}) = \mathbb{P} \{ \omega : U(\cdot) \in \mathcal{B} \}$  for any Borel subset  $\mathcal{B}$  of  $X$ .

This theorem is a consequence of the approximation Theorem 4.1 stated below and its proof is postponed to Section 4, after the statement of the Wong-Zakai approximation.

**Remark 3.2.** Theorem 3.1 is *conditional* in the sense that it gives the description of the support of solutions which satisfy some additional conditions concerning their properties in the space  $\mathcal{H}$ . We do not know whether these conditions can be derived from the basic requirements which we already have imposed on the model. We discuss this issue in the Appendix (see Section 7), where we show that conditions (i) and (ii) in the statement of Theorem 3.1 hold for several important cases of hydrodynamical models, which include 2D Navier-Stokes equations, 2D MHD equations and shell models of turbulence.

#### 4. WONG-ZAKAI TYPE APPROXIMATION THEOREM

Let  $u$  and  $u^n$  be solutions to (2.31) and (2.33) respectively. The main result of this section proves that the  $X$  norm of the difference  $u^n - u$  converges to 0 in probability. This is the key point of the support characterization stated in section 3.

**Theorem 4.1.** *Let conditions (B), (S), (DS) and (GR) hold and  $h \in L^2([0, T], H_0)$ . Let  $u$  be the solution to (2.31) and suppose that conditions (i) and (ii) of Theorem 3.1 hold. For every integer  $n \geq 1$  let  $u^n$  be the solution to (2.33). Then for every  $\lambda > 0$  we have*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0, T]} |u(t) - u^n(t)|^2 + \int_0^T \|u(s) - u^n(s)\|^2 ds \geq \lambda \right) = 0. \quad (4.1)$$

This convergence allows to deduce both inclusions characterizing the support of the distribution of the solution  $U$  to (3.1). Note that the convergence (4.1) remains valid if one supposes that  $h \in \mathcal{A}_M$  for some  $M > 0$ .

**Proof of Theorem 3.1.** The argument is similar to that introduced in [15].

In the definition of the evolution equation (2.33) let  $\tilde{\sigma} = \Xi$ ,  $\sigma = 0$ ,  $G = 0$ . Then if  $v_h$  is the solution to (3.2), where  $\phi = \tilde{W}_n$  is the (random) element of  $L^2(0, T; H_0)$  defined by (2.13), then we have that  $u^n = v_{\tilde{W}_n}$  for  $u^n$  defined by (2.33). Under this choice of  $\tilde{\sigma}$ ,  $\sigma$  and  $G$  for the solution  $u$  to (2.31), we obviously have that  $U(t) = u(t)$ , where  $U$  solves (3.1). Therefore the Wong-Zakai approximation stated in Theorem 4.1 implies that

$$\lim_n \mathbb{P}(\|v_{\tilde{W}_n} - U\|_X \geq \lambda) = 0 \quad \text{for any } \lambda > 0,$$

where  $\|\cdot\|_X$  is the norm defined by (2.29). Therefore  $\text{Support}(\mathbb{P} \circ U^{-1}) \subset \bar{\mathcal{L}}$ .

Conversely, fix  $h \in L^2(0, T; H_0)$ , let  $n \geq 1$  be an integer,  $\tilde{\sigma} = -\sigma$  and  $G = \sigma := \Xi$ . Let  $T_n^h : \Omega \rightarrow \Omega$  be defined by

$$T_n^h(\omega) = W(\omega) - \widetilde{W}^n(\omega) + \int_0^\cdot h(s)ds. \quad (4.2)$$

Then for every fixed integer  $n \geq 1$ , by Girsanov's theorem there exists a probability  $\mathbb{Q}_n^h \ll \mathbb{P}$  such that  $T_n^h$  is a  $\mathbb{Q}_n^h$ -Brownian motion with values in  $H$  and the same covariance operator  $Q$ . Indeed, the proof is easily decomposed in two steps, using Theorem 10.14 and Proposition 10.17 in [8]. First, since  $(\widetilde{W}_t^n, t \in [0, T])$  is  $H_0$ -valued and  $(\mathcal{F}_t)$ -adapted, we check that the following infinite dimensional version of the Benes criterion holds: for some  $\delta > 0$  we have that  $\sup_{0 \leq s \leq T} \mathbb{E}(\exp(\delta |\widetilde{W}_n(s)|_0^2)) < +\infty$ . This is a straightforward consequence of the inequality for some standard Gaussian random variable  $Z$ :

$$\begin{aligned} \sup_{0 \leq s \leq T} \mathbb{E}(\exp(\delta |\widetilde{W}_n(s)|_0^2)) &\leq \prod_{1 \leq j \leq n} \sup_{0 \leq s \leq T} \mathbb{E}(\exp(\delta T^{-2} 2^{2n} |\beta_j(\underline{s}_n) - \beta_j(s)|^2)) \\ &\leq \left( \mathbb{E}(\exp(\delta T^{-1} 2^n |Z|^2)) \right)^n < +\infty \end{aligned}$$

for  $\delta > 0$  small enough. Therefore, the measure with density  $L_T^1 = \exp\left(\int_0^T \widetilde{W}^n(s) dW(s) - \frac{1}{2} \int_0^T |\widetilde{W}^n(s)|_0^2 ds\right)$  with respect to  $\mathbb{P}$  is a probability  $\mathbb{Q}_1 \ll \mathbb{P}$ , such that the process  $(W_1(s) := W(s) - \widetilde{W}^n(s), 0 \leq s \leq t)$  is a  $\mathbb{Q}_1$  Brownian motion with values in  $H$ , and the same covariance operator  $Q$ . Then using once more these two results, since  $h \in L^2([0, T], H_0)$ , the measure with density  $L_T^2 = \exp\left(-\int_0^T h(s) dW_1(s) - \frac{1}{2} \int_0^T |h(s)|^2 ds\right)$  with respect to  $\mathbb{Q}_1$  is a probability  $\mathbb{Q}_2 \ll \mathbb{P}$ , such that the process

$$W_2(t) = W_1(t) + \int_0^t h(s)ds = W(t) - \widetilde{W}^n(t) + \int_0^t h(s)ds$$

is a  $H$ -valued Brownian motion under  $\mathbb{Q}_2$ , with covariance operator  $Q$ . Clearly  $\mathbb{Q}_n^h = \mathbb{Q}_2$ .

Let  $U$  denote the solution to (3.1); then, since  $T_n^h$  can be seen as a transformation of the standard Wiener space with Brownian motion  $W(t)$ , we deduce that  $U(\cdot)(T_n^h(\omega)) = u^n(\cdot)(\omega)$  in distribution on  $[0, T]$ . Thus Theorem 4.1 implies that for every  $\varepsilon > 0$ ,

$$\limsup_n \mathbb{P}(\{\omega : \|U(T_n^h(\omega)) - v_h\|_X < \varepsilon\}) > 0.$$

Let  $n_0 \geq 1$  be an integer such that

$$\mathbb{Q}_{n_0}^h(\{\omega : \|U(\omega) - v_h\|_X < \varepsilon\}) \equiv \mathbb{P}(\{\omega : \|U(T_{n_0}^h(\omega)) - v_h\|_X < \varepsilon\}) > 0.$$

Since  $\mathbb{Q}_{n_0}^h \ll \mathbb{P}$ , this implies  $\mathbb{P}(\{\omega : \|U(\omega) - v_h\|_X < \varepsilon\}) > 0$  which yields:

$$\overline{\mathcal{L}} \subset \text{Support}(\mathbb{P} \circ U^{-1}).$$

This completes the proof of Theorem 3.1.

**First step of the proof of the Wong-Zakai approximation Theorem 4.1.** Let  $M > 0$  be such that  $h \in \mathcal{A}_M$ . Without loss of generality we may and do assume in the sequel that  $0 < \lambda \leq 1$ . Fix  $N \geq 1$ ,  $m \geq 1$  and  $\lambda \in ]0, 1]$ . Let us introduce the following stopping times which will enable us to bound several norms for  $u$  and  $u^n$ :

$$\tau_N^{(1)} = \inf \left\{ t > 0 : \sup_{s \in [0, t]} |u(s)|^2 + \int_0^t \|u(s)\|^2 ds \geq N \right\} \wedge T,$$

$$\tau_n^{(2)} = \inf \left\{ t > 0 : \sup_{s \in [0, t]} |u(s) - u^n(s)|^2 + \int_0^t \|u(s) - u^n(s)\|^2 ds \geq \lambda \right\} \wedge T,$$

$$\tau_n^{(3)} = \inf \left\{ t > 0 : \left[ \sup_{j \leq n} \sup_{s \in [0, t]} \left| \dot{\beta}_j^n(s) \right| \right] \vee \left[ n^{-\frac{1}{2}} \sup_{s \in [0, t]} \left| \dot{W}^n(s) \right|_{H_0} \right] \geq \alpha n^{1/2} 2^{n/2} \right\} \wedge T,$$

and

$$\tau_m^{(4)} = \inf \left\{ t > 0 : \sup_{s \in [0, t]} \|u(s)\|_{\mathcal{H}} \geq m \right\} \wedge T.$$

In the sequel, the constants  $N$  and  $m$  will be chosen to make sure that, except on small sets,  $\tau_N^{(1)}$  and  $\tau_m^{(4)}$  are equal to  $T$ ; once this is done, only the dependence in  $n$  will be relevant. Thus once  $N$  and  $m$  have been chosen in terms of the limit process  $u$ , we let

$$\tau_n = \tau_N^{(1)} \wedge \tau_n^{(2)} \wedge \tau_n^{(3)} \wedge \tau_m^{(4)}. \quad (4.3)$$

One can see from the definition of  $\tau_N^{(1)}$  and  $\tau_n^{(2)}$  that

$$\sup_{s \in [0, \tau_n]} (|u(s)|^2 \vee |u^n(s)|^2) + \int_0^{\tau_n} (\|u(s)\|^2 \vee \|u^n(s)\|^2) ds \leq 2(N+1); \quad (4.4)$$

the definition of  $\tau_n^{(3)}$  yields

$$\sup_{s \leq \tau_n} \left( \left[ \sup_{j \leq n} \left| \dot{\beta}_j^n(s) \right| \right] \vee \left[ n^{-\frac{1}{2}} \sup_{s \leq t} \left| \dot{W}^n(s) \right|_{H_0} \right] \right) \leq \alpha n^{1/2} 2^{n/2}. \quad (4.5)$$

Furthermore, the definition of  $\tau_m^{(4)}$  implies

$$\sup_{s \in [0, \tau_n]} \|u(s)\|_{\mathcal{H}} \leq m. \quad (4.6)$$

We use the following obvious properties; their standard proof is omitted.

**Lemma 4.2.** *Let  $\Psi(t) \equiv \Psi(\omega, t)$  be a random, a.s. continuous, nondecreasing process on the interval  $[0, T]$ . Let  $\tau_\lambda = \inf\{t > 0 : \Psi(t) \geq \lambda\} \wedge T$ . Then*

$$\mathbb{P}(\Psi(T) \geq \lambda) = \mathbb{P}(\Psi(\tau_\lambda) \geq \lambda).$$

Let  $\tau_*$  be a stopping time such that  $0 \leq \tau_* \leq T$  and  $\mathbb{P}(\tau_* < T) \leq \varepsilon$ . Then

$$\mathbb{P}(\Psi(T) \geq \lambda) \leq \mathbb{P}(\Psi(\tau_\lambda \wedge \tau_*) \geq \lambda) + \varepsilon.$$

Apply Lemma 4.2 with  $\tau_* = \tau_N^{(1)} \wedge \tau_n^{(3)} \wedge \tau_m^{(4)}$  and

$$\Psi(t) = \sup_{s \in [0, t]} |u(s) - u^n(s)|^2 + \int_0^t \|u(s) - u^n(s)\|^2 ds.$$

Since a.s.  $u \in \mathcal{C}([0, T], H)$  and  $\int_0^T \|u(s)\|^2 ds < +\infty$ , the map  $\Psi$  is a.s. continuous and

$$\begin{aligned} \{\tau_* < T\} &\subset \{\tau_N^{(1)} < T\} \cup \{\tau_n^{(3)} < T\} \cup \{\tau_m^{(4)} < \tau_N^{(1)}\} \\ &\subset \left\{ \sup_{s \in [0, \tau_N^{(1)}]} |u(s)|^2 + \int_0^{\tau_N^{(1)}} \|u(s)\|^2 ds \geq N \right\} \cup \left\{ \sup_{s \in [0, \tau_N^{(1)}]} \|u(s)\|_{\mathcal{H}} \geq m \right\} \cup \Omega_n(T)^c, \end{aligned}$$

where  $\Omega_n(t)$  is given by (2.14). Therefore, by Chebyshev's inequality, from (2.32) and (3.3) we deduce that

$$\mathbb{P}(\tau_* < T) \leq C_1 N^{-1} + C_2(N) m^{-q} + \mathbb{P}(\Omega_n(T)^c).$$

Hence, given  $\varepsilon > 0$ , one may choose  $N$  and then  $m$  large enough to have  $C_1 N^{-1} + C_2(N) m^{-q} < \frac{\varepsilon}{2}$ . Using Lemma 2.1 we deduce that there exists  $n_0 \geq 1$  such that for all

integers  $n \geq n_0$ ,  $\mathbb{P}(\Omega_n(T)^c) < \frac{\varepsilon}{2}$ . Thus Lemma 4.2 shows that in order to prove (4.1) in Theorem 4.1, we only need to prove the following: Fix  $N, m > 0$ ; for every  $\lambda > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0, \tau_n]} |u(t) - u^n(t)|^2 + \int_0^{\tau_n} \|u(s) - u^n(s)\|^2 ds \geq \lambda \right) = 0, \quad (4.7)$$

where  $\tau_n$  is defined by (4.3). To check this convergence, it is sufficient to prove

$$\lim_{n \rightarrow \infty} \left[ \mathbb{E} \left( \sup_{t \in [0, \tau_n]} |u(t) - u^n(t)|^2 \right) + \mathbb{E} \int_0^{\tau_n} \|u(s) - u^n(s)\|^2 ds \right] = 0. \quad (4.8)$$

The proof of this last convergence result is given in Section 6. It relies on some precise control of times increments which is proven in the next section.

## 5. TIME INCREMENTS

Let  $h \in \mathcal{A}_M$ ,  $\xi$  be an  $\mathcal{F}_0$ -measurable  $H$ -valued random variable such that  $\mathbb{E}|\xi|^4 < \infty$  and let  $u$  be the solution to (2.31). For any integer  $N \geq 1$  and  $t \in [0, T]$  set

$$\tilde{G}_N(t) = \left\{ \sup_{0 \leq s \leq t} |u(s)| \leq N \right\}. \quad (5.1)$$

The following lemma refines the estimates proved in [6], Lemma 4.3 and the ideas are similar; see also [9], Lemma 4.2.

**Proposition 5.1.** *Let  $\phi_n, \psi_n : [0, T] \mapsto [0, T]$  be non-decreasing piecewise continuous functions such that*

$$0 \vee (s - k_0 T 2^{-n}) \leq \phi_n(s) \leq \psi_n(s) \leq (s + k_1 T 2^{-n}) \wedge T \quad (5.2)$$

for some integers  $k_0, k_1 \geq 0$ . Assume that  $h \in \mathcal{A}_M$  and  $\xi$  is a  $\mathcal{F}_0$ -measurable,  $H$ -valued random variable such that  $\mathbb{E}|\xi|^4 < \infty$ . Let  $\tilde{G}_N(t)$  be given by (5.1) and  $u$  be the solution to (2.31). There exists a constant  $C(N, M, T)$  such that

$$I_n = \mathbb{E} \int_0^T 1_{\tilde{G}_N(\psi_n(s))} |u(\psi_n(s)) - u(\phi_n(s))|^2 ds \leq C(N, M, T) 2^{-3n/4} \quad (5.3)$$

for every  $n = 1, 2, \dots$

*Proof.* We at first consider the case  $\phi_n(s) = 0 \vee (s - k_0 T 2^{-n})$  for some  $k_0 \geq 0$ ; then

$$I_n = \mathbb{E} \int_0^{t_{k_0}} 1_{\tilde{G}_N(\psi_n(s))} |u(\psi_n(s)) - \xi|^2 ds + I'_n$$

where  $t_{k_0} = k_0 T 2^{-n}$  and

$$I'_n = \mathbb{E} \int_{t_{k_0}}^T 1_{\tilde{G}_N(\psi_n(s))} |u(\psi_n(s)) - u(\phi_n(s))|^2 ds.$$

Therefore, using the definition (5.1) one can see that

$$I_n \leq C_{N,T} 2^{-n} + I'_n. \quad (5.4)$$

Furthermore, Itô's formula yields

$$|u(\psi_n(s)) - u(\phi_n(s))|^2 = 2 \int_{\phi_n(s)}^{\psi_n(s)} (u(r) - u(\phi_n(s)), du(r)) + \int_{\phi_n(s)}^{\psi_n(s)} |(\sigma + \tilde{\sigma})(u(r))|_{L_Q}^2 dr,$$

so that  $I'_n = \sum_{1 \leq i \leq 6} I_{n,i}$ , where

$$I_{n,1} = 2 \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (u(r) - u(\phi_n(s)), (\sigma + \tilde{\sigma})(u(r)) dW(r)) \right),$$

$$\begin{aligned}
I_{n,2} &= \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} |(\sigma + \tilde{\sigma})(u(r))|_{L_Q}^2 dr \right), \\
I_{n,3} &= 2 \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (G(u(r)) h(r), u(r) - u(\phi_n(s))) dr \right), \\
I_{n,4} &= -2 \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \langle A u(r), u(r) - u(\phi_n(s)) \rangle dr \right), \\
I_{n,5} &= -2 \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \langle B(u(r)), u(r) - u(\phi_n(s)) \rangle dr \right), \\
I_{n,6} &= -2 \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (R(u(r)), u(r) - u(\phi_n(s))) dr \right).
\end{aligned}$$

Clearly  $\tilde{G}_N(\psi_n(s)) \subset \tilde{G}_N(r)$  for  $r \leq \psi_n(s)$ . This means that  $|u(r)| \vee |u(\phi_n(s))| \leq N$  in the above integrals. We use this observation in the considerations below.

The Burkholder-Davis-Gundy inequality and (2.16) yield

$$\begin{aligned}
|I_{n,1}| &\leq 6 \int_{t_{k_0}}^T ds \mathbb{E} \left( \int_{\phi_n(s)}^{\psi_n(s)} |(\sigma + \tilde{\sigma})(u(r))|_{L_Q}^2 1_{\tilde{G}_N(r)} |u(r) - u(\phi_n(s))|^2 dr \right)^{\frac{1}{2}} \\
&\leq 6 \sqrt{2(K_0 + K_1 N^2)} \int_{t_{k_0}}^T ds \mathbb{E} \left( \int_{\phi_n(s)}^{\psi_n(s)} 1_{\tilde{G}_N(r)} |u(r) - u(\phi_n(s))|^2 dr \right)^{\frac{1}{2}}. \quad (5.5)
\end{aligned}$$

This implies that

$$|I_{n,1}| \leq C_N \int_0^T |\psi_n(s) - \phi_n(s)|^{1/2} ds \leq C_N T \sqrt{k_0 + k_1} 2^{-\frac{n}{2}}. \quad (5.6)$$

In a similar way using (2.16) again we deduce that

$$|I_{n,2}| \leq C_N \int_0^T |\psi_n(s) - \phi_n(s)| ds \leq C_N T (k_0 + k_1) 2^{-n}. \quad (5.7)$$

The growth condition (2.27) yields

$$|I_{n,3}| \leq C_N \int_0^T ds \int_{\phi_n(s)}^{\psi_n(s)} |h(r)|_0 dr \leq C_N \int_0^T ds \int_{0 \vee (s - k_0 T 2^{-n})}^{(s + k_1 T 2^{-n}) \wedge T} |h(r)|_0 dr,$$

and Fubini's theorem implies

$$|I_{n,3}| \leq C_N \int_0^T |h(r)|_0 dr 2^{-n} \leq C(N, T, M) 2^{-n}. \quad (5.8)$$

Using Schwarz's inequality we deduce that

$$I_{n,4} \leq 2 \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} dr \left[ -\|u(r)\|^2 + \|u(r)\| \|u(\phi_n(s))\| \right] \right). \quad (5.9)$$

The antisymmetry relation (2.2) and inequality (2.7) yield

$$\begin{aligned}
|\langle B(u(r)), u(r) - u(\phi_n(s)) \rangle| &= |\langle B(u(r)), u(\phi_n(s)) \rangle| \\
&\leq \frac{1}{2} \|u(r)\|^2 + C |u(r)|^2 \|u(\phi_n(s))\|_{\mathcal{H}}^4.
\end{aligned}$$

Therefore,

$$|I_{n,5}| \leq \mathbb{E} \left( \int_{t_{k_0}}^T ds 1_{\tilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \|u(r)\|^2 dr \right)$$

$$+ 2C\mathbb{E}\left(\int_0^T ds 1_{\tilde{G}_N(\psi_n(s))} \|u(\phi_n(s))\|_{\mathcal{H}}^4 \int_{\phi_n(s)}^{\psi_n(s)} |u(r)|^2 dr\right).$$

Using this inequality, (5.9) and (2.3), we deduce:

$$\begin{aligned} I_{n,4} + I_{n,5} &\leq \mathbb{E} \int_{t_{k_0}}^T 1_{\tilde{G}_N(\psi_n(s))} \|u(\phi_n(s))\|^2 |\psi_n(s) - \phi_n(s)| ds \\ &\quad + C_N \mathbb{E} \int_{t_{k_0}}^T 1_{\tilde{G}_N(\psi_n(s))} \|u(\phi_n(s))\|_{\mathcal{H}}^4 |\psi_n(s) - \phi_n(s)| ds \\ &\leq C(N, T) 2^{-n} \mathbb{E} \int_{t_{k_0}}^T 1_{\tilde{G}_N(\psi_n(s))} \|u(\phi_n(s))\|^2 ds = C(N, T) 2^{-n} \mathbb{E} \int_{t_{k_0}}^T \|u(s - t_{k_0})\|^2 ds. \end{aligned}$$

Hence, this last inequality and (2.32) imply

$$I_{n,4} + I_{n,5} \leq C(N, T) 2^{-n}. \quad (5.10)$$

A similar easier computation based on the growth condition (2.28) on  $R$  yields

$$|I_{n,6}| \leq 4R_0 N(1 + N) \int_0^T |\psi_n(s) - \phi_n(s)| ds \leq C(T, N) (k_0 + k_1) 2^{-n}. \quad (5.11)$$

Thus by (5.4), (5.6)–(5.8), (5.10) and (5.11), when  $\phi_n(s) = 0 \vee (s - k_0 T 2^{-n})$  we obtain:

$$I_n = \mathbb{E} \int_0^T 1_{\tilde{G}_N(\psi_n(s))} |u(\psi_n(s)) - u(\phi_n(s))|^2 ds \leq C(N, M, T) 2^{-n/2}. \quad (5.12)$$

In order to obtain (5.3), we need to improve the bound for  $I_{n,1}$  (see (5.6)). We make it using (5.12) and we again assume at first that  $\phi_n(s) = 0 \vee (s - k_0 T 2^{-n})$ . Let us denote by  $\chi_{i,n}(s) = \underline{(s + (i + 1)T 2^{-n})}_n$  the step function defined with the help of relations (2.11) and set

$$\begin{aligned} \mathcal{I}_n^{(i,-)} &= \int_{t_{k_0}}^{T-t_{k_1}} ds \mathbb{E} \int_{s+iT 2^{-n}}^{\chi_{i,n}(s)} 1_{\tilde{G}_N(r)} |u(r) - u(\phi_n(s))|^2 dr, \\ \mathcal{I}_n^{(i,+)} &= \int_{t_{k_0}}^{T-t_{k_1}} ds \mathbb{E} \int_{\chi_{i,n}(s)}^{s+(i+1)T 2^{-n}} 1_{\tilde{G}_N(r)} |u(r) - u(\phi_n(s))|^2 dr. \end{aligned}$$

The inequality (5.5) implies that

$$\begin{aligned} |I_{n,1}| &\leq C_N \int_{t_{k_0}}^T ds \mathbb{E} \left( \int_{\phi_n(s)}^{\psi_n(s)} 1_{\tilde{G}_N(r)} |u(r) - u(\phi_n(s))|^2 dr \right)^{\frac{1}{2}} \\ &\leq C_{N,T} 2^{-n} + C_{N,T} \left[ \sum_{-k_0 \leq i < k_1} \left( \mathcal{I}_n^{(i,-)} + \mathcal{I}_n^{(i,+)} \right) \right]^{1/2}. \end{aligned} \quad (5.13)$$

For any  $r$  from the interval  $[s + iT 2^{-n}, \chi_{i,n}(s)]$  we have  $\bar{r}_n = \chi_{i,n}(s)$ ; therefore

$$\begin{aligned} \mathcal{I}_n^{(i,-)} &\leq 2 \int_{t_{k_0}}^{T-t_{k_1}} ds \mathbb{E} \left[ \int_{s+iT 2^{-n}}^{\chi_{i,n}(s)} 1_{\tilde{G}_N(r)} |u(r) - u(\bar{r}_n)|^2 dr \right. \\ &\quad \left. + T 2^{-n} 1_{\tilde{G}_N(\chi_{i,n}(s))} |u(\chi_{i,n}(s)) - u(\phi_n(s))|^2 \right]. \end{aligned}$$

Thus, using Fubini's theorem and (5.12) we can conclude that

$$\mathcal{I}_n^{(i,-)} \leq C(N, M, T) 2^{-3n/2}.$$

Similarly

$$\begin{aligned} \mathcal{I}_n^{(i,+)} &\leq 2 \int_{t_{k_0}}^{T-t_{k_1}} ds \mathbb{E} \left[ \int_{\chi_{i,n}(s)}^{s+(i+1)T2^{-n}} 1_{\tilde{G}_N(r)} |u(r) - u(\underline{x}_n)|^2 dr \right. \\ &\quad \left. + T2^{-n} 1_{\tilde{G}_N(\chi_{i,n}(s))} |u(\chi_{i,n}(s)) - u(\phi_n(s))|^2 \right] \leq C(N, M, T)2^{-3n/2}. \end{aligned}$$

Hence (5.13) implies  $I_{n,1} \leq C(N, M, T)2^{-3n/4}$ . This inequality and the above upper estimates for  $I_{n,i}$  with  $i \neq 1$  prove (5.3) in the case  $\phi_n(s) = \phi_n^*(s) := 0 \vee (s - k_0 T 2^{-n})$ . In the general case, we can write

$$\begin{aligned} &1_{\tilde{G}_N(\psi_n(s))} |u(\psi_n(s)) - u(\phi_n(s))|^2 \\ &\leq 2 \left( 1_{\tilde{G}_N(\psi_n(s))} |u(\psi_n(s)) - u(\phi_n^*(s))|^2 + 1_{\tilde{G}_N(\phi_n(s))} |u(\phi_n(s)) - u(\phi_n^*(s))|^2 \right); \end{aligned}$$

this concludes the proof of Proposition 5.1 for functions  $\phi_n$  and  $\psi_n$  which satisfy (5.2).  $\square$

We also need a similar for the time increments of the approximate solutions  $u^n$ .

**Proposition 5.2.** *Let  $\phi_n, \psi_n : [0, T] \mapsto [0, T]$  be non-decreasing piecewise continuous functions such that condition (5.2) is satisfied for some positive integers  $k_0$  and  $k_1$ . Fix  $M > 0$ , let  $h \in \mathcal{A}_M$  and  $\xi$  be a  $\mathcal{F}_0$ -measurable,  $H$ -valued random variable such that  $\mathbb{E}|\xi|^4 < \infty$ . Let  $u^n$  be the solution to (2.33); for  $N > 0$  set*

$$G_N^n(t) = \left\{ \sup_{0 \leq s \leq t} |u^n(s)| \leq N \right\} \cap \left\{ \int_0^t \|u^n(s)\|^2 ds \leq N \right\} \cap \Omega_n(t), \quad (5.14)$$

where  $\Omega_n(t)$  is defined in (2.14) and let  $\tau_n$  be the stopping time defined in (4.3). There exists a constant  $C(N, M, T)$  such that

$$\tilde{I}_n = \mathbb{E} \int_0^{\tau_n} 1_{G_N^n(\psi_n(s))} |u^n(\psi_n(s)) - u^n(\phi_n(s))|^2 ds \leq C(N, M, T)n^{3/2}2^{-3n/4} \quad (5.15)$$

for every  $n = 1, 2, \dots$

*Proof.* We use the same idea as in the proof of Proposition 5.1 and at first suppose that  $\phi_n(s) = 0 \vee (s - k_0 T 2^{-n})$  for some  $k_0 \geq 0$ . Let  $t_{k_0} = k_0 T 2^{-n}$ ; then we have

$$\tilde{I}_n \leq C_{N,T} 2^{-n} + \tilde{I}'_n, \quad (5.16)$$

where

$$\tilde{I}'_n = \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} 1_{G_N^n(\psi_n(s))} |u^n(\psi_n(s)) - u^n(\phi_n(s))|^2 ds. \quad (5.17)$$

Itô's formula applied to  $|u^n(\cdot) - u^n(\phi_n(s))|^2$  implies that  $\tilde{I}'_n = \sum_{1 \leq i \leq 6} \tilde{I}_{n,i}$ , where

$$\begin{aligned} \tilde{I}_{n,1} &= 2\mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (u^n(r) - u^n(\phi_n(s)), \sigma(u^n(r)) dW(r)), \\ \tilde{I}_{n,2} &= \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} |\sigma(u^n(r))|_{L_Q}^2 dr, \\ \tilde{I}_{n,3} &= 2\mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (u^n(r) - u^n(\phi_n(s)), \tilde{\sigma}(u^n(r)) \overset{\circ}{W}(r)) dr, \\ \tilde{I}_{n,4} &= 2\mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (u^n(r) - u^n(\phi_n(s)), G(u^n(r))h(r)) dr, \\ \tilde{I}_{n,5} &= 2\mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (u^n(r) - u^n(\phi_n(s)), (\varrho + \frac{1}{2}\tilde{\varrho} - R)(u^n(r))) dr, \end{aligned}$$

$$\tilde{I}_{n,6} = -2 \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} (u^n(r) - u^n(\phi_n(s)), Au^n(r) + B(u^n(r))) dr.$$

Estimates for  $\tilde{I}_{n,2}$ ,  $\tilde{I}_{n,4}$ ,  $\tilde{I}_{n,5}$  are obvious. Indeed, we can first extend outward integration to the time interval  $[t_{k_0}, T]$  and then use the growth conditions (2.16), (2.24) and (2.26). This yields the estimate

$$|\tilde{I}_{n,i}| \leq C(N, T) 2^{-n}, \quad i = 2, 4, 5. \quad (5.18)$$

Note that (5.18) holds as soon as  $0 \leq \psi_n(s) - \phi_n(s) \leq C2^{-n}$  for some constant  $C > 0$ , and does not require the specific form of  $\phi_n$ . Schwarz's inequality and Condition **(B)** imply

$$\begin{aligned} -(u^n(r) - u^n(\phi_n(s)), Au^n(r) + B(u^n(r))) &\leq C_1 \|u(\phi_n(s))\|^2 + C_2 |u(r)|^2 \|u(\phi_n(s))\|_{\mathcal{H}}^4 \\ &\leq C_0 \|u(\phi_n(s))\|^2 [1 + |u(r)|^2 |u(\phi_n(s))|^2]. \end{aligned}$$

Therefore, if  $\phi_n(s) = (s - t_{k_0}) \vee 0$  we deduce

$$\begin{aligned} |\tilde{I}_{n,6}| &\leq C(N, T) \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \|u^n(\phi_n(s))\|^2 dr \\ &\leq C(N, T) 2^{-n} \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} 1_{G_N^n(\psi_n(s))} \|u^n(s - t_{k_0})\|^2 ds \\ &\leq C(N, T) 2^{-n} \mathbb{E} \int_{(\tau_n \wedge t_{k_0} - t_{k_0})^+}^{(\tau_n - t_{k_0})^+} \|u^n(s)\|^2 ds \\ &\leq C(N, T) 2^{-n} \mathbb{E} \int_0^{\tau_n} \|u^n(s)\|^2 ds \leq C(N, T) 2^{-n}. \end{aligned} \quad (5.19)$$

Using (4.3) and the upper bound of  $\psi_n(s) - \phi_n(s)$  (and not the specific form of  $\phi_n$ ), we deduce

$$|\tilde{I}_{n,3}| \leq C_N n 2^{-\frac{n}{2}} \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} |u^n(r) - u^n(\phi_n(s))| dr \leq C(T, N) n 2^{-\frac{n}{2}}. \quad (5.20)$$

Since for  $s \leq t$  we have  $G_N^n(t) \subset G_N^n(s)$ , the local property of stochastic integrals, the linear growth condition (2.16) and Schwarz's inequality imply that

$$\begin{aligned} |\tilde{I}_{n,1}| &\leq 2\sqrt{T} \left( \mathbb{E} \int_{t_{k_0} \wedge \tau_n}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \right. \\ &\quad \times \left. \left| \int_{\phi_n(s) \wedge \tau_n}^{\psi_n(s) \wedge \tau_n} 1_{G_N^n(r)} (\sigma(u^n(r)), u^n(r) - u^n(\phi_n(s))) dW(r) \right|^2 \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{T} \left( \int_0^T ds \mathbb{E} \int_{\phi_n(s)}^{\psi_n(s)} 1_{G_N^n(r)} 1_{[0, \tau_n]}(r) |\sigma(u^n(r))|_{L^Q}^2 |u^n(r) - u^n(\phi_n(s))|^2 dr \right)^{\frac{1}{2}} \end{aligned} \quad (5.21)$$

$$\leq C(N, T) \left( \int_0^T ds \int_{\phi_n(s)}^{\psi_n(s)} dr \right)^{\frac{1}{2}} = C(N, T) 2^{-n/2}. \quad (5.22)$$

The inequalities (5.16) - (5.22) yield for  $\phi_n(s) = (s - t_{k_0}) \vee 0$ :

$$\tilde{I}_n = \mathbb{E} \int_0^{\tau_n} 1_{G_N^n(\psi_n(s))} |u^n(\psi_n(s)) - u^n(\phi_n(s))|^2 ds \leq C(N, M, T) n 2^{-n/2}. \quad (5.23)$$

In order to obtain the final estimate in (5.15) we need to improve the upper estimates of  $\tilde{I}_{n,1}$  and  $\tilde{I}_{n,3}$ . This can be done in a way similar to that used in the proof of previous Proposition. One can easily see that (5.23) holds when  $\phi_n \leq \psi_n$  satisfy the assumptions

of the Proposition and  $\phi_n$  is piece-wise constant. Then (5.20) and Schwarz's inequality obviously imply that

$$|\tilde{I}_{n,3}| \leq C_N n^{3/2} 2^{-3n/4}.$$

Thus, to conclude the proof we need to deal with the improvement of  $I_{n,1}$ . Let the function  $\phi_n$  be piece-wise constant; then given  $r \in [\phi_n(s), \psi_n(s)]$ , we have  $\phi_n(s) \in \{r - i2^{-n} : 0 \leq i \leq k_0\}$ . Therefore, using the inequality (5.21), Fubini's theorem and (5.23) applied with the functions  $\phi_{n,i}(r) = \underline{r - iT2^{-n}}$  and  $\psi_n(r) = r$  we deduce

$$\begin{aligned} |\tilde{I}_{n,1}| &\leq C(N, T) \left( \sum_{0 \leq i \leq k_0} \mathbb{E} \int_0^{\tau_n} dr 1_{G_N^n(r)} |u^n(\psi_n(r)) - u^n(\phi_{n,i}(r))|^2 \int_r^{(r+k_0T2^{-n}) \wedge T} ds \right)^{\frac{1}{2}} \\ &\leq (k_0 + 1)^{\frac{1}{2}} C(N, M, T) n^{\frac{1}{2}} 2^{-3n/4}; \end{aligned}$$

this concludes the proof of (5.15) when  $\phi_n$  is piece-wise constant. To deduce that this inequality holds for arbitrary functions  $\phi_n$  and  $\psi_n$  satisfying (5.2), apply (5.15) for  $\tilde{\phi}_n(s) = (s - t_{k_0})_n$  and either  $\tilde{\psi}_n = \phi_n$  or  $\tilde{\psi}_n = \psi_n$ ; this concludes the proof.  $\square$

Proposition 5.2 implies that

$$\mathbb{E} \int_0^{\tau_n} 1_{G_N^n(s)} \left( |u^n(s) - u^n(s_n)|^2 + |u^n(s) - u^n(\underline{s}_n)|^2 \right) ds \leq C(T, N, M) n^{3/2} 2^{-3n/4} \quad (5.24)$$

where  $G_N^n(t)$  is defined by (5.14). This is precisely what we need below.

## 6. PROOF OF CONVERGENCE RESULT

The aim of this section is to prove Theorem 4.1; in this section  $N$  and  $m$  are fixed. For every integer  $n \geq 1$ ,  $\tau_n$  is the stopping time defined by (4.3) and we prove (4.8). In the estimates below, constants may change from line to line, but we indicate their dependence on parameters when it becomes important. From equations (2.33) and (2.31) we deduce:

$$\begin{aligned} u^n(t) - u(t) &= - \int_0^t \left[ A[u^n(s) - u(s)] + B(u^n(s)) - B(u(s)) + R(u^n(s)) - R(u(s)) \right] ds \\ &\quad + \int_0^t [G(u^n(s)) - G(u(s))] h(s) ds + \int_0^t [\sigma(u^n(s)) - \sigma(u(s))] dW(s) \\ &\quad + \int_0^t \left[ \tilde{\sigma}(u^n(s_n)) \tilde{W}^n(s) ds - \tilde{\sigma}(u(s)) dW(s) \right] - \int_0^t \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)) ds \\ &\quad + \int_0^t [\tilde{\sigma}(u^n(s)) - \tilde{\sigma}(u^n(s_n))] \tilde{W}^n(s) ds. \end{aligned} \quad (6.1)$$

Let  $\underline{t}_n$  and  $\bar{t}_n$  be defined by (2.11), let  $\mathcal{E}_n$  denote projector in  $L^2(0, T)$  on the subspace of step functions defined by

$$(\mathcal{E}_n f)(t) = \left( T^{-1} 2^n \int_{\underline{t}_n}^{\bar{t}_n} f(s) ds \right) \cdot 1_{[\underline{t}_n, \bar{t}_n]}(t)$$

and let  $\delta_n : L^2(0, T) \mapsto L^2(0, T)$  denote the shift operator defined by

$$(\delta_n f)(t) = f((t + T2^{-n}) \wedge T) \quad \text{for } t \in [0, T].$$

Using (2.12) and (2.13) we deduce

$$\int_0^t \tilde{\sigma}(u^n(s_n)) \tilde{W}^n(s) ds = \int_0^t \mathcal{E}_n \left[ (\delta_n [1_{[0,t]}](s) \tilde{\sigma}(u^n(\underline{s}_n)) \circ \Pi_n \right] dW(s). \quad (6.2)$$

Hence

$$\begin{aligned} u^n(t) - u(t) &= - \int_0^t \left[ A(u^n(s) - u(s)) + B(u^n(s)) - B(u(s)) + R(u^n(s)) - R(u(s)) \right] ds \\ &+ \int_0^t [G(u^n(s)) - G(u(s))] h(s) ds + \int_0^t \left( [\sigma + \tilde{\sigma}](u^n(s)) - [\sigma + \tilde{\sigma}](u(s)) \right) dW(s) \\ &+ \int_0^t [\tilde{\sigma}(u^n(s)) - \tilde{\sigma}(u^n(s_n))] \dot{\tilde{W}}^n(s) ds - \int_0^t \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)) ds + \int_0^t \tilde{\Sigma}_n(s) dW(s), \end{aligned}$$

where

$$\tilde{\Sigma}_n(s) = \mathcal{E}_n \left[ (\delta_n 1_{[0,t]}) (s) \tilde{\sigma}(u^n(\underline{x}_n)) \circ \Pi_n \right] - \tilde{\sigma}(u^n(s)). \quad (6.3)$$

Itô's formula implies that

$$\begin{aligned} |u^n(t) - u(t)|^2 + 2 \int_0^t \|u^n(s) - u(s)\|^2 ds &= -2 \int_0^t \langle B(u^n(s)) - B(u(s)), u^n(s) - u(s) \rangle ds \\ &+ 2 \int_0^t \left( [G(u^n(s)) - G(u(s))] h(s) - [R(u^n(s)) - R(u(s))] \right), u^n(s) - u(s) ds \\ &+ 2 \int_0^t \left( [\tilde{\sigma}(u^n(s)) - \tilde{\sigma}(u^n(s_n))] \dot{\tilde{W}}^n(s) - \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)) \right), u^n(s) - u(s) ds \\ &+ \int_0^t |\Sigma_n(s)|_{L^Q}^2 ds + 2 \int_0^t \left( \Sigma_n(s) dW(s), u^n(s) - u(s) \right), \end{aligned}$$

where

$$\Sigma_n(s) = \tilde{\Sigma}_n(s) + (\sigma + \tilde{\sigma})(u^n(s)) - (\sigma + \tilde{\sigma})(u(s)). \quad (6.4)$$

Using (2.15) and (4.3), we have  $\dot{\tilde{W}}_n(s) = \dot{W}_n(s)$  on the set  $\{s \leq \tau_n\}$ ; let

$$\begin{aligned} Z_n^{(0)}(t) &= \int_0^t (\Sigma_n(s) dW(s), u^n(s) - u(s)), \quad (6.5) \\ Z_n^{(1)}(t) &= \int_0^t |\mathcal{E}_n \left[ (\delta_n 1_{[0,t]}) (s) \tilde{\sigma}(u^n(\underline{x}_n)) \circ \Pi_n \right] - \tilde{\sigma}(u^n(s))|_{L^Q}^2 ds, \\ Z_n^{(2)}(t) &= \int_0^t \left( [\tilde{\sigma}(u^n(s)) - \tilde{\sigma}(u^n(s_n))] \dot{W}_n(s) - \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)) \right), u^n(s) - u(s) ds. \end{aligned}$$

The equation (2.8) with  $\eta = 1/2$  and condition **(GR)** yield

$$\begin{aligned} |u^n(t \wedge \tau_n) - u(t \wedge \tau_n)|^2 + \int_0^{t \wedge \tau_n} \|u^n(s) - u(s)\|^2 ds &\leq 2 \sum_{0 \leq i \leq 2} Z_n^{(i)}(t \wedge \tau_n) \quad (6.6) \\ &+ 2 \int_0^{t \wedge \tau_n} \left( 2L + \sqrt{L} |h(s)|_0 + R_1 + C_{1/2} \|u(s)\|_{\mathcal{H}}^4 \right) |u^n(s) - u(s)|^2 ds. \end{aligned}$$

For every integer  $n \geq 1$  and every  $t \in [0, T]$ , set

$$T_n(t) = \sup_{0 \leq s \leq t \wedge \tau_n} |u^n(s) - u(s)|^2 + \int_0^{t \wedge \tau_n} \|u^n(s) - u(s)\|^2 ds. \quad (6.7)$$

Using (4.6) and Gronwall's lemma we conclude that for all  $t \in [0, T]$

$$\mathbb{E} T_n(t) \leq C \sum_{0 \leq i \leq 2} \mathbb{E} \left( \sup_{s \leq t \wedge \tau_n} |Z_n^{(i)}(s)| \right). \quad (6.8)$$

**6.1. Estimate for  $Z_n^{(0)}$ .** The Burkholder-Davies-Gundy inequality, equations (6.3), (6.4) and (2.17) imply that for any  $\eta > 0$  there exists  $C_\eta > 0$  such that

$$\begin{aligned} \mathbb{E} \left( \sup_{s \leq t \wedge \tau_n} \left| Z_n^{(0)}(s) \right| \right) &\leq 3 \mathbb{E} \left\{ \int_0^{t \wedge \tau_n} |u^n(s) - u(s)|^2 |\Sigma_n(s)|_{L_Q}^2 ds \right\}^{1/2} \\ &\leq 3 \mathbb{E} \left\{ \sup_{s \leq t \wedge \tau_n} |u^n(s) - u(s)| \left[ \int_0^{t \wedge \tau_n} |\Sigma_n(s)|_{L_Q}^2 ds \right]^{1/2} \right\} \\ &\leq \eta \mathbb{E} T_n(t) + C_\eta \mathbb{E} \left( \int_0^{t \wedge \tau_n} |\Sigma_n(s)|_{L_Q}^2 ds \right) \\ &\leq \eta \mathbb{E} T_n(t) + 2 C_\eta \mathbb{E} Z_n^{(1)}(t \wedge \tau_n) + 4 L C_\eta \int_0^t \mathbb{E} |u^n(s \wedge \tau_n) - u(s \wedge \tau_n)|^2 ds. \end{aligned}$$

Thus if  $\eta = \frac{1}{2}$ , (6.8) and Gronwall's lemma imply that for some constant  $C$  which does not depend on  $n$ ,

$$\mathbb{E} T_n(t) \leq C \left( \mathbb{E} \sup_{s \leq t \wedge \tau_n} Z_n^{(1)}(s) + \mathbb{E} \sup_{s \leq t \wedge \tau_n} |Z_n^{(2)}(s)| \right). \quad (6.9)$$

**6.2. Estimate of  $Z_n^{(1)}$ .** The convergence of  $Z_n^{(1)}$  is stated in the following assertion.

**Lemma 6.1.** *Fix  $M > 0$ ,  $h \in \mathcal{A}_M$  and let  $Z_n^{(1)}(t)$  be defined by (6.5); then for fixed  $N$  and  $m$  we have:*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \sup_{t \in [0, T]} Z_n^{(1)}(t \wedge \tau_n) \right) = 0. \quad (6.10)$$

*Proof.* For  $k = 0, \dots, 2^n - 1$  let  $\Omega_{n,k} = \{t_k < t \wedge \tau_n \leq t_{k+1}\}$ , where as above we set  $t_k = kT2^{-n}$ . We consider  $Z_n^{(1)}(t \wedge \tau_n)$  separately on each set  $\Omega_{n,k}$ .

We start with the case  $\omega \in \Omega_{n,k}$  for  $k \geq 2$ . Then  $(\delta_n 1_{[0, t \wedge \tau_n]})(s) = 1$  for  $s \leq t_{k-1}$  and

$$\begin{aligned} Z_n^{(1)}(t \wedge \tau_n) &= \int_0^{t \wedge \tau_n} |\mathcal{E}_n [(\delta_n 1_{[0, t \wedge \tau_n]})(s) \tilde{\sigma}(u^n(\underline{s}_n)) \circ \Pi_n] - \tilde{\sigma}(u^n(s))|_{L_Q}^2 ds \\ &\leq \sum_{0 \leq i \leq k-2} \int_{t_i \wedge \tau_n}^{t_{i+1} \wedge \tau_n} |\mathcal{E}_n [\tilde{\sigma}(u^n(\underline{s}_n)) \circ \Pi_n] - \tilde{\sigma}(u^n(s))|_{L_Q}^2 ds \\ &\quad + 2 \int_{t_{k-1} \wedge \tau_n}^{t_{k+1} \wedge \tau_n} \left( |\tilde{\sigma}(u^n(\underline{s}_n)) \circ \Pi_n|_{L_Q}^2 + |\tilde{\sigma}(u^n(s))|_{L_Q}^2 \right) ds. \end{aligned}$$

Thus using (2.16) and (4.4) we deduce that for some constant  $C = C(K_0, K_1, N, T)$  which does not depend on  $n$ :

$$\begin{aligned} Z_n^{(1)}(t \wedge \tau_n) &\leq C 2^{-n} + \sum_{0 \leq i \leq k-2} \int_{t_i \wedge \tau_n}^{t_{i+1} \wedge \tau_n} |\tilde{\sigma}(u^n(\underline{s}_n)) \circ \Pi_n - \tilde{\sigma}(u^n(s))|_{L_Q}^2 ds \\ &\leq C 2^{-n} + 2T \sup_{|u| \leq 2(N+1)} |\tilde{\sigma}(u) \circ \Pi_n - \tilde{\sigma}(u)|_{L_Q}^2 \\ &\quad + 2 \sum_{0 \leq i \leq k-2} \int_{t_i \wedge \tau_n}^{t_{i+1} \wedge \tau_n} |\tilde{\sigma}(u^n(t_i)) - \tilde{\sigma}(u^n(s))|_{L_Q}^2 ds \\ &\leq C 2^{-n} + 2T \sup_{|u| \leq 2(N+1)} |\tilde{\sigma}(u) \circ \Pi_n - \tilde{\sigma}(u)|_{L_Q}^2 + 2L \int_0^{t \wedge \tau_n} |u^n(\underline{s}_n) - u^n(s)|^2 ds \\ &\leq C 2^{-n} + 2T \sup_{|u| \leq 2(N+1)} |\tilde{\sigma}(u) \circ \Pi_n - \tilde{\sigma}(u)|_{L_Q}^2 + 2L \int_0^{\tau_n} 1_{G_N^c}(s) |u^n(\underline{s}_n) - u^n(s)|^2 ds, \end{aligned}$$

where  $G_N^n(s)$  is defined by (5.14). Furthermore, given  $\omega \in \Omega_{n,1} \cup \Omega_{n,2}$  we have:

$$Z_n^{(1)}(t \wedge \tau_n) \leq 2 \int_0^{t_2 \wedge t \wedge \tau_n} \left( |\tilde{\sigma}(u^n(\underline{s}_n)) \circ \Pi_n|_{L^Q}^2 + |\tilde{\sigma}(u^n(s))|_{L^Q}^2 \right) ds \leq C 2^{-n},$$

where  $C = C(K_0, K_1, N, T)$  does not depend on  $n$ . This yields

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in [0, T]} Z_n^{(1)}(t \wedge \tau_n) \right) &\leq C 2^{-n} + 2T \sup_{|u| \leq 2(N+1)} |\tilde{\sigma}(u) \circ \Pi_n - \tilde{\sigma}(u)|_{L^Q}^2 \\ &\quad + 2L \mathbb{E} \int_0^{\tau_n} 1_{G_N^n(s)} |u^n(\underline{s}_n) - u^n(s)|^2 ds; \end{aligned}$$

therefore, (6.10) follows from (2.18) and (5.24).  $\square$

### 6.3. Estimate of $Z_n^{(2)}$ .

**6.3.1. Main splitting.** The identities (2.19), (2.13) and (2.15) yield

$$\begin{aligned} Z_n^{(2)}(t \wedge \tau_n) &= \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( [\tilde{\sigma}_j(u^n(s)) - \tilde{\sigma}_j(u^n(s_n))] \dot{\beta}_j^n(s), u^n(s) - u(s) \right) ds \\ &\quad - \int_0^{t \wedge \tau_n} \left( \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)), u^n(s) - u(s) \right) ds. \end{aligned} \quad (6.11)$$

For every  $j = 1, \dots, n$  Taylor's formula implies that

$$\begin{aligned} \tilde{\sigma}_j(u^n(s)) - \tilde{\sigma}_j(u^n(s_n)) &= D\tilde{\sigma}_j(u^n(s_n)) [u^n(s) - u^n(s_n)] \\ &\quad + \int_0^1 (1 - \mu) d\mu \left\langle D^2 \tilde{\sigma}_j(u^n(s_n) + \mu [u^n(s) - u^n(s_n)]); u^n(s) - u^n(s_n), u^n(s) - u^n(s_n) \right\rangle, \end{aligned}$$

where  $\langle D^2 \tilde{\sigma}_j(v); v_1, v_2 \rangle$  denotes the value of the second Fréchet derivative  $D^2 \tilde{\sigma}_j(v)$  on elements  $v_1$  and  $v_2$ . Therefore condition (2.21) and the bound (4.4) imply that for every  $t \in [0, T]$ ,

$$\left| Z_n^{(2)}(t \wedge \tau_n) \right| \leq T_n(t, 1) + \left| \tilde{Z}_n^{(2)}(t) \right|, \quad (6.12)$$

where

$$T_n(t, 1) = C_2(2N + 1) \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} |u^n(s) - u^n(s_n)|^2 |\dot{\beta}_j^n(s)| |u^n(s) - u(s)| ds,$$

and

$$\begin{aligned} \tilde{Z}_n^{(2)}(t) &= \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( D\tilde{\sigma}_j(u^n(s_n)) [u^n(s) - u^n(s_n)], u^n(s) - u(s) \right) \dot{\beta}_j^n(s) ds \\ &\quad - \int_0^{t \wedge \tau_n} \left( \left( \varrho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)), u^n(s) - u(s) \right) ds. \end{aligned} \quad (6.13)$$

For  $G_N^n(t)$  defined by (5.14), one has

$$T_n(t, 1) \leq C_N \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} 1_{G_N^n(s)} |u^n(s) - u^n(s_n)|^2 |\dot{\beta}_j^n(s)| |u^n(s) - u(s)| ds.$$

Therefore, (4.3), the inequalities (4.5) and (5.24) yield for some constant  $C := C(N, M, T)$

$$\mathbb{E} \left( \sup_{t \in [0, T]} T_n(t, 1) \right) \leq \tilde{C}_N n^{\frac{3}{2}} 2^{\frac{n}{2}} \mathbb{E} \int_0^{\tau_n} 1_{G_N^n(s)} |u^n(s) - u^n(s_n)|^2 ds \leq C n^3 2^{-\frac{n}{4}}. \quad (6.14)$$

To bound  $\tilde{Z}_n^{(2)}$ , rewrite  $u^n(s) - u^n(s_n)$  in (6.13) using the evolution equation (2.33). This yields the following decomposition:

$$\tilde{Z}_n^{(2)}(t) = \sum_{2 \leq i \leq 6} T_n(t, i), \quad (6.15)$$

where

$$\begin{aligned} T_n(t, 2) &= \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( D\tilde{\sigma}_j(u^n(s_n)) \mathcal{I}_n(s, s_n) \dot{\beta}_j^n(s), u^n(s) - u(s) \right) ds \\ &\quad - \int_0^{t \wedge \tau_n} \left( \left( \rho + \frac{1}{2} \tilde{\varrho} \right) (u^n(s)), u^n(s) - u(s) \right) ds \end{aligned} \quad (6.16)$$

with

$$\mathcal{I}_n(s, s_n) := \int_{s_n}^s \sigma(u^n(r)) dW(r) + \int_{s_n}^s \tilde{\sigma}(u^n(r)) \dot{W}^n(r) dr, \quad (6.17)$$

$$T_n(t, 3) = - \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( D\tilde{\sigma}_j(u^n(s_n)) \left[ \int_{s_n}^s A u^n(r) dr \right] \dot{\beta}_j^n(s), u^n(s) - u(s) \right) ds,$$

$$T_n(t, 4) = - \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( D\tilde{\sigma}_j(u^n(s_n)) \left[ \int_{s_n}^s B(u^n(r)) dr \right] \dot{\beta}_j^n(s), u^n(s) - u(s) \right) ds,$$

$$T_n(t, 5) = \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( D\tilde{\sigma}_j(u^n(s_n)) \left[ \int_{s_n}^s G(u^n(r)) h(r) dr \right] \dot{\beta}_j^n(s), u^n(s) - u(s) \right) ds,$$

$$T_n(t, 6) = - \sum_{1 \leq j \leq n} \int_0^{t \wedge \tau_n} \left( D\tilde{\sigma}_j(u^n(s_n)) \left[ \int_{s_n}^s \tilde{R}(u^n(r)) dr \right] \dot{\beta}_j^n(s), u^n(s) - u(s) \right) ds,$$

with  $\tilde{R}(u) = R(u) + \rho(u) + \frac{1}{2} \tilde{\varrho}(u)$ . The most difficult term to deal with is  $T_n(t, 2)$  and therefore we devote several separate subsections below to upper estimate it. Let us start with the easier case  $3 \leq i \leq 6$ .

**6.3.2. Bound for  $T_n(t, i)$ ,  $3 \leq i \leq 6$ .** By duality we obtain

$$|T_n(t, 3)| = \sum_{1 \leq j \leq n} \left| \int_0^{t \wedge \tau_n} \dot{\beta}_j^n(s) \left( \int_{s_n}^s A^{1/2} u^n(r) dr, A^{1/2} [D\tilde{\sigma}_j(u^n(s_n))]^* [u^n(s) - u(s)] \right) ds \right|.$$

Therefore, using (2.22), (4.4) and (4.5) we deduce that for every  $\tilde{t} \in [0, T]$

$$\sup_{t \in [0, \tilde{t}]} |T_n(t, 3)| \leq C_3 (2N + 2) \alpha n^{3/2} 2^{n/2} \int_0^{\tilde{t} \wedge \tau_n} \left( \int_{s_n}^s \|u^n(r)\| dr \right) \|u^n(s) - u(s)\| ds.$$

For any  $\eta > 0$ , Schwarz's inequality yields

$$\sup_{t \in [0, \tilde{t}]} |T_n(t, 3)| \leq \eta \int_0^{\tilde{t} \wedge \tau_n} \|u^n(s) - u(s)\|^2 ds + C n^3 \int_0^{\tau_n} \int_{s_n}^s \|u^n(r)\|^2 dr ds.$$

for some constant  $C := C(N, T, \eta)$ . Finally, Fubini's theorem and (4.4) imply that

$$\mathbb{E} \left( \sup_{t \in [0, T]} |T_n(t, 3)| \right) \leq \eta T_n(T) + C(N, T, \eta) n^3 2^{-n}. \quad (6.18)$$

Similarly, using (2.6) and (4.5) we obtain

$$|T_n(t, 4)| = \sum_{1 \leq j \leq n} \left| \int_0^{t \wedge \tau_n} \dot{\beta}_j^n(s) \left( \left[ \int_{s_n}^s B(u^n(r)) dr \right], [D\tilde{\sigma}_j(u^n(s_n))]^* [u^n(s) - u(s)] \right) ds \right|$$

$$\leq C\alpha n^{3/2}2^{n/2} \int_0^{t\wedge\tau_n} ds \int_{s_n}^s \|u^n(r)\|_{\mathcal{H}}^2 dr \sup_{1\leq j\leq n} \|[D\tilde{\sigma}_j(u^n(s_n))]^*[u^n(s) - u(s)]\|.$$

Thus the inequalities (2.22), (4.6) and (4.4) yield that for some constant  $C := C(N, m, T)$ :

$$\mathbb{E}\left(\sup_{t\in[0,T]} |T_n(t, 4)|\right) \leq C_3(2N + 2)\alpha m^2 n^{3/2}2^{-n/2} \int_0^{\tau_n} \|u^n(s) - u(s)\| ds \leq Cn^{3/2}2^{-n/2}. \quad (6.19)$$

Using (2.20), (4.4) and (4.5) we deduce

$$|T_n(t, 5)| \leq C_1(2N + 2)\alpha n^{3/2}2^{n/2} \int_0^{t\wedge\tau_n} \left(\int_{s_n}^s |G(u^n(r))h(r)| dr\right) |u^n(s) - u(s)| ds.$$

Therefore, (2.27), (4.4) and Fubini's theorem yield for some constant  $C := C(N, M, T)$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |T_n(t, 5)|\right) \leq C_N n^{3/2}2^{n/2} \mathbb{E} \int_0^{\tau_n} \int_{s_n}^s |h(r)| dr ds \leq Cn^{3/2}2^{-n/2}. \quad (6.20)$$

Similarly, relying on (2.24), (2.28), (4.5) and (4.4) we deduce

$$\mathbb{E}\left(\sup_{t\in[0,T]} |T_n(t, 6)|\right) \leq C_{K,R_0,N} n^{3/2}2^{-n/2}. \quad (6.21)$$

Thus, collecting the relations in (6.9)–(6.21), and choosing  $\eta > 0$  small enough in (6.18), we obtain the following assertion:

**Proposition 6.2.** *Let the assumptions of Theorem 4.1 be satisfied,  $T_n(t)$  be defined by (6.7); then we have:*

$$\mathbb{E}T_n(T) \leq \gamma_n(N, M, m, T) + C\mathbb{E}\left(\sup_{t\in[0,T]} |T_n(t, 2)|\right),$$

where  $\lim_{n\rightarrow\infty} \gamma_n(N, M, m, T) = 0$  and  $T_n(t, 2)$  is defined by (6.16).

**6.3.3. Splitting of  $T_n(t, 2)$ .** Let  $T_n(t, 2)$  be defined by (6.16); then we have the following decomposition:

$$T_n(t, 2) = \sum_{1\leq i\leq 7} S_n(t, i), \quad (6.22)$$

where

$$S_n(t, 1) = \sum_{j=1}^n \int_0^{t\wedge\tau_n} \dot{\beta}_j^n(s) \left( D\tilde{\sigma}_j(u^n(s_n)) \mathcal{I}_n(s, s_n), [u^n(s) - u(s)] - [u^n(s_n) - u(s_n)] \right) ds,$$

$$S_n(t, 2) = - \int_0^{t\wedge\tau_n} \left( (\varrho + \frac{1}{2}\tilde{\varrho})(u^n(s)) - (\varrho_n + \frac{1}{2}\tilde{\varrho}_n)(u^n(s)), u^n(s) - u(s) \right) ds,$$

$$S_n(t, 3) = - \int_0^{t\wedge\tau_n} \left( (\varrho_n + \frac{1}{2}\tilde{\varrho}_n)(u^n(s)) - (\varrho_n + \frac{1}{2}\tilde{\varrho}_n)(u^n(s_n)), u^n(s) - u(s) \right) ds,$$

$$S_n(t, 4) = - \int_0^{t\wedge\tau_n} \left( (\varrho_n + \frac{1}{2}\tilde{\varrho}_n)(u^n(s_n)), [u^n(s) - u(s)] - [u^n(s_n) - u(s_n)] \right) ds,$$

$$S_n(t, 5) = \sum_{1\leq j\leq n} \int_0^{t\wedge\tau_n} \dot{\beta}_j^n(s) \left( D\tilde{\sigma}_j(u^n(s_n)) \left[ \int_{s_n}^s [\sigma(u^n(r)) - \sigma(u^n(s_n))] dW(r) + \int_{s_n}^s [\tilde{\sigma}(u^n(r)) - \tilde{\sigma}(u^n(s_n))] \dot{\tilde{W}}^n(r) dr \right], u^n(s_n) - u(s_n) \right) ds,$$

$$S_n(t, 6) = \int_0^{t\wedge\tau_n} \left( \sum_{1\leq j\leq n} \dot{\beta}_j^n(s) D\tilde{\sigma}_j(u^n(s_n)) [\sigma(u^n(s_n))(W(s) - W(s_n))] - \varrho_n(u^n(s_n)), \right.$$

$$u^n(s_n) - u(s_n) \Big) ds, \quad (6.23)$$

$$S_n(t, 7) = \int_0^{t \wedge \tau_n} \left( \sum_{1 \leq j \leq n} \dot{\beta}_j^n(s) D\tilde{\sigma}_j(u^n(s_n)) \left[ \tilde{\sigma}(u^n(s_n)) \left( \int_{s_n}^s \dot{\tilde{W}}^n(r) dr \right) \right] - \frac{1}{2} \tilde{\varrho}_n(u^n(s_n)) \right) u^n(s_n) - u(s_n) \Big) ds. \quad (6.24)$$

The most difficult terms to deal with are  $S_n(t, 6)$  and  $S_n(t, 7)$ . We start with the simpler ones  $S_n(t, i)$ ,  $i = 1, \dots, 5$ .

**6.3.4. Bound for  $S_n(t, 1)$ .** Let  $\mathcal{I}_n(s, s_n)$  be defined in (6.17); using (2.20), (4.4) and (4.5) we obtain

$$|S_n(t, 1)| \leq C_1 (2N + 2) \alpha n^{3/2} 2^{n/2} \int_0^{t \wedge \tau_n} |\mathcal{I}_n(s, s_n)| (|u^n(s) - u^n(s_n)| + |u(s) - u(s_n)|) ds.$$

Thus for  $\mathcal{N}_n = \int_0^{\tau_n} (|u^n(s) - u^n(s_n)|^2 + |u(s) - u(s_n)|^2) ds$ , Schwarz's inequality yields

$$\mathbb{E} \left( \sup_{t \in [0, T]} |S_n(t, 1)| \right) \leq C_N n^{3/2} 2^{n/2} [\mathbb{E} \mathcal{N}_n]^{1/2} \left[ \mathbb{E} \int_0^{\tau_n} |\mathcal{I}_n(s, s_n)|^2 ds \right]^{1/2}. \quad (6.25)$$

Since

$$\mathbb{E} \mathcal{N}_n \leq \mathbb{E} \int_0^{\tau_n} 1_{G_N^n(s)} (|u^n(s) - u^n(s_n)|^2 + |u(s) - u(s_n)|^2) ds$$

with  $G_N^n(s)$  defined by (5.14), using (5.3) with  $\phi_n(s) = s_n$  and  $\psi_n(s) = s$  we deduce:

$$\mathbb{E} \mathcal{N}_n \leq C(N, M, T) n^{3/2} 2^{-3n/4}. \quad (6.26)$$

Furthermore, the local property of the stochastic integral and (4.5) yield

$$\begin{aligned} & \mathbb{E} \int_0^{\tau_n} |\mathcal{I}_n(s, s_n)|^2 ds \\ & \leq C \int_0^T \left[ \mathbb{E} \left| \int_{s_n}^s 1_{G_N^n(r)} \sigma(u^n(r)) dW(r) \right|^2 + \mathbb{E} \left| \int_{s_n}^s 1_{G_N^n(r)} \tilde{\sigma}(u^n(r)) \dot{W}^n(r) dr \right|^2 \right] ds \\ & \leq C \int_0^T \left[ \mathbb{E} \int_{s_n}^s 1_{G_N^n(r)} |\sigma(u^n(r))|_{L_Q}^2 dr + \alpha^2 n^2 \mathbb{E} \int_{s_n}^s 1_{G_N^n(r)} |\tilde{\sigma}(u^n(r))|_{L_Q}^2 dr \right] ds. \end{aligned}$$

Thus by (2.16) and the definition of the set  $G_N^n(s)$  given in (5.14), we deduce:

$$\mathbb{E} \int_0^{\tau_n} |\mathcal{I}_n(s, s_n)|^2 ds \leq C(N, M, T) n^2 2^{-n}. \quad (6.27)$$

Consequently the inequalities (6.25) – (6.27) yield

$$\mathbb{E} \left( \sup_{t \in [0, T]} |S_n(t, 1)| \right) \leq C(N, M, T) n^{13/4} 2^{-3n/8}. \quad (6.28)$$

**6.3.5. Bound for  $S_n(t, 2)$ .** The inequality (4.4) implies that

$$\sup_{t \in [0, T]} |S_n(t, 2)| \leq C(N) T \sup_{|u| \leq 2(N+1)} \{ |\varrho_n(u) - \varrho(u)| + |\tilde{\varrho}_n(u) - \tilde{\varrho}(u)| \}.$$

Therefore, the locally uniform convergence (2.26) of  $\rho_n$  to  $\rho$  and  $\tilde{\rho}_n$  to  $\tilde{\rho}$  respectively yields

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \sup_{t \in [0, T]} |S_n(t, 2)| \right) = 0. \quad (6.29)$$

6.3.6. **Bound for  $S_n(t, 3)$ .** The local Lipschitz property (2.25) and (4.4) imply

$$\begin{aligned} |S_n(t, 3)| &\leq 2\bar{C}_{2N+2}\sqrt{N+1}\int_0^{\tau_n}|u^n(s)-u^n(s_n)|ds \\ &\leq C(N, T)\left[\int_0^{\tau_n}1_{G_N^n(s)}|u^n(s)-u^n(s_n)|^2ds\right]^{1/2}, \end{aligned}$$

where  $G_N^n(s)$  is defined by (5.14). Thus (5.24) yields

$$\mathbb{E}\left(\sup_{t\in[0, T]}|S_n(t, 3)|\right)\leq C(N, M, T)n^{3/4}2^{-3n/8}. \quad (6.30)$$

6.3.7. **Bound for  $S_n(t, 4)$ .** The local growth condition (2.24), relations (5.3) and (5.24), and also Schwarz's inequality imply

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0, T]}|S_n(t, 4)|\right) &\leq 2\bar{K}_{2N+2}\mathbb{E}\int_0^{\tau_n}(|u^n(s)-u^n(s_n)|+|u(s)-u(s_n)|)ds \\ &\leq 2\bar{K}_{2N+2}\sqrt{T}\left[\mathbb{E}\int_0^{\tau_n}\left(1_{G_N^n(s)}|u^n(s)-u^n(s_n)|^2+1_{\tilde{G}_N(s)}|u(s)-u(s_n)|^2\right)ds\right]^{1/2} \\ &\leq C(N, M, T)n^{3/4}2^{-3n/8}. \end{aligned} \quad (6.31)$$

6.3.8. **Bound for  $S_n(t, 5)$ .** The local bound (2.20) together with the inequalities (4.4) and (4.5) yield

$$\begin{aligned} |S_n(t, 5)| &\leq C_1(2N+2)\alpha n^{3/2}2^{n/2}\int_0^{\tau_n}\left\{\left|\int_{s_n}^s[\sigma(u^n(r))-\sigma(u^n(s_n))]dW(r)\right| \right. \\ &\quad \left. +\left|\int_{s_n}^s[\tilde{\sigma}(u^n(r))-\tilde{\sigma}(u^n(s_n))]\tilde{W}^n(r)dr\right|\right\}ds. \end{aligned}$$

Using Schwarz's inequality, (2.17) and (4.5), we deduce

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0, T]}|S_n(t, 5)|\right) &\leq C_N n^{3/2}2^{n/2}\left\{\int_0^T\mathbb{E}\left|\int_{s_n\wedge\tau_n}^{s\wedge\tau_n}1_{G_N^n(r)}[\sigma(u^n(r))-\sigma(u^n(s_n))]dW(r)\right|^2ds \right. \\ &\quad \left. +\alpha^2n^22^nL\mathbb{E}\int_0^{\tau_n}\left|\int_{s_n}^s|u^n(r)-u^n(s_n)|dr\right|^2ds\right\}^{1/2} \\ &\leq C_N n^{3/2}2^{n/2}\sqrt{L}\left\{\int_0^Tds\mathbb{E}\int_{s_n\wedge\tau_n}^{s\wedge\tau_n}1_{G_N^n(r)}|u^n(r)-u^n(s_n)|^2dr \right. \\ &\quad \left. +\alpha^2n^2T\mathbb{E}\int_0^{\tau_n}ds\int_{s_n}^s|u^n(r)-u^n(s_n)|^2dr\right\}^{1/2} \\ &\leq C_N n^{3/2}2^{n/2}\sqrt{L(1+\alpha^2n^2T)}\left\{\int_0^Tds\mathbb{E}\int_{s_n\wedge\tau_n}^{s\wedge\tau_n}1_{G_N^n(r)}|u^n(r)-u^n(s_n)|^2dr\right\}^{1/2}. \end{aligned}$$

Fubini's theorem and (5.24) imply that

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0, T]}|S_n(t, 5)|\right) &\leq \sqrt{L}C(N, T)n^{5/2}2^{n/2} \\ &\quad \times\left(\mathbb{E}\int_0^{\tau_n}1_{G_N^n(r)}\left[|u^n(r)-u^n(r_n)|^2+|u^n(r)-u^n(\underline{r}_n)|^2\right]2T2^{-n}dr\right)^{\frac{1}{2}} \\ &\leq C(N, M, T)n^{13/4}2^{-3n/8}. \end{aligned} \quad (6.32)$$

Proposition 6.2 and the relations in (6.28)–(6.32) imply the following assertion:

**Proposition 6.3.** *Let the assumptions of Theorem 4.1 be satisfied and let  $T_n(t)$  be defined by (6.7); then we have:*

$$\mathbb{E}T_n(T) \leq \gamma_n^*(N, M, m, T) + C\mathbb{E}\left(\sup_{t \in [0, T]} |S_n(t, 6)| + \sup_{t \in [0, T]} |S_n(t, 7)|\right),$$

where  $\lim_{n \rightarrow \infty} \gamma_n^*(N, M, m, T) = 0$ ,  $S_n(t, 6)$  and  $S_n(t, 7)$  are defined by (6.23) and (6.24).

The upper estimates of  $S_n(t, 6)$  and  $S_n(t, 7)$  are the key ingredients of the proof; they justify the drift correction term in the definition of  $u^n$ .

### 6.3.9. Bound for $S_n(t, 6)$ .

**Lemma 6.4.** *Let the assumptions of Theorem 4.1 be satisfied and  $S_n(t, 6)$  be given by (6.23). Then there exists a constant  $C(N, T)$  such that*

$$\mathbb{E}\left(\sup_{t \in [0, T]} |S_n(t, 6)|\right) \leq C(N, T) n 2^{-\frac{n}{2}}. \quad (6.33)$$

*Proof.* For  $t \in [0, T]$  set

$$\begin{aligned} U_j^n(s) &= \dot{\beta}_j^n(s) D\tilde{\sigma}_j(u^n(s_n))(\sigma(u^n(s_n)) [W(s) - W(s_n)]), \\ \Delta_n(s) &= \left( \sum_{1 \leq j \leq n} U_j^n(s) - \varrho_n(u^n(s_n)), u^n(s_n) - u(s_n) \right). \end{aligned}$$

We also have an obvious decomposition

$$\sum_{1 \leq j \leq n} U_j^n(s) - \varrho_n(u^n(s_n)) = \sum_{1 \leq i \leq 3} V_n^{(i)}(s),$$

where (2.10) yields

$$\begin{aligned} V_n^{(1)}(s) &= \sum_{1 \leq j \leq n} D\tilde{\sigma}_j(u^n(s_n)) \sigma(u^n(s_n)) [W(s) - W(\underline{s}_n)] \dot{\beta}_j^n(s), \\ V_n^{(2)}(s) &= \sum_{1 \leq j \leq n} \sum_{l \neq j} D\tilde{\sigma}_j(u^n(s_n)) \sigma_l(u^n(s_n)) [\beta_l(\underline{s}_n) - \beta_l(s_n)] 2^n T^{-1} [\beta_j(\underline{s}_n) - \beta_j(s_n)], \\ V_n^{(3)}(s) &= \sum_{1 \leq j \leq n} D\tilde{\sigma}_j(u^n(s_n)) \sigma_j(u^n(s_n)) \left[ 2^n T^{-1} (\beta_j(\underline{s}_n) - \beta_j(s_n))^2 - 1 \right]. \end{aligned}$$

The obvious identity

$$\mathbf{1}_{\{s \leq \tau_n\}} = \mathbf{1}_{\{s_n \leq \tau_n\}} - \mathbf{1}_{\{s_n \leq \tau_n < s\}} \quad (6.34)$$

yields the following decomposition, where  $G_N^n(t)$  is defined by (5.14):

$$S_n(t, 6) = \int_0^{t \wedge \tau_n} \Delta_n(s) \mathbf{1}_{G_N^n(s_n)} ds = \sum_{1 \leq i \leq 3} S_n^{(i)}(t) - S_n^{(4)}(t), \quad (6.35)$$

where,

$$\begin{aligned} S_n^{(i)}(t) &= \int_0^t \mathbf{1}_{\{s_n \leq \tau_n\}} \mathbf{1}_{G_N^n(s_n)} (V_n^{(i)}(s), u^n(s_n) - u(s_n)) ds, \quad i = 1, 2, 3, \\ S_n^{(4)}(t) &= \int_0^t \mathbf{1}_{\{s_n \leq \tau_n < s\}} \mathbf{1}_{G_N^n(s_n)} \Delta_n(s) ds. \end{aligned}$$

We note that  $S_n^{(i)}(t) = 0$  for every  $i = 1, 2, 3$  and  $t \leq t_2$ .

*Bound for  $S_n^{(4)}$ .* Set  $t_{-1} = t_0 = 0$ ; using twice Schwarz's inequality, we deduce

$$\mathbb{E}\left(\sup_{t \in [0, T]} |S_n^{(4)}(t)|\right) \leq \sum_{0 \leq k < 2^n} \mathbb{E} \int_{t_k}^{t_{k+1}} \mathbf{1}_{\{t_{k-1} \leq \tau_n \leq t_{k+1}\}} \mathbf{1}_{G_N^n(s_n)} |\Delta_n(s)| ds$$

$$\begin{aligned}
&\leq \left\{ 2 \sum_{0 \leq k < 2^n} \mathbb{E} 1_{\{t_k \leq \tau_n \leq t_{k+1}\}} \right\}^{1/2} \left\{ \sum_{0 \leq k < 2^n} \mathbb{E} \left( \int_{t_k}^{t_{k+1}} 1_{G_N^n(s_n)} |\Delta_n(s)| ds \right)^2 \right\}^{1/2} \\
&\leq \sqrt{2} \left\{ T 2^{-n} \mathbb{E} \int_0^T 1_{G_N^n(s_n)} |\Delta_n(s)|^2 ds \right\}^{1/2}.
\end{aligned}$$

Schwarz's inequality, (2.16), (2.20), (2.24) and the definition (5.14) of the set  $G_N^n(s_n)$  yield

$$\begin{aligned}
1_{G_N^n(s_n)} |\Delta_n(s)|^2 &\leq C(N) \left( 1 + \sum_{1 \leq j \leq n} |W(s) - W(s_n)|_0 |\dot{\beta}_j^n(s)| \right)^2 \\
&\leq C(N) \left( 1 + n |W(s) - W(s_n)|_0^2 \sum_{1 \leq j \leq n} |\dot{\beta}_j^n(s)|^2 \right).
\end{aligned}$$

Therefore, Schwarz's inequality implies that for some constants  $C(N, T)$ ,  $c_1, c_2$ , one has:

$$\begin{aligned}
&\mathbb{E} \left( \sup_{t \in [0, T]} |S_n^{(4)}(t)| \right) \\
&\leq C(N, T) 2^{-n/2} \left\{ 1 + n \sum_{1 \leq j \leq n} \int_0^T \left[ \mathbb{E} |W(s) - W(s_n)|_0^4 \right]^{1/2} \left[ \mathbb{E} |\dot{\beta}_j^n(s)|^4 \right]^{1/2} ds \right\}^{1/2} \\
&\leq C(N, T) 2^{-n/2} \left\{ 1 + n^2 \left[ c_1 \frac{T^2}{2^{2n}} \right]^{1/2} \left[ T^{-4} 2^{4n} c_2 \frac{T^2}{2^{2n}} \right]^{1/2} \right\}^{1/2} \leq C(N, T) n 2^{-\frac{n}{2}}. \quad (6.36)
\end{aligned}$$

*Bound for  $S_n^{(1)}$ .* Using duality and Fubini's theorem, we can write

$$\begin{aligned}
S_n^{(1)}(t) &= \sum_{1 \leq j \leq n} \int_0^t 1_{\{s_n \leq \tau_n\}} 1_{G_N^n(s_n)} \dot{\beta}_j^n(s) \\
&\quad \times \int_{\underline{s}_n}^s ([D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))]^* (u^n(s_n) - u(s_n)), dW(r)) ds \\
&= \sum_{1 \leq j \leq n} \int_0^t \left( \int_r^{\bar{r}_n} 1_{\{s_n \leq \tau_n\}} 1_{G_N^n(s_n)} \dot{\beta}_j^n(s) \right. \\
&\quad \left. \times [D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))]^* (u^n(s_n) - u(s_n)) ds, dW(r) \right).
\end{aligned}$$

Since  $\dot{\beta}_j^n(s)$  is  $\mathcal{F}_{\underline{s}_n} = \mathcal{F}_{L_n}$  adapted for  $r \leq s \leq \bar{r}_n$ , the process  $S_n^{(1)}$  is a martingale. Therefore, the Burkholder-Davies-Gundy and Schwarz inequalities, (2.16), (2.20) and (5.14) imply

$$\begin{aligned}
&\mathbb{E} \left( \sup_{t \in [0, T]} |S_n^{(1)}(t)| \right) \leq c_0 \mathbb{E} \left\{ \int_0^T \left| \sum_{1 \leq j \leq n} \int_r^{\bar{r}_n} 1_{\{s_n \leq \tau_n\}} 1_{G_N^n(s_n)} \dot{\beta}_j^n(s) \right. \right. \\
&\quad \left. \left. \times [D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))]^* (u^n(s_n) - u(s_n)) ds \right|^2 dr \right\}^{1/2} \\
&\leq \frac{C(N, T) \sqrt{n}}{2^{n/2}} \mathbb{E} \left\{ \int_0^T dr \sum_{1 \leq j \leq n} \int_r^{\bar{r}_n} 1_{\{s_n \leq \tau_n\}} 1_{G_N^n(s_n)} |D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))|^2 |\dot{\beta}_j^n(s)|^2 ds \right\}^{1/2} \\
&\leq C(N, T) \sqrt{n} 2^{-n/2} \left\{ \int_0^T dr \sum_{1 \leq j \leq n} \mathbb{E} \int_r^{\bar{r}_n} |\dot{\beta}_j^n(s)|^2 ds \right\}^{1/2} \leq C(N, T) n 2^{-n/2}. \quad (6.37)
\end{aligned}$$

*Bound for  $S_n^{(2)}$ .* For  $j = 1, \dots, n$ ,  $l \neq j$ ,  $i = 1, \dots, 2^n - 1$ , set

$$\Phi_{j,l}(i) = \left( D\tilde{\sigma}_j(u^n(t_i))\sigma_l(u^n(t_i)), u^n(t_i) - u(t_i) \right) 1_{\{t_i \leq \tau_n\}} 1_{G_N^n(t_i)},$$

and for  $k = 3, \dots, 2^n$ , let

$$M_k = \sum_{2 \leq i \leq k} \sum_{1 \leq j \leq n} \sum_{l \neq j} \Phi_{j,l}(i-1) (\beta_l(t_i) - \beta_l(t_{i-1})) (\beta_j(t_i) - \beta_j(t_{i-1})).$$

Then the random variable  $\Phi_{j,l}(i-1)$  is  $\mathcal{F}_{t_{i-1}}$  measurable, and since for  $l \neq j$  the sigma-field  $\mathcal{F}_{t_{i-1}}$  and the random variables  $\beta_j(t_i) - \beta_j(t_{i-1})$  and  $\beta_l(t_i) - \beta_l(t_{i-1})$  are independent, the process  $(M_k, \mathcal{F}_{t_k}, 2 \leq k < 2^n)$  is a discrete martingale. Furthermore, for the cases (a)  $i < i'$  and  $l' \neq j'$ , (b)  $i' < i$  and  $l \neq j$  or (c)  $i = i'$  and  $(\min(j, l), \max(j, l)) \neq (\min(j', l'), \max(j', l'))$ , one has

$$\begin{aligned} & \mathbb{E}[\Phi_{j,l}(i-1) \Phi_{j',l'}(i'-1) (\beta_l(t_i) - \beta_l(t_{i-1})) (\beta_j(t_i) - \beta_j(t_{i-1})) \\ & \quad \times (\beta_{l'}(t_{i'}) - \beta_{l'}(t_{i'-1})) (\beta_{j'}(t_{i'}) - \beta_{j'}(t_{i'-1}))] = 0. \end{aligned}$$

Therefore, Schwarz's and Doob's inequalities yield

$$\begin{aligned} & \mathbb{E}\left(\max_{2 \leq k < 2^n} |M_k|\right)^2 \leq \mathbb{E}\left(\max_{2 \leq k < 2^n} M_k^2\right) \leq 4\mathbb{E}(M_{2^{n-1}}^2) \\ & \leq 12 \sum_{2 \leq i < 2^n} \sum_{1 \leq j \leq n} \sum_{l \neq j} \mathbb{E}(\Phi_{j,l}(i-1)^2) \mathbb{E}(|\beta_l(t_i) - \beta_l(t_{i-1})|^2) \mathbb{E}(|\beta_j(t_i) - \beta_j(t_{i-1})|^2). \end{aligned}$$

Furthermore, using (2.20), (2.16) and (5.14) we deduce that for every  $i, j, l$

$$\mathbb{E}(\Phi_{j,l}(i-1)^2) \leq q_l C_1(N)^2 (K_0 + K_1 N^2) (2N)^2,$$

which implies

$$\mathbb{E}\left(\max_{2 \leq k < 2^n} |M_k|\right) \leq C(N, T) n 2^{-n/2}. \quad (6.38)$$

A similar easier computation shows that

$$\begin{aligned} & \mathbb{E}\left(\max_{2 \leq k < 2^n} \sup_{t_k \leq t \leq t_{k+1}} \left| \sum_{1 \leq j \leq n} \sum_{l \neq j} \Phi_{j,l}(k-1) \frac{2^n(t-t_k)}{T} (\beta_l(t_k) - \beta_l(t_{k-1})) (\beta_j(t_k) - \beta_j(t_{k-1})) \right|\right) \\ & \leq \mathbb{E}\left(\max_{2 \leq k < 2^n} \left| \sum_{1 \leq j \leq n} \sum_{l \neq j} \Phi_{j,l}(k-1) (\beta_l(t_k) - \beta_l(t_{k-1})) (\beta_j(t_k) - \beta_j(t_{k-1})) \right|\right) \\ & \leq \left\{ \mathbb{E}\left(\max_{2 \leq k < 2^n} \left| \sum_{1 \leq j \leq n} \sum_{l \neq j} \Phi_{j,l}(k-1) (\beta_l(t_k) - \beta_l(t_{k-1})) (\beta_j(t_k) - \beta_j(t_{k-1})) \right|^2\right) \right\}^{1/2} \\ & \leq \left\{ \sum_{2 \leq k < 2^n} \mathbb{E} \left| \sum_{1 \leq j \leq n} \sum_{l \neq j} \Phi_{j,l}(k-1) (\beta_l(t_k) - \beta_l(t_{k-1})) (\beta_j(t_k) - \beta_j(t_{k-1})) \right|^2 \right\}^{1/2} \\ & \leq \left\{ \sum_{2 \leq k < 2^n} \sum_{1 \leq j \leq n} \sum_{l \neq j} \mathbb{E}(\Phi_{j,l}(k-1)^2) \mathbb{E}(|\beta_l(t_k) - \beta_l(t_{k-1})|^2) \mathbb{E}(|\beta_j(t_k) - \beta_j(t_{k-1})|^2) \right\}^{1/2} \\ & \leq C(N, T) n 2^{-n/2}. \end{aligned} \quad (6.39)$$

Furthermore,

$$\begin{aligned} & \mathbb{E}\left(\sup_{t \in [0, T]} |S_n^{(2)}(t)|\right) \leq \mathbb{E}\left(\sup_k \sup_{t \in [t_k, t_{k+1}]} |S_n^{(2)}(t)|\right) \\ & \leq \mathbb{E} \sup_{k \geq 3} \left| \sum_{2 \leq i < k} \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{s_n \leq \tau_n\}} \mathbf{1}_{G_N^n(s_n)} (V_n^{(2)}(s), u^n(s_n) - u(s_n)) ds \right| \\ & \quad + \mathbb{E} \sup_{k \geq 2} \left[ \sup_{t \in [t_k, t_{k+1}]} \left| \int_{t_k}^t \mathbf{1}_{\{s_n \leq \tau_n\}} \mathbf{1}_{G_N^n(s_n)} (V_n^{(2)}(s), u^n(s_n) - u(s_n)) ds \right| \right]. \end{aligned}$$

This inequality, (6.38) and (6.39) immediately yield

$$\mathbb{E}\left(\sup_{t \in [0, T]} |S_n^{(2)}|\right) \leq C(N, T) n 2^{-\frac{n}{2}}. \quad (6.40)$$

*Bound of  $S_n^{(3)}$ .* The argument is similar to the previous one, based on a different discrete martingale. For  $i = 1, \dots, 2^n - 1$ ,  $j = 1, \dots, n$ , set

$$\Phi_j(i) = T 2^{-n} \left( D\tilde{\sigma}_j(u^n(t_i)) \sigma_j(u^n(t_i)), u^n(t_i) - u(t_i) \right) 1_{\{t_i \leq \tau_n\}} 1_{G_N^n(t_i)}.$$

Then  $\Phi_j(i-1)$  is  $\mathcal{F}_{t_{i-1}}$ -measurable and independent of the centered random variable  $Y_{ij} = 2^n T^{-1} |\beta_j(t_i) - \beta_j(t_{i-1})|^2 - 1$ . Furthermore, for  $(i, j) \neq (i', j')$  one has

$$\mathbb{E}(\Phi_j(i-1) Y_{ij} \Phi_{j'}(i'-1) Y_{i'j'}) = 0.$$

Using (2.20), (2.16) and (5.14), we deduce that for all  $i, j$ ,  $\mathbb{E}(|\Phi_j(i-1)|^2) \leq C_{N, T} 2^{-2n}$ . For  $k = 2, \dots, 2^n$ , set

$$N_k = \sum_{2 \leq i \leq k} \sum_{1 \leq j \leq n} \Phi_j(i-1) Y_{ij}.$$

The process  $(N_k, \mathcal{F}_{t_k})$  is a discrete martingale; thus Schwarz's and Doob's inequality yield

$$\begin{aligned} \mathbb{E}\left(\max_{2 \leq k \leq 2^n} |N_k|\right) &\leq 2 \left\{ \mathbb{E}(|N_{2^n}|^2) \right\}^{\frac{1}{2}} \\ &\leq 2 \left\{ \sum_{2 \leq i \leq 2^n} \sum_{1 \leq j \leq n} \mathbb{E}(\Phi_j(i-1)^2) \mathbb{E}(|Y_{ij}|^2) \right\}^{\frac{1}{2}} \leq C_{T, N} n^{\frac{1}{2}} 2^{-\frac{n}{2}}. \end{aligned} \quad (6.41)$$

Finally, a similar argument shows that

$$\begin{aligned} &\mathbb{E}\left(\max_{1 \leq k < 2^n} \sup_{t_k \leq t \leq t_{k+1}} \left| 2^n T^{-1} (t - t_k) \sum_{1 \leq j \leq n} \Phi_j(k-1) Y_{kj} \right|\right) \\ &\leq \left( \mathbb{E} \sum_{1 \leq k < 2^n} \left| \sum_{1 \leq j \leq n} \Phi_j(k-1) Y_{kj} \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{1 \leq k < 2^n} \mathbb{E} \sum_{1 \leq j \leq n} |\Phi_j(k-1) Y_{kj}|^2 \right)^{\frac{1}{2}} \leq C(N, T) n^{\frac{1}{2}} 2^{-\frac{n}{2}}. \end{aligned} \quad (6.42)$$

The inequalities (6.41) and (6.42) imply that

$$\mathbb{E}\left(\sup_{t \in [0, T]} |S_n^{(3)}|\right) \leq C(N, T) n^{\frac{1}{2}} 2^{-\frac{n}{2}}. \quad (6.43)$$

Using (6.35) and collecting the upper estimates in (6.35), (6.37), (6.40) and (6.43), we conclude the proof of Lemma 6.4.  $\square$

### 6.3.10. Bound for $S_n(t, 7)$ .

**Lemma 6.5.** *Let the assumptions of Theorem 4.1 be satisfied and  $S_n(t, 7)$  be defined by (6.24). There exists a constant  $C(N, T)$  such that*

$$\mathbb{E}\left(\sup_{t \in [0, T]} |S_n(t, 7)|\right) \leq C(N, T) n^2 2^{-\frac{n}{2}}. \quad (6.44)$$

*Proof.* For  $s \in [0, T]$ ,  $j = 1, \dots, n$ , set

$$\begin{aligned} \tilde{U}_j^n(s) &= D\tilde{\sigma}_j(u^n(s_n)) \left[ \tilde{\sigma}(u^n(s_n)) \left( \int_{s_n}^s \tilde{W}^n(r) dr \right) \right] \dot{\beta}_j^n(s), \\ \tilde{\Delta}_n(s) &= \left( \sum_{1 \leq j \leq n} \tilde{U}_j^n(s) - \frac{1}{2} \tilde{\partial}_n(u^n(s_n)), u^n(s_n) - u(s_n) \right). \end{aligned}$$

We obviously have that

$$\sum_{1 \leq j \leq n} \tilde{U}_j^n(s) - \frac{1}{2} \tilde{Q}_n(u^n(s_n)) = \sum_{1 \leq i \leq 3} \tilde{V}_n^{(i)}(s),$$

where

$$\begin{aligned} \tilde{V}_n^{(1)}(s) &= \sum_{1 \leq j \leq n} D\tilde{\sigma}_j(u^n(s_n)) \tilde{\sigma}(u^n(s_n)) [W_n(s_n) - W_n((s_n - T2^{-n}) \vee 0)] \dot{\beta}_j^n(s), \\ \tilde{V}_n^{(2)}(s) &= \sum_{1 \leq j \leq n} \sum_{l \neq j} D\tilde{\sigma}_j(u^n(s_n)) \tilde{\sigma}_l(u^n(s_n)) (s - \underline{s}_n) [\beta_l(\underline{s}_n) - \beta_l(s_n)] \frac{2^{2n}}{T^2} [\beta_j(\underline{s}_n) - \beta_j(s_n)], \\ \tilde{V}_n^{(3)}(s) &= \sum_{1 \leq j \leq n} \tilde{\sigma}_j(u^n(s_n)) \tilde{\sigma}_j(u^n(s_n)) \left[ \frac{2^{2n}}{T^2} (s - \underline{s}_n) [\beta_j(\underline{s}_n) - \beta_j(s_n)]^2 - \frac{1}{2} \right]. \end{aligned}$$

Using (6.34) we deduce the following decomposition of  $S_n(t, 7)$ :

$$S_n(t, 7) = \int_0^{t \wedge \tau_n} \tilde{\Delta}_n(s) ds = \sum_{1 \leq i \leq 3} \tilde{S}_n^{(i)}(t) - \tilde{S}_n^{(4)}(t), \quad (6.45)$$

where

$$\begin{aligned} \tilde{S}_n^{(i)}(t) &= \int_{t_2}^t 1_{\{s_n \leq \tau_n\}} 1_{G_N^n(s_n)} (\tilde{V}_n^{(i)}(s), u^n(s_n) - u(s_n)) ds, \quad i = 1, 2, 3, \\ \tilde{S}_n^{(4)}(t) &= \int_0^t 1_{\{s_n \leq \tau_n < s\}} 1_{G_N^n(s_n)} \tilde{\Delta}_n(s) ds. \end{aligned}$$

We note that  $\tilde{S}_n^{(i)}(t) = 0$  for  $i = 1, 2, 3$  and  $t \leq t_2$ .

*Bound for  $\tilde{S}_n^{(4)}$ .* The proof is similar to that of the upper estimate of  $S_n^{(4)}$ . Schwarz's inequality implies

$$\mathbb{E} \left( \sup_{t \in [0, T]} |\tilde{S}_n^{(4)}(t)| \right) \leq \left\{ 2T2^{-n} \sum_{0 \leq k < 2^{n-1}} \mathbb{E} \int_{t_k}^{t_{k+1}} 1_{G_N^n(t_k)} \left| \tilde{\Delta}_n(s) \right|^2 ds \right\}^{1/2}.$$

The inequalities (2.16), (2.20), (2.24), the definition (5.14) of the set  $G_N^n(s)$  and Schwarz's inequality yield for  $t_k \leq s < t_{k+2}$ :

$$1_{G_N^n(t_k)} \left| \tilde{\Delta}_n(s) \right|^2 \leq C(N) \left( 1 + n \left| \int_{s_n}^s \dot{\tilde{W}}^n(r) dr \right|^2 \sum_{1 \leq j \leq n} |\dot{\beta}_j^n(s)|^2 \right).$$

Therefore, Fubini's theorem and Schwarz's inequality imply

$$\begin{aligned} &\mathbb{E} \left( \sup_{t \in [0, T]} |\tilde{S}_n^{(4)}(t)| \right) \\ &\leq C_{N, T} 2^{-n/2} \left\{ T + n \sum_{1 \leq j \leq n} \sum_{0 \leq k < 2^n} \mathbb{E} \int_{t_{k-1}}^{t_{k+1}} \left| \int_{s_n}^s \dot{\tilde{W}}^n(r) dr \right|_0^2 |\dot{\beta}_j^n(s)|^2 ds \right\}^{1/2} \\ &\leq C_{N, T} 2^{-n/2} \left\{ 1 + n2^{-n} \sum_{1 \leq j \leq n} \sum_{0 \leq k < 2^n} \mathbb{E} \left[ \int_{t_{k-1} \vee 0}^{t_{k+1}} \left| \dot{\tilde{W}}^n(r) \right|_0^2 dr \int_{t_k}^{t_{k+1}} |\dot{\beta}_j^n(s)|^2 ds \right] \right\}^{1/2} \\ &\leq C_{N, T} 2^{-\frac{n}{2}} \left\{ 1 + n2^{-2n} \sum_{1 \leq j \leq n} \sum_{0 \leq k < 2^n} \left[ \int_{t_{k-1} \vee 0}^{t_{k+1}} \mathbb{E} \left| \dot{\tilde{W}}^n(r) \right|_0^4 dr \right]^{\frac{1}{2}} \left[ \int_{t_k}^{t_{k+1}} \mathbb{E} |\dot{\beta}_j^n(s)|^4 ds \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \end{aligned}$$

Since for every  $s \in [0, T]$   $\mathbb{E}|\tilde{W}^n(s)|_0^4 \leq C(T)n^4 2^{2n}$  and  $\mathbb{E}|\dot{\beta}_j^n(s)|^4 \leq C(T)2^{2n}$ , we deduce the existence of some constant  $C(N, T)$  such that

$$\mathbb{E}\left(\sup_{t \in [0, T]} |\tilde{S}_n^{(4)}(t)|\right) \leq C(N, T)n^2 2^{-n/2}. \quad (6.46)$$

*Bound for  $\tilde{S}_n^{(1)}$ .* For  $j = 1, \dots, n$  let

$$\varphi_j(s) = 1_{\{\underline{s}_n \leq \tau_n\}} 1_{G_N^n(\underline{s}_n)} \left( D\tilde{\sigma}_j(u^n(\underline{s}_n)) [\tilde{\sigma}(u^n(\underline{s}_n))(W_n(\underline{s}_n) - W_n(s_n))] \right), u^n(\underline{s}_n) - u(\underline{s}_n).$$

Then  $\varphi_j(s)$  is  $\mathcal{F}_{\underline{s}_n}$  measurable and for  $t \geq t_2$ ,

$$\tilde{S}_n^{(1)}(t) = \sum_{1 \leq j \leq n} \int_{t_1}^{t_n} \varphi_j(s) d\beta_j(s) + \sum_{1 \leq j \leq n} \varphi_j(t - T2^{-n}) 2^n T^{-1} (t - \underline{t}_n) [\beta_j(\underline{t}_n) - \beta_j(t_n)].$$

For fixed  $j$  the process  $(\varphi_j(t_k)(\beta_j(t_{k+1}) - \beta_j(t_k)), 0 \leq k < 2^n)$  is a martingale increments. Therefore, the Burkholder and Schwarz inequalities, (2.20), (2.16) and (5.14), yield

$$\begin{aligned} & \mathbb{E}\left(\sup_{t \in [0, T]} |\tilde{S}_n^{(1)}(t)|\right) \\ & \leq C \left\{ \mathbb{E} \int_0^T \sum_{1 \leq j \leq n} \varphi_j(s)^2 ds \right\}^{\frac{1}{2}} + C \mathbb{E} \left( \sum_{1 \leq j \leq n} \max_{1 \leq k < 2^n} |\varphi_j(t_k)| |\beta_j(t_{k+1}) - \beta_j(t_k)| \right) \\ & \leq C_{N, T} \left\{ n \mathbb{E} \int_0^T |W_n(\underline{s}_n) - W_n(s_n)|_0^2 ds \right\}^{\frac{1}{2}} \\ & \quad + C_{N, T} \mathbb{E} \left\{ n \sum_{1 \leq k < 2^n} \sum_{1 \leq j \leq n} |W_n(t_k) - W_n(t_{k-1})|_0^2 |\beta_j(t_{k+1}) - \beta_j(t_k)|^2 \right\}^{\frac{1}{2}} \\ & \leq C_{N, T} \sqrt{n} \left[ 2^{-\frac{n}{2}} + \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k < 2^n} \mathbb{E} |W_n(t_k) - W_n(t_{k-1})|_0^2 \mathbb{E} |\beta_j(t_{k+1}) - \beta_j(t_k)|^2 \right\}^{\frac{1}{2}} \right] \\ & \leq C(N, T) n 2^{-n/2}. \end{aligned} \quad (6.47)$$

*Bound for  $\tilde{S}_n^{(2)}$ .* For  $i = 1, \dots, 2^n - 1$ ,  $j = 1, \dots, n$  and  $l \neq j$  set

$$\tilde{\Phi}_{j, l}(i) = 2^{2n} T^{-2} 1_{\{t_i \leq \tau_n\}} 1_{G_N^n(t_i)} \left( D\tilde{\sigma}_j(u^n(t_i)) \tilde{\sigma}_l(u^n(t_i)), u^n(t_i) - u(t_i) \right).$$

Then  $\tilde{\Phi}_{j, l}(i)$  is  $\mathcal{F}_{t_i}$  measurable and since for  $l \neq j$ ,  $\mathcal{F}_{t_{i-1}}$ ,  $\beta_j(t_i) - \beta_j(t_{i-1})$  and  $\beta_l(t_i) - \beta_l(t_{i-1})$  are independent, if one sets  $Z_{j, l}(i) = (\beta_l(t_i) - \beta_l(t_{i-1})) (\beta_j(t_i) - \beta_j(t_{i-1}))$ , the following process  $(\tilde{M}_k, 2 \leq k \leq 2^n)$  is a  $(\mathcal{F}_{t_k})$  centered martingale:

$$\begin{aligned} \tilde{M}_k &= \sum_{2 \leq i \leq k} \sum_{1 \leq j \leq n} \sum_{l \neq j} \int_{t_i}^{t_{i+1}} \tilde{\Phi}_{j, l}(i-1) (s - t_i) Z_{j, l}(i) ds \\ &= T^2 2^{-(1+2n)} \sum_{2 \leq i \leq k} \sum_{1 \leq j \leq n} \sum_{l \neq j} \tilde{\Phi}_{j, l}(i-1) Z_{j, l}(i). \end{aligned}$$

Furthermore, if  $i < i'$  and  $l' \neq j'$ , or  $i' < i$  and  $l \neq j$ , or  $i = i'$  and  $(\min(j, l), \max(j, l)) \neq (\min(j', l'), \max(j', l'))$ , one has

$$\mathbb{E}[\tilde{\Phi}_{j, l}(i-1) Z_{j, l}(i) \tilde{\Phi}_{j', l'}(i'-1) Z_{j', l'}(i')] = 0.$$

Hence Doob's, Schwarz's inequalities together with (2.20), (2.16) and (5.14) yield

$$\mathbb{E}\left(\max_{2 \leq k < 2^n} |\tilde{M}_k|\right)^2 \leq \mathbb{E}\left(\max_{2 \leq k < 2^n} |\tilde{M}_k|^2\right) \leq 4\mathbb{E}(\tilde{M}_{2^n-1}^2)$$

$$\begin{aligned}
&\leq C_T 2^{-4n} \sum_{2 \leq i < 2^n} \sum_{1 \leq j \leq n} \sum_{l \neq j} \mathbb{E}(|\tilde{\Phi}_{j,l}(i-1)|^2) \mathbb{E}(|\beta_l(t_i) - \beta_l(t_{i-1})|^2) \mathbb{E}(|\beta_j(t_i) - \beta_j(t_{i-1})|^2) \\
&\leq C(N, T) n 2^{-n}.
\end{aligned} \tag{6.48}$$

A computation similar to that performed in (6.39) proves that

$$\begin{aligned}
&\mathbb{E} \left( \max_{2 \leq k < 2^n} \sup_{t_k \leq t \leq t_{k+1}} \left| \sum_{1 \leq j \leq n} \sum_{l \neq j} \int_{t_k}^t \tilde{\Phi}_{j,l}(k-1)(s-t_k) Z_{j,l}(k) ds \right| \right) \\
&\leq T^2 2^{-2n} \left\{ \sum_{2 \leq k < 2^n} \sum_{1 \leq j \leq n} \sum_{l \neq j} 2^{-4n} \mathbb{E}(|\tilde{\Phi}_{j,l}(k-1)|^2) \right. \\
&\quad \left. \times \mathbb{E}(|\beta_l(t_k) - \beta_l(t_{k-1})|^2) \mathbb{E}(|\beta_j(t_k) - \beta_j(t_{k-1})|^2) \right\}^{\frac{1}{2}} \\
&\leq C(N, T) n 2^{-n/2}.
\end{aligned} \tag{6.49}$$

The inequalities (6.48) and (6.49) yield

$$\mathbb{E} \left( \sup_{t \in [0, T]} |\tilde{S}_n^{(2)}(t)| \right) \leq C(N, T) n 2^{-n/2}. \tag{6.50}$$

*Bound for  $\tilde{S}_n^{(3)}$ .* Finally, for  $i = 1, \dots, 2^n - 1$  and  $j = 1, \dots, n$ , set

$$\begin{aligned}
\tilde{\Phi}_j(i) &= 1_{\{t_i \leq \tau_n\}} 1_{G_N^n(t_i)} \left( D\tilde{\sigma}_j(u^n(t_i)) \tilde{\sigma}_j(u^n(t_i)), u^n(t_i) - u(t_i) \right), \\
Z_j(i) &= \int_{t_i}^{t_{i+1}} \left[ \frac{2^{2n}}{T^2} (s-t_i) (\beta_j(t_{i+1}) - \beta_j(t_i))^2 - \frac{1}{2} \right] ds.
\end{aligned}$$

Then the random variables  $Z_j(i)$  and  $\tilde{\Phi}_j(i)$  are independent,  $\mathbb{E}(Z_j(i)) = 0$  and  $\mathbb{E}(Z_j(i)^2) \leq C_T 2^{-2n}$ . Furthermore, for  $(i, j) \neq (i', j')$ ,  $\mathbb{E}(\tilde{\Phi}_j(i) Z_j(i) \tilde{\Phi}_{j'}(i') Z_{j'}(i')) = 0$ . The process defined for  $k = 1, \dots, 2^n - 1$  by  $\tilde{N}_k = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq n} \tilde{\Phi}_j(i) Z_j(i)$  is a discrete  $(\mathcal{F}_{t_{k+1}})$  martingale. Doob's and Schwarz's inequalities, (2.20), (2.16) and (5.14) imply that

$$\begin{aligned}
&\mathbb{E} \left( \sup_{t \in [0, T]} |\tilde{S}_n^{(3)}(t)| \right) \leq \mathbb{E} \left( \max_{1 \leq k < 2^n} |\tilde{N}_k| \right) \\
&\quad + \mathbb{E} \left( \max_{1 \leq k < 2^n} \sup_{t_k \leq t \leq t_{k+1}} \left| \sum_{1 \leq j \leq n} \tilde{\Phi}_j(k) \int_{t_k}^t \left[ \frac{2^{2n}}{T^2} (s-t_k) (\beta_j(t_{k+1}) - \beta_j(t_k))^2 - \frac{1}{2} \right] ds \right| \right) \\
&\leq C \mathbb{E}(|\tilde{N}_{2^n-1}|^2)^{\frac{1}{2}} + \left( 2^n n \max_{1 \leq k < 2^n} \max_{1 \leq j \leq n} \mathbb{E}(\tilde{\Phi}_j(k)^2) \mathbb{E}(Z_j(k)^2) \right)^{\frac{1}{2}} \\
&\leq C \left( n 2^n \max_{2 \leq k < 2^n} \max_{1 \leq j \leq n} \mathbb{E}(\tilde{\Phi}_j(k)^2) \mathbb{E}(Z_j(k)^2) \right)^{\frac{1}{2}} \leq C(N, T) n^{\frac{1}{2}} 2^{-\frac{n}{2}}.
\end{aligned} \tag{6.51}$$

The relations in (6.45) – (6.51) conclude the proof of Lemma 6.5.  $\square$

Now using Proposition 6.3, Lemmas 6.4 and 6.5, we obtain (4.8); this completes the proof of Theorem 4.1.

## 7. APPENDIX

We consider some additional properties of the solution to (2.31). The aim of this section is to introduce some more properties on the coefficients  $\sigma$ ,  $\tilde{\sigma}$ ,  $G$  and  $R$  which will ensure that the property (3.3) holds. Let  $\bar{C}$  denote a constant such that

$$|u| \leq \bar{C} \|u\|, \forall u \in V. \tag{7.1}$$

### 7.1. Exponential moments.

**Proposition 7.1.** *Let  $h(t) \in S_M$  be deterministic, suppose that the operators  $G$  and  $\sigma + \tilde{\sigma}$  are uniformly bounded and that the linear growth of  $R$  is small enough, i.e., there exist positive constants  $K_0$ ,  $R_0$  and  $\tilde{R}_0$  such that:*

$$|G(u)|_{L(H_0, H)}^2 \leq K_0, |(\sigma + \tilde{\sigma})(u)|_{L_Q}^2 \leq K_0, |R(u)| \leq R_0 + \tilde{R}_0|u| \text{ with } \tilde{R}_0 < \bar{C}^{-2} \quad (7.2)$$

for every  $u \in H$ . Let  $u(t)$  be the solution to (2.31) such that the initial condition has some exponential moment, i.e.,  $\mathbb{E} \exp(\alpha_0 |\xi|^2) < \infty$  for some  $\alpha_0 > 0$ . Then there exist constants  $\alpha_1 \in ]0, \alpha_0]$ ,  $\beta(\alpha) > 0$  and  $c_i > 0, i = 1, 2$  such that for  $0 < \alpha < \alpha_1$  and  $t \in [0, T]$ :

$$\mathbb{E} \exp \left( \alpha |u(t)|^2 + \beta(\alpha) \int_0^t \|u(s)\|^2 ds \right) \leq e^{c_1 t + c_2 M} \mathbb{E} \exp(\alpha |\xi|^2). \quad (7.3)$$

The same estimate holds for Galerkin approximations  $u_n$  of  $u$  with constants  $c_1, c_2$  which do not depend on  $n$ .

*Proof.* Let  $\sigma_0 = \sigma + \tilde{\sigma}$ ,  $\Phi_0(t) = \exp(\alpha |u(t)|^2)$  and  $\Phi(t) = \Phi_0(t) \exp\left(\beta \int_0^t \|u(s)\|^2 ds\right)$ . By Itô's formula we have for every  $t \in [0, T]$ :

$$d\Phi(t) = [\beta \|u(t)\|^2 \Phi_0(t) dt + d\Phi_0(t)] \exp\left(\beta \int_0^t \|u(s)\|^2 ds\right)$$

and

$$d\Phi_0(t) = \alpha \Phi_0(t) \left[ 2(u(t), du(t)) + |\sigma_0(u(t))|_{L_Q}^2 dt + 2\alpha |\sigma_0^*(u(t))u(t)|_{H_0}^2 dt \right].$$

Therefore, if  $I(t) = 2\alpha \int_0^t \Phi(s) (u(s), \sigma_0(u(s)) dW(s))$ ,

$$\begin{aligned} d\Phi(t) = & \Phi(t) \left[ -(2\alpha - \beta) \|u(t)\|^2 + 2\alpha (-R(u(t)) + G(u(t))h(t), u(t)) + \alpha |\sigma_0(u(t))|_{L_Q}^2 \right. \\ & \left. + 2\alpha^2 |\sigma_0^*(u(t))u(t)|_{H_0}^2 \right] dt + I(t). \end{aligned}$$

For any integer  $n \geq 1$ , let  $\tau_n = \inf\{t : \sup_{0 \leq s \leq t} |u(s)|^2 + \int_0^t \|u(s)\|^2 ds \geq n\} \wedge T$ . Then we have  $\mathbb{E}(I(t \wedge \tau_n)) = 0$  for  $t \in [0, T]$ . Since  $|u(t)| \leq \bar{C} \|u(t)\|$ , if  $\tilde{R}_0$  from (7.2) is such that  $\tilde{R}_0 < \bar{C}^{-2}$ , for  $\alpha_1 \leq \alpha_0$  small enough and  $0 < \alpha < \alpha_0$ , we have  $1 - (\tilde{R}_0 + 2^{-1}\alpha K_0)\bar{C}^2 > 0$ . For  $0 < \beta < \beta(\alpha)$  with  $\beta(\alpha)$  small enough, and for  $\epsilon$  small enough, Fubini's theorem implies:

$$\begin{aligned} \sup_{0 \leq s \leq t} \mathbb{E} \Phi(s \wedge \tau_n) & \leq \exp(\alpha |\xi|^2) + \mathbb{E} \int_0^{t \wedge \tau_n} \Phi(s) [R_0 \epsilon^{-1} + K_0 \epsilon^{-1} |h(s)|_0^2 + \alpha K_0] ds \\ & \leq \exp(\alpha |\xi|^2) + \int_0^t \mathbb{E} \Phi(s \wedge \tau_n) [R_0 \epsilon^{-1} + K_0 \epsilon^{-1} |h(s)|_0^2 + \alpha K_0] ds. \end{aligned}$$

Since  $\Phi(\cdot \wedge \tau_n)$  is bounded, Gronwall's lemma implies that there exist constants  $c_1, c_2$  depending on  $K_0, R_0$  and  $\alpha$  such that for every  $t \in [0, T]$ ,

$$\sup_n \sup_{0 \leq t \leq T} \mathbb{E} \Phi(t \wedge \tau_n) \leq \exp(\alpha |\xi|^2) \exp(c_1 T + c_2 M).$$

Using (2.32) and the monotone convergence theorem, we conclude the proof by letting  $n \rightarrow \infty$ .  $\square$

**7.2. Properties in  $\mathcal{H}$ .** Now we are in position to state the conditions which guarantee the validity of conditions (i) and (ii) in Theorems 3.1 and 4.1.

**Condition (BS+)** Let condition (B) hold with  $\mathcal{H} = \text{Dom}(A^{1/4})$  and suppose that there exists a constant  $K > 0$  such that for  $u \in \mathcal{H}$ :

$$|A^{1/4}\sigma(t, u)|_{L^2(Q(H_0, H))}^2 + |A^{1/4}\tilde{\sigma}(t, u)|_{L^2(Q(H_0, H))}^2 \leq K(1 + \|u\|_{\mathcal{H}}^2). \quad (7.4)$$

**Condition (GR1)** There exist constants  $\bar{K}_0$  and  $\bar{R}_0$  such that for every  $u \in \mathcal{H}$ :

$$|A^{1/4}G(u)|_{L^2(H_0, H)}^2 \leq \bar{K}_0(1 + \|u\|_{\mathcal{H}}^2), \quad |A^{1/4}R(u)| \leq \bar{R}_0(1 + \|u\|_{\mathcal{H}}). \quad (7.5)$$

**Proposition 7.2.** *Assume that conditions (BS+), (GR1), as well as (2.16) and (2.17) from condition (S) are satisfied. Let the hypotheses of Proposition 7.1 be in force and let  $u$  be the solution to (2.31). Assume in addition that  $\mathbb{E}\|\xi\|_{\mathcal{H}}^2 < \infty$ . Then there exist  $q > 0$  and  $q_* > 0$  such that*

$$\mathbb{E}\left(\text{ess sup}_{[0, T]} \|u(t)\|_{\mathcal{H}}^q\right) + \mathbb{E}\left(\left|\int_0^T |A^{3/4}u(\tau)|^2 d\tau\right|^{q_*}\right) < \infty. \quad (7.6)$$

*Proof.* We consider the Galerkin approximations  $u_n$  and, to ease notations, we skip the index  $n$ . Let  $\sigma_0 = \sigma + \tilde{\sigma}$  and for  $t \in [0, T]$  set

$$I(t) := \sup_{0 \leq s \leq t} 2 \left| \int_0^s (A^{1/4}\sigma_0(u(r))dW(r), A^{1/4}u(r)) \right|.$$

Using Itô's formula for  $\|u(t)\|_{\mathcal{H}}^2 = |A^{1/4}u(t)|^2$  and usual upper estimates, we deduce that

$$\begin{aligned} \sup_{s \leq t} \|u(s)\|_{\mathcal{H}}^2 + 2 \int_0^t |A^{3/4}u(s)|^2 ds &\leq \|\xi\|_{\mathcal{H}}^2 + 2 \int_0^t |\langle B(u(s), u(s)), A^{1/2}u(s) \rangle| ds \\ &+ I(t) + \int_0^t 4K(1 + \|u(s)\|_{\mathcal{H}}^2) ds + 2 \int_0^t |(-R(u(s)) + G(u(s))h(s), A^{1/2}u(s))| ds. \end{aligned}$$

The inequality (2.6) and condition (GR1) imply

$$\begin{aligned} |\langle B(u, u), A^{1/2}u \rangle| &\leq C_0 \|u\|_{\mathcal{H}} \|u\| |A^{3/4}u| \leq |A^{3/4}u|^2 + C_0^2 2^{-2} \|u\|_{\mathcal{H}}^2 \|u\|^2, \\ |(-R(u) + G(u)h, A^{1/2}u)| &\leq c_0(1 + |h|_0)(1 + \|u\|_{\mathcal{H}}^2), \end{aligned}$$

where  $c_0$  depends on  $\bar{K}_0$  and  $\bar{R}_0$ . Hence, for  $X(t) = \sup\{\|u(s)\|_{\mathcal{H}}^2 : 0 \leq s \leq t\}$ , we deduce

$$X(t) + \int_0^t |A^{3/4}u(s)|^2 ds \leq \|\xi\|_{\mathcal{H}}^2 + I(t) + c_1 + c_2 \int_0^t [1 + |h(s)|_0 + \|u(s)\|^2] X(s) ds, \quad (7.7)$$

where the constant  $c_1$  depends on  $K, \bar{K}_0, \bar{R}_0, T, M$  and  $c_2$  depends on  $\bar{K}_0$  and  $\bar{R}_0$ . Gronwall's lemma yields

$$X(t) \leq [c_1 + \|\xi\|_{\mathcal{H}}^2 + I(t)] \exp\left(c_2 \int_0^t [1 + |h(s)|_0 + \|u(s)\|^2] ds\right).$$

This implies that for  $\delta > 0$ :

$$\mathbb{E}|X(t)|^\delta \leq C(M, T) \left[ \mathbb{E}(c_1 + \|\xi\|_{\mathcal{H}}^2 + I(t))^{2\delta} \right]^{1/2} \left[ \mathbb{E} \exp\left(2c_2\delta \int_0^t \|u(s)\|^2 ds\right) \right]^{1/2}.$$

Thus Proposition 7.1 implies that for  $\delta$  small enough we have:

$$\mathbb{E}|X(t)|^\delta \leq C(M, T) \mathbb{E} \left[ \exp(2c_2\delta|\xi|^2) \right]^{1/2} \left[ 1 + \mathbb{E}\|\xi\|_{\mathcal{H}}^2 + \mathbb{E}I(t) \right]^{1/2}.$$

The Burkholder-Davies-Gundy inequality, relations (7.4) and (2.32) yield

$$\mathbb{E}I(t) \leq 6 \mathbb{E} \left\{ \int_0^t |A^{1/4}u(r)|^2 |A^{1/4}[\sigma + \tilde{\sigma}](u(r))|_{L^2}^2 dr \right\}^{1/2}$$

$$\leq 6 \mathbb{E} \left\{ 4K \int_0^t \|u(r)\|_{\mathcal{H}}^2 (1 + \|u(r)\|_{\mathcal{H}}^2) dr \right\}^{\frac{1}{2}} \leq c_4(T, K, C).$$

Thus there exists constants  $q > 0$  and  $c := c(K, T, M, C)$  such that

$$\sup_{n \geq 1} \mathbb{E} \left( \sup_{0 \leq s \leq T} \|u_n(s)\|_{\mathcal{H}}^q \right) = c < +\infty \quad (7.8)$$

for the Galerkin approximations  $u_n$ . As  $n \rightarrow +\infty$ , after limit transition we deduce that the first term in the left hand-side of (7.6) is finite.

To prove that the second term is finite as well, note that (7.7) implies that for every  $n$ :

$$\int_0^t |A^{\frac{3}{4}} u_n(s)|^2 ds \leq C + \|\xi\|_{\mathcal{H}}^2 + I(t) + c_2 \text{ess sup}_{0 \leq s \leq T} \|u_n(s)\|_{\mathcal{H}}^2 \int_0^t [1 + |h(s)|_0^2 + \|u_n(s)\|^2] ds.$$

Thus we can use (7.8) and complete the proof of (7.6) by a similar argument.  $\square$

We prove that the process  $u$  solving (2.31) belongs to  $\mathcal{C}([0, T], \mathcal{H})$  a.s.

**Proposition 7.3.** *Let the conditions of Proposition 7.2 be satisfied and let  $u$  be the solution to (2.31). Then the process  $u$  belongs to  $\mathcal{C}([0, T], \mathcal{H})$  a.s.*

*Proof.* Let  $\sigma_0 = \sigma + \tilde{\sigma}$ ; then for fixed  $\delta > 0$ , we have  $e^{-\delta A} u \in C([0, T], \mathcal{H})$ . Indeed, (7.4) and (2.32) imply that  $\mathbb{E} \int_0^T |A^{\frac{1}{4}} e^{-\delta A} \sigma_0(u(s))|_{L_Q}^2 ds < +\infty$ , so that  $\int_0^t e^{-\delta A} \sigma_0(u(s)) dW(s) \in \mathcal{C}([0, T], \mathcal{H})$ . Since for  $\delta > 0$  the operator  $e^{-\delta A}$  maps  $H$  to  $V$  and  $V'$  to  $\mathcal{H}$ , we deduce that almost surely the maps  $A^{\frac{1}{4}} e^{-\delta A} \int_0^t [B((u(s)) + R(u(s)))] ds$  and  $A^{\frac{1}{4}} e^{-\delta A} \int_0^t G((u(s)) h(s)) ds$  belong to  $\mathcal{C}([0, T], \mathcal{H})$ . Therefore it is sufficient to prove that

$$\lim_{\delta \rightarrow 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} \|u(t) - e^{-\delta A} u(t)\|_{\mathcal{H}}^{2p} \right) = 0 \quad (7.9)$$

for some  $p > 0$ . Let  $T_\delta = Id - e^{-\delta A}$  and apply Itô's formula to  $\|T_\delta u(t)\|_{\mathcal{H}}^2$ . This yields

$$\begin{aligned} \|T_\delta u(t)\|_{\mathcal{H}}^2 &= \|T_\delta \xi\|_{\mathcal{H}}^2 - 2 \int_0^t |A^{\frac{3}{4}} u(s)|^2 ds + 2I(t) + \int_0^t |A^{\frac{1}{4}} T_\delta \sigma_0(u(s))|_{L_Q}^2 ds \\ &\quad - 2 \int_0^t \langle B(u(s)) + R(u(s)) - G(u(s))h(s), A^{\frac{1}{2}} T_\delta^2 u(s) \rangle ds, \end{aligned} \quad (7.10)$$

where  $I(t) = \int_0^t (A^{\frac{1}{4}} T_\delta \sigma_0(u(s)) dW(s), A^{\frac{1}{4}} T_\delta u(s))$ . The Burkholder-Davies-Gundy and Schwarz inequalities together with (7.4) imply that for any  $p > 0$ :

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |I(t)|^p &\leq C_p \mathbb{E} \left( \int_0^T \|T_\delta u(s)\|_{\mathcal{H}}^2 |A^{\frac{1}{4}} T_\delta \sigma_0(u(s))|_{L_Q}^2 ds \right)^{p/2} \\ &\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T} \|T_\delta u(t)\|_{\mathcal{H}}^{2p} + \frac{C_p^2}{2} \mathbb{E} \left( \int_0^T |A^{\frac{1}{4}} T_\delta \sigma_0(u(s))|_{L_Q}^2 ds \right)^p. \end{aligned}$$

Hence (7.10) yields for  $0 < p < 1$  the existence of a constant  $c_p$  such that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \|T_\delta u(t)\|_{\mathcal{H}}^{2p} &\leq c_p \left[ \|T_\delta \xi\|_{\mathcal{H}}^{2p} + \mathbb{E} \left| \int_0^T |A^{\frac{1}{4}} T_\delta \sigma_0(u(s))|_{L_Q}^2 ds \right|^p \right. \\ &\quad \left. + \mathbb{E} \left( \int_0^T \left| \langle B(u(s)) + R(u(s)) - G(u(s))h(s), A^{\frac{1}{2}} T_\delta^2 u(s) \rangle \right| ds \right)^p \right]. \end{aligned}$$

Since for every  $u \in \mathcal{H}$ ,  $\|T_\delta u\|_{\mathcal{H}} \rightarrow 0$  as  $\delta \rightarrow 0$  and  $\sup_{\delta > 0} |T_\delta|_{L(\mathcal{H}, \mathcal{H})} \leq 1$ , we deduce that if  $\{\varphi_k\}$  denotes an orthonormal basis in  $H$ , then  $|A^{\frac{1}{4}} T_\delta \sigma_0(u(s)) Q^{1/2} \varphi_k|^2 \rightarrow 0$  for every  $k$

and almost every  $(\omega, s) \in \Omega \times [0, T]$ . Since  $\sup_{\delta > 0} \|e^{-\delta A}\|_{L(\mathcal{H})} < +\infty$ , (7.4) implies

$$\sup_{\delta > 0} |A^{\frac{1}{4}} T_\delta \sigma_0(u)|_{L^Q}^2 \leq C(1 + \|u\|_{\mathcal{H}}^2) \in L^1(\Omega \times [0, T]).$$

Therefore, the Lebesgue dominated convergence theorem yields

$$\mathbb{E} \int_0^T |A^{\frac{1}{4}} T_\delta \sigma_0(u(s))|_{L^Q}^2 ds \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Furthermore, using (2.6) we deduce

$$\begin{aligned} \int_0^T \left| \langle B(u(s)), A^{\frac{1}{2}} T_\delta^2 u(s) \rangle \right| ds &\leq C \int_0^T \|u(s)\|_{\mathcal{H}} \|u(s)\| |A^{\frac{3}{4}} T_\delta^2 u(s)| ds \\ &\leq C \operatorname{ess\,sup}_{[0, T]} \|u(s)\|_{\mathcal{H}} \left[ \int_0^T \|u(s)\|^2 ds \right]^{1/2} \left[ \int_0^T |A^{\frac{3}{4}} T_\delta^2 u(s)|^2 ds \right]^{1/2}. \end{aligned}$$

Thus, using Proposition 7.2 for  $p > 0$  small enough and Hölder's inequality, we obtain

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \left| \langle B(u(s)), A^{\frac{1}{2}} T_\delta^2 u(s) \rangle \right| ds \right]^p &\leq C \left[ \mathbb{E} \operatorname{ess\,sup}_{[0, T]} \|u(s)\|_{\mathcal{H}}^{2p} \right]^{1/2} \left[ \mathbb{E} \left( \int_0^T \|u(s)\|^2 ds \right)^{2p} \right]^{1/4} \\ &\times \left[ \mathbb{E} \left( \int_0^T |A^{\frac{3}{4}} T_\delta^2 u(s)|^2 ds \right)^{2p} \right]^{1/4} \leq C \left[ \mathbb{E} \left( \int_0^T |A^{\frac{3}{4}} T_\delta^2 u(s)|^2 ds \right)^{2p} \right]^{1/4}. \end{aligned}$$

Given  $u \in \operatorname{Dom}(A^{\frac{3}{4}})$  we have  $|A^{\frac{3}{4}} T_\delta^2 u| \rightarrow 0$  as  $\delta \rightarrow 0$  while  $|A^{\frac{3}{4}} T_\delta^2 u| \leq 2|A^{\frac{3}{4}} u|$ . Hence the dominated convergence theorem yields  $\mathbb{E} \left[ \int_0^T \left| \langle B(u(s)), A^{\frac{1}{2}} T_\delta^2 u(s) \rangle \right| ds \right]^p \rightarrow 0$  as  $\delta \rightarrow 0$ . A similar argument can be applied to the term  $\int_0^T \left| \langle R(u(s)) - G(u(s))h(s), A^{\frac{1}{2}} T_\delta^2 u(s) \rangle \right| ds$ . Thus we obtain that (7.9) holds with  $p > 0$  small enough.  $\square$

**7.3. Examples of models.** The properties above hold for several important cases which includes 2D Navier-Stokes equations and MHD equations in the periodic domains, as well as shell models of turbulence. For more details concerning the models mentioned in this section we refer to [6] and to the references therein.

**7.3.1. 2D Navier-Stokes equations in the periodic domains.** In this case we have relation  $(B(u, u), Au) = 0$  (see, e.g., [7]) which makes it possible to obtain the additional properties which are needed for Theorem 3.1 to hold. These extra properties hold in a stronger form and do not require any additional hypotheses concerning the model.

**7.3.2. 2D Navier-Stokes equations and 2D Boussinesq model for the Bénard convection with Dirichlet boundary conditions.** In these cases the relation  $(B(u, u), Au) = 0$  is not true. However, we can use some results on interpolation of intersections (see [19]) to obtain that  $\operatorname{Dom}(A^{1/4}) \subset [L_4(D)]^2$ . This implies hypotheses **(B)** with  $\mathcal{H} = \operatorname{Dom}(A^{1/4})$ . Thus we can apply the above results proved in this Appendix.

**7.3.3. 2D MHD equations and 2D magnetic Bénard problem in bounded domains.** In this case, even in the periodic case, the relation  $(B(u, u), Au) = 0$  is not true. However a direct analysis based on results of interpolation of intersections ([19]) makes it possible to prove that  $\operatorname{Dom}(A^{1/4}) \subset [L_4(D)]^2$ . As above, we can apply Theorem 3.1 under the conditions imposed in this Appendix.

7.3.4. *Shell models of turbulence.* We can consider either the GOY model or the Sabra model, or else the so-called dyadic model. Since (see [6, Sect.2.1.6]) in all these models we have that

$$|\langle B(u, v), w \rangle| \leq C \|u\| A^{1/2} \|v\| \|w\|, \quad \forall u, w \in H, \quad \forall v \in \text{Dom}(A^{1/2}).$$

Thus condition **(B)** holds with  $\mathcal{H} = \text{Dom}(A^s)$  for any choice of  $s \in [0, 1/4]$ . In particular we can choose  $\mathcal{H} = H$ . In this case conditions (i) and (ii) in Theorem 3.1 trivially hold.

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(I. Chueshov) DEPARTMENT OF MECHANICS AND MATHEMATICS, KHARKOV NATIONAL UNIVERSITY,  
4 SVOBODY SQUARE, 61077, KHARKOV, UKRAINE

*E-mail address*, I. Chueshov: `chueshov@univer.kharkov.ua`

(A. Millet) SAMOS-MATISSE, CENTRE D'ÉCONOMIE DE LA SORBONNE (UMR 8174), UNIVERSITÉ  
PARIS 1, CENTRE PIERRE MENDÈS FRANCE, 90 RUE DE TOLBIAC, F- 75634 PARIS CEDEX 13, FRANCE  
*and* LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES (UMR 7599), UNIVERSITÉS PARIS 6-  
PARIS 7, BOÎTE COURRIER 188, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE

*E-mail address*, A. Millet: `amillet@univ-paris1.fr` *and* `annie.millet@upmc.fr`