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# Properties of periodic solutions near their oscillation threshold for a class of hyperbolic partial differential equations with localized nonlinearity

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## Abstract

The periodic solutions of a type of nonlinear hyperbolic partial differential equations with a localized nonlinearity are investigated. For instance, these equations are known to describe several acoustical systems with fluid-structure interaction. It also encompasses particular types of delay differential equations. These systems undergo a bifurcation with the appearance of a small amplitude periodic regime. Assuming a certain regularity of the oscillating solution, several of its properties around the bifurcation are given: bifurcation point, dependence of both the amplitude and period with respect to the bifurcation parameter, and law of decrease of the Fourier series components. In addition, this study is based on a Fourier domain analysis and the harmonic balance method has been extended to the class of infinite dimensional problems hereby considered. Estimates on the errors made if the Fourier series is truncated are provided.

## 1 Introduction

### 1.1 Physical motivations

The starting point of the present work is due to physical motivations. The first objective is to justify a thirty-year-old conjecture assumed in order to obtain information on the bifurcations encountered in woodwind musical instruments. It is now established [1, 2] that woodwind instruments are nonlinear systems having a transition between a steady state and an oscillating regime, for a certain value of the pressure in the mouth of the musician. These instruments possess a fluid-structure interaction which implies the presence of partial differential equation (PDE) along with a nonlinearity. Analytical calculations have been done for small oscillations around the bifurcation, giving valuable information on the oscillation with respect to the parameters of the instrument. But these calculations are possible only if a conjecture, first presented by Worman [3], is assumed. Suppose the instrument emits a periodic sound of small amplitude, and the acoustic pressure  $p_0$  inside the mouthpiece is oscillating with period  $\tau$ . Let  $P(\omega_q)$  be the Fourier component of  $p_0$  associated with the angular frequency  $\omega_q = 2q\pi/\tau$ . Worman's conjecture states:

$$|P(\omega_q)| \leq |P(\omega_1)|^{|q|}, \text{ for all } q \in \mathbb{Z}^*,$$

where, since it is of small amplitude,  $|P(\omega_1)| < 1$ . This key hypothesis is not obvious and arguments to justify it have been given in the theoretical work of Grand et al [4]. However, the Fourier series has been truncated in the calculations of this latter work and the control of the remaining tail was not investigated. The first aim of the present study is to show that the weaker assumption of a twice continuously differentiable solution for the model of woodwind instruments implies Worman's conjecture. This former regularity assumption, allowing to control the tail of the Fourier series, is quite natural from a physical point of view.

In addition to this first motivation, the hypotheses on the equations are weak enough to hold for a broader class of physical systems. The present approach is especially suitable for vibrating systems and the ones described by equations in the frequency domain with the formalism of transfer functions. This study proposes an extension of the harmonic balance method, widespread in engineering [5, 6], and of the underlying Galerkin procedure, to infinite dimensional systems governed by partial differential equations with a localized nonlinearity. This generalization holds for the case of small amplitude solutions which

are twice continuously differentiable with respect to time. The present study justifies in a mathematical way the frequency domain expression, the truncation of the Fourier series, and the analysis around the bifurcation threshold.

## 1.2 Mathematical framework

Properties of the solutions of a nonlinear system around a bifurcation point can be obtained by general techniques and theorems such as the reduction to the center manifold and normal forms [9], [10] or the Hopf bifurcation theorem [11]. Nevertheless, in the case of PDEs involving the wave equation, which makes them hyperbolic, the hypotheses required for the use of these powerful theorems may not be satisfied. In such systems, the norm of the resolvent associated to the linearized operator in a neighbourhood of a bifurcation may not decrease when the spectral parameter goes to infinity. This is the case of woodwind musical instruments. The present study shows that although the hypotheses of the Hopf theorem are not satisfied, several of the results given by this theorem can be retrieved, provided the periodic solution is of small amplitude and twice continuously differentiable. The existence of such a solution is an hypothesis of the present work. Notice however that several studies show that nonlinear systems with PDEs possess such regular solutions, for example the case of the wave equation with nonlinear terms and damping, see e.g. [12, 13, 14].

After having settled the hypotheses satisfied by the systems under study, a correspondence between expressions of the model in the time and the frequency domain is established in lemma 3.1. Then theorems 4.1 gives required conditions for an oscillation to appear and theorems 4.1 and 6.1, with corollary 6.2 give the dependence of both the Fourier harmonics and the frequency on the bifurcation parameter, near its birth point. As expected, it is similar to the finite dimensional case and what is given by the finite dimensional Hopf bifurcation theorem. In addition, this study gives also a valuable estimation of the law of decrease of the harmonics of the solution with respect to the multiple of the fundamental frequency in theorem 5.1. This justifies the process of truncation used in the harmonic balance method and gives an error term on it. The last section comes back to the physical motivations and is devoted to an application in the field of musical acoustics. It introduces the equations modeling a reed woodwind instrument and shows how the mathematical results can be applied to this example.

## 2 The system

Let  $t \in T \subset \mathbb{R}$  denote the time and  $X \subset \mathbb{R}^n$  the space of configurations, for  $n \in \mathbb{N}$ . Let  $\mathcal{X} = L^2(T \times X)$  be the separable Hilbert space of square integrable functions associated to the domain  $T \times X$ . Only time periodic solutions are investigated and  $T \times X$  is supposed to be a compact domain of  $\mathbb{R}^{n+1}$ . The nonlinearity takes place at a particular location  $x_0 \in X$  in space and the infinite dimensional dynamical system under study is described formally by

$$Ap = \delta(x - x_0)B\mathcal{G}(\gamma, p), \quad (1)$$

where  $p \in \mathcal{X}$  is the state of the system, the real number  $\gamma$  is to be called the bifurcation parameter,  $A$  is a linear operator associated to a partial differential equation. The right hand side of (1) models the localized nonlinearity with  $\mathcal{G}$  being a nonlinear function of  $p$ ,  $B$  a linear operator acting on  $\mathcal{G}$  and the symbol  $\delta$  being the Dirac distribution. As a consequence, (1) must be understood in the distribution sense. Since  $A$  is a differential operator one can always find a solution  $p$ , well defined as a distribution (see e.g. Schwartz [15] section VII-10 and the examples therein). Let  $\tilde{p}$  be the Green function (distribution), solution of  $A\tilde{p} = \delta(x)\delta(t)$ . Thus  $p$  is given by the convolution  $p = \tilde{p} * \delta(x - x_0)B\mathcal{G}(\gamma, p)$ . Let  $p_0 \in L^2(T)$  be the value of  $p$  at  $x_0$ . Let us assume that  $B$  and  $\mathcal{G}$  are decoupled from the space variable i.e. they are such that:

$$\int_X \delta(x - x_0)B\mathcal{G}(\gamma, p)dx = B\mathcal{G}(\gamma, p_0).$$

Let us further assume that  $A$  and  $\mathcal{G}$  possess the properties allowing to write:

$$\begin{aligned} p(x, t) &= \int_{X \times T} \tilde{p}(x - y, t - \tau)\delta(y - x_0) B\mathcal{G}(\gamma, p)|_{y, \tau} dy d\tau \\ &= \int_T \tilde{p}(x - x_0, t - \tau) B\mathcal{G}(\gamma, p_0)|_{\tau} d\tau. \end{aligned} \quad (2)$$

As a consequence of the above equation, there exists an operator  $\tilde{Z} : L^2(T) \rightarrow \mathcal{X}$  such that:

$$p = \tilde{Z}\mathcal{G}(\gamma, p_0). \quad (3)$$

Notice that the action of  $\tilde{Z}$  implies a convolution with respect to time, which becomes a multiplication by Fourier transform. Expressing (3) at  $x = x_0$ , one can introduce the operator  $Z = \tilde{Z}|_{x_0}$ , where  $Z : L^2(T) \rightarrow L^2(T)$ , and the study is hence reduced to the investigation of the oscillating solutions belonging to  $L^2(T)$  of:

$$p_0 = Z\mathcal{G}(\gamma, p_0). \quad (4)$$

Equation (1) is a formal expression whereas (4) is rigorously defined in the following and is the starting point of the study.

The nonlinear vector valued function  $\mathcal{G} : (\mathbb{R} \times L^2(T)) \rightarrow L^2(T)$  has the particular shape:

$$\mathcal{G}(\gamma, p_0) = Lp_0 + \sum_{n=2}^{\infty} R_n(\gamma, p_0), \quad (5)$$

where the linear operator  $L$  depends on  $\gamma$  and each  $R_n$  corresponds to a monomial of degree  $n$  in  $p_0$ . More precisely, for each  $n$ , let  $Z_{n,1}, Z_{n,2}, \dots, Z_{n,n}$  be bounded linear operators from  $L^2(T)$  to itself. Each operator may depend on  $\gamma$ . It is assumed that the nonlinear terms can be written as

$$R_n(\gamma, p_0) = (Z_{n,1}p_0)(Z_{n,2}p_0) \cdots (Z_{n,n}p_0). \quad (6)$$

This expression of the nonlinearity allows to cover the case of woodwind musical instruments as it will be shown in section 7 as well as different types of delay differential equations.

*Remark 2.1.* Let some dynamical system be described by the standard notation:

$$\frac{dp_0}{dt} = L_2p_0 + g(\gamma, p_0) \quad (7)$$

where  $L_2$  is the linear part and  $g$  the nonlinear mapping. See [20] for examples of assumptions on these objects, [9] for the infinite dimensional case, or [11] for the case of the Hopf bifurcation. The connection with this study is made by writing formally  $Z^{-1}(1 - ZL) = L_2 - d/dt$ . By Fourier transform the time derivative operator becomes the product by some complex number  $i\omega$ ,  $\omega \in \mathbb{R}$ . Looking for a pure imaginary eigenvalue of  $L_2$  is the same problem as searching for a zero eigenvalue of  $Z^{-1}(1 - ZL)$ . In order to prove the existence of a Hopf bifurcation, the norm of the resolvent  $(L_2 \pm i\omega)^{-1}$  must be bounded by  $1/|\omega|$  when the spectral parameter  $\pm i\omega$  tends to complex infinity (see [9]), a property which does not arise in hyperbolic PDEs. Here the inverse of  $Z^{-1}(1 - ZL)$  is only required to be bounded for each  $\gamma$  in a neighborhood of the bifurcation point, except at this critical point. However, this decrease of the norm of the resolvent is replaced here by the assumption of a twice continuously differentiable periodic solution.

Let  $L^\infty(T)$  be the space of bounded functions on  $T$  and let  $\mathcal{B}_\varepsilon \subset L^\infty(T)$  be the ball of radius  $\varepsilon > 0$  centered at zero. Let  $C^2(T)$  be the set of functions twice continuously differentiable with respect to  $t \in T$  and define

$$\Lambda = \mathcal{B}_\varepsilon \cap C^2(T).$$

The solution of (4) under investigation satisfies the following hypothesis:

**H1** For all  $\varepsilon \in (0, 1)$  there exists a non-empty, open, simply connected set  $\Omega_1 \subset \mathbb{R}$ , where for all  $\gamma \in \Omega_1$  the system has a unique non trivial real-valued periodic solution belonging to  $\Lambda$ , continuous with respect to  $\gamma$ . Its period will be denoted by  $\tau = \tau(\gamma)$ . There exists a  $\gamma_0 \in \mathbb{R}$ , with  $\gamma_0 \in \Omega$  the closure of  $\Omega_1$ , such that the periodic solution tends to zero when  $\gamma$  tends to  $\gamma_0$ . The point  $\gamma_0$  will be called the critical point or the bifurcation point.

As said in the introduction, and stated in **H1**, the regularity of the solution is an hypothesis. Nevertheless, several publications have shown regularity properties of the solutions of nonlinear hyperbolic PDEs. For instance [12] shows that a twice continuously differentiable solution with respect to time exists for a model of wave equation with damping and source term, this solution being a distribution

with respect to the space variable. In [13], for a slightly different model, the solution is twice continuously differentiable with respect to time and, for all time fixed in a bounded interval, it belongs to a Sobolev space. The periodic solutions of the nonlinear wave equation are even more regular for some class of nonlinear terms as shown in [14] and references therein. To the knowledge of the author, no such results have been settled in the case of a nonlinear function localized at one point in space. However, the model described by (1), where  $A$  represents the wave equation in a tube with damping, has been investigated theoretically [16, 17] and numerically [18, 19] by physicists, giving results compatible with this regularity hypothesis, and motivating the present approach.

Let us introduce the notation

$$J_\gamma = Z^{-1}(1 - ZL),$$

and assume:

- H2** 1)  $J_\gamma$  is a closed operator acting in  $L^2(T)$ . It is diagonalizable and its eigenvectors consist of the vectors of the Fourier basis of  $L^2(T)$ .
- 2) Let  $Q_e, Q_{-e}$  be the orthogonal projections associated to the vectors  $\psi_e, \psi_{-e}$ , respectively, of the Fourier basis.  $J_\gamma$  is such that

$$Q_e J_\gamma Q_e = \beta_e(\gamma) Q_e, \quad Q_{-e} J_\gamma Q_{-e} = \beta_{-e}(\gamma) Q_{-e}$$

where  $\beta_e(\gamma), \beta_{-e}(\gamma)$  are two complex conjugate eigenvalues of  $J_\gamma$  such that  $\beta_e(\gamma_0) = 0$ . Moreover, for all  $\gamma \in \Omega$ , the operator  $Q J_\gamma Q$ , where  $Q = (1 - (Q_e + Q_{-e}))$ , possesses a bounded inverse.

On the nonlinear part, the hypotheses are:

- H3** For all  $\gamma \in \Omega$  and  $n, m \in \mathbb{N}$ ,  $n \geq 2$ ,  $m \in [1, n]$ ,  $Z_{n,m}(\gamma) : L^2(T) \rightarrow L^2(T)$  is a bounded linear operator. Moreover, as for  $J_\gamma$ , the Fourier basis is its eigenbasis. For all  $n \geq 2$ ,  $\gamma \in \Omega$  and  $p \in \Lambda$ , by the Hölder inequality, the quantity  $R_n(\gamma, p_0)$  belongs to  $L^2(T)$ , since the  $\{Z_{n,m}\}$  are bounded and  $\|p_0\|_\infty \leq \varepsilon$ , where  $\|\cdot\|_\infty$  denotes the norm associated to  $L^\infty(T)$ . Eventually, it is supposed that for all  $\gamma \in \Omega$  and  $p_0 \in \Lambda$ , the series of nonlinear terms is convergent :

$$\lim_{M \rightarrow \infty} \sum_{n=2}^M R_n(\gamma, p_0) = \overline{R}(\gamma, p_0) \in L^2(T).$$

*Remark 2.2.* If for all  $\gamma \in \Omega$ ,  $\mathcal{G}$  is an analytic function in its second variable, then it satisfies the above hypothesis: all the  $Z_{n,m}$  are identity operators except when  $m = 1$ , where  $Z_{n,1}$  is the term of the Taylor expansion  $Z_{n,1} = \mathcal{G}^{(n)}(\gamma, 0)/n!$ , with  $\mathcal{G}^{(n)}$  denoting the  $n$ th derivative.

Since the solution is periodic in time, it is natural to work in the Fourier space. Let  $S^\tau$  be the circle of perimeter  $\tau$ , then  $T = S^\tau$ . Let us define  $\ell^2$ , the Hilbert space of sequences which are square summable. Let  $\mathcal{F}$  denotes the Fourier transform operator from  $L^2(T)$  to  $\ell^2$ . The Fourier transform of  $p_0$  will be denoted  $\widehat{p}_0 = \mathcal{F}p_0 \in \ell^2$  or, using the physicists convention, by a capital letter  $P = \widehat{p}_0$ . For all  $n \in \mathbb{Z}$  and  $\omega_n = \frac{2\pi n}{\tau}$ , the Fourier transform is defined by :

$$P_n = \widehat{p}(\omega_n) = (\mathcal{F}p_0)(\omega_n) = \frac{1}{\tau} \int_{S^\tau} p_0(t) e^{-i\omega_n t} dt,$$

and the inverse transform at time  $t \in S^\tau$  is:

$$(\mathcal{F}^{-1}P)(t) = \sum_{n \in \mathbb{Z}} P_n e^{i\omega_n t}.$$

Using integration by part in the expression of the Fourier transform, the hypothesis of twice continuously differentiable solutions of **H1** implies the existence of  $a_0 > 0$  such that:

$$\forall q \neq 0, \quad |P_q| \leq \frac{a_0}{|q|^2}. \quad (8)$$

Moreover, from **H1** and the definition of the Fourier transform, the small amplitude condition gives for all  $\gamma \in \Omega_0$ :

$$\sup_{q \in \mathbb{Z}} |P_q| \leq \varepsilon. \quad (9)$$

### 3 Expression in the frequency domain

Many physical systems having an oscillating behavior, for instance in mechanics, acoustics or electrical engineering are described by relations taking place in the frequency domain. This is the formalism of transfer functions. Moreover, investigating oscillations turns out to be extremely efficient when using a frequency domain approach. In finite dimension, the harmonic balance method or the describing function method are such techniques. In infinite dimension, these techniques have not been proved to hold. This section, with lemma 3.1, shows how the infinite dimensional system under study can be written in the frequency domain. It is the first step toward its analysis using frequency domain techniques.

The key point is to express the nonlinear relationship in the Fourier basis. It will become a combination of convolutions. Let  $\widehat{g}, \widehat{h} \in \ell^2$  be such that their harmonics decrease like in (8), in consequence they belong to  $\ell^1$ , the space of summable sequences. Thus, the convolution  $\widehat{g} * \widehat{h}$  defined at  $q \in \mathbb{Z}$  by :

$$(\widehat{g} * \widehat{h})(q) = \sum_{n \in \mathbb{Z}} \widehat{h}_{q-n} \widehat{g}_n,$$

is an  $\ell^1$  function by Young's inequality, see e.g. [21, Sec. IX.4]. For each  $Z_{n,m}$  defined in **H3**, let us introduce the notation:

$$\widehat{Z_{n,m}} = \mathcal{F} Z_{n,m} \mathcal{F}^{-1}.$$

From **H3**, since each  $Z_{n,m}$  is diagonal in the Fourier domain, one has for the value of  $P$  associated to the vector  $\psi_q$  of the Fourier basis:

$$(\mathcal{F} Z_{n,m} p)(q) = \widehat{Z_{n,m}}(q) P_q$$

where  $\widehat{Z_{n,m}}(q)$  is a complex number.

More generally, let  $p, u_1, u_2, \dots, u_{N-1}$  be  $N \geq 2$  real periodic functions depending on time  $t \in S^\tau$ . Let these functions describe the state of some physical system under study. Let  $P, U^{(1)}, U^{(2)}, \dots, U^{(N-1)}$  the Fourier transform associated to  $p, u_1, u_2, \dots, u_{N-1}$  respectively. Suppose they are related together through  $N - 1$  linear relations, in the frequency domain, involving transfer functions:

$$U^{(m)} = \widehat{Z_m} P, \quad (10)$$

where the  $\{\widehat{Z_m}\}$  are bounded operators. All may depend on the bifurcation parameter. Let us suppose there is also one nonlinear relation, often expressed in the time domain:

$$\mathcal{N}(\gamma, p, u_1, u_2, \dots, u_{N-1}) = 0. \quad (11)$$

If this relationship involves polynomial terms between the variables, such as  $p^2, pu_1$  or  $u_1^2 u_2$ , the following lemma can be used to express this relation in the frequency domain, with convolution products. For example, a cubic term involving three functions  $p, u_1 = Z_1 p, u_2 = Z_2 p$ , with  $p$  satisfying **H1** and  $Z_1, Z_2$  satisfying **H3** would give :

$$\widehat{pu_1 u_2} = \widehat{p} * \widehat{u_1} * \widehat{u_2} = \widehat{p} * (\widehat{Z_1 p}) * (\widehat{Z_2 p}).$$

And for all  $q \in \mathbb{Z}$ :

$$\left\{ \widehat{p} * (\widehat{Z_1 p}) * (\widehat{Z_2 p}) \right\}(\omega_q) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \widehat{Z_1}(\omega_n - \omega_m) \widehat{Z_2}(\omega_m) P_{q-n} P_{n-m} P_m.$$

The whole relation (11) can be stated under a set of harmonic balance equations, in the frequency domain, given by (13).

**Lemma 3.1.** *Suppose  $p_0 \in \Lambda$  and  $P$  denotes its Fourier transform. Under the hypotheses **H2**, **H3**, for all  $\gamma \in \Omega$ , all  $n \in \mathbb{N}, n \geq 2$ ,*

$$\mathcal{F} R_n(\gamma, p_0) = S(g_n, n, P),$$

where  $S(g_n, n, P) \in \ell^1$ . For all  $q \in \mathbb{Z}$  the elements of  $S(g_n, n, P)$  are defined by :

$$S_q(g_n, n, P) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \dots \sum_{k_{n-1} \in \mathbb{Z}} g_n(q, k_1, \dots, k_{n-1}) P_{q-k_1} P_{k_1-k_2} \dots P_{k_{n-1}}, \quad (12)$$

where for all  $n$ ,  $g_n$  is a complex-valued function of  $n$  variables depending on combinations of  $Z_{n,m}$ :

$$g_n(q, k_1, \dots, k_{n-1}) = \widehat{Z}_{n,1}(q - k_1) \widehat{Z}_{n,2}(k_1 - k_2) \cdots \widehat{Z}_{n,n}(k_{n-1}).$$

Moreover, the infinite sum gives:

$$\mathcal{F} \sum_{n=2}^{\infty} R_n(\gamma, p_0) = \sum_{n=2}^{\infty} S(g_n, n, P) \in \ell^2.$$

In consequence, solving the system described by (4) is equivalent to solving:

$$\widehat{J}_\gamma P = \sum_{n=2}^{\infty} S(g_n, n, P). \quad (13)$$

Notice that although not specified, the convolution  $S$  depends on  $\gamma$ .

*Proof of lemma 3.1.* Let  $G^{(1)}, G^{(2)}, \dots, G^{(n+1)} \in \ell^1$ ,  $n \in \mathbb{N}$ ,  $n \geq 1$ , then the  $n$ -product of convolution  $G^{(1)} * G^{(2)} * \dots * G^{(n+1)}$  is an  $\ell^1$  function, by Young inequality. Suppose  $g, h \in L^2(T)$  such that their respective Fourier transform  $G, H$  are  $\ell^1$  functions. Then Fubini's theorem can be used to get:

$$\begin{aligned} \sum_{q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} H_{q-n} G_n e^{i\omega_q t} &= \sum_{q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} H_{q-n} e^{i\omega_{q-n} t} G_n e^{i\omega_n t} = \sum_{n \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} H_{q-n} e^{i\omega_{q-n} t} G_n e^{i\omega_n t} \\ &= \sum_{n \in \mathbb{Z}} h(t) G_n e^{i\omega_n t} = h(t) g(t). \end{aligned} \quad (14)$$

Notice that the functions  $h, g$  are continuous by the Riemann-Lebesgue lemma. Reciprocally, for  $h, g \in L^2(T)$  twice continuously differentiable,  $H$  and  $G$  are  $\ell^1$  functions (see explanation of (8)) and this equality holds. In consequence, Eq. (14) shows the correspondence between multiplication in the time domain and convolution in the frequency domain of a finite number of products, in the case of twice continuously differentiable functions on the compact domain  $T$ . Since **H3** holds, all the  $\{Z_{n,m}\}$  are bounded and for all  $n \geq 2$ ,  $g_n$  is bounded. The bound may depend on  $n$ . Using Hölder inequality, one gets:

$$\|R_n(\gamma, p)\|_{L^2(T)} \leq \|Z_{n,1} \cdots Z_{n,n}\| \cdot (\|p\|_\infty)^{n-1} \cdot \|p\|_{L^2(T)},$$

which shows that  $R_n(\gamma, p)$  belongs to  $L^2(T)$ . The symbol  $\|\cdot\|$  denotes the operator norm. Suppose  $S(1, n, P) \in \ell^1$ , then

$$S(1, n+1, P) = P * S(1, n, P) \in \ell^1$$

and since  $S(1, 2, P) \in \ell^1$ , by induction, for all  $n \geq 2$ ,  $S(1, n, P) \in \ell^1$ . Then, one has:

$$\|S(g_n, n, P)\|_{\ell^1} \leq \sup |g_n| \cdot \|S(1, n, P)\|_{\ell^1}.$$

In consequence, (14) shows that for  $n \geq 2$  fixed,  $S(g_n, n, P)$  is the Fourier transform of  $R_n(\gamma, p)$ . Let us study the limit  $n \rightarrow \infty$ . Since the Fourier transform is bounded, and even unitary one has

$$\overline{S}(P) = \mathcal{F} \overline{R}(\gamma, p) \in \ell^2,$$

and the quantity:

$$\left\| \overline{S}(P) - \sum_{n=2}^M S(g_n, n, P) \right\|_{\ell^2} \leq \left\| \overline{R}(\gamma, p) - \sum_{n=2}^M R_n(\gamma, p) \right\|_{L^2(T)}$$

tends to zero as  $M$  tends to infinity from **H3**. □

## 4 Preliminary results and properties at threshold

This section demonstrates how the method based on the frequency domain approach can be used to investigate a bifurcation in a nonlinear system. It shows how to handle the hypothesis and the Fourier harmonics in order to control infinite sums and convolutions. The first result stated in the following theorem concerns the determination of the birth point of the oscillation and the need of a zero eigenvalue

for the linear part. Notice that it could have been directly given by mean of the Hartman-Grobman theorem, however, it was not the case so as to demonstrate how the frequency approach works. The second result of the theorem gives an estimation of the harmonics and shows the importance of the first harmonic. This serves as an introduction to Th. 5.1 of section 5 which will go further in this estimation. Finally, this section settles many important intermediate results which will be used in the other parts of the study.

**Theorem 4.1.** *Suppose **H1**, **H2** and **H3** hold. Then the bifurcation point  $\gamma_0$  where the oscillating solution starts and the frequency  $\omega_e$  of oscillation at this point are given by the characteristic equation:*

$$\beta_e(\gamma_0) = 0.$$

Moreover, for all  $\gamma \in \Omega$  where  $\varepsilon$  is small enough,  $|P_e| = \varepsilon$  and there exists  $c_0 > 0$  such that for all  $n \in \mathbb{Z} \setminus \{e, -e\}$ ,  $|P_n| \leq c_0 \varepsilon^2$ .

*Remark 4.2.* The quantities  $e$ ,  $\varepsilon$  and  $\gamma$  are dependent of each other and this dependence will be made explicit in the section 6.

Before stating the proof of this theorem, several lemmas giving bounds on the different convolution products encountered are presented. Let us start by showing a property of the convolution which will be crucial in the following.

**Lemma 4.3.** *Let  $a_1, a_2$  be positive constants,  $\lambda \in (0, 1)$ ,  $\theta_1, \theta_2$  be strictly positive parameters and  $\alpha_1, \alpha_2 \in (0, 1]$ . Let  $f, g : \mathbb{Z} \rightarrow \mathbb{C}$  be two bounded decreasing functions such that*

$$\forall q \in \mathbb{Z}, \quad |f(q)| \leq \lambda^{\theta_1}, \quad |g(q)| \leq \lambda^{\theta_2},$$

and  $\forall q \in \mathbb{Z}^*$ ,

$$|f(q)| \leq \frac{a_1}{|q|^{1+\alpha_1}}, \quad |g(q)| \leq \frac{a_2}{|q|^{1+\alpha_2}}.$$

Then,

i) there exists  $c_1 > 0$  such that

$$\forall q \in \mathbb{Z}, \quad |(f * g)(q)| \leq c_1 \cdot \lambda^n, \quad n = \max\left\{\theta_1 + \frac{\theta_2}{2} \frac{\alpha_2}{1 + \alpha_2}, \theta_2 + \frac{\theta_1}{2} \frac{\alpha_1}{1 + \alpha_1}\right\}$$

ii) there exists a small  $\alpha > 0$  and  $c_2 > 0$  such that  $\forall q \in \mathbb{Z}^*$

$$|(f * g)(q)| \leq c_2 \cdot \frac{1}{|q|^m}, \quad m = \frac{1 + \alpha_1 + \alpha_2}{2} - \alpha.$$

As a consequence, for any number  $M \geq 2$  of functions  $f_1, f_2, \dots, f_M$  satisfying the hypothesis for some  $\{\theta_i\}_i$  and  $\{\alpha_i\}_i$ ,  $i \in [1, M]$ , the multiple convolution  $f_1 * f_2 * \dots * f_M$  is a bounded function with bound depending on a positive power of  $\lambda$ .

*Proof.* Let us introduce the notation  $f_q = f(q)$  for  $f : \mathbb{Z} \rightarrow \mathbb{C}$ . Let  $f, g$  be two functions satisfying the hypothesis. Let  $\sigma_1, \sigma_2 \in (0, 1]$  such that  $(1 + \alpha_1)(1 - \sigma_1) + (1 + \alpha_2)(1 - \sigma_2) > 1$ . For all  $q \in \mathbb{Z}$ ,

$$\begin{aligned} \left| \sum_{n \in \mathbb{Z}} f_{q-n} g_n \right| &\leq 2\lambda^{(\theta_1 + \theta_2)} + \sup_{n \in \mathbb{Z}^*, n \neq q} |f_{q-n}|^{\sigma_1} |g_n|^{\sigma_2} \sum_{n \in \mathbb{Z}^*, n \neq q} |f_{q-n}|^{1-\sigma_1} |g_n|^{1-\sigma_2} \\ &\leq 2\lambda^{(\theta_1 + \theta_2)} + \lambda^{\theta_1 \sigma_1 + \theta_2 \sigma_2} a_1^{1-\sigma_1} a_2^{1-\sigma_2} \sum_{n \in \mathbb{Z}^*, n \neq q} \frac{1}{|q-n|^{(1+\alpha_1)(1-\sigma_1)} |n|^{(1+\alpha_2)(1-\sigma_2)}} \\ &\leq c(\sigma_1, \sigma_2) \lambda^{\theta_1 \sigma_1 + \theta_2 \sigma_2}, \end{aligned}$$

where  $c(\sigma_1, \sigma_2) > 0$  depends on  $\sigma_1, \sigma_2$  and tends to infinity when  $\sigma_1, \sigma_2$  are such that  $[(1 + \alpha_1)(1 - \sigma_1) + (1 + \alpha_2)(1 - \sigma_2)]$  tends to unity. The last step in the formula is obtained using Hölder's inequality. Fix  $\sigma_1 = 1$ , then the convergence criterion implies  $\sigma_2 < \frac{\alpha_2}{1 + \alpha_2}$ . If  $\sigma_2$  is fixed to the particular value

$$\sigma_2 = \frac{1}{2} \frac{\alpha_2}{1 + \alpha_2}, \tag{15}$$



there exists a  $c_1 > 0$  such that the convolution is bounded by:

$$\left| \sum_{n \in \mathbb{Z}} f_{q-n} g_n \right| \leq c_1 \lambda^m, \quad m = \theta_1 + \theta_2 \frac{1}{2} \frac{\alpha_2}{1 + \alpha_2}.$$

And similarly for  $\sigma_2 = 1$ , one can obtain the value  $m = \theta_2 + \theta_1 \frac{1}{2} \frac{\alpha_1}{1 + \alpha_1}$ .

For *ii*), for  $q \in \mathbb{Z}^*$ , one has the estimation :

$$\left| \sum_{n \in \mathbb{Z}} f_{q-n} g_n \right| \leq \frac{\lambda^{\theta_1}}{|q|^{1+\alpha_2}} + \frac{\lambda^{\theta_2}}{|q|^{1+\alpha_1}} + \sum_{n \in \mathbb{Z}^*, n \neq q} \frac{1}{|h_q(n)|},$$

where

$$h_q(n) = |q - n|^{1+\alpha_1} |n|^{1+\alpha_2}.$$

For  $n \neq 0$ ,  $n \neq q$ , the derivative  $h'_q$  is :

$$h'_q(n) = (-\text{sgn}(q - n)(1 + \alpha_1)|n| + \text{sgn}(n)(1 + \alpha_2)|q - n|) |q - n|^{\alpha_1} |n|^{\alpha_2}.$$

The function  $h'_q$  has the following sign for  $q > 0$ :

$$\begin{array}{c|cccccc} n & -\infty & 0 & \frac{q(1+\alpha_2)}{2+\alpha_1+\alpha_2} & q & \infty \\ \hline h'_q & - & | & + & 0 & - & | & + \end{array}$$

If  $q < 0$ , taking  $(-n)$  instead of  $n$  leads to the same result. Hence, for  $n, q \in \mathbb{Z}$ ,  $n \neq 0$ ,  $n \neq q$  and  $|q| > 1$ , the minimum of the function  $h_q$  is:

$$\min_{n \in \mathbb{Z}^*, n \neq q} h_q(n) = ||q| - 1|^\beta, \quad \beta = \min \{1 + \alpha_1, 1 + \alpha_2\}.$$

For  $q = 0$  or  $\pm 1$ , the minimum is 1. As a consequence, one can write for  $\sigma \in [0, 1]$  :

$$\sum_{n \in \mathbb{Z}^*, n \neq q} \frac{1}{|h(n)|} \leq \sup_{n \in \mathbb{Z}^*, n \neq q} \frac{1}{(|q - n||n|)^{\beta\sigma}} \sum_{n \in \mathbb{Z}^*, n \neq q} \frac{1}{|q - n|^{1+\alpha_1-\beta} |n|^{1+\alpha_2-\beta}} \frac{1}{(|q - n||n|)^{\beta(1-\sigma)}},$$

and if  $\sigma < \frac{1+\alpha_1+\alpha_2}{2\beta}$  and  $\sigma \leq 1$  the sum converges for  $|q| > 1$  and this leads to:

$$\left| \sum_{n \in \mathbb{Z}^*, n \neq q} f_{q-n} g_n \right| \leq \frac{c(\sigma)}{||q| - 1|^{\beta\sigma}}, \quad c(\sigma) > 0.$$

Since  $\alpha_1, \alpha_2 \leq 1$ , one has  $\frac{1+\alpha_1+\alpha_2}{2\beta} \leq 1$ , then the condition for convergence of the sum yields  $\beta\sigma < \frac{1+\alpha_1+\alpha_2}{2}$ . For the case  $|q| > 1$ , one has  $|q|/2 \leq ||q| - 1|$  and there exists a constant  $c_2$  and a small  $\alpha > 0$  such that

$$\left| \sum_{n \in \mathbb{Z}^*, n \neq q} f_{q-n} g_n \right| \leq \frac{c_2}{|q|^{m-\alpha}},$$

where  $m = \frac{1+\alpha_1+\alpha_2}{2}$ . The constant  $c_2$  can be chosen large enough to include the case  $q = \pm 1$ .  $\square$

The next lemma states upper bounds on the value of the function  $S$  defined in (12).

**Lemma 4.4.** *Suppose there exists  $a_0 > 0$ , and  $\varepsilon \in (0, 1)$  such that for all  $q \in \mathbb{Z}$ ,*

$$i) |P_q| \leq \varepsilon,$$

$$ii) \text{ for all } q \in \mathbb{Z}^* \quad |P_q| \leq \frac{a_0}{|q|^2}.$$

*Then for all  $n \in \mathbb{N}$ ,  $n \geq 2$ ,  $q \in \mathbb{Z}$ , there exists  $C > 0$ , such that*

$$|S_q(1, n, P)| \leq \varepsilon^{\frac{5}{4}} (C\varepsilon^{1/4})^n. \quad (16)$$

*Proof.* Let us first investigate the case  $n = 2$ . The hypotheses *i*) and *ii*) allow to apply lemma 4.3 with  $\theta_1 = \theta_2 = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ . This gives:

$$\forall q \in \mathbb{Z}, \quad |S_q(1, 2, P)| = |(P * P)(q)| \leq C_1 \cdot \varepsilon^{5/4}, \quad \forall q \in \mathbb{Z}^*, \quad |S_q(1, 2, P)| \leq C_2 \cdot \frac{1}{|q|^{\frac{3}{2} - \eta_2}}, \quad (17)$$

where  $C_1, C_2 > 0$  and  $\eta_2 > 0$ . This latter parameter can be chosen as small as wanted. Let us remind that the functions  $S(1, i, P)$ ,  $i \geq 3$ , can be written for all  $q \in \mathbb{Z}$

$$S(1, i, P) = P * S(1, i - 1, P). \quad (18)$$

Using lemma 4.3 with this formula and the estimation (17), this yields an estimate of  $S(1, 3, P)$ :

$$\forall q \in \mathbb{Z}, \quad |S_q(1, 3, P)| \leq C_1^2 \cdot \varepsilon^{(\frac{5}{4} + \frac{1}{4})}, \quad (19)$$

$$\forall q \in \mathbb{Z}^*, \quad |S_q(1, 3, P)| \leq C_2^2 \cdot \frac{1}{|q|^{1 + \alpha_3}}, \quad (20)$$

where  $\alpha_3 = (3/2 - \eta_2)/2 - \eta_3$ , with  $\eta_3 > 0$ . Similarly, by induction, each quantity  $S(1, n, P)$  is bounded:

$$\forall q \in \mathbb{Z}, \quad |S_q(1, n, P)| \leq C_1^n \cdot \varepsilon^{\theta_n}, \quad (21)$$

$$\forall q \in \mathbb{Z}^*, \quad |S_q(1, n, P)| \leq C_2^n \cdot \frac{1}{|q|^{\alpha_n}}, \quad (22)$$

with  $\theta_n = 5/4 + n/4$  and  $\alpha_n = \alpha_{n-1}/2 - \eta_n$ ,  $\eta_n > 0$ . Remark that for all  $n \in \mathbb{N}$   $\alpha_n > 0$  and

$$\lim_{n \rightarrow \infty} \alpha_n = 0.$$

□

One can now give a first estimate on the harmonics  $P_q$  of the solution of (13):

**Lemma 4.5.** *Suppose **H1**, **H3** hold. For  $\varepsilon$  small enough, relation (13) implies the following properties:*

- i) if hypothesis 1) of **H2** holds and the operator  $J_\gamma$  is invertible for all  $\gamma \in \Omega_1$ ,  $J_\gamma$  must not be invertible at  $\gamma_0$ .*
- ii) if **H2** holds, then for  $e \in \mathbb{Z}^*$  defined in this hypothesis,  $P_e \neq 0$  and  $\varepsilon$  can be chosen to be  $\varepsilon = |P_e|$ . Moreover, there exists a constant  $c_0 > 0$  such that, for all  $q \in \mathbb{Z}$ ,  $q \neq e$ ,  $|P_q| \leq c_0 \varepsilon^2$ .*
- iii) if **H2** holds, for all  $\beta \in \mathbb{Z}$  such that  $\beta \notin \{me\}_{m \in \mathbb{Z}}$ ,  $P_\beta = 0$ .*

*Proof.* Suppose  $J_\gamma$  has a bounded inverse for all  $\gamma \in \Omega$ , then the complex-valued function  $\widehat{J_\gamma}^{-1}$  is bounded. Since **H3** holds, the functions  $\{g_n\}$  are bounded. Let us write  $\tilde{g}_n$  for the supremum of  $|g_n|$ . Equation (13) gives for all  $q \in \mathbb{Z}$ ,  $M \geq 2$ :

$$\begin{aligned} |P_q| \leq & \sup_k |\widehat{J_\gamma}^{-1}(k)| \times \sup_{n \in [2, M]} \tilde{g}_n \times \left| \sum_{n=2}^M S_q(1, n, P) \right| + \\ & + \sup_k |\widehat{J_\gamma}^{-1}(k)| \times \left| \sum_{n=M}^{\infty} S_q(g_n, n, P) \right|. \end{aligned} \quad (23)$$

From **H3** and lemma 3.1, the second term of the previous expression tends to zero as  $M$  tends to infinity. Let us choose  $M$  large enough such that

$$\left| \sum_{n=M}^{\infty} S_q(g_n, n, P) \right| \leq \sup_{n \in [2, M]} \tilde{g}_n \times \left| \sum_{n=2}^M S_q(1, n, P) \right|,$$

then there exists a constant  $A > 0$  such that

$$|P_q| \leq A \times \left| \sum_{n=2}^M S_q(1, n, P) \right|. \quad (24)$$

Since **H1** holds, the hypotheses of lemma 4.4 are satisfied and one gets :

$$\forall n \geq 2, q \in \mathbb{Z}, |S_q(1, n, P)| \leq (C\varepsilon^\theta)^n \varepsilon^{1+\theta}, \quad \theta = 1/4, \quad (25)$$

where  $C$  is given by lemma 4.4. For  $\varepsilon$  small enough such that  $\varepsilon < 1/C^4$ , for all  $q \in \mathbb{Z}$ :

$$\begin{aligned} |P_q| &\leq A\varepsilon^{1+\theta} \sum_{n=2}^M (C\varepsilon^\theta)^n \\ &\leq d_1 \varepsilon^{1+\theta} = \left( d_1^{\frac{1}{1+\theta}} \varepsilon \right)^{1+\theta}, \end{aligned} \quad (26)$$

where  $d_1 > 0$  is independent of  $M$ .

For  $i$ ), using this new bound as hypothesis for lemma 4.4 (replacing  $\varepsilon$  by  $d_1 \varepsilon^{1+\theta}$ ), gives for all  $q \in \mathbb{Z}$ :

$$\forall n \geq 2, \quad |S_q(1, n, P)| \leq (C d_1^\theta \varepsilon^{\theta^2} \varepsilon^\theta)^n d_1^{1+\theta} \varepsilon^{(1+\theta)^2}. \quad (27)$$

If  $\varepsilon < 1/(d_1^{1/\theta})$ , a new, stronger, bound on  $P_q$  is obtained for all  $q$ , i.e.:

$$|P_q| \leq d_1 d_1^{1+\theta} \varepsilon^{(1+\theta)^2} = \left( d_1^{\frac{1}{1+\theta} + \frac{1}{(1+\theta)^2}} \varepsilon \right)^{(1+\theta)^2}. \quad (28)$$

Let us show that doing this reasoning again, an infinite number of times, will lead to  $P_q = 0$ . Let  $u_i$  be the series :

$$u_i = \frac{1}{1+\theta} \sum_{k=0}^i \frac{1}{(1+\theta)^k}, \quad i \in \mathbb{N}.$$

The series  $u_i$  converges to a finite limit  $D_\theta$  when  $i$  tends to infinity, since  $\theta > 0$ . Let us introduce for all  $i \in \mathbb{N}^*$  the real value  $\eta_i = (1+\theta)^i u_i / ((1+\theta)^i - 1)$ . As a consequence, there exists a  $D$  such that for all  $i \in \mathbb{N}^*$  one has  $d_1^{\eta_i} \leq D$ . Suppose  $\varepsilon \leq 1/D$  and

$$|P_q| \leq (d_1^{u_i} \varepsilon)^{(1+\theta)^i}, \quad (29)$$

then

$$\forall q \in \mathbb{Z}, \forall n \geq 2, \quad |S_q(1, n, P)| \leq \left( C d_1^{\theta(1+\theta)^i u_i} \varepsilon^{\theta(1+\theta)^i} \right)^n (d_1^{u_i} \varepsilon)^{(1+\theta)^{i+1}}. \quad (30)$$

Since  $\varepsilon \leq 1/D$ , one has  $d_1^{\theta(1+\theta)^i u_i} \varepsilon^{\theta(1+\theta)^i - \theta} \leq 1$  and the term inside the first parenthesis is lower than  $C\varepsilon^\theta$ . One can write:

$$|P_q| \leq d_1 (d_1^{u_i} \varepsilon)^{(1+\theta)^{i+1}} = (d_1^{u_{i+1}} \varepsilon)^{(1+\theta)^{i+1}}. \quad (31)$$

In consequence, for  $\varepsilon$  small enough such that  $\varepsilon < 1/D$  and  $\varepsilon < 1/C^{1/\theta}$ , the bound converges to zero when  $i$  tends to infinity. Hence the only possible solution of (13) is  $P_q = 0$  for all  $q \in \mathbb{Z}$ .

For  $ii$ ), let  $s$  be the set  $s = \{e, -e, q-e, q+e\}$ , where  $e$  is the integer defined in **H2** and  $q \in \mathbb{Z}$ . Here  $J_\gamma$  is not invertible for all  $\gamma \in \Omega$ . One can replace  $J_\gamma^{-1}$  by  $(QJ_\gamma Q)^{-1}$  which is bounded and estimates (24) and (26) hold here, for all  $q \neq \pm e$ :

$$|P_q| \leq d_1 \varepsilon^{1+\theta}. \quad (32)$$

The function  $S(1, 2, P)$  at  $q \in \mathbb{Z}$  can be written as:

$$S_q(1, 2, P) = 2P_{q-e}P_e + 2P_{q+e}P_{-e} + \sum_{\substack{i \in \mathbb{Z} \\ i \notin s}} P_{q-i}P_i. \quad (33)$$

Using (32), the first two terms are bounded by a quantity proportional to  $\varepsilon^{2+\theta}$  in the case where  $q \neq \pm 2e$  and  $q \neq 0$  and  $\varepsilon^2$  otherwise. The sum over  $n$  contains only terms bounded by  $d_1 \varepsilon^{1+\theta}$  so has a bound

proportional to  $\varepsilon$  to the power  $(1 + \theta)^2$ , consequence of lemma 4.4 (recall that  $\theta = 1/4$ ). Similarly, for  $q \in \mathbb{Z}$  and  $n \geq 3$ :

$$\begin{aligned} S_q(1, n, P) = (P * S(1, n - 1, P))(q) = & P_e S_{q-e}(1, n - 1, P) + P_{-e} S_{q+e}(1, n - 1, P) + \\ & + P_{q-e} S_e(1, n - 1, P) + P_{q+e} S_{-e}(1, n - 1, P) + \\ & + \sum_{\substack{i \in \mathbb{Z} \\ i \neq s}} P_{q-i} S_i(1, n - 1, P). \end{aligned} \quad (34)$$

This is just (33) where the  $P_i$  have been replaced by  $S_i(1, n - 1, P)$ . So, similarly, using lemma (4.4), for  $\varepsilon$  small enough, it implies that there exists  $C > 0$  such that:

$$\forall n \geq 2, q \in \mathbb{Z}, q \neq e, |S_q(1, n, P)| \leq C\varepsilon^{(1+\theta)^2}. \quad (35)$$

The quantity  $QJQ$  is invertible, from **H2**, then for all  $q \neq e$  one has the estimate (24) and there exists a constant  $D > 0$ , such that, from (35):

$$|P_q| \leq D\varepsilon^{(1+\theta)^2}. \quad (36)$$

Similarly to the proof of *i*), this procedure can be repeated several times and gives each time a larger power of  $\varepsilon$ , until the first two terms of (33) and four terms of (34) become the largest terms. This implies that for all  $q \in \mathbb{Z}$ ,  $q \neq e$ ,  $|P_q| \leq |P_e|$ . If  $P_e = 0$ , then no oscillation exists. If  $P_e \neq 0$ , the largest term  $P_e$  can be chosen to be  $|P_e| = \varepsilon$  and there exists  $D_2 > 0$  such that for all  $q \in \mathbb{Z}$ ,  $q \neq e$

$$|P_q| \leq D_2\varepsilon^2. \quad (37)$$

The above estimate is given and limited by the case  $q = 2e$ .

For *iii*), let us define the set  $s_e = \{me\}_{m \in \mathbb{Z}^*}$ . If  $\beta \notin s_e$ , then for each  $n \in \mathbb{Z}$  fixed, at least one of the quantities  $(\beta - n)$  or  $n$  does not belong to  $s_e$ . Suppose for all  $n \notin s_e$ ,

$$|P_n| \leq \varepsilon^a, \quad a \geq 1. \quad (38)$$

The first two terms of (33) are bounded by  $\varepsilon^{a+1}$  and from *i*) of lemma 4.3, one can write:

$$\left| \sum_{\substack{i \in \mathbb{Z} \\ i \neq s}} P_{\beta-i} P_i \right| \leq c_1 \varepsilon^m, \quad m = a + \frac{1}{4}.$$

It is similar for (34). Following the procedure used in the proof of *ii*), one will find that  $a$  is increasing at each step. Eventually,  $a$  tends to infinity and implies for  $\varepsilon$  small enough:

$$P_\beta = 0.$$

□

*Proof of Theorem 4.1.* The proof is a direct application of the results given in lemma 4.5. Notice first that in order to have a solution satisfying **H1**, the point *i*) of this lemma requires that  $J_\gamma$  possesses at least one zero eigenvalue at  $\gamma_0 \in \Omega$ , showing the necessity of a hypothesis such as **H2**. As a consequence, the bifurcation point is the point for which an eigenvalue of  $J_\gamma$  reaches zero. The point *ii*) of the same lemma states that the Fourier series of the appearing oscillation must include a non zero value for  $P_e$ , where  $e$  is given in **H2**. The point *iii*) implies that  $P_e$  is the term of the Fourier series associated to the frequency of oscillation, since only harmonics associated with frequencies multiple of  $e$  can be different from zero. Eventually, the bound on the Fourier components given in *ii*) of lemma 4.5 finishes the proof. □

## 5 Decrease of the harmonics

This section concerns the estimation of the amplitude of the Fourier components of the solution. It is dedicated to theorem 5.1 and its proof.

Before stating the result, notice that the index  $e$  of **H2** can be chosen to be  $e = 1$  without loss of generality. This is justified by *ii*) and *iii*) of lemma 4.5 which asserts that the fundamental frequency is given by  $\omega_e$  and there exists only harmonics associated to multiple of this frequency.

**Theorem 5.1.** *Under the hypothesis of Theorem 4.1, for all  $\gamma \in \Omega$  where  $|P_1|$  is small enough, there exists  $k = k(|P_1|) \geq 2$  such that the oscillating solution of the system (13) possesses the following properties: for all  $q \in \mathbb{Z}^*$ ,*

$$\begin{aligned} |P_q| &\leq c_q |P_1|^{|q|}, \quad \text{for all } |q| \leq k \\ |P_q| &\leq c_k |P_1|^k, \quad \text{for all } |q| \geq k, \end{aligned} \quad (39)$$

where  $c_q, c_k$  are positive constants. The rank  $k$  increases when  $|P_1|$  decreases and tends to infinity when  $|P_1|$  tends to zero.

*Remark 5.2.* If some of the transfer functions  $\{\widehat{Z}_{i,j}\}$  change sign or decrease fast enough when their variable tends to infinity, the decrease of the harmonics amplitude with frequency may be faster than the estimation given in the above theorem.

*Proof of theorem 5.1.* The reasoning of the proof is similar to the one used in the proof of lemma 4.5. The operator  $Q = (1 - Q_1 - Q_{-1})$  commute with the Fourier transform since  $Q_1, Q_{-1}$  are associated to vectors of the Fourier basis. Then the operator  $Q\widehat{J}_\gamma Q$  has a bounded inverse for all  $\gamma \in \Omega$ . Since **H3** holds, the functions  $\{g_n\}$  are bounded. Let us write  $\tilde{g}_n$  for the supremum of  $|g_n|$ . Equation (13) gives for all  $q \in \mathbb{Z} \setminus \{1, -1\}$ ,  $M \geq 2$ :

$$\begin{aligned} |P_q| &\leq \sup_k |(Q\widehat{J}_\gamma Q)^{-1}(k)| \times \sup_{n \in [2, M]} \tilde{g}_n \times \left| \sum_{n=2}^M S_q(1, n, P) \right| + \\ &+ \sup_k |(Q\widehat{J}_\gamma Q)^{-1}(k)| \times \left| \sum_{n=M}^{\infty} S_q(g_n, n, P) \right|. \end{aligned} \quad (40)$$

From **H3** and lemma 3.1, the second term of the previous expression tends to zero as  $M$  tends to infinity. Let us choose  $M$  large enough such that

$$\left| \sum_{n=M}^{\infty} S_q(g_n, n, P) \right| \leq \sup_{n \in [2, M]} \tilde{g}_n \times \left| \sum_{n=2}^M S_q(1, n, P) \right|,$$

then there exists a constant  $A > 0$  such that

$$|P_q| \leq A \times \left| \sum_{n=2}^M S_q(1, n, P) \right| \quad \forall q \neq \pm 1. \quad (41)$$

Let  $s$  be the set  $s = \{0, 1, -1, 2, -2, q-1, q+1, q-2, q+2\}$ . The function  $S(1, 2, P)$  at  $q \in \mathbb{Z}$  can be written as:

$$S_q(1, 2, P) = 2P_{q-1}P_1 + 2P_{q+1}P_{-1} + 2P_{q-2}P_2 + 2P_{q+2}P_{-2} + \sum_{\substack{n \in \mathbb{Z} \\ n \notin s}} P_{q-n}P_n. \quad (42)$$

From Th. 4.1, for all  $|q| \geq 3$  the first four terms of (42) are bounded by a quantity proportional to  $\varepsilon^3$ . Likewise, but for all  $q \in \mathbb{Z}$ , each term of the sum over  $n$  contains two quantities, each one is bounded by  $c_0\varepsilon^2$  for some  $c_0 > 0$ , then lemma 4.3 gives the estimate:

$$\sum_{\substack{n \in \mathbb{Z} \\ n \notin s}} |P_{q-n}P_n| \leq c_1 c_0^2 \varepsilon^{2+\theta}, \quad \text{with } c_1 > 0, \quad (43)$$

with  $\theta$  chosen such that  $\theta \in (0, 1/4)$ . As a consequence, for  $|q| \geq 3$ , there exists a constant  $c_{2,1}$  such that  $|S_q(1, 2, P)| \leq c_{2,1}\varepsilon^{2+\theta}$ . In the case where  $q = 0$ ,  $q = \pm 1$  and  $q = \pm 2$ , some of the first terms of (42) are bounded by  $\varepsilon^2$ , so for  $q \in \{0, \pm 1, \pm 2\}$ , there exists  $c_{2,2} > 0$  such that  $|S_q(1, 2, P)| \leq c_{2,2}\varepsilon^2$ . Let us now give a bound on the  $S(1, n, P)$  by induction. Assume that at rank  $n-1 \geq 2$  the following statements are true:

- for  $q \in \{0, \pm 1, \pm 2\}$ ,  $|S_q(1, n-1, P)| \leq C_1\varepsilon^2$ ,

- for  $q$ ,  $|q| \geq 3$ ,  $|S_q(1, n-1, P)| \leq C_2 \varepsilon^{2+\theta}$ ,

for some  $C_1, C_2 > 0$  and  $\theta \in (0, 1/4)$ . Then at rank  $n$ , for  $q \in \mathbb{Z}$  :

$$S_q(1, n, P) = (P * S(1, n-1, P))(q) = U_q^{n-1} + \sum_{\substack{i \in \mathbb{Z} \\ i \notin s}} P_{q-i} S_i(1, n-1, P). \quad (44)$$

where, for  $s_2 = \{0, \pm 1, \pm 2\}$ ,

$$U_q^{n-1} = \sum_{k \in s_2} P_k S_{q-k}(1, n-1, P) + P_{q-k} S_k(1, n-1, P).$$

Firstly, this latter expression is bounded by a quantity proportional to  $\varepsilon^3$  for all  $q$ . Secondly, for all  $q \in \mathbb{Z}$  the sum over  $i$  of (44) is bounded from lemma 4.3:

$$\sum_{\substack{i \in \mathbb{Z} \\ i \notin s}} \left| \frac{P_{q-i} S_i(1, n-1, P)}{c_0 C_2} \right| \leq c_1 \varepsilon^{2+\theta+\tilde{\theta}},$$

where  $\tilde{\theta}$  can take value up to  $1/2$  so that it can be chosen to be twice larger than  $\theta$ . Let us choose

$$\sum_{\substack{i \in \mathbb{Z} \\ i \notin s}} |P_{q-i} S_i(1, n-1, P)| \leq c_1 c_0 C_2 \varepsilon^{2\theta} \varepsilon^{2+\theta}.$$

For  $\varepsilon$  small enough,  $c_1 c_0 C_2 \varepsilon^{2\theta} \leq C_2$  and by induction, it implies:

$$\forall n \geq 2, q \in \mathbb{Z}, q \notin s_2, |S_q(1, n, P)| \leq C_2 \varepsilon^{2+\theta},$$

and (41) implies that there exists a constant  $D > 0$ , such that for all  $|q| \geq 3$ :

$$|P_q| \leq D \varepsilon^{2+\theta}. \quad (45)$$

Similarly to the proof of *ii*) of lemma 4.5, this procedure can be repeated several times and gives each time a larger power of  $\varepsilon$ , until the first terms of (42) and some of the terms of  $U_q^n$  become the largest terms. The bound on these latter terms is here proportional to  $\varepsilon^3$ . In conclusion, there exist  $D_2, D_3 > 0$  such that one has:

$$\begin{aligned} \text{for } q \in s_2, \quad |P_q| &\leq D_2 \varepsilon^2, \\ \text{for } q \notin s_2, \quad |P_q| &\leq D_3 \varepsilon^3. \end{aligned} \quad (46)$$

For the general case, suppose there exists a rank  $r \geq 3$  and a function  $f : \mathbb{Z} \rightarrow \mathbb{R}_+$  such that :

- for all  $q$ ,  $|q| \leq r$ ,  $|P_q| \leq f(|q|) \varepsilon^{|q|}$ ,
- for all  $q$ ,  $|q| \geq r$ ,  $|P_q| \leq f(r) \varepsilon^r$ .

Let us show it is true at rank  $r+1$  under appropriate conditions. The steps are the same as for the case  $r=3$  treated above. In the following, only the case  $q$  positive is treated as the case  $q$  negative will lead to the same estimates. For  $q \in [2, r+1]$ , let us introduce the set  $s_q = \mathbb{Z} \cap [1, q-1]$  and the function  $\tilde{f}$  defined by :

$$\forall q \geq 2, \quad \tilde{f}(q) = \sum_{i \in s_q} f(q-i) f(i), \quad \tilde{f}(1) = 1. \quad (47)$$

Notice that the function  $\tilde{f}$  is monotone increasing on  $\mathbb{N}$ . Let us introduce for  $q \geq 2$  and  $n \geq 3$

$$U_q^2 = \sum_{i \in s_q} P_{q-i} P_i, \quad U_q^n = \sum_{i \in s_q} P_{q-i} S_i(1, n-1, P).$$

Instead of (42), one has to estimate the quantity:

$$S_q(1, 2, P) = U_q^2 + \sum_{i \notin s_q} P_{q-i} P_i. \quad (48)$$

For  $q \leq r$ , the quantity  $U_q^2$  is bounded by:

$$|U_q^2| \leq \varepsilon^q \sum_{i \in s_q} f(q-i)f(i) = \tilde{f}(q)\varepsilon^q,$$

and the sum over  $i$  of (48) is bounded by  $c_1 f(r)^2 \varepsilon^{q+\theta}$  from lemma 4.3. Hence, for  $\varepsilon$  small enough, satisfying the condition

$$c_1 f(r)^2 \varepsilon^\theta < 1, \quad (49)$$

there exists  $d_{2,1} > 0$  such that:

$$\forall q \leq r, \quad |S_q(1, 2, P)| \leq d_{2,1} \tilde{f}(q) \varepsilon^q. \quad (50)$$

For  $q \geq r+1$ , let us introduce the set  $t_r = \mathbb{Z} \cap [-r, r]$  and for  $n \geq 3$

$$V_q^2 = \sum_{i \in t_r} P_{q-i} P_i, \quad V_q^n = \sum_{i \in t_r} P_{q-i} S_i(1, n-1, P).$$

Instead of (42), one has to estimate the quantity:

$$S_q(1, 2, P) = V_q^2 + \sum_{i \notin t_r} P_{q-i} P_i. \quad (51)$$

The quantity  $V_q^2$  contains a small number of terms bounded by  $\varepsilon^{r+1}$ , this number is independent of  $q$ , and the others are bounded by  $\varepsilon^{r+2}$  or lower bounds. So for  $\varepsilon$  small enough, there exists a  $d_0 > 0$  such that:

$$|V_q^2| \leq d_0 f(r) \varepsilon^{r+1}.$$

For the sum over  $i$  of (51), each term contains two quantities where at least one is bounded by  $f(r) \varepsilon^r$ . Lemma 4.3 gives:

$$\left| \sum_{i \notin t_q} P_{q-i} P_i \right| \leq c_1 f(r)^2 \varepsilon^{r+\theta}.$$

Let us introduce

$$f_m(r+1) = \max\{f^2(r), \tilde{f}(r+1)\},$$

then there exists  $d_{2,2} > 0$  such that for all  $q \geq r+1$ :

$$|S_q(1, 2, P)| \leq d_{2,2} f_m(r+1) \varepsilon^{r+\theta}. \quad (52)$$

Next, one has to treat the case of multiple convolutions. Similarly to the case  $r=3$ , suppose at rank  $n-1 \geq 2$  there exist a function  $h_{n-1} : \mathbb{Z} \rightarrow \mathbb{R}_+$  such that:

- for  $q \in [-r, r]$ ,  $|S_q(1, n-1, P)| \leq d_{n-1,1} h_{n-1}(|q|) \varepsilon^{|q|}$ ,
- for  $q$ ,  $|q| \geq r+1$ ,  $|S_q(1, n-1, P)| \leq d_{n-1,2} h_{n-1}(r+1) \varepsilon^{r+\theta}$ ,

for some  $d_{n-1,1}, d_{n-1,2} > 0$  and  $\theta \in (0, 1/4)$ . Then, at rank  $n$ , for  $|q| \leq r$ :

$$S_q(1, n, P) = (P * S(1, n-1, P))(q) = U_q^{n-1} + \sum_{\substack{i \in \mathbb{Z} \\ i \notin s_q}} P_{q-i} S_i(1, n-1, P), \quad (53)$$

and for  $|q| \geq r+1$ :

$$S_q(1, n, P) = (P * S(1, n-1, P))(q) = V_q^{n-1} + \sum_{\substack{i \in \mathbb{Z} \\ i \notin t_r}} P_{q-i} S_i(1, n-1, P). \quad (54)$$

It is similar to the previous case, where  $n=2$ , provided  $\varepsilon$  is small enough. More precisely, it is required that for all  $n \in [3, M+1]$ , one has  $\varepsilon$  such that

$$h_{n-1}(r+1) \varepsilon < 1.$$

Then the quantity  $U_q^n$  is bounded by

$$|U_q^n| \leq h_n(q)\varepsilon^q, \quad \text{where} \quad h_n(q) = \sum_{i \in s_q} f(q-i)h_{n-1}(i),$$

The term  $V_q^n$  is bounded by a quantity proportional to  $h(r)\varepsilon^{r+1}$  for  $\varepsilon$  small enough, the sum over  $i$  of (53) (resp. (54)) can be shown to be bounded by a quantity proportional to  $\varepsilon^{|q|+\theta}$  (resp.  $\varepsilon^{r+\theta}$ ). This finishes the estimates on the multiple convolutions. The last step of the proof is to use (41) and proceed like in the case  $r = 2$  treated in the beginning of the proof. Then the function  $f$  will have to satisfy

$$f(q) \geq d_2 f_m(\min\{q, r\}) + \sum_{i=3}^M d_i h_i(\min\{q, r\}),$$

with some  $d_i > 0$  for all  $i \geq 2$ . Eventually,  $\varepsilon$  must be small enough such that  $f(q)\varepsilon < 1$ . But  $f$  is increasing with  $q$  and for  $\varepsilon$  fixed, there exists a rank  $k = k(\varepsilon)$  where the above requirement is not satisfied. Eventually, Th. 4.1 gives  $|P_1| = \varepsilon$  and this concludes the proof.  $\square$

## 6 Bifurcation point and stability

In this section, the regime of the system around the bifurcation point is investigated. The direction of the bifurcation, the amplitude of the first Fourier component and the influence of  $\gamma$  on the oscillation frequency are given. The results are similar to the ones arising in the finite dimensional Hopf bifurcation. As for the Hopf theorem, an additional hypothesis on the system is required, i.e. the transversality condition stated in **H4**.

Without loss of generality, as explained in the previous part, the index  $e$  of **H2** is assumed to be  $e = 1$  in the following. Suppose **H2** holds, then for  $\gamma \in \Omega$ ,  $\beta_2(\gamma)$ , the eigenvalue of  $J_\gamma$  associated to  $\omega_2$ , is not null. Assume **H3** and let us introduce the notation:

$$D_1(\gamma) = [\beta_2(\gamma)]^{-1} g_2(2, 1) (g_2(1, 2) + g_2(1, -1)) - (g_3(1, 0, 1) + g_3(1, 0, -1) + g_3(1, 2, 1)). \quad (55)$$

Let us denote by  $\mathcal{Re}[z]$  the real part and  $\mathcal{Im}[z]$  the imaginary part of the complex number  $z$ . In this section, the following additional hypothesis is assumed:

**H4** There exists an open set  $\Gamma$  with  $\gamma_0 \in \Gamma$  such that the complex-valued functions  $D_1$  and  $\beta_1$  are respectively continuously differentiable and twice continuously differentiable with respect to  $\gamma$  on  $\Gamma$ . Moreover,  $D_1(\gamma_0) \neq 0$  and

$$\mathcal{Re}\left[\frac{1}{D_1(\gamma_0)} \frac{d\beta_1}{d\gamma}(\gamma_0)\right] \neq 0.$$

**Theorem 6.1.** *Suppose **H1**, **H2**, **H3**, **H4** hold. Then,*

i) *The quantity*

$$\alpha = \frac{1}{D_1(\gamma_0)} \frac{d\beta_1}{d\gamma}(\gamma_0), \quad (56)$$

*is a real number, different from zero. If its sign is positive (resp. negative) the bifurcation is direct (resp. inverse), i.e. the oscillating solution described in **H1** exists for  $\gamma > \gamma_0$  (resp.  $\gamma < \gamma_0$ ).*

ii) *for all  $\gamma \in \Gamma$  such that  $|\gamma - \gamma_0|$  is small enough, the modulus of the first Fourier component of the oscillating solution is given by:*

$$|P_1| = \sqrt{|\alpha|} \sqrt{|\gamma - \gamma_0|} + \mathcal{O}(|\gamma - \gamma_0|^{3/2}). \quad (57)$$

*Proof.* For  $\varepsilon$  small enough, from Th. 5.1 and lemma 4.3, one can write:

$$\begin{aligned} S(g_2, 2, 1) &= (g_2(1, 2) + g_2(1, -1)) P_{-1} P_2 + \mathcal{O}(\varepsilon^4), \\ S(g_3, 3, 1) &= (g_3(1, 0, 1) + g_3(1, 0, -1) + g_3(1, 2, 1)) |P_1|^2 P_1 + \mathcal{O}(\varepsilon^4), \\ \forall n \geq 4, \quad |S(g_n, n, 1)| &\leq c_1 \varepsilon^4, \quad \text{with } c_1 > 0. \end{aligned}$$



One has also:

$$\begin{aligned} S(g_2, 2, 2) &= g_2(2, 1)P_{-1}P_1 + \mathcal{O}(\varepsilon^4), \\ \forall n \geq 3, \quad |S(g_n, n, 2)| &\leq c_2\varepsilon^4, \\ \forall n \geq 2, q \geq 3, \quad |S(g_n, n, q)| &\leq c_3\varepsilon^4. \end{aligned}$$

Then it yields for  $P_2$ :

$$\begin{aligned} J_\gamma P_2 &= \beta_2(\gamma)P_2 = \sum_{i=2}^{\infty} S(g_i, i, 2) = g_2(2, 1)P_{-1}P_1 + \mathcal{O}(\varepsilon^4) \\ P_2 &= [\beta_2(\gamma)]^{-1}g_2(2, 1)P_{-1}P_1 + \mathcal{O}(\varepsilon^4), \end{aligned}$$

For  $P_1$ :

$$\begin{aligned} J_\gamma P_1 &= \sum_{i=2}^{\infty} S(g_i, i, 1) \\ &= (g_2(1, 2) + g_2(1, -1))P_{-1}P_2 + (g_3(1, 0, 1) + g_3(1, 0, -1) + g_3(1, 2, 1))|P_1|^2P_1 + \mathcal{O}(\varepsilon^4), \end{aligned}$$

which yields

$$P_1 \{ \beta_1(\gamma) - D_1(\gamma)|P_1|^2 \} + \mathcal{O}(\varepsilon^4) = 0,$$

where  $D_1$  is the differentiable function defined in (55). For the oscillating solution, where  $P_1 \neq 0$ , one can write

$$|P_1|^2 D_1(\gamma) = \beta_1(\gamma) + \mathcal{O}(\varepsilon^4),$$

From the hypotheses of **H4** and **H2**, one has around  $\gamma_0$  the following Taylor expansions:

$$D_1(\gamma) = D_1(\gamma_0) + \mathcal{O}(|\gamma - \gamma_0|), \quad \beta_1(\gamma) = \beta_1'(\gamma_0)(\gamma - \gamma_0) + \mathcal{O}(\gamma - \gamma_0)^2.$$

and

$$|P_1|^2 = \frac{\beta_1'(\gamma_0)}{D_1(\gamma_0)}(\gamma - \gamma_0) + \mathcal{O}(\gamma - \gamma_0)^2 + \mathcal{O}(\varepsilon^4). \quad (58)$$

For  $\varepsilon$  small enough, one has  $|P_1| = \varepsilon$  from Th 4.1 and since the numerator is not null from **H4**, equation (58) shows that  $(\gamma - \gamma_0)$  is proportional to  $\varepsilon^2$ . Moreover, the left hand side of (58) is a positive quantity, implying:

$$\begin{cases} \mathcal{R}e[\alpha](\gamma - \gamma_0) > 0 \\ \mathcal{I}m[\alpha] = 0 \end{cases}. \quad (59)$$

Hence the sign of  $\mathcal{R}e[\alpha]$  determines the direction of the bifurcation (direct or inverse).  $\square$

The last result of this section concerns the influence of  $\gamma$  on the oscillation frequency,  $\omega_1$ , around the threshold. It requires two further hypotheses, one on the linear part of the system and one on the coefficients in front of the quadratic and cubic terms, contained in the function  $D_1$ . The proof is in fact an application of the implicit function theorem. For the next result,  $\beta_1$ ,  $D_1$  shall be written as functions of two variables:  $\beta(\omega_1, \gamma) = \beta_1(\gamma)$ ,  $D(\omega_1, \gamma) = D_1(\gamma)$ .

**Corollary 6.2.** *Under the hypotheses of Th. (6.1), suppose the function  $\mathcal{I}m[D\beta]$  is twice continuously differentiable with respect to its variable  $\omega_1$  and  $\gamma$  in an open set around the point  $(\omega_1, \gamma_0)$ , and*

$$\left. \frac{\partial}{\partial \omega_1} \mathcal{I}m[D\beta_1] \right|_{(\omega_1, \gamma_0)} \neq 0, \quad \left. \frac{\partial^2}{\partial \omega_1^2} \mathcal{I}m[D\beta_1] \right|_{(\omega_1, \gamma_0)} \neq 0.$$

*Suppose further that  $\beta$  is differentiable with respect to both variables  $\omega_1$  and  $\gamma$  at  $(\omega_1, \gamma_0)$ . Then the expression of the angular frequency  $\omega_1$  for  $\gamma$  in a neighbourhood of  $\gamma_0$  is given by:*

$$\omega_1(\gamma) = \omega_1(\gamma_0) + \omega_1'(\gamma_0) \cdot (\gamma - \gamma_0) + \mathcal{O}(\gamma - \gamma_0)^2, \quad (60)$$

where  $\omega_1(\gamma_0)$  is the solution of  $\mathcal{I}m[\beta(\omega_1(\gamma_0), \gamma_0)] = 0$  and

$$\omega_1'(\gamma_0) = -\frac{\mathcal{I}m\left[D\frac{\partial\beta}{\partial\gamma}\right](\omega_1(\gamma_0), \gamma_0)}{\mathcal{I}m\left[D\frac{\partial\beta}{\partial\omega}\right](\omega_1(\gamma_0), \gamma_0)}.$$

*Proof.* The Taylor expansion in Eq. (60) is a consequence of the implicit function theorem applied to the twice differentiable function  $\mathcal{I}m[D\beta]$  around  $\gamma_0$ . The value of  $\omega_1(\gamma_0)$  is given by Th. 4.1. The hypothesis **H2 2)** and the point  $i$ ) of Th. 6.1, imply that

$$\mathcal{I}m\left[\frac{dD_1\beta_1}{d\gamma}\right](\gamma_0) = \mathcal{I}m\left[D_1(\gamma_0)\frac{d\beta_1}{d\gamma}(\gamma_0)\right] = 0. \quad (61)$$

The implicit function theorem and the hypotheses of differentiability allow to write

$$\mathcal{I}m\left[D\frac{d\beta_1}{d\gamma}\right](\gamma_0) = \frac{d\omega}{d\gamma}(\gamma_0) \cdot \mathcal{I}m\left[D\frac{\partial\beta}{\partial\omega}\right](\omega_1(\gamma_0), \gamma_0) + \mathcal{I}m\left[D\frac{\partial\beta}{\partial\gamma}\right](\omega_1(\gamma_0), \gamma_0).$$

Putting this latter expression in Eq. (61) concludes the proof.  $\square$

## 7 Example of the woodwind musical instruments

A reed woodwind musical instrument, such as a clarinet or a saxophone, is an example of a system which can be described by an hyperbolic PDE and a localized polynomial nonlinearity. In acoustics, the major challenge is to understand the influence on the produced sound of physical parameters such as the pressure in the mouth of the musician, the stiffness of the reed, the shape of the mouthpiece and of the resonator. Assuming it can be done, a first order calculation by linearizing the system gives the bifurcation point and the frequency of oscillation at this point. But the investigation of these systems requires to go further and to obtain the shape of the Fourier harmonics with respect to the bifurcation parameter. That is why the frequency domain approach is perfectly appropriate here. For small oscillations, one can obtain analytical formulae giving information on the solution of the system, but, as said in the introduction this requires until now a hypothesis on the Fourier series of the solution. Let  $p_0 \in L^2(T)$  be the acoustic pressure inside the mouthpiece of the wind instrument under study and let us suppose it is oscillating. The hypothesis made by acousticians, see e.g. [4], [22], [16], [17], is that the  $n$ -th harmonic of this pressure obeys:

$$|P_n| \leq |P_1|^{|n|}, \quad n \neq 0. \quad (62)$$

This hypothesis is now going to be proved in the framework of the mathematical results obtained in the previous sections.

The equations describing this system are detailed in e.g. [1], [2]. Although the instrument possesses a three dimensional shape, the model is one dimensional and  $X = [0, \ell]$  where  $\ell$  is the length of the instrument. The dimensionless variables associated to the system are the acoustic pressure,  $p$ , and the volume flow,  $u$ , inside the instrument, both belonging to  $\mathcal{X}$ , and  $h \in L^2(T)$  the reed tip opening localized  $x = 0$ . The pressure inside the mouth of the instrumentalist  $\gamma$  is the bifurcation parameter. Let us call  $p_0$  and  $u_0$  the pressure and volume flow at  $x = 0$ . The acoustic pressure depends on the volume flow entering the mouthpiece through the relation:

$$Ap = \delta(x)Bu, \quad (63)$$

where  $B = \partial/\partial t$  and  $A$  is the differential operator associated to the wave equation inside the instrument, with losses at the boundaries and along the resonator (visco-thermal losses). By using the Green functions formalism, it can be written as

$$p(t, x) = \int_{T \times X} g(t - \tau, x - y)Bu(\tau, y)\delta(y)dyd\tau = \int_T g(t - \tau, x)Bu(\tau, 0)d\tau, \quad (64)$$

where  $g$  is the Green function solution of  $Ap = \delta(x)\delta(t)$ . See e.g. [2], Chapter II.5, section 1.2.1 for an explicit expression of this Green function in the case of the clarinet. Let  $U = \mathcal{F}u_0$ , the above equation expressed in the Fourier space, and at  $x = 0$ , yields:

$$P = \widehat{Z}U, \quad (65)$$

where  $\widehat{Z}$  is called a transfer function. This is the equation (4) in the case of wind instruments, in the frequency domain. Indeed,  $U$  is expressed in terms of  $P$  through a nonlinear relation in the following. Notice that the quantity  $\widehat{Z}$  is called by acousticians the input impedance of the bore and can be measured thanks to physical experiments. It is bounded, provided losses are taken into account in the model. See [2], chapter II.7, for examples of impedances of various wind instruments. The Fourier basis is the eigenbasis of this operator, thus in the Fourier space,  $\widehat{Z}$  is a multiplication operator. For all  $q \in \mathbb{Z}$ ,

$$P_q = \widehat{Z}(\omega_q)U_q, \quad (66)$$

where the complex-valued function  $\widehat{Z}(\cdot)$  is never null except maybe at  $\omega_0 = 0$ .

The nonlinear relationship is localized at the tip of the mouthpiece and relates  $u_0$ ,  $p_0$  and  $h$ . Under some physical hypotheses, the Bernoulli law at the entry of the instrument gives the nonlinear relationship at time  $t \in \mathbb{R}$ :

$$u_0^2(t) = \zeta^2 (1 - \gamma + h(t))^2 (\gamma - p_0(t)), \quad (67)$$

where  $\zeta$  is a positive constant,  $0 < \gamma < 1$ . At rest, when there is no action from the instrumentalist,  $\gamma = 0$ , and the equilibrium position of the instrument yields  $u_0 = 0$ ,  $p_0 = 0$  and  $h = 0$ . It is supposed that for all  $t$ ,  $p_0(t)$  and  $h(t)$  are small enough such that  $\gamma - p_0(t) \geq 0$  and  $1 - \gamma + h(t) \geq 0$  (small oscillations).

Let  $H$  denote the Fourier transform of  $h$ . The reed opening is related to  $p_0$  via a linear relation expressed in the frequency domain:

$$H_q = \widehat{D}(q)P_q, \quad \text{for all } q \in \mathbb{Z}. \quad (68)$$

The complex-valued function  $\widehat{D}$  is bounded. Let us introduce  $Q_0$  the orthogonal projection associated to the constant part of the Fourier decomposition. From (66), one has

$$U_q = \widehat{Y}(q)(1 - Q_0)P_q + U_0Q_0, \quad \text{for all } q \in \mathbb{Z}. \quad (69)$$

In the physical system under study, the function  $\widehat{Y}(1 - Q_0) : \mathbb{Z} \rightarrow \mathbb{C}$  is bounded. Nevertheless, the value  $\widehat{Y}(0)$  is not defined as it is assumed that  $P_0 = 0$  and  $U_0 > 0$ . It gives a good opportunity to show how to cope with this problem and satisfy the hypothesis of boundedness of the operators despite it: in this case the system will be written as a set of two equations, with a specific one for  $U_0$ , see below how to obtain (72a) and (72b). Behind the function  $\widehat{Y}$  is hidden the hyperbolicity given by the wave equation inside the resonator and the contribution of  $B$ . It reveals itself when looking at the limit of  $|\widehat{Y}(q)|$  when  $q$  tends to infinity: it tends to a strictly positive constant.

Let us assume the pressure  $p_0$  satisfies hypothesis **H1**. Let us recall that since  $\widehat{D}$  is bounded, one has for example, for all  $q \in \mathbb{Z}$ ,

$$\widehat{hp}(q) = (H * P)(q) = ((\widehat{D}P) * P)(q) = \sum_{n \in \mathbb{Z}} \widehat{D}_{q-n}P_{q-n}P_n,$$

as shown by lemma 3.1. If, like in [17], the following notations are introduced:

$$\begin{aligned} u_{00} &= \zeta^2 \gamma (1 - \gamma)^2, & A_q &= 2\zeta^2 \gamma (1 - \gamma) \widehat{D}(q) - \zeta^2 (1 - \gamma)^2, \\ \mathbf{B}_{q,n} &= \zeta^2 \gamma \widehat{D}(q - n) \widehat{D}(n) - 2\zeta^2 (1 - \gamma) \widehat{D}(q - n), \\ C_{q,n,m} &= -\zeta^2 \widehat{D}(q - n) \widehat{D}(n - m), \end{aligned} \quad (70)$$

and for  $n \neq 0$ ,  $n \neq q$ ,

$$\mathbf{H}_{q,n} = \mathbf{B}_{q,n} - \widehat{Y}(q - n) \widehat{Y}(n), \quad (71)$$

then the relation (67), together with (68), (69) gives (see [17] for more details):

$$(2U_0 \widehat{Y}(q) - A_q)P_q = \sum_{n \in \mathbb{Z}} \mathbf{H}_{q,n} P_{q-n} P_n + \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} C_{q,n,m} P_{q-n} P_{n-m} P_m, \quad \text{for } q \neq 0 \quad (72a)$$

$$U_0^2 = u_{00} + \sum_{n \in \mathbb{Z}} \mathbf{H}_{0,n} |P_n|^2 + \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} C_{0,n,m} P_{-n} P_{n-m} P_m, \quad \text{for } q = 0. \quad (72b)$$

Lemma 3.1 justifies this expression as  $\widehat{Y}(1 - Q_0)$  and  $\widehat{D}$  are bounded and satisfy **H3**. As said previously, Eq.(72b) appears because  $\widehat{Y}$  is not defined at 0. In order to obtain an expression similar to (13), one has to replace  $U_0$  in (72a) by its value given in (72b). For this purpose it is assumed that  $U_0$  has a positive value, allowing to express it as a square root. It is assumed that the system encounters a bifurcation for  $\gamma_0 > 0$  which implies  $u_{00} > 0$ . Thus, for a small enough acoustic pressure, this square root can be expanded as a Taylor series. Replacing  $U_0$  by this expression in (72a) leads to a nonlinearity of the form presented in lemma 3.1. As a consequence, theorems 4.1 and 5.1 can be used. The above relations are complex to write in the canonical form given in (13), however, for instance one has for the linear part:

$$\widehat{J}_\gamma(q) = (2\sqrt{u_{00}}\widehat{Y}(q) - A_q). \quad (73)$$

This linear part satisfies **H2**, provided  $(2\sqrt{u_{00}}\widehat{Y}(q) - A_q) = 0$  for only two values  $q = \pm s$ . Notice that the stationary solutions  $U_0 = \pm\sqrt{u_{00}}$ , where for all  $n \in \mathbb{Z}$ ,  $P_n = 0$ , exist for all  $\gamma \in (0, 1)$ . Consequently, from (73), the bifurcation parameter and the frequency of oscillation at the bifurcation will be given, according to Th. 4.1, by the characteristic equation:

$$2\sqrt{u_{00}}\widehat{Y}(s) - A_s = 0.$$

The main objective is now attained: hypotheses **H2** and **H3** hold and if one supposes a solution of the type described in **H1**, the hypothesis (62) can be justified in the framework of the present study by applying Th. 5.1. The results presented in the publications cited in this section, which required at first the assumption (62), can be obtained by assuming the weaker assumption of twice continuous differentiability of the solution. Let us notice a slight difference between the result of theorem 5.1 and the relation (62). The fact that there exists a value  $k$  in Th. 5.1 where (62) stops to be valid does not change the results of the previously cited publications. Indeed, these results can be retrieved by knowing estimates on the first harmonics and involve only a limited number of harmonics in calculus. Moreover,  $k$  can be taken as large as needed by limiting the range of amplitude for which the results are valid. Eventually, the hypotheses required in section 6 are satisfied, see [17], and the quantity  $D_1(\gamma_0)$  defined in this latter section can be deduced from equations (72). This leads to

$$|P_1| = \sqrt{|\alpha|}\sqrt{\gamma - \gamma_0} + \mathcal{O}(|\gamma - \gamma_0|^{3/2}),$$

where

$$\alpha = \frac{\beta'(\gamma_0)}{D_1(\gamma_0)},$$

confirming the results of [17].

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