Risk aggregation in Solvency II: How to converge the approaches of the internal models and those of the standard formula?

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ABSTRACT

Two approaches may be considered in order to determine the Solvency II economic capital: the use of a standard formula or the use of an internal model (global or partial). However, the results produced by these two methods are rarely similar, since the underlying hypothesis of marginal capital aggregation is not verified by the projection models used by companies. We demonstrate that the standard formula can be considered as a first order approximation of the result of the internal model. We therefore propose an alternative method of aggregation that enables to satisfactorily capture the diversification among the various risks that are considered, and to converge the internal models and the standard formula.

KEYWORDS : Economic capital, Solvency II, nested simulations, standard formula, risk aggregation, equity, risk factors, diversification

1. Introduction

For the purpose of the new solvency repository of the European Union for the insurance industry, Solvency II, insurance companies are now required to determine the amount of their equity, adjusted to the risks that they incur. Two types of approach are possible for this calculation: the use of a standard formula or the use of an internal model[1].

The "standard formula" method consists in determining a capital for each elementary risk and to aggregate these elements using correlation parameters. However, the internal model enables to measure the effects of diversification by creating a simultaneous projection of all of the risks incurred by the company. Since these two methods lead in practice to different results (see Derien et al. (2009) for an analysis for classical loss distributions and copulas), it seems crucial to explain the nature of the observed deviations. This is essential, not only in terms of certification of the internal model (in relation to the regulator), but also at an internal level in the Company's Risk Management strategy, as the calculation of the standard formula must in any case be carried out, independently of the use of a partial internal model. One must therefore be able to explain to the management the reason for these differences, in a manner that is understood by all, including the top ranks of the management and the shareholders.

In this paper, we shall be analysing the validity conditions of a "standard formula" approach for both the calculation of the marginal capital and the calculation of the global capital. We shall demonstrate that under certain hypotheses that are often satisfied in models used by companies, the *marginal* capitals according to the standard formula are very close, and sometimes identical, to those obtained with the internal model. However, we shall also demonstrate that the standard formula generally fails in terms of elementary capital aggregations and shows deviations in relation to the global capital calculated with the internal model that can be significant. These differences observed in the results are mainly caused by two phenomena:

- the level of equity is not adjusted in terms of underlying risk factors,
- the "standard formula" method does not take into account the "cross-effects" of the different risks that are being considered.

In the event of the hypotheses inherent to the "standard formula" approach not being satisfied, we present an alternative aggregation technique that will enable to adequately comprehend the diversification among risks. The advantage of this method is that risk aggregation with the standard formula may be regarded as the first-order term of a multivariate McLaurin expansion series of the “equity" with respect to the risk factors. In some instances, risk aggregation with internal models may be approximated by using higher order terms in addition in the expansion series. In any case, this way of considering things enables to explain to the management the main reasons of the difference between the result of the standard formula and the result obtained with the internal model.

[1] A combination of these methods may be envisaged in the case of partial internal models.

In the first part we shall discuss the issues surrounding the calculation of economic capital in the Solvency II environment. We shall then formalise the "standard formula" and "internal model" approaches and explain the differences on the base of projections of a savings type portfolio. In the last section, we shall offer a description of our alternative aggregation method and apply it to the portfolio being considered. Finally, we shall examine the field of application and limitations of this approach by using another portfolio with a risk profile that makes it more complex to apply our method.

2. The calculation of the Solvency II economic capital

In this Section we offer some reminders concerning the notion of Solvency Economic Capital II and we describe the "standard formula" approaches and the technique of "nested simulations" implemented for the purposes of an internal model.

1. General Information

For a detailed presentation of the Solvency II economic capital calculation problematic, the reader may consult Devineau and Loisel (2009). It is useful to remember that the Solvency II economic capital corresponds to the amount in equity available to a company facing financial bankruptcy with a one year horizon and a confidence level of 99.5%. This definition of the capital rests on three notions:

- **Financial bankruptcy :** situation where the market value of the Company's assets is inferior to the economic value of the liabilities (negative equity),
- **One year horizon:** necessity of being able to carry out the distribution of the equity within one year,
- **The 99.5% threshold:** the required level of Solvency.

The Solvency II capital is based on the economic balance sheet of the company as from date t=0 and as of date t=1.

We offer here an explanation of the following notations:

- A_t the market value of the asset at *t,*
- L_t the fair value of liabilities at *t,*
- E_t the equity at *t.*

The balance sheet at $t \in \{0,1\}$ takes on the following form:

Economic balance sheet at t	
A_t	E_t L_t

At the initial date of the assets' value, the liabilities and the equity of the company are determinist figures, whereas at t=1, they are random variables that depend on random (financial, demographic...) factors that took place during the first year.

The value of each item in the balance sheet corresponds to the expected value under the risk-neutral probability Q of discounted future cash-flows.

Denote :

- $(F_t)_{t\geq 0}$ the filtration that permits to characterise the available information for each date,
- δ_u the discount factor that is expressed with the instantaneous risk free interest rate r_u : $\delta_u = e^{-\int_0^u r_h dh}$,
- P_t the cash-flows of the liabilities (claims, commissions, expenses) for the period t,
- R_t the profit of the company for period t.

Equity E_0 and the fair value of the liabilities at the start date, L_0, are calculated in the following manner:

$$L_0 = E_Q\left[\sum_{u\geq 1} \delta_u . P_u \,|F_0\right],$$

and

$$E_0 = E_Q\left[\sum_{u\geq 1} \delta_u . R_u \,|F_0\right].$$

In order to determine the equity E_1 and the fair value of the liabilities L_1 at t=1, a "real-world" conditioning must be introduced for the first period. The E_1 and L_1 variables are calculated with the expected value under the risk-neutral probability of the discounted future cash-flows, dependent of the "real-world" information of the first year (designated as F_1^{RW}).

This leads to the following calculations:

$$E_1 = R_1 + E_Q\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} R_u \,|F_1^{RW}\right],$$

and

$$L_1 = E_Q\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} P_u \,|F_1^{RW}\right].$$

The economic capital is then evaluated with the following relation: $C = E_0 - P(0,1).q_{0.5\%}(E_1)$, where P(0,1) is the price at time 0 of a zero-coupon bond with maturity 1 year.

The quantity $P(0,1).q_{0.5\%}(E_1)$ appears as a (mathematical) surplus that needs to be added to the initial equity in order to guarantee the following condition: $P(E_1 < 0) = 0.5\%$.

2. The standard formula

In this paper, we shall use the term "standard formula" to describe any method that aims to calculate the economic capital at the level of each "elementary risk" (stock, interest rate, mortality rate,...) and then to aggregate these capitals with correlation matrices.

A "standard formula" method may either rest on a single level of aggregation or implement successive aggregations, as is the case for the QIS (see: CEIOPS *QIS 4 Technical Specifications 2008*). In fact, this method consists in aggregating, in a first stage, the elementary capitals within different risk modules ("market" module, "life" module, "non-life" module,...) This phase corresponds to an intra-modular aggregation. The capitals of each module are then aggregated, so as to obtain the global economic capital (inter-modular aggregation). It should be noted that both the GCAE (2005) and Filipovic (2008) underline the limits of such an approach[2].

A "standard formula" type method corresponds to a bottom-up approach (i.e. starting with the elementary risks and ending with the calculation of the global capital). The calculation of the elementary capitals implies the use of an ALM model that provides a financial balance sheet as from the start date. This model enables, amongst other things, to calculate the amount of "central" equity, i.e. the equity according to the conditions on the calculation date, as well as the equity resulting from an instantaneous shock of these conditions.

More precisely, to calculate the elementary capital C_R for the purpose of risk R, an instantaneous shock is delivered to the R factor, and the equity E_0^R is determined after the shock. This amount is then subtracted from the central equity E_0 in order to obtain the economic capital for the purpose of R.

In order to determine E_0^R, the calculations must be reconditioned with a new filtration $(F_t^R)_{t\geq 0}$in mind, which derives from the instantaneous shock on the R factor. The ALM model is used to estimate the following quantity:

$$E_0^R = E_{Q_R}\left[\sum_{u\geq 1} \delta_u . R_u \left| F_0^R \right.\right],$$

where Q_R corresponds to the risk-neutral probability that is applied to filtration $(F_t^R)_{t\geq 0}$.

The elementary capital is then represented as

$C_R = E_0 - E_0^R$.

The following diagram illustrates the calculation method of the elementary capital in terms of Risk R:

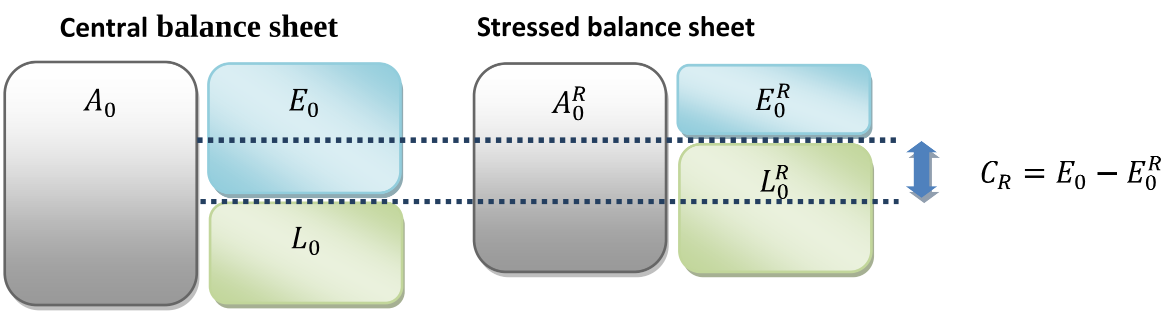

Figure 1 : calculation of the elementary capital in terms of risk R with the "standard formula" method.

Note that A_0^R (resp. L_0^R) represents the market value of the assets (resp. the fair value of the liabilities) at *0* after the shock on the R factor.

[2] Filipovic (2008) demonstrates that the correlation factors that enable to carry out the inter-modular aggregation are entity specific. Therefore, since it is impossible to use a "benchmark" correlation matrix, this approach loses its universal characteristic.

In order to estimate quantities E_0 and E_0^R, Monte-Carlo simulations are carried out. The following notation should be introduced at this point, in order to formalise the calculations performed according to the ALM method.

Write:

- R_u^s (resp. R'^s_u) the result of date $u \geq 1$ for the simulation $s \in \{1, \dots, S\}$ according to Q (resp. under Q_R),
- δ_u^s (resp. δ'^s_u) the discount factor of the $u \geq 1$ date for the s simulation under Q (resp. under Q_R).

The amounts of E_0 and E_0^R are then estimated in the following manner:

$$\hat{E}_0 = \frac{1}{S}\sum_{s=1}^{S}\sum_{u\geq 2} \delta_u^s . R_u^s,$$

and

$$\hat{E}_0^R = \frac{1}{S}\sum_{s=1}^{S}\sum_{u\geq 2} \delta'^s_u . R'^s_u.$$

Comment: for the purpose of coherence with the definition of the Solvency II economic capital, the instantaneous shocks delivered to the various elementary risks are homogeneous in terms of extreme deviations (i.e. the 0.5% or 99.5% threshold depending on the "sense" of risk) according to the physical probability.

The elementary capitals are then aggregated with correlation matrices. Let us define:

- R_m the set of risks of module m,
- C_i the capital for the purpose of risk i,
- $\rho_{i,j}^{R_m}$ the correlation coefficient that enables to aggregate the capitals of risks i and j belonging to module m,
- SCR_m the economic capital (designated as Solvency Capital Requirement) of module m,
- M the set of modules,
- $\rho_{i,j}^{M}$ the correlation coefficient that enables to aggregate the capitals of modules i and j,
- $BSCR$ the global economic capital (designated as Basic Solvency Capital Requirement) before operational risks and adjustments.

A QIS type aggregation is based on two main stages:

- An intra-modular aggregation: for each risk module m, the economic capital SCR_m is calculated in the following manner :

$$SCR_m = \sqrt{\sum\nolimits_{(i,j)\in R_m^2} \rho_{i,j}^{R_m} . C_i . C_j},$$

- An inter-modular aggregation: the BSCR global capital is obtained by aggregating the capitals of the different modules.

$$BSCR = \sqrt{\sum\nolimits_{(i,j)\in M^2} \rho_{i,j}^{M} . SCR_i . SCR_j}.$$

Hereunder is the mapping that was chosen for the calculation of the economic capital QIS 4:

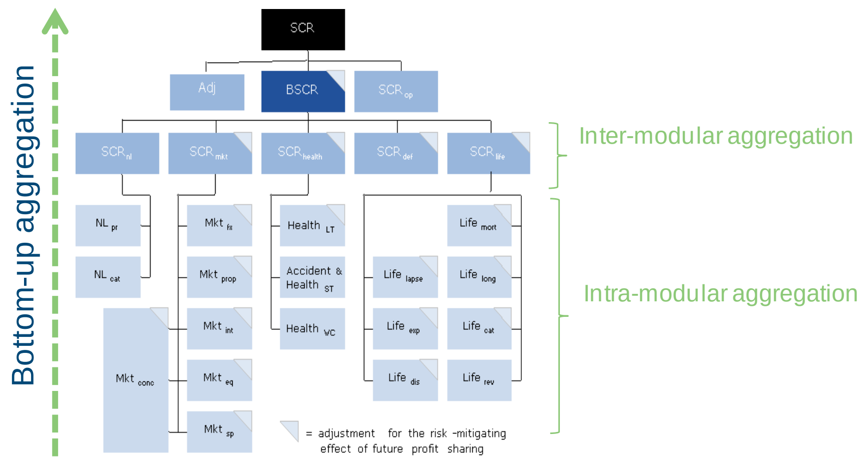

Figure 2 : mapping of the risks of the QIS 4

Comment: in a "standard formula" approach, the calculations are often carried out at the initial date. Therefore, the economic capital does not rest on the distribution of equity at the end of the first year but rather on the elementary capitals determined at t=0.

On the other hand, an internal model that performs NS projections (Nested Simulations) enables to calculate the economic capital by complying with all the Solvency II criteria.

3. The Nested Simulations (NS) method

As we have seen above, the Solvency II economic capital is described in relation to the 0.5% percentile of the distribution of equity at the end of the first year and of the amount of equity at the start date. The link between these various elements is provided by the following relationship: $C = E_0 - P(0,1).\,q_{0.5\%}(E_1)$.

There are generally no operational issues in the determination of the E_0 amount; all that is needed to obtain this quantity is an ALM model that enables to carry out "market consistent" calculations at t=0. However, it is more delicate to obtain the distribution of the E_1 variable, and the calculation of the equity at t=1 is required, conditional on the hazards of the "real-world". The "Nested simulations" technique (NS) enables to address this problematic. To this date, this application is one of the most compliant methods with the Solvency II criteria for annuity products. Devineau and Loisel (2009) offer a detailed description.

This method consists in carrying out, through an internal model, "real-world" simulations on the first period (called primary simulations) and launching, at the end of each one of these simulations, a set of new simulations (called secondary simulations), in order to determine the distribution of the equity of the company at $t = 1$. The secondary simulations have to be "market consistent"; in most cases these are risk-neutral simulations.

In order to formalise the calculations carried out in a NS approach, let us define

- $\mathrm{R}_{\mathrm{u}}^{\mathrm{p,s}}$ the profit of the $u > 1$ date for the primary simulation, $\mathrm{p} \in \{1, \dots, \mathrm{P}\}$, and for the secondary simulation $\mathrm{s} \in \{1, \dots, \mathrm{S}\}$,
- $\mathrm{R}_1^{\mathrm{p}}$ the result of the first period for the primary simulation p,
- $\delta_{\mathrm{u}}^{\mathrm{p,s}}$ the discount factor of the $u > 1$ date for the primary simulation p, and for the secondary simulation s,
- δ_1^{p} the discount factor of the first period for the primary simulation p,
- $\mathrm{F}_1^{\mathrm{p}}$ the information of the first year contained in the primary simulation p,
- $\mathrm{E}_1^{\mathrm{p}}$ the equity at the end of the first period for the primary simulation p,
- $\mathrm{L}_1^{\mathrm{p}}$ the fair value of liabilities at the end of the first period for the primary simulation p,
- $\mathrm{A}_1^{\mathrm{p}}$ the market value of the assets at the end of the first period for the primary simulation p.

This application may be seen in the following diagram:

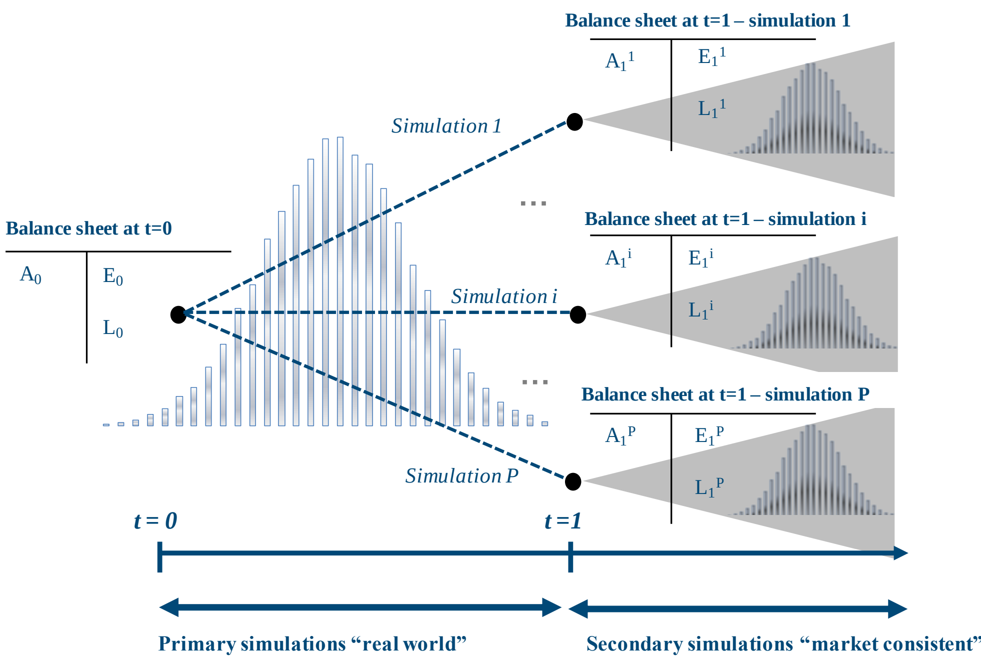

Figure 3 : obtaining the distribution of equity with the NS method.

The equity at t=1, for the primary simulation p, satisfies:

$$E_1^p = R_1^p + E\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} R_u \,\middle|\, F_1^p\right].$$

For the calculation of E_1^p, the following estimator is considered:

$$\hat{E}_1^p = R_1^p + \frac{1}{S} \sum_{s=1}^{S} \sum_{u\geq 2} \frac{\delta_u^{p,s}}{\delta_1^p} R_u^{p,s}.$$

The determination of the $q_{0.5\%}(E_1)$, quantity is generally based on the $E_1^{([0.5\% \times P])}$estimator. In other words, the "worst value" $[0.5\% \times P]$ of the $\left(E_1^p\right)_{p=1,\ldots,P}$sample is taken as estimator of $q_{0.5\%}(E_1)$.

The economic capital is then evaluated with the estimator: $\hat{C} = \hat{E}_0 - P(0,1).\ E_1^{([0.5\% \times P])}$.

3. Formalising the "standard formula" and "NS" approaches

In this section, we propose a formalisation of the "standard formula" and NS approaches. First we shall introduce the notion of risk factors, which we associate with "standard formula" shocks and with the primary simulations of a NS projection. Then we shall adapt the definition of the economic capital calculation so as to return to an analysis over a single period, which enables to compare the results of the "standard formula" and those of the internal model. Finally, we shall establish the theoretical framework that legitimises the marginal and global capitals obtained with the "standard formula" method. The partial internal models presented herein are of the same type as those used by companies. We are aware of the limits of these models. It would be a good idea to perfect them, but that is not the object of this paper: our aim is to study the risk aggregation issues in partial internal models typically used by insurance companies.

1. Risk factors

Risk factors are elements that enable to summarise the intensity of the risk for each primary simulation in an NS projection. For example, let us suppose that the stock price is modelled according to a geometric Brownian motion; in this case, the risk factor that one can consider is that of an increase of the Brownian motion of the diffusion over the period in question. Very low values for these increases correspond to cases where the stock price may undergo very strong downward shocks (adverse situations in terms of solvency).

It is possible to extract the risk factors from a table of economic scenarios for the first period by specifying an underlying model for each risk and by evaluating the parameters of each model. We shall describe this approach as an "*a posteriori* determination method"[3].

In the example that we offer as part of the fourth section "Application: comparison of the standard formula and NS approaches", we follow an “a posteriori” approach based on the first year "real-world" table used for NS projections.

From now on in this Section, write:

[3] When the company has a precise knowledge of the underlying risks' modelling and simulates its own trajectories, it is sufficient to export all the simulated hazards when the primary trajectories are generated. Amongst other things, this enables to realise the increase of Brownian motions of the diffusions (interest rate, stock,...). In this case, the factors are known before the modelling.

- S_t the stock price at time t,
- Z^S a random variable distributed according to Normal-Inverse Gaussian distribution $NIG(\alpha, \beta, \delta, \mu)$,
- Z^{ZC} a standard normal random variable,
- $P(t, T)$ the price at t of a zero-coupon bond with maturity $T > t$,
- μ_T^P the real-world return of the zero-coupon bond with maturity T,
- σ_T^P the real-world volatility of the zero-coupon bond with maturity T,
- ρ the Pearson's correlation coefficient of variables Z^S and Z^{ZC}.

We shall suppose that the evolution of the value of stock price and of the price of zero-coupon bonds in a "real-world" environment for the first year is described by

$$S_1 = S_0 e^{Z^S}, \qquad (1)$$

and

$$P(1,T) = P(0,T) e^{\mu_T^P + \sigma_T^P . Z^{ZC}}. \quad (2)$$

Relation (1) corresponds to a modelling of the stock price according to an exponential NIG-Levy process. For a detailed description of this type of model, see Papapantoleon (2008).
Relation (2) is derived from a linear volatility HJM (Heath-Jarrow-Morton) type model[4].

- ***<u>Calibration of the parameters</u>***

Let:
- S_1^p be the stock price at date *1* in primary simulation p,
- $P^p(1, T)$ be the price at t of a zero-coupon bond with maturity T in simulation p.

The interest rate parameters are evaluated from the economic scenarios' table of the first period.

$$\hat{\sigma}_T^P = \sqrt{\frac{1}{P-1} \sum_{p=1}^{P} \left(ln(P^p(1,T)/P(0,T)) - \frac{1}{P} \sum_{p=1}^{P} ln(P^p(1,T)/P(0,T)) \right)^2},$$

$$\hat{\mu}_T^P = \frac{1}{P} \sum_{p=1}^{P} ln(P^p(1,T)/P(0,T)).$$

In order to estimate the parameters of the stock price model, we present hereunder a reminder of the properties of a $Z^S \approx NIG(\alpha, \beta, \delta, \mu)$ distribution. With $\gamma = \sqrt{\alpha^2 - \beta^2}$, we obtain:
- $E(Z^S) = \mu + \frac{\delta\beta}{\gamma}$,
- $V(Z^S) = \frac{\delta\alpha^2}{\gamma^3}$,
- $S(Z^S) = \frac{1}{V(Z^a)^{3/2}} E\left(\left(Z^S - E(Z^S) \right)^3 \right) = \frac{3\beta}{\alpha\sqrt{\delta\gamma}}$,

[4] See Devineau et Loisel (2009).

- and $K(Z^S) = \frac{1}{V(Z^a)^2} E\left(\left(Z^S - E(Z^S)\right)^4\right) - 3 = \frac{3\left(1+\frac{4\beta^2}{\alpha^2}\right)}{\delta\gamma}$,

where $S(Z^S)$ (resp. $K(Z^S)$) represents the skewness coefficient (resp. Kurtosis excess coefficient) of the Z^S distribution.

Let $\hat{E}$ (resp. $\hat{V}$, $\hat{S}$, $\hat{K}$) be the empirical estimator of the expected value (resp. the variance, the skewness, the excess of kurtosis excess) calculated for the $\left(ln\left(S_1^p/S_0\right)\right)_{p=1,\dots,P}$ sample.

The $f_{NIG}(x)$density of $NIG(\alpha,\beta,\delta,\mu)$is expressed as follows:

$$f_{NIG}(x) = \frac{\alpha}{\pi} exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right)\frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}}.$$

where K_1 is a Bessel function of the third kind with parameter 1.

First, an estimation of the moments of parameters α,β,δ,μ is to be carried out by minimisation of the criteria

$$\Sigma(\alpha,\beta,\delta,\mu) = \left(\frac{E(Z^S) - \hat{E}}{\hat{E}}\right)^2 + \left(\frac{V(Z^S) - \hat{V}}{\hat{V}}\right)^2 + \left(\frac{S(Z^S) - \hat{S}}{\hat{S}}\right)^2 + \left(\frac{K(Z^S) - \hat{K}}{\hat{K}}\right)^2.$$

We shall then determine the estimator of maximum likelihood for α,β,δ,μ by initialising the optimization algorithm with the moments' estimator obtained above.

- ***<u>Extraction of stock and zero-coupon bond related risk factors</u>***

For each primary simulation *p*, we shall establish the $(\varepsilon_S^p, \varepsilon_{ZC}^p)$ pair of centred and reduced random events, using the estimators presented above:

$$\varepsilon_S^p = \frac{ln\left(\frac{S_1^p}{S_0}\right) - \hat{\mu} - \frac{\hat{\delta}\hat{\beta}}{\hat{\gamma}}}{\sqrt{\frac{\hat{\delta}\hat{\alpha}^2}{\hat{\gamma}^3}}},$$

$$\varepsilon_{ZC}^p = \frac{1}{T-1}\sum_{t=2}^{T}\left[\frac{ln\left(\frac{P^s(1,t)}{P(0,t)}\right) - \hat{\mu}_T^P}{\hat{\sigma}_T^P}\right],$$

and

$$\hat{\rho} = \frac{1}{P}\sum_{p=1}^{P} \varepsilon_S^p . \varepsilon_{ZCB}^p.$$

2. The global and marginal NS projections

The NS method described above enables us to determine the global economic capital of the company. However, in order to compare the NS and "standard formula" approaches, it might be useful to know, in addition to the global capitals, the value of the elementary capitals. This will enable to determine if the differences noted between the two methods are due to elementary capitals or to the aggregation method (or both).

Definitions:

- We shall use the term **marginal scenarios for risk R** to describe a set of primary simulations, for which all the hazards are cancelled out, except for the hazards pertaining to R.
- We shall use the term **marginal NS in terms of risk R** to describe any NS projection for which the primary scenarios are the marginal scenarios of risk R.

It is thus possible to determine the 0.5% level percentile of the equity distribution at t=1 conditional on risk R, by performing a marginal NS. Where $E_1^{R,([0.5\% \times P])}$ is the estimator of the said percentile.
It is then easy to obtain the marginal economic capital $\widehat{C_R}$ in terms of risk R from the following relation: $\widehat{C_R} = \hat{E}_0 - P(0,1).\ E_1^{R,([0.5\% \times P])}$.

3. "Standard formula" vs internal model

The results of the standard formula and the internal model can be analysed on two levels:

- Marginal level: comparison of the "standard formula" capital determined by stress test and the capital calculated according a marginal NS,
- Global level: in the case where the marginal capitals obtained with the "standard formula" are very close or identical to those obtained with the internal model, comparison of the "standard formula" aggregation method and the NS method.

Hereunder is a recall of the diagram showing the marginal capital calculation in terms of *R* using the standard formula:

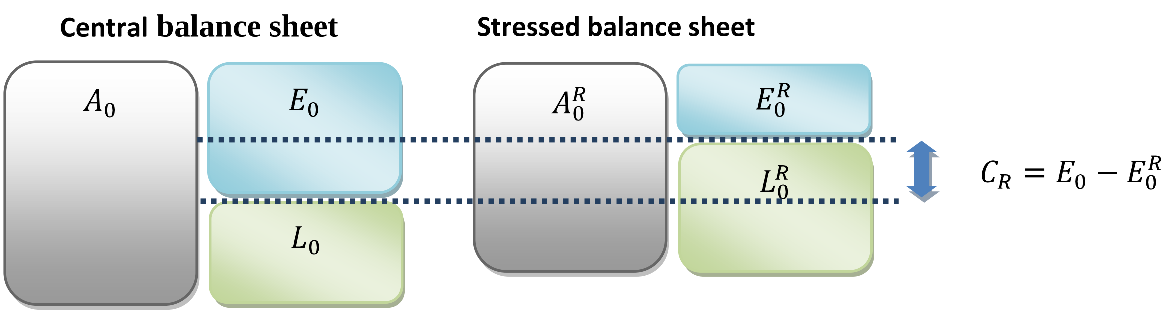

Figure 4 : calculation of the elementary capital in terms of risk R with the "standard formula" method.

Hereunder we also present a figure showing the calculation of the capital in terms of risk R using a marginal NS:

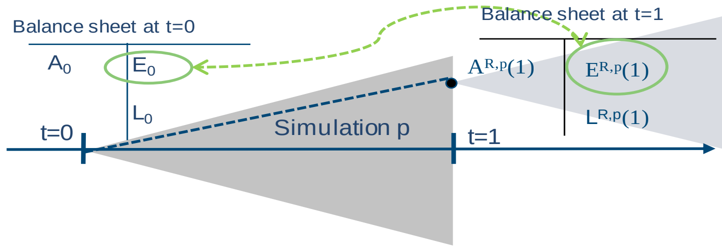

Figure 5 : calculation of the NS marginal capital relating to risk R.

Where the primary simulation p is the simulation associated with the 0.5% level percentile of the $E^R(1)$ variable that represents the distribution of equity at t=1, conditional only on risk R.

Two fundamental differences are observed in terms of marginal capitals between the "standard formula" and internal model approaches:

- Calculation timing: the "standard formula" approach consists in comparing the value of the equity before and after the shock at t=0, whereas the calculation using the marginal NS is based on the discounted percentile of the equity at then end of the first period.
- The "standard formula" method uses a valorisation after shock (notion of percentile on the R risk factor), whereas the "Marginal NS" method rests on marginal simulations of equity (notion of percentile on the distribution of equity).

In order to compare the results of the "internal model" and those obtained with the single period "standard formula" approach, we shall slightly amend the latter by modifying our definition of economic capital:

- We shall then place ourselves in a single period context and we shall describe the following value as economic capital:
$$C = E_1(0) - q_{0.5\%}(E_1),$$
where $E_1(0)$ represents the value of equity at $t = 1$, when all the hazards of the first period have been cancelled out. This relation enables to define the global capital and the marginal capital, the calculation of which can be carried out using a "NS (global or marginal)" method.

- Rather than performing an instantaneous stress test for the determination of the marginal capital in terms of risk R with the standard formula, we shall apply the corresponding shock to the first period by cancelling out all the other sources of randomness. The marginal capital will thus be the difference between the central value $E_1(0)$ and the level of equity at $t = 1$, conditional on the "standard formula" shock (noted E_1^R):
$$C_R = E_1(0) - E_1^R.$$

The following diagram illustrates the change of shock timing in the "standard formula" method:

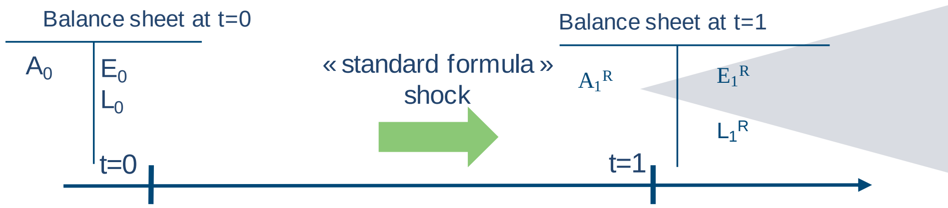

Figure 6 : adapting of the shock timing in the "standard formula" method.

On the basis of these adjustments, we shall propose in the following section a theoretical analysis of the "standard formula" approach.

4. Theoretical analysis of the "standard formula" approach

In this section we describe the theoretical framework required to calculate the economic capital with the "standard formula" method.

4.1. Case of an elementary capital

In this Section, we shall assume that risk R may be entirely characterised by a risk factor that we shall denote as ε_R.

Note that in a marginal NS projection in terms of R, the value of equity at t=1 is a function of the risk factor ε_R.

In other words, if ε_R^p designates the value of the risk factor in the primary situation p, then: $E_1^{R,p} = f(\varepsilon_R^p)$.

By taking α=0.5% or α=99.5% depending on the "sense" of risk *R*, then the calculation of the C_R^{SF} economic capital can be described as

$$C_R^{SF} = E_1(0) - f\big(q_\alpha(\varepsilon_R)\big)$$

whereas an approach of the marginal NS type would give the following C_R^{NS} capital:

$$C_R^{NS} = E_1(0) - q_{0.5\%}\big(f(\varepsilon_R)\big).$$

In the above expression, the percentile is considered on the "equity" function of the ε_R factor and not on the factor itself.

The analysis of the "standard formula" vs the internal model therefore consists in comparing elements $f(q_\alpha(\varepsilon_R))$ and $q_{0.5\%}\big(f(\varepsilon_R)\big)$.

In order to compare these elements, let us introduce the following ***H0*** hypothesis:

H0 : the amount of equity at t=1 is a monotonic function of the risk factor $\boldsymbol{\varepsilon_R}$.

According to **H0**, there are two scenarios. These are as follows:

- f is a decreasing function[5] :

$$C_R^{NS} = E_1(0) - q_{0.5\%}\big(f(\,\varepsilon_{\mathrm{R}})\big) \;= E_1(0) - f\big(q_{99.5\%}(\,\varepsilon_{\mathrm{R}})\big).$$

- f is an increasing function:

$$C_R^{NS} = E_1(0) - q_{0.5\%}\big(f(\,\varepsilon_{\mathrm{R}})\big) \;= E_1(0) - f\big(q_{0.5\%}(\,\varepsilon_{\mathrm{R}})\big).$$

H0 is a very strong hypothesis. In some cases, equity may be penalised for both very low and very high values of the risk factor ε_{R}.
As an example of this, consider an annuity product with a significant guaranteed interest rate and with a dynamic lapses' rule. The equity will be degraded for both low and high values of the "interest rate" risk factor and its monotonic nature will not be verified. It is possible to relax the **H0** hypothesis by considering the **$\mathbf{H0_{bis}}$** hypothesis, which we shall designate as hypothesis of predominance.

$\mathbf{H0_{bis}}$: hypothesis of predominance

If one assumes that the "equity" function:

- is decreasing (resp. increasing) beyond the q-percentile, where $q < 98\%$, say, (resp. before q-percentile with $q < 2\%$, say) of the risk factor,
- and takes on higher values when the factor is below (resp. above) the q-percentile,

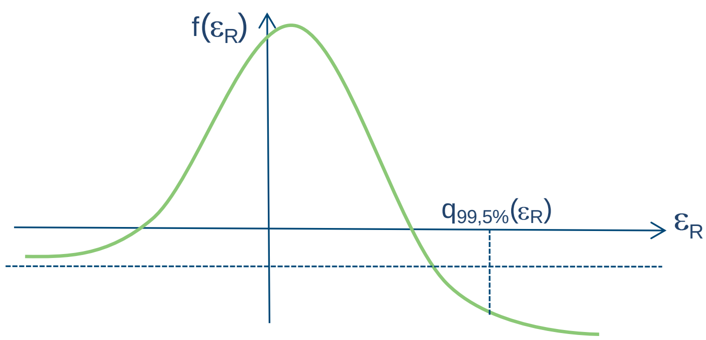

Figure 7 : profile of the "equity" function according to the hypothesis of predominance.

Then: $q_{0.5\%}(f(\,\varepsilon_{\mathrm{R}})) = \; f\big(q_{99.5\%}(\,\varepsilon_{\mathrm{R}})\big)\;\Big(\mathrm{resp.}\, q_{0.5\%}(f(\,\varepsilon_R)) = \; f\big(q_{0.5\%}(\,\varepsilon_R)\big)\Big).$

The hypothesis of predominance consists in considering that the situations of bad solvency are explained by extreme values taken on by the risk factor "in any direction" (upwards or downwards). Statistical issues about tests of Hypothesis **$\mathbf{H0_{bis}}$** are left for future research.

The monotonic hypothesis, also called the hypothesis of the predominance of the "equity" function in terms of the risk factor, justifies the fact that the percentile approach on equity is equivalent to the percentile approach on risk factor.

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[5] With $\alpha = P\Big(f(X) \le q_\alpha\big(f(X)\big)\Big) \Leftrightarrow \alpha = P\Big[X \ge f^{-1}\Big(q_\alpha\big(f(X)\big)\Big)\Big] \Leftrightarrow 1-\alpha = P\Big[X < f^{-1}\Big(q_\alpha\big(f(X)\big)\Big)\Big].$

Where $q_{1-\alpha}(X) = f^{-1}\Big(q_\alpha\big(f(X)\big)\Big) \Leftrightarrow q_\alpha\big(f(X)\big) = f\big(q_{1-\alpha}(X)\big)$

4.2. Analysis of the risk aggregation method

The technique of risk aggregation using a correlation matrix rests on a Markowitz mean-variance type approach. This method of aggregation is described, amongst other authors, by Saita (2004) and by Rosenberg and Schuermann (2004). The latter describe in their paper the case of the VaR of a portfolio containing three assets; the approach can be broadened to the calculation of the VaR of equity, depending on the different risk factors.

Aggregation techniques are often based on the notion of an economic capital that corresponds to the difference between the percentile and the expected value of a reference distribution (value of the portfolio, amount of losses, equity level,...). In our case, and using, for the purpose of simplifying the notations, *E* to describe the end of period equity, this definition leads to the following amount C of economic capital:

$$C = \mu_E - q_{0.5\%}(E),$$

where μ_E is the expected value of variable *E.*

We shall use this hypothesis to demonstrate the "standard formula" aggregation method under certain hypotheses.

A pre-requirement for the application of this method is that the global variable (annuity of the asset portfolio, equity of the company) is a linear function in terms of drivers (annuities of the portfolio's assets, risk factors,...). This is indeed the hypothesis that will enable to calculate de variance of the global variable in relation to the variance and covariance of the drivers.

We shall then assume that the company is exposed to three risks, *X, Y, Z* and that the distribution of equity at t=1 is linear for each one of these factors:

$$E = C + a.X + b.Y + c.Z,$$

with a, b and $c \in R^*$.

Hereunder we shall use notation μ_M (resp. σ_M) to describe the expected value (resp. the standard deviation) of a random variable *M*.
The $\rho_{M,N}$ coefficient will describe the linear correlation (Pearson's coefficient) between the two variables *M* and *N*.

We shall assume in this Section that variables *E, X, Y* and *Z* have finite one order and two order moments.
First, let us calculate the variance of *E* :

$$\sigma_E^2 = a^2.\sigma_X^2 + b^2.\sigma_Y^2 + c^2.\sigma_Z^2 + 2a.b.\rho_{X,Y}.\sigma_X.\sigma_Y + 2a.c.\rho_{X,Z}.\sigma_X.\sigma_Z + 2b.c.\rho_{Y,Z}.\sigma_Y.\sigma_Z.$$

Let *M* be a random variable with expected value μ_M and standard deviation σ_M. We shall use $\widetilde{M}$ to describe the reduced and centred variable

$$\widetilde{M} = \frac{M - \mu_M}{\sigma_M}.$$

We obtain the following relation:

$$q_\alpha(M) = \mu_M + \sigma_M . q_\alpha(\widetilde{M}).$$

Consequentially, by using the expression of the variance of *E* in relation to the variance and correlation coefficient of each one of the 3 drivers *X, Y* and *Z*, one obtains:

$$q_\alpha(E) = \mu_E$$
$$+q_\alpha(\tilde{E}).\sqrt{a^2.\sigma_X^2 + b^2.\sigma_Y^2 + c^2.\sigma_Z^2 + 2a.b.\rho_{X,Y}.\sigma_X.\sigma_Y + 2a.c.\rho_{X,Z}.\sigma_X.\sigma_Z + 2b.c.\rho_{Y,Z}.\sigma_Y.\sigma_Z}.$$

In the case of an extreme percentile (α=0.5%), the value of the normalised distribution's percentile is therefore negative:

$$q_\alpha(E) = \mu_E$$
$$-\sqrt{\begin{matrix} a^2.\sigma_X^2.\left[q_\alpha(\tilde{E})\right]^2 + b^2.\sigma_Y^2.\left[q_\alpha(\tilde{E})\right]^2 + c^2.\sigma_Z^2.\left[q_\alpha(\tilde{E})\right]^2 \\ +2\rho_{X,Y}.a\sigma_X.\left[q_\alpha(\tilde{E})\right] \ .b\sigma_Y.\left[q_\alpha(\tilde{E})\right] \\ +2\rho_{X,Z}.a\sigma_X.\left[q_\alpha(\tilde{E})\right] \ .c\sigma_Z.\left[q_\alpha(\tilde{E})\right] \ +2\rho_{Y,Z}.b\sigma_Y.\left[q_\alpha(\tilde{E})\right] \ .c\sigma_Z.\left[q_\alpha(\tilde{E})\right] \end{matrix}}.$$

In Appendix 2 we recall that when the (X, Y, Z) random vector is elliptic[6], one obtains the following result:

$$q_\alpha(E) = \mu_E - \sqrt{\begin{matrix} C_X^2 + C_Y^2 + C_Z^2 + 2.sg(ab).\rho_{X,Y}.C_X.C_Y + 2.sg(ac).\rho_{X,Z}.C_X.C_Z \\ +2.sg(bc).\rho_{Y,Z}.C_Y.C_Z \end{matrix}}$$

which leads to the *C* capital hereunder:

$$C = \mu_E - q_\alpha(E) = \sqrt{\begin{matrix} C_X^2 + C_Y^2 + C_Z^2 + 2.sg(ab).\rho_{X,Y}.C_X.C_Y + 2.sg(ac).\rho_{X,Z}.C_X.C_Z \\ +2.sg(bc).\rho_{Y,Z}.C_Y.C_Z \end{matrix}}$$

Where C_X (resp. C_Y, C_Z) corresponds to the economic capital in terms of risk *X* (resp. *Y*, *Z*), and $sg(x)$ is the sign of $x \in R^*$.

In the event of all the coefficients (a, b, c) all having the same sign, the QIS aggregation relation is found.

$$C = \mu_E - VaR_E(\alpha) = \sqrt{C_X^2 + C_Y^2 + C_Z^2 + 2\rho_{X,Y}.C_X.C_Y + 2\rho_{X,Z}.C_X.C_Z + 2\rho_{Y,Z}.C_Y.C_Z}$$

Comment: it is always possible to return to risk factor coefficients that have the same sign, even if this entails considering the opposites of the risk factors. However, in this instance, the correlations change sign.

[6] Gaussian and multivariate Student distributions are well known examples of elliptic distributions. For a detailed description of these distributions, see Appendix 1.

It should be reminded that the establishment of this relation required the hypotheses hereunder.

H1 : the *E* variable is a linear function of variables *X*, *Y* and *Z,*

H2 : the (X, Y, Z) vector follows an elliptic distribution (e.g. normal or Student distribution).

Comments:

- The H1 hypothesis ensures the standard nature of the correlation coefficient. Indeed, if the "equity" function is not linear in terms of risk factors, the linear correlations of the marginal distributions of equity are, generally speaking, different from those of the factors[7]. These parameters are no longer "market" values since they become "company" values (and therefore the "standard formula" approach loses its universal nature).

- The H2 hypothesis imposes a constraint on both the marginal distributions and the copula that links them. In other words, all marginal distributions must be identical and belong to the same family as the copula.
  In practice, this means considering the two most standard cases:
  - marginal distributions and Gaussian copulas
  - marginal distributions and Student copulas

4. Application: comparing the "standard formula" and "NS" approaches

In this Section we shall present, for a savings type portfolio, a comparison of economic capitals obtained, on one hand with the "standard formula", and on the other with the internal model. To begin with, we shall restore the results obtained from global and marginal NS projections, and we shall compare these to the aggregated and elementary capitals obtained with the "standard formula". We shall then propose a deviations' analysis that will enable to explain in large part the noted differences.

1. Description of the portfolio and of the model

The portfolio that we consider in this study is a savings' portfolio with no guaranteed interest rate. We have projected this portfolio using an internal model that performs ALM stochastic projections and the calculation of equity after one year. This projection tool enables the modelling of the profit sharing mechanism, as well as the modelling of behaviours in terms of dynamic lapses of the insured parties when the interest rates handed out by the company are deemed insufficient in relation to the reference interest rate offered by the competitors.

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[7] The linear correlation is not invariant by increasing transformations, contrary to a Kendall tau rank correlation coefficient.

In this study, are considered only the stock and interest rate related risks. The tables of economic scenarios that are used were updated on December 31, 2008. Let's note that the implicit "stock" and "interest rate" volatility parameters have been assumed as being identical for each set of "risk-neutral" secondary simulations. However, one should note that it is possible, in a NS application, to jointly project the risk factors and implicit volatilities on the first period, and to reprocess the market consistent secondary tables in relation, inter alia, to simulated volatilities. This approach would make it possible to take the implicit volatility risk into account, as suggested by the CRO Forum (2009).

The company's economic balance sheet at t=0 is as follows:

Asset market value - A_0	360 754
Fair value of the liabilities - L_0	353 394
Equity - E_0	7 360

Table: economic balance sheet of the company at t=0 (in M€)

The investment strategy at time 0 is as follows :

Cash	5%
Stock	15%
Bonds	80%

Table: distribution of the assets at market value at t=0

2. Results

2.1. Risk factors

The extraction of risk factors according to the method described above leads to the following cloud:

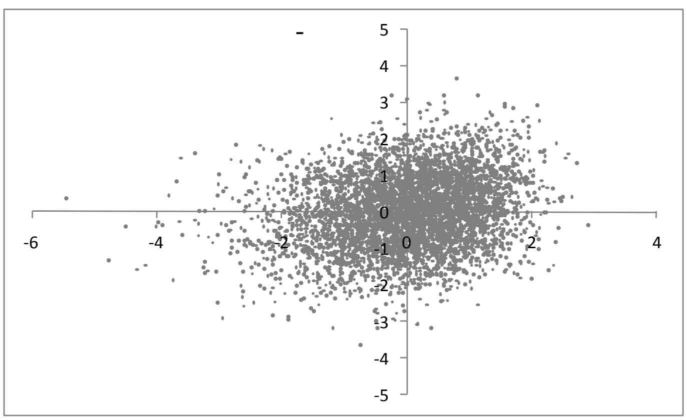

Figure 8 : risk factor pairs relating to stock (abscissa) and zero-coupon bonds (ordinate)

Each point in the cloud corresponds to a primary simulation. In the graph hereunder, we present descriptive statistics pertaining to stock and zero-coupon bonds (noted ZC) related risk factors.

	Stock risk factor $(\varepsilon_S^p)_{p=1,\dots,P}$	ZC risk factor $(\varepsilon_{ZC}^p)_{p=1,\dots,P}$
Expected value	0.0	0.0
Std error	1.0	1.0
Skewness	-0.5	0.0
Kurtosis	0.7	-0.1

Table 1 : statistical indicators of samples $(\varepsilon_S^p)_{p=1,\dots,P}$ et $(\varepsilon_{ZC}^p)_{p=1,\dots,P}$

Pearson's correlation coefficient for stock factors and zero-coupon bonds' factors is the following: $\hat{\rho} = 21.9\%$.

These two distributions are centred and reduced but the stock distribution shows kurtosis and skewness coefficients that are significantly different from those found in a normal distribution. This is due to the fact that the log-increase of the stock price follows a Normal-Inverse Gaussian distribution. The graph hereunder shows that the ε_S variable takes on more extreme negative values than the ε_{ZC} variable. Indeed, the ε_S distribution is asymmetrical with a heavy tail, whereas the ε_{ZC} follows a normal standard distribution.

2.2. Distribution of equity and first calculations

The distribution of equity as provided by NS stochastic projections is as follows:

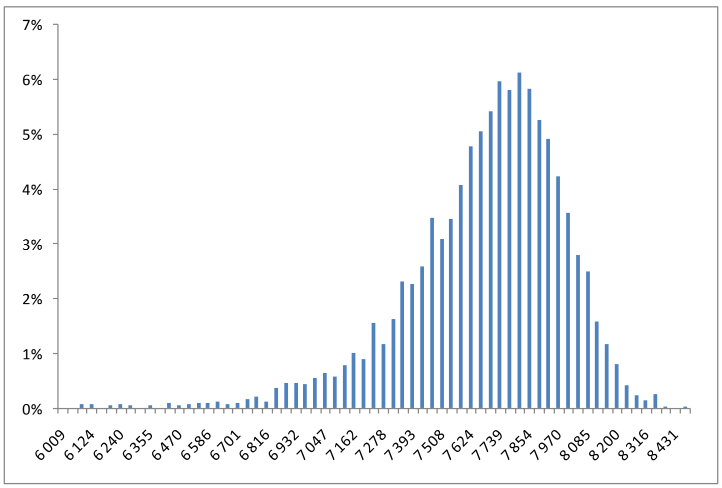

Figure 9 : distribution of the E_1 variable (in M€)

The NS method enables to estimate the economic capital using the estimator hereunder:

$$C_{NS} = \widehat{E}_1(0) - E_1^{([0.5\%\times P])},$$

where $\widehat{E}_1(0)$ is an estimator of $E_1(0)$.

The following value is obtained: $C_{NS} = 1209.7$, where C_S (resp. C_{ZC}) is the economic capital in terms of "stock" risks (resp. "interest rates"). By definition we have:

$$C_S = E_1(0) - q_{0.5\%}(E_1^S),$$
$$C_{ZC} = E_1(0) - q_{0.5\%}(E_1^{ZC}).$$

These two quantities can be estimated using marginal NS projections with the following estimators:

$$C_{NS}^S = \hat{E}_1(0) - E_1^{S,([0.5\% \times P])},$$

and

$$C_{NS}^{ZC} = \hat{E}_1(0) - E_1^{ZC,([0.5\% \times P])}.$$

Hereunder are the results of the estimation:

C_{NS}^S	C_{NS}^{ZC}
567.0	737.7

Table 2 : calculations of the NS marginal capitals

3. Analysis of the deviations

3.1. Comparison of stand-alone capitals

As has been demonstrated above, the comparison of "standard formula" and internal model approaches means to compare respectively the elements $f(q_\alpha(\varepsilon_R))$ and $q_{0.5\%}(f(\varepsilon_R))$, where f is the "equity" function and α=0.5% or α=99.5% depend on the sense of risk *R.*

Since the "stock" related risk is a decreasing risk, elements $f(q_{0.5\%}(\varepsilon_S))$ and $q_{0.5\%}(f(\varepsilon_S))$are compared hereunder.

For the purpose of our research, the following equality is used:

$$f\left(\varepsilon_S^{[0.5\% \times P]}\right) = E_1^{S,[0.5\% \times P]} = 7135.1\ .$$

Therefore, the "standard formula" approach (equity governed by the risk factor percentile) and the internal model approach (percentile on the distribution of equity) coincide.

Hereunder, we present the profile of E_1^S in relation to the value of the risk factor ε_S:

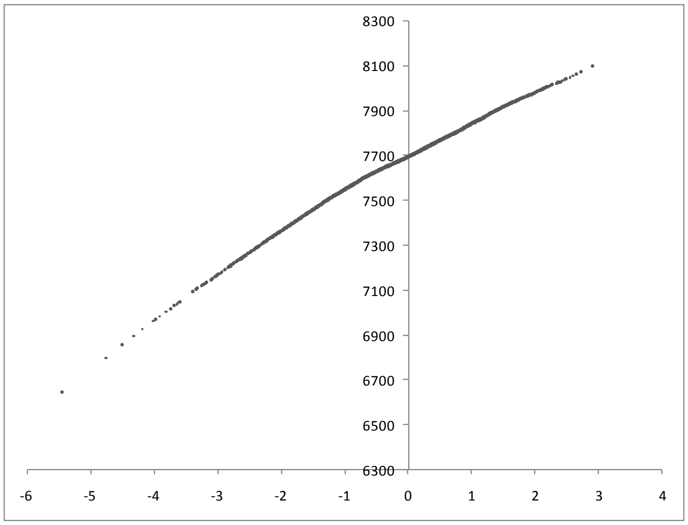

Figure 10 : value of E_1^S in relation to the level of risk factor ε_S

As this is an increasing function, Hypothesis ***H0*** is verified and the "standard formula" and internal model approaches are equivalent.

The graph hereunder presents the marginal equity E_1^{ZC} in terms of the value of the risk factor ε_{ZC} :

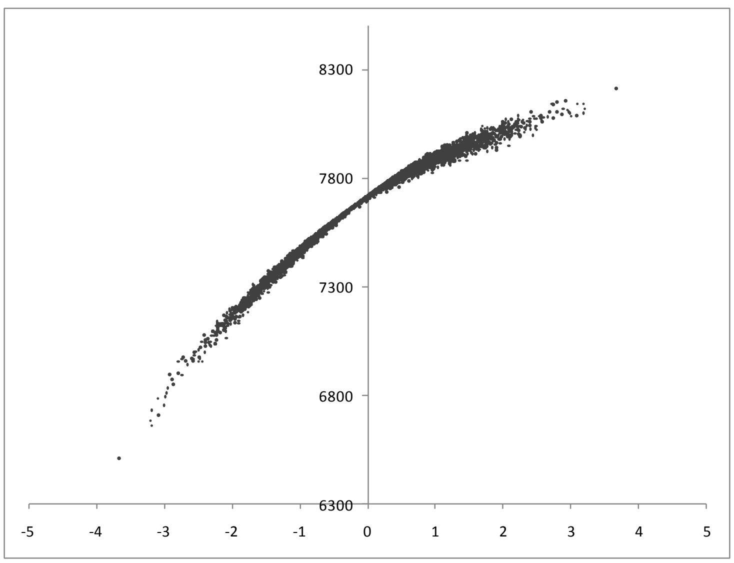

Figure 11 : Value of E_1^{ZC} in relation to the level of risk factor ε_{ZC}

One notes that it is the very low values for ε_{ZC} that lead to the most adverse situations in terms of solvency. One should remember that a low ε_{ZC} value corresponds to the case where the price of zero-coupons falls and therefore the interest rates increase. This corresponds to the product under consideration as it is exposed to an increase of the interest rate (triggering of a wave of dynamic lapses).

For the purpose of this study, we must therefore compare the elements $f(q_{0.5\%}(\varepsilon_{\text{ZC}}))$ and $q_{0.5\%}\big(f(\varepsilon_{\text{ZC}})\big)$.

We find $f\left(\varepsilon_{\mathrm{ZC}}^{[0.5\%\times\mathrm{P}]}\right) = 6997.2$ and $\mathrm{E}_1^{\mathrm{ZC},[0.5\%\times\mathrm{P}]} = 6971.7$.
There is a 0.4% difference between these two amounts.

Although these two values are very close, they are not identical since the extraction of zero-coupon bonds related risk factors induces a specification error. The deformation of the price of zero-coupon bonds is summarised independently from the maturities by a single random variable, whereas the underlying model is generally far more complex.

However, one may observe that the value of marginal equity rises globally along with the $\varepsilon_{\mathrm{ZC}}$ risk factor.

The linear nature of the E_1^S variable in terms of the "stock" risk factor is acceptable with regard to graph 10. However, graph 11 contradicts the linear nature of E_1^{ZC} in terms of risk factor $\varepsilon_{\mathrm{ZC}}$. The ***H1*** hypothesis (assuming a linear relation between equity and risk factors) is therefore not verified and the aggregation of the "standard formula" is compromised in such a context.

3.2. Comparison of global capitals

In this Section we compare the result obtained by aggregation of marginal capitals with the result provided by the internal model.

One should remember that the "standard formula" global capital is calculated in the following manner:

$$C_{SF} = \sqrt{{C_S^{SF}}^2 + {C_{ZC}^{SF}}^2 + 2\rho.C_S^{SF}.C_{ZC}^{SF}},$$

where $\mathrm{C}_{\mathrm{S}}^{\mathrm{SF}}$ (resp. $\mathrm{C}_{\mathrm{ZC}}^{\mathrm{SF}}$) corresponds to the "standard formula" marginal capital in terms of stock related risk (resp. zero-coupon bonds) and ρ represents the correlation between variables ε_{S} and $\varepsilon_{\mathrm{ZC}}$.

The table hereunder enables to compare the "standard formula" capitals with the internal model capitals:

C_S^{SF}	C_{ZC}^{SF}	C_{SF}	C_{NS}
567.0	743.1	1028.8	1209.7

Table 3 : comparison of "standard formula" capitals and internal model capitals

The difference between C_{SF} and C_{NS} is of 15%.

This difference is mainly due to the fact that the hypotheses ***H1*** and ***H2*** are not respected, a fact that justifies the aggregation by standard formula. We have insisted above on the fact that the "equity" function is not linear in terms of risk factors. The ***H1*** hypothesis is therefore not verified. Furthermore, in this study, the distributions of reduced and centred factors ε_{S} and $\varepsilon_{\mathrm{ZC}}$ are different. This contradicts the ***H2*** hypothesis that assumes that the $(\mathrm{E}_1, \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{ZC}})$ vector is elliptic in nature.

In the following section, we propose an analysis of the differences due to aggregation methods.

3.3. Parametric form and analysis of differences

a. Introduction of a parametric form

We have underlined above the non-linearity of the function that links "equity" to the zero-coupon bonds' factor. To strengthen our analysis, we shall first refine our choice of regression variables.

To achieve this, the following linear regression is considered:

$$E_1 = int + A_1.\varepsilon_S + A_2.\varepsilon_S^2 + A_3.\varepsilon_{ZC} + A_4.\varepsilon_S.\varepsilon_{ZC} + A_5.\varepsilon_S.\varepsilon_{ZC}^2 + A_6.\varepsilon_{ZC}^2 + A_7.\varepsilon_{ZC}^3 + U,$$

where U is a centred distribution that is independent from the pair of risk factors $(\varepsilon_S, \varepsilon_{ZC})$.

Following an estimation of the parameters, one obtains a R^2 equal to 99.6%. Hereunder we restore the QQ-plot "E_1 vs expected values of $\hat{E}_1$ " that adequately translates the distributions:

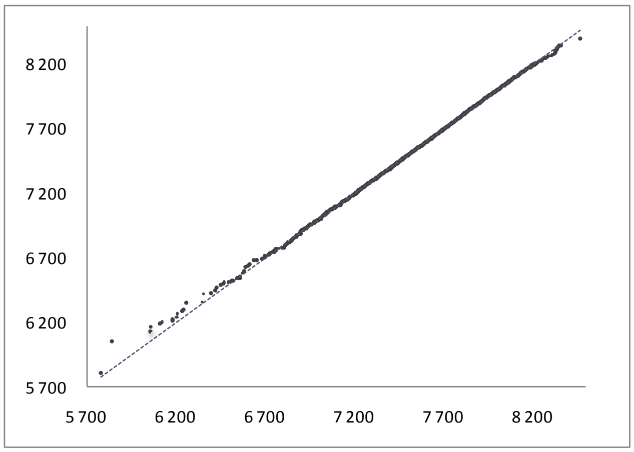

Figure 12 : QQ-plot $\hat{E}_1$ (abscissa) vs E_1 (ordinate)

The Kolmogorov-Smirnov test does not reject the goodness of fit of the distributions by giving a P-value equal to 76%.

The goodness of fit enables us to obtain an amount of economic capital C_{param} based on the $\hat{E}_1$ variable that is very close to the C_{NS} amount:

C_{NS}	C_{param}	Relative error
1209.7	1201.6	0.7%

Table 4 : comparison of the NS capital and the capital obtained with the parametric form

with $C_{param} = E_1(0) - q_{0.5\%}(\hat{E}_1)$.

The use of a parametric form will enable us to specify more accurately the deformations that occur during a "standard formula" type aggregation.

It should be noted that it is also possible to compare economic capitals that result from marginal equity distributions with the results obtained with the parametric form. To achieve this, the parametric marginal equity is considered[8] :

$$E_1^{S,param} = int + A_1.\varepsilon_S + A_2.\varepsilon_S^2,$$

$$E_1^{ZC,param} = int + A_1.\mu_S + A_2.{\mu_S}^2 + A_3.\varepsilon_{ZC} + A_4.\mu_S.\varepsilon_{ZC} + A_5.\mu_S.\varepsilon_{ZC}^2 + A_6.\varepsilon_{ZC}^2 + A_7.\varepsilon_{ZC}^3,$$

where μ_S is the stock's real-world return.

The QQ-plots hereunder for E_1^S (resp. E_1^{ZC}) vs $E_1^{S,param}$ (resp. $E_1^{ZC,param}$) show a very good fit for the distributions:

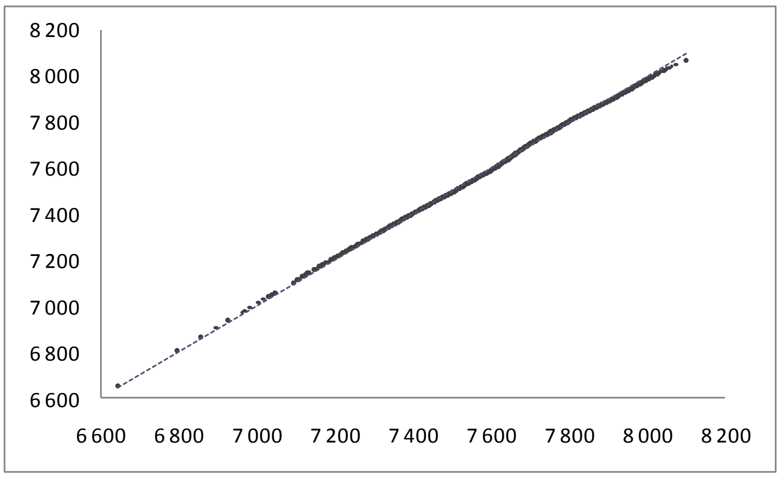

Figure 13 : QQ-plot $E_1^{S,param}$ (abscissa) vs E_1^S (ordinate)

[8] The stock (resp. zero-coupon bonds) parametric marginal equity is obtained by cancelling out the zero-coupon bond random factor (resp. by substituting the real-world return μ_S for the risk factor ε_S) in the parametric form presented above.

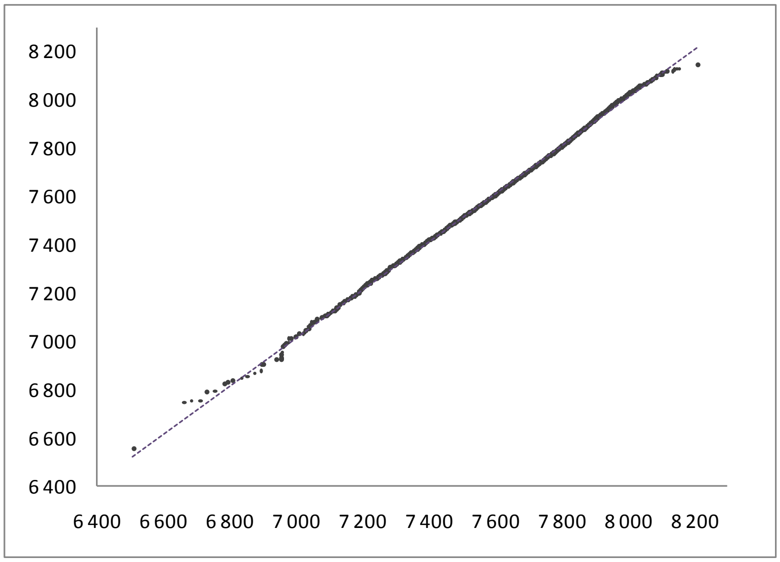

Figure 14 : QQ-plot $E_1^{ZC,param}$ (abscissa) vs E_1^{ZC} (ordinate)

The goodness of fit is also measured by the P-value of the KS-test, equal to 39% (resp. 19%) for "Stock equity" ("resp. zero-coupon bonds equity").

Let C_{param}^{S} (resp. C_{param}^{ZC}) be the "stock" marginal capital (resp. zero-coupon bonds) calculated with the variable $E_1^{S,param}$ (resp. $E_1^{ZC,param}$). One obtains:

$$C_{param}^{S} = E_1(0) - q_{0.5\%}\left(E_1^{S,param}\right),$$

and

$$C_{param}^{ZC} = E_1(0) - q_{0.5\%}\left(E_1^{ZC,param}\right).$$

The parametric approach provides an estimation of the marginal capitals that is very close to the results obtained with marginal NS projections:

C_{NS}^{S}	C_{param}^{S}	Difference
567,0	555,9	2,0%

Table 5 : comparison of "NS" and "parametric" stock marginal capitals

C_{NS}^{ZC}	C_{param}^{ZC}	Difference
737,7	723,6	1,9%

Table 6 : comparison of "NS" and "parametric" zero-coupon bonds' marginal capitals

Since the results of the NS projections are very close to those obtained with the parametric form, we shall use the latter as basis in the rest of this section. The parametric structure will indeed enable us to explain the deviations noted between "standard formula" economic capitals and "internal model" economic capitals, based on cross-terms of the $\varepsilon_S.\varepsilon_{ZC}$ or $\varepsilon_S.\varepsilon_{ZC}^2$ type.

b. Analysis of the deviations

Consider the following variable:
$E_1^{prod,param} = -int - A_1.\mu_S - A_2.{\mu_S}^2 - A_4.\mu_S.\varepsilon_{ZC} - A_5.\mu_S.\varepsilon_{ZC}^2 + A_4.\varepsilon_S.\varepsilon_{ZC} + A_5.\varepsilon_S.\varepsilon_{ZC}^2$.

The $E_1^{prod,param}$ variable specifically integrates the cross-terms $\varepsilon_S.\varepsilon_{ZC}$ ou $\varepsilon_S.\varepsilon_{ZC}^2$.
We shall designate the economic capital in terms of cross-effects as C_{param}^{prod} . It is defined by the following relation:

$$C_{param}^{prod} = E_1(0) - q_{0.5\%}\left(E_1^{prod,param}\right).$$

We obtain a linear relation between the E_1 variable and the marginal distributions of vector $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$:

$$E_1 = E_1^{S,param} + E_1^{ZC,param} + E_1^{prod,param}.$$

If the distribution of vector $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$ belongs to the same family of elliptic distributions, the global capital (noted C_2) may be calculated as follows:

$$C_2 = \sqrt{\begin{array}{c} {C_{param}^{S}}^2 + {C_{param}^{ZC}}^2 + {C_{param}^{prod}}^2 + 2.\rho_{S,ZC}.C_{param}^{S}.C_{param}^{ZC} + 2.\rho_{S,prod}.C_{param}^{S}.C_{param}^{prod} \\ +2.\rho_{ZC,prod}.C_{param}^{ZC}.C_{param}^{prod} \end{array}},$$

Where:

- $\rho_{S,ZC}$ is the linear correlation between variables $E_1^{S,param}$ and $E_1^{ZC,param}$,
- $\rho_{S,prod}$ is the linear correlation between variables $E_1^{S,param}$ and $E_1^{prod,param}$,
- $\rho_{ZC,prod}$ is the linear correlation between variables $E_1^{ZC,param}$ and $E_1^{prod,param}$.

Comment: a "standard formula" type method fails to capture the cross-terms $\varepsilon_S.\varepsilon_{ZC}$ or $\varepsilon_S.\varepsilon_{ZC}^2$, since isolating the risks implies cancelling out one of the two factors (ε_S ou ε_{ZC}).

A "standard formula" method therefore consists in performing the following calculation:

$$C_{SF} = \sqrt{C_S^2 + C_{ZC}^2 + 2.\rho_{S,ZC}.C_S.C_{ZC}}\ .$$

This approach therefore underestimates the risk when:

$$\rho_{S,prod}.C_{param}^{S}.C_{param}^{prod} + \rho_{ZC,prod}.C_{param}^{ZC}.C_{param}^{prod} > 0.$$

Hereunder are the obtained results:

Stand-alone capitals:

C_{param}^{S}	C_{param}^{ZC}	C_{param}^{prod}
555.9	723.6	227.6

Table 7 : marginal capitals associated to variables $E_1^{S,param}$, $E_1^{ZC,param}$ and $E_1^{prod,param}$

Correlation matrix:

	$E_1^{S,param}$	$E_1^{ZC,param}$	$E_1^{prod,param}$
$E_1^{S,param}$	1	21.5%	40.2%
$E_1^{ZC,param}$	21.5%	1	32.5%
$E_1^{prod,param}$	40.2%	32.5%	1

Table 8 : correlation matrix of vector $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$

Capital aggregated using the previous correlations

Capital C_{SF} « standard formula » on $\left(E_1^{S,param}, E_1^{ZC,param}\right)$	1002.9
Capital C_2 on $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$	1225.4
C_{param}	1201.6

Table 9 : capitals associated with risks $\left(E_1^{S,param}, E_1^{ZC,param}\right)$ and $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$

The difference between the C_{param} capital and the capital obtained by aggregation of risks $\left(E_1^{S,param}, E_1^{ZC,param}\right)$ is significant (16.5%). This is due to the fact that the risk inherent to cross-terms is not integrated in the calculation. By taking this risk into account in the C_2 calculation based on $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$, the difference is reduced from 16.5% to 6.3% in relation to the reference capital C_{param}.
However, as the linear hypothesis is verified (E_1 is a linear function of variables $E_1^{S,param}$, $E_1^{ZC,param}$ and $E_1^{prod,param}$), the residual error is explained by the non-elliptic nature of the distribution.

Consider the QQ-plot of standard distributions (i.e. centred and reduced) of $E_1^{S,param}$ and $E_1^{prod,param}$:

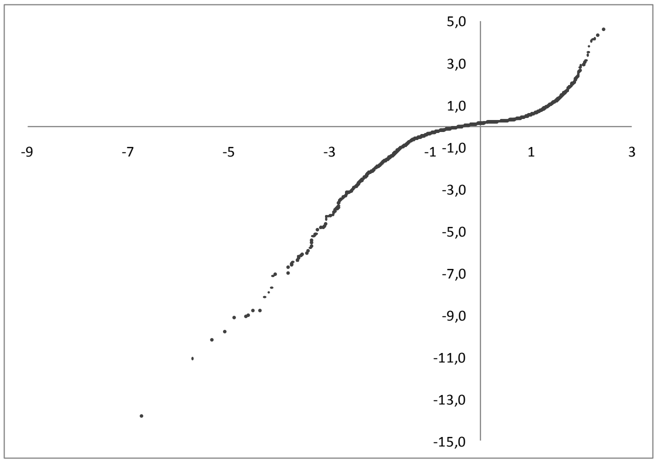

Figure 15 : QQ-plot of standard distributions of $E_1^{S,param}$ and $E_1^{prod,param}$

The above graph reveals that the reduced and centred marginal distributions of vector $\left(E_1^{S,param}, E_1^{ZC,param}, E_1^{prod,param}\right)$ do not follow identical distributions. The latter's elliptic nature is therefore contradicted.

The following diagram offers a summary of the deviations between capitals obtained with the standard formula and those calculated with the internal model:

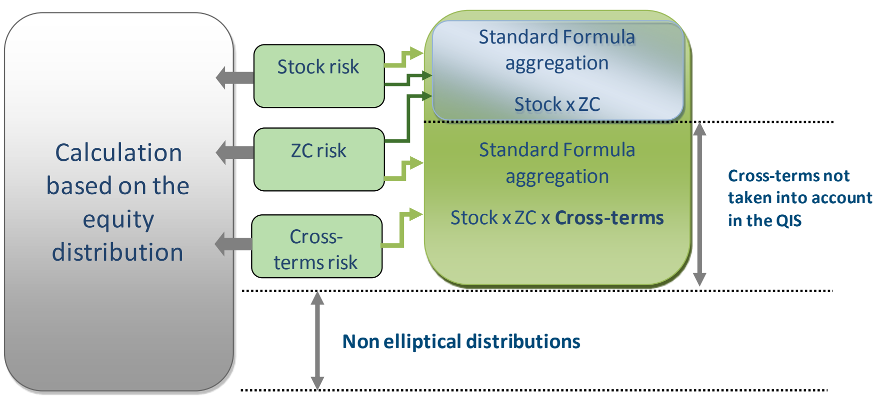

Figure 16 : summary of the differences between "standard formula" capitals and internal model capitals

5. Alternative aggregation method

The principle behind this method is to infer the results obtained with the parametric model[9] in the risk aggregation method. We have observed above that the calculation of the global capital by aggregation in a non-linear situation lead to a different amount than that found with the "internal model" C_{NS}. This is essentially due to the $\varepsilon_S.\varepsilon_{ZC}$ or $\varepsilon_S.\varepsilon_{ZC}^2$ cross-terms that are not taken into account (as they are cancelled out in succession) in the "standard formula" approach. The sole use of marginal capitals is therefore not sufficient to satisfactorily measure the effects of diversification.
In order to capture this phenomenon, without necessarily using an entirely integrated NS internal model (relatively complex modelling), we propose a method that is easily implemented and based on an ALM projection tool that enables to carry out valorisations only at t=0.

1. Description of the method

We shall detail here the principle stages of the alternative method:

Stage 0 - determination of the marginal capitals:
Calculation of the stand-alone capitals of each risk factor (by variation of the equity at t=0 due to an immediate shock on a risk factor).

Stage 1 - obtaining an equity distribution:

- **Step 1** : establishment of risk factors' tuples (stock, interest rate, mortality,...) These tuples are not necessarily vectors created from simulations and they can be established "manually". Each tuple represents a deformation of the initial conditions.
- **Step 2**: calculation of the amounts of the equity in relation to each tuple, using the projection model at t=0.
- **Step 3**: calibration of a parametric form of the "equity" variable on the previous tuples.
- **Step 4**: simulation of the risk factors (modelling of marginal distributions and of the copula that links them together).
- **Step 5**: obtaining the equity "distribution" using the previous simulations and the parametric form calibrated in Step 3.

Stage 2 - adjustment of the correlations that reveal the "non-linear" diversification:
Consider three risks, *X, Y* and *Z* to describe this point.
The capital calculated on the basis of the distribution in step 5 is noted C_{param} and the elementary capitals calculated from the parametric form (by cancelling out all the other risks) are noted

[9] On the basis of a calibration that requires few observations (see hereunder).

$C_{param}^X, C_{param}^Y, C_{param}^Z$. If R is the correlation matrix that enables to reproduce the non-linear diversification, one obtains:

$$C_{param} = \sqrt{\begin{pmatrix} C_{param}^X & C_{param}^Y & C_{param}^Z \end{pmatrix} . R . \begin{pmatrix} C_{param}^X \\ C_{param}^Y \\ C_{param}^Z \end{pmatrix}} . \qquad (*)$$

The minimal standard R, for which (*) is respected, is found. This leads to the following optimisation program:

$$R^* = ArgMin_R \|R\| \text{ under the constraint (*),}$$

with $\|D\| = \sqrt{tr(D.D')}$.

Stage 3 - calculation of the global capital that integrates the "non-linear" diversification:
If $C_X^{SF}, C_Y^{SF}, C_Z^{SF}$ are the elementary capitals calculated in stage 0, the global capital C_{SF}^* is determined with the following relation:

$$C_{SF}^* = \sqrt{\begin{pmatrix} C_X^{SF} & C_Y^{SF} & C_Z^{SF} \end{pmatrix} . R^* . \begin{pmatrix} C_X^{SF} \\ C_Y^{SF} \\ C_Z^{SF} \end{pmatrix}} .$$

Comments:

- When only risks *X* and *Y* are considered, the constraint (*) has a single R^* solution:

$$R^* = \frac{{C_{param}}^2 - {C_{param}^X}^2 - {C_{param}^Y}^2}{2 . C_{param}^X . C_{param}^Y} .$$

- The coefficients of the R^* matrix enable to "reproduce" the effects of the diversification that are due to the parametric form but they do not correspond, generally speaking, to the correlation coefficients. These adjustment factors are used in order to integrate the marginal capitals by using the standard formula, but in no case are these Pearson's correlation coefficients of underlying variables[10].

2. Implementing the alternative method

In the section concerning the analysis of deviations, the calibration of the function linking equity and risk factors is based on all of the 5000 primary simulations. A comprehensive NS projection was therefore carried out to calibrate this function.

However, it is often operationally difficult to carry out such a large number of simulations as these imply significant computation times. Devineau and Loisel (2009) have developed an acceleration algorithm that enables to reduce the number of primary simulations in a NS calculation.

[10] In some cases, they can be greater than 1 in absolute value.

The principle consists in calculating for each primary simulation the norm[11] that is associated with the underlying risk factors and then performing NS projections in a decreasing order until reaching the stability of the $[0.5\% \times P]$worst values of equity.

The NS accelerator converges after 300 primary simulations on the portfolio under consideration. In this section, we have taken the parametric structure introduced above and calibrated it on the basis of the 300 pairs of factors on the biggest norms. This enables to adjust the parametric form on the extreme percentiles of the equity distribution, as the calculation of the economic capital rests on these elements.

Hereunder is the point cloud used in the calibration process:

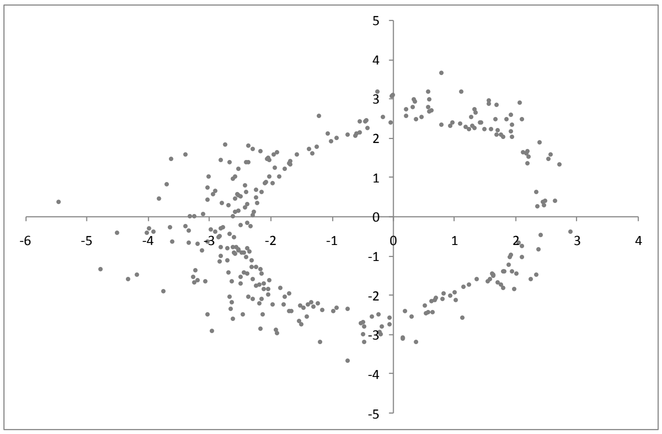

Figure 17 : sample of the calibration process

After having estimated the coefficients in a parametric form, we determined its value for each of the pairs $\left(\varepsilon_S^p, \varepsilon_{ZC}^p\right)_{p=1,\ldots,P}$. Using the parametric distribution of equity, we then calculated the C_{param} capital:

$C_{param} = 1219.6.$

We determined the parametric marginal capitals in the same manner:

C_{param}^S	C_{param}^{ZC}
555.7	729.5

Table 10 : calculations of "parametric" marginal capitals

Finally, the following relation:

$$R^* = \frac{{C_{param}}^2 - {C_{param}^S}^2 - {C_{param}^{ZC}}^2}{2.C_{param}^S . C_{param}^{ZC}},$$

---

[11] The norm of a $(\varepsilon_S, \varepsilon_{ZC})$ pair corresponds to $\sqrt{\varepsilon_S{}^2 + \varepsilon_{ZC}{}^2 - 2.\rho.\varepsilon_S.\varepsilon_{ZC}}$, where ρ is Pearson's coefficient of ε_S et ε_{ZC}.

enabled us to measure the adjustment factor: $\mathrm{R}^* = 79.8\%$.

The following table lists the results that were obtained:

C_{NS}	C_{SF}^*	Difference
1209.7	1243.3	2.8%

Table 11 : adjusted comparison of "standard formula" capitals and NS capitals

Here $C_{SF}^* = \sqrt{{C_S^{SF}}^2 + {C_{ZC}^{SF}}^2 + 2.R^*.C_S^{SF}.C_{ZC}^{SF}}$ is the "standard formula" capital, calculated with an adjustment factor R^* that enables to integrate the non-linear diversification.

One observes that the difference between C_{NS} and C_{param}capitals is only of 0.6% and that the differences between marginal capitals are inferior to 2%. By using R^*in order to aggregate the stand-alone capital due to "standard formula" shocks, we obtain a global capital C_{SF}^* that is relatively close to the C_{NS} capital determined by NS projections (deviation of 2.8%).

3. Limits and points of attention

In some cases the alternative aggregation method can lead to disappointing results. Depending on the complexity of the modelled products, it can become difficult to adjust a parametric form to the "equity" function. For the purpose of illustrating this point, we carried out an additional study of an annuity product with a revalorisation of the guarantees indexed on inflation. This product was used in a projection with an internal model fed with economic scenarios calibrated as of the 31/12/2008. For this study, in addition to the risks and interest rates, we also factored in the inflation risk.

We describe ε_I as the inflation risk factor retrieved from the economic table according to an application similar to that used for the other risk factors. Consider the following profile of parametric form:

$$E_1 = int + A_1.\varepsilon_S + A_2.\varepsilon_S^2 + A_3.\varepsilon_{ZC} + A_4.\varepsilon_{ZC}^2 + A_5.\varepsilon_S.\varepsilon_{ZC} + \\ +A_6.\varepsilon_I \ + A_7.\varepsilon_I \ \varepsilon_{ZC} + A_8.\varepsilon_S \ \varepsilon_I \ .$$

We estimated the coefficients of this function with the least squares method on over 500 simulations of the biggest standards[12] and we then determined its value for each pair $\left(\varepsilon_S^p, \varepsilon_{ZC}^p, \varepsilon_I^p\right)_{p=1,\ldots,P}$. The parametric equity distribution provided the following C_{param} capital:

$$C_{param} = 578.7.$$

We obtained the following parametric marginal capitals:

[12] The number of observations was increased to strengthen the estimation. This parametric form uses more regressors than in the case presented above.

C^{S}_{param}	C^{ZC}_{param}
559.8	149.9

Table 12 : calculations of "parametric" marginal capitals

These values enabled us to calculate the adjustment factor: $\mathrm{R}^{*} = -0.5\%$.
The following table lists the results that were obtained:

C_{NS}	C^{*}_{SF}	Difference
627.2	580.0	7.5%

Table 13 : adjusted comparison of "standard formula" capitals and NS capitals

Here, the difference between C_{NS} and C_{param}capitals is 7.7%. This situation results from a bad match between the parametric distribution of equity and the distribution obtained with the NS calculation, as shown in the QQ-plot hereunder:

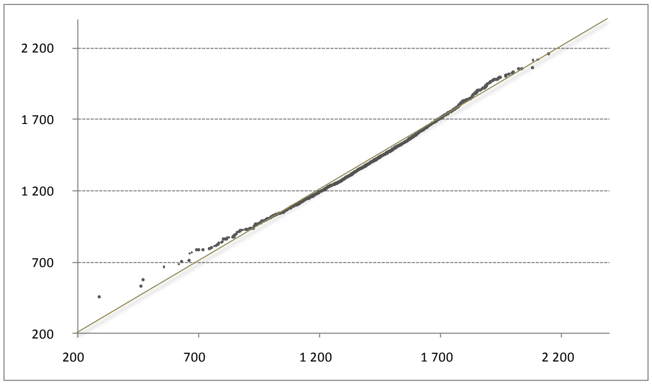

Figure 18 : QQ-plot E_1 (abscissa) vs $\widehat{E}_1$ (ordinate)

The use of the R^{*}determined with the imperfectly adjusted parametric form leads to a CSF* capital that is significantly lower than the reference C_{NS} capital (7.5% difference).
In order to improve this result, the choice of regressors adapted to this type of product should be refined.

6. Conclusion

In this paper, we have presented a formalisation of the "standard formula method" and of the NS approach. Having established a theoretic context for the application of the "standard formula" method, we have demonstrated that the internal models used by companies do not generally guarantee the validity of the hypotheses required for this type of aggregation. Indeed, even if the profile of the equity variable in relation to risk factors leads to marginal capital values that are very close, and even similar, for these two methods, the levels of the global capital may differ greatly. We have shown that the aggregation error committed by the standard formula is essentially due to two phenomena:

- the level of equity is not adjusted in terms of underlying risk factors,
- the "standard formula" method does not take into account the "cross-effects" of the different risks that are being considered.

To address this issue, we have developed an alternative technique of aggregation that uses very few simulations to satisfactorily capture the main part of the diversification among risks. This method aims to adjust the correlation coefficients, so as to obtain "standard formula" results and internal models that are as close as possible, and to explain the deviations. The quality of the adjustment depends in theory on the convergence rate of the McLaurin expansion series of the equity variable for a compact and convex set that includes all the values of risk factors that lead to net situations included within the best estimate and a percentile at a level greater than 99.5%. The analysis of this conversion rate and the associated estimation problems are to be analysed in future studies. We would also like to add that studies concerning the integration of other risks, such as mortality are currently under way and are to be published in the future. Finally, dynamic correlation models incorporating correlation crises (see Biard et al., 2008 and Fisher et al., 2008) could be introduced to better capture diversification effect on longer time horizons.

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Appendix 1: Elliptic distributions

The family of elliptic distributions is a category of multivariate distributions that share the main properties of normal distributions while enabling to model extreme dependence and other forms of non Gaussian dependence. For a detailed presentation of these distributions, see Embrechts, Lindskog and Mac Neil (2003).

Definition

If X is a random vector with a dimension n. If $\mu \in R^n et\, \Sigma$ is a positive-definite matrix of dimensions $n \times n$ for which the characteristic function $\varphi_{X-\mu}$of $X-\mu$ is a function of quadratic form $t'.\Sigma.t$, i.e. $\varphi_{X-\mu}(t) = \phi(t'.\Sigma.t)$, then X is said to be an elliptic distribution.

The notation is then $X \sim E_n(\mu, \Sigma, \phi)$.

The function ϕ is called the characteristic generator of the distribution.

Characterisation theorem

One obtains $X \sim E_n(\mu, \Sigma, \phi)$ with $rg(\Sigma) = k$ only if there is a random variable $R \geq 0$ independent from the random vector U of dimension k uniformly distributed on the unit sphere $\{z \in R^k \;/ : z'.z = 1\}$ and an A matrix of dimension $n \times k$ with $AA' = \Sigma$ for which :

$$X =_L \mu + RAU.$$

Consequence : in dimension 1, the family of elliptic distributions corresponds exactly to that of symmetrical distributions.

Theorem : If $X \sim E_n(\mu, \Sigma, \phi)$, B is a matrix of dimension $q \times n$ and $b \in R^q$. One obtains:

$$b + BX \sim E_n(b + B\mu, B\Sigma B', \phi)$$

Corollary : If $X \sim E_n(\mu, \Sigma, \phi)$ and $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ where X_1 (resp. X_2) and μ_1 (resp. μ_2) are dimension vectors r (resp. $n-r$) et Σ_{11} (resp. Σ_{22}) is a matrix of dimension $r \times r$ (resp. $(n-r) \times (n-r)$).

Then:

$$X_1 \sim E_r(\mu_1, \Sigma_{11}, \phi) \text{ and } X_2 \sim E_{n-r}(\mu_2, \Sigma_{22}, \phi).$$

Consequences :

- The marginal distributions $X_1, X_2, \dots, X_n$ of an elliptic vector $X = (X_1, \dots, X_n)'$ are elliptic and of the same type (i.e. same characteristic generator).
- Any linear combination, $Y = C + \sum_{i=1}^{n} a_i.X_i$ of the marginal distribution of an elliptic vector is elliptic and has the same characteristic generator.
- With $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ (resp. $\tilde{Y}$) the marginal distributions (resp. variable Y), centred and reduced. The above results enable to demonstrate that variables $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ et $\tilde{Y}$ are elliptic and have the same generator. They therefore follow similar distributions

Appendix 2: Proof of the aggregation formula

In this section we prove the aggregation formula presented in part 3.4.2.

Proposition : assuming that a random vector (X, Y, Z) is elliptic and that the equity distribution is linear according to *X*, *Y* et *Z*, i.e. in the form $E = C + a.X + b.Y + c.Z$ where $a, b, c \in R^*$
Then:

$$C = \sqrt{C_X^2 + C_Y^2 + C_Z^2 + 2.sg(ab).\rho_{X,Y}.C_X.C_Y + 2.sg(ac).\rho_{X,Z}.C_X.C_Z + 2.sg(bc).\rho_{Y,Z}.C_Y.C_Z}$$

Proof
Under the hypotheses stated in the proposition, the standard distributions of the variables *X*, *Y*, *Z* and *E* are identical[13].
It is easily demonstrated that for[14] any $\alpha \in]0;1[: q_\alpha(\tilde{E}) = q_\alpha(\tilde{X}) = q_\alpha(\tilde{Y}) = q_\alpha(\tilde{Z})$, $\forall a, b, c \in R^*$.

Let us examine the calculation of marginal capitals. Without loss of generality, let us consider the economic capital C_X . There is the relation: $C_X = E(C + a.X) - q_\alpha(C + a.X)$.
As the *X* factor is centred, one obtains: $C_X = -q_\alpha(a.X)$.
Therefore:

- *if* $a \leq 0 : C_X = -q_\alpha(a.X) = -a.q_{1-\alpha}(X) = a.q_\alpha(X) = a.\sigma_X.q_\alpha(\tilde{X}) = a\sigma_X.q_\alpha(\tilde{E})$,
- *if* $a \geq 0 : C_X = -q_\alpha(a.X) = -a.q_\alpha(X) = -a\sigma_X.q_\alpha(\tilde{E})$.

Let $C_X = -sg(a).a.\sigma_X.q_\alpha(\tilde{E})$.
Similarly, one obtains :

$$\mathrm{C_Y = -sg(b).b.\sigma_Y.q_\alpha(\tilde{E})},$$

And,

$$\mathrm{C_Z = -sg(c).c.\sigma_Z.q_\alpha(\tilde{E})}.$$

Consider the expression presented in part 3.4.2 :

$$q_\alpha(E) = \mu_E - \sqrt{\begin{array}{c} a^2.\sigma_X^2.\left[q_\alpha(\tilde{E})\right]^2 + b^2.\sigma_Y^2.\left[q_\alpha(\tilde{E})\right]^2 + c^2.\sigma_Z^2.\left[q_\alpha(\tilde{E})\right]^2 \\ +2\rho_{X,Y}.a\sigma_X.\left[q_\alpha(\tilde{E})\right] \quad .b\sigma_Y.\left[q_\alpha(\tilde{E})\right] \\ +2\rho_{X,Z}.a\sigma_X.\left[q_\alpha(\tilde{E})\right] \quad .c\sigma_Z.\left[q_\alpha(\tilde{E})\right] \quad + 2\rho_{Y,Z}.b\sigma_Y.\left[q_\alpha(\tilde{E})\right] \quad .c\sigma_Z.\left[q_\alpha(\tilde{E})\right] \end{array}}.$$

[13] Refer to the previous appendix for a description of the properties of elliptic distributions.
[14] It might be useful to remember that an elliptic random variable is symmetrical.

With the results presented above, one obtains:

$$q_\alpha(E) = \mu_E - \sqrt{\begin{array}{c} {C_X}^2 + {C_Y}^2 + {C_Z}^2 \\ +2\rho_{X,Y}.\left(-sg(a).a.\sigma_X.q_\alpha(\tilde{E})\right).\left(-sg(b).b.\sigma_Y.q_\alpha(\tilde{E})\right).sg(ab) \\ +2\rho_{X,Z}.\left(-sg(a).a.\sigma_X.q_\alpha(\tilde{E})\right).\left(-sg(c).c.\sigma_Z.q_\alpha(\tilde{E})\right).sg(ac) \\ +2\rho_{Y,Z}.\left(-sg(b).b.\sigma_Y.q_\alpha(\tilde{E})\right).\left(-sg(c).c.\sigma_Z.q_\alpha(\tilde{E})\right).sg(bc) \end{array}}$$

$$= \mu_{FP} - \sqrt{C_X^2 + C_Y^2 + C_Z^2 + 2.sg(ab).\rho_{X,Y}.C_X.C_Y + 2.sg(ac).\rho_{X,Z}.C_X.C_Z + 2.sg(bc).\rho_{Y,Z}.C_Y.C_Z}.$$

Therefore:

$$C = \mu_E - q_\alpha(E)$$

$$= \sqrt{C_X^2 + C_Y^2 + C_Z^2 + 2.sg(ab).\rho_{X,Y}.C_X.C_Y + 2.sg(ac).\rho_{X,Z}.C_X.C_Z + 2.sg(bc).\rho_{Y,Z}.C_Y.C_Z}$$