



HAL
open science

Stability analysis for sampled-data systems with a time-varying period

Alexandre Seuret

► **To cite this version:**

Alexandre Seuret. Stability analysis for sampled-data systems with a time-varying period. 48th IEEE Conference on Decision and Control, CDC 2009, Dec 2009, Shangai, China. pp.6. hal-00403252v1

HAL Id: hal-00403252

<https://hal.science/hal-00403252v1>

Submitted on 9 Jul 2009 (v1), last revised 21 Sep 2009 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Stability analysis for sampled-data systems with a time-varying period

Alexandre Seuret
alexandre.seuret@gipsa-lab.inpg.fr

Department of Automatic Control, GIPSA-Lab
NeCS Team, Grenoble, France

Abstract—This paper deals with a new analysis of the stability of linear systems with sampled-data inputs. Inspired by the input-delay approach and the stability of impulsive systems, the proposed method provides novel stability conditions. The stability analysis concerns both constant and time-varying sampling periods. More precisely, this article focus on ensuring stability of systems under two successive sampling periods. This result allows considering one of the periods greater than the theoretical bound, based on an estimation of the convergence rate. The delay-dependent conditions are expressed using computable simple linear matrix inequalities. Several examples show the efficiency and the limitation of such stability criteria.

I. INTRODUCTION

In the last decades, a large attention has been taken to Networked Control Systems (NCS) (see [5], or [15]). Such systems are a control systems containing several distributed plants which are connected through a communication network. In such applications, a heavy temporary load of computation in a processor can corrupt the sampling period of a certain controller. Nevertheless, the sampling period can be scheduled in the design in order to avoid this load. In both cases, the variation of the sampling period will affect the stability properties.

Sampled-data systems have already been studied in the literature [2], [16], [17] and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However the case of asynchronous samplings still leads to several open problems. Recently, several articles drive the problem of time-varying periods based on a discrete-time approach, [4], [13]. Recent papers considered the modelling of continuous-time systems with sampled-data control in the form of continuous-time systems with delayed control input. In [3], a Lyapunov-Krasovskii approach is introduced. Improvements are provided in [8], using the small gain theorem and in [9] by an impulsive systems approach. Nevertheless, these results are still more conservative than the ones from discrete-time approaches. However a discrete-time approach is less interesting in the case of uncertain or time-varying parameters.

The article proposes novel stability conditions to ensure stability of linear and time-varying systems. Improved stability conditions based on the continuous-time approach and the stability of impulsive systems developed in [9] is provided.

More especially, this paper cope with both stability and performances of the systems.

This article is organized as follows. the next section formulate the problem. Section III and IV respectively deals with the analysis of asymptotic and exponential stability. Section V cope with the stability of a systems under two sampling periods. Some examples and simulations are provided and show the efficiency of the method in Section VI.

Notations. Throughout the article, for a n -dimensional state vector x and a non-negative delay τ , x_t denotes a function such that $x_t(\theta) = x(t - \theta)$ for all $\theta \in [-\tau, 0]$. The superscript 'T' stands for the matrix transposition. The notation $P > 0$ for $P \in \mathbb{R}^{n \times n}$ means that P is a symmetric and positive definite matrix. The symbols I and 0 represent the identity and the zero matrices of the appropriate dimension.

II. PROBLEM FORMULATION

Consider the linear system with a sampled-data input:

$$\dot{x}(t) = Ax(t) + Bu(t_k) \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state variable and the input vector. The matrices A and B are constant and of appropriate dimension. We are looking for a piecewise-constant control law of the form $u(t) = u_d(t_k)$, $t_k \leq t < t_{k+1}$, where u_d is a discrete-time control signal and $0 = t_0 < t_1 < \dots < t_k < \dots$ are the sampling instants. Our objective is to ensure the stability of the system together with a given state-feedback controller of the form:

$$u(t) = Kx(t_k), \quad t_k \leq t < t_{k+1}. \quad (2)$$

Assume that the difference between two successive sampling instants satisfies

$$0 < t_{k+1} - t_k \leq \tau_m \quad \forall k \geq 0. \quad (3)$$

Several authors investigated in guaranteing the stability of such a system. In [3], a first approach was introduced. It allows assimilating sampling effects as the ones of a particular delay. Substituting (2) into (1), we obtain the following closed-loop system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - \tau(t)), \\ \tau(t) &= t - t_k, \quad t_k \leq t < t_{k+1}. \end{aligned} \quad (4)$$

where $A_d = BK$.

From (3), it follows that $\tau(t) \leq \tau_m$ since $\tau(t) \leq t_{k+1} - t_k$. We will further consider (4) as the system with uncertain and bounded delay. However the stability conditions were designed to deal with all kind of delay functions. As sampled-data systems are systems subject to a particular delay, this approach was finally conservative. In [9], the authors introduce a new type of Lyapunov-Krasovskii which depends linearly on the delay function. This allows obtaining less conservative results but some conservatism remains. In this article, a new Laypunov functional was introduced to especially consider sampled delays. This method leads to less conservative result.

III. ASYMPTOTIC STABILITY ANALYSIS

A. Time-varying sampling period

Consider system (1) with a time-varying sampling period satisfying (3). The following theorem holds:

Theorem 1: Assume that there exist symmetric positive definite matrices P, R and $S \in \mathbb{R}^{n \times n}$ and a matrix $N \in \mathbb{R}^{2n \times n}$ such that satisfy:

$$\begin{aligned} \Pi_1 + \tau_m \Pi_2 &< 0, \\ \begin{bmatrix} \Pi_1 & \tau_m N \\ * & -\tau_m R \end{bmatrix} &< 0, \end{aligned} \quad (5)$$

where

$$\begin{aligned} \Pi_1 &= M_1^T P M_3 + M_3^T P M_1 - M_2^T S M_2 \\ &\quad - N M_2 - M_2^T N^T, \\ \Pi_2 &= M_2^T S M_3 + M_3^T S M_2 + M_3^T R M_3 \end{aligned}$$

and the matrices M_i , for $i = 1, 2, 3$ are given by:

$$\begin{aligned} M_1 &= \begin{bmatrix} I & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} I & -I \end{bmatrix}, \\ M_3 &= \begin{bmatrix} A & A_d \end{bmatrix}, \end{aligned}$$

The system (1) is thus asymptotically for any time-varying period less than τ_m .

Proof: Inspired from the technic introduced in [9], consider the following form of functional:

$$\begin{aligned} V(x_t) &= x^T(t) P x(t) + (\tau_m - \tau(t)) \zeta_0^T(t) S \zeta_0(t) \\ &\quad + (\tau_m - \tau(t)) \int_{t_k}^t \dot{x}^T(s) R \dot{x}(s) ds \end{aligned} \quad (6)$$

where $\zeta_0(t) = x(t) - x(t_k)$, consequently, $\dot{\zeta}_0(t) = \dot{x}(t)$. Just before the sampling instant t_k , the Lyapunov-Krasovskii functionals $V(t_k)$ is strictly greater than $x(t_k)^T P x(t_k)$. Just after the sampling instant, the two last terms of the functional are zero (since $t = t_k$ and $\tau(t_k) = 0$). Then it means that the functional is decreasing discontinuously at each sampled intervals (see [9] for more details). To prove the stability of the system, one has to ensure that V is decreasing within each period. An expression of the derivative of V during one sampling period is derived:

$$\begin{aligned} \dot{V}(x_t) &= 2x^T(t) P \dot{x}(t) + 2(\tau_m - \tau(t)) \zeta_0^T(t) S \dot{x}(t) \\ &\quad + (\tau_m - \tau(t)) \dot{x}^T(t) R \dot{x}(t) - \zeta_0^T(t) S \dot{\zeta}_0(t) \\ &\quad - \int_{t_k}^t \dot{x}^T(s) R \dot{x}(s) ds \end{aligned}$$

The next step of the proof consists in rewriting the expression of \dot{V} using the vector $\xi(t) = [x^T(t) \quad x^T(t_k)]^T$. It is

easy to see that $x(t) = M_1 \xi(t)$, $\dot{x}(t) = A x(t) + B x(t_k) = M_3 \xi(t)$ and $x(t) - x(t_k) = M_2 \xi(t)$. This leads to:

$$\begin{aligned} \dot{V}(x_t) &= \xi^T(t) [2M_1^T P M_3 - M_2^T S M_2 \\ &\quad + \tau_m (2M_3^T S M_2 + M_3^T R M_3) \\ &\quad + \tau(t) (-M_3^T R M_3 - 2M_3^T S M_2)] \xi(t) \\ &\quad - \int_{t_k}^t \dot{x}^T(s) R \dot{x}(s) ds \end{aligned}$$

We introduce the following term $2\xi^T(t) N M_2 \xi(t) = 2\xi^T(t) N \int_{t_k}^t \dot{x}(s) ds$. Using a classical bounding ensuring that:

$$\begin{aligned} 2\xi^T(t) N M_2 \xi(t) &\leq \tau(t) \xi^T(t) N R^{-1} N^T \xi(t) \\ &\quad + \int_{t_k}^t \dot{x}^T(s) R \dot{x}(s) ds, \end{aligned}$$

the following inequality is obtained:

$$\begin{aligned} \dot{V}(x_t) &\leq \xi^T(t) [\Pi_1 + \tau_m \Pi_2 \\ &\quad + \tau(t) (N R^{-1} N^T - \Pi_2)] \xi(t) \end{aligned}$$

This inequality has to be satisfied for all values of the delay $\tau(t) \in [0, \tau_m[$. As this equation depends linearly on the delay function, it is necessary and sufficient to ensure the negativity of the matrix at $\tau(t) = 0$ and τ_m . Applying the Schur complement on the vertice $\tau(t) = \tau_m$ leads to the stability conditions of Theorem 1. ■

The stability conditions of Theorem 1 are very similar to the ones from [9]. The only difference comes from the integral term with the matrix R . In [9], the authors base the functional on the classical double integral term of a Lyapunov-Krasovskii function $\int_{-\tau(t)}^0 \int_{t+\theta}^t \dot{x}^T(s) R \dot{x}(s) ds d\theta$. By an integration by part, this term is expressed as $\int_{t_k}^t (\tau_m - t + s) \dot{x}^T(s) R \dot{x}(s) ds$ or equivalently as $\int_{t_k}^t (\tau_m - \tau(t) + \tau(s)) \dot{x}^T(s) R \dot{x}(s) ds$. To understand the difference between this functional and the one used in Theorem 1, two aspects are considered. The first one is that the term $\dot{x}^T(t) R \dot{x}(t)$ in the derivative of V is multiplied by the constant gain τ_m in [9] while it is multiplied by $\tau_m - \tau(t)$. This is less conservative since it depends on the delay variations. The second aspect to see the reduction of the conservatism is to split the integral into two terms. First, the one use in Theorem 1 and the second one $\int_{t_k}^t \tau(s) \dot{x}^T(s) R \dot{x}(s) ds$. The derivative of the second term is $\tau(t) \dot{x}^T(t) R \dot{x}(t)$ which is only positive definite. This term makes the conditions more conservative than the ones from Theorem 1.

B. Constant sampling period

Following the line of [9], Theorem 1 can be improved in the case of constant sampling period.

Theorem 2: Assume that there exist symmetric positive definite matrices P, R and $S \in \mathbb{R}^{n \times n}$ and a matrix $N \in \mathbb{R}^{2n \times n}$ such that satisfy:

$$\bar{\Pi}_1 + \tau_m \bar{\Pi}_2 < 0, \quad \begin{bmatrix} \bar{\Pi}_1 & \tau_m N \\ * & -\tau_m R \end{bmatrix} < 0 \quad (7)$$

where

$$\begin{aligned} \bar{\Pi}_1 &= \Pi_1 - (M_1 - M_2)^T U M_2 - M_2^T U^T (M_1 - M_2) \\ \bar{\Pi}_2 &= \Pi_2 + (M_1 - M_2)^T U M_3 - M_3^T U^T (M_1 - M_2) \end{aligned}$$

The system (1) is thus asymptotically stable for the constant sampling period less than τ_m .

Proof: The proof follows the line of Theorem 1. Consider the functional:

$$V_1(x_t) = V(x_t) + 2(\tau_m - \tau(t))x^T(t_k)U\zeta_0(t)$$

Note that the additional terms of the functional are not necessary positive. However the fact that the sampling period is known allows ensuring that the term $2(\tau_m - \tau(t))x^T(t_k)U\zeta_0(t)$ is continuous and equal to zero at all sampling instants. This is the reason why this term can only be considered in the particular the case of constant sampling period. The differentiation of V_1 along the trajectories of (1) leads to:

$$\dot{V}_1(x_t) = \dot{V}(x_t) + 2(\tau_m - \tau(t))x^T(t_k)U\dot{\zeta}_0(t) - 2x^T(t_k)U\zeta_0(t)$$

Noting that $x(t_k) = (M_1 - M_2)\xi(t)$, the derivative of V_1 satisfies:

$$\dot{V}_1(x_t) = \xi^T(t) [\bar{\Pi}_1 + \tau_m \bar{\Pi}_2 + \tau(t)(NR^{-1}N^T - \bar{\Pi}_2)] \xi(t)$$

Applying the same technic as in Theorem 1, system (1) is asymptotically stable for all constant sampling periods, τ_m , that satisfy (7). ■

C. System with polytopic type uncertainties

An extension to the case of uncertainties in the system parameters can be dealt by considering system (1) and with A and A_d from the uncertain polytope given by $\forall t \in \mathbb{R}^+$, $\Omega(t) = \sum_{k=1}^M \lambda_k(t)\Omega_k$ where $\forall t \in \mathbb{R}^+$, $\sum_{k=1}^M \lambda_k(t) = 1$, $\forall k = 1, \dots, M$, $0 \leq \lambda_k(t)$. The Ω vertices of the polytope are described by $\Omega_k = [A(k) \ A_d(k)]$. Since the conditions of Theorems 1 and 2 are non linear with respect matrices A and A_d because of the term $M_3^T R M_3$, a direct extension to the case of polytopic systems is not straightforward. However they can be easily adapted as it is exposed in the following theorem:

Theorem 3: Assume that there exist symmetric positive definite matrices P , R and $S \in \mathbb{R}^{n \times n}$ and a matrix $N \in \mathbb{R}^{2n \times n}$ such that satisfy:

$$\begin{bmatrix} \Pi_1^i + \tau_m \Pi_2^i & \tau_m M_3^{iT} R \\ * & -\tau_m R \end{bmatrix} < 0, \quad (8)$$

$$\begin{bmatrix} \Pi_1^i & \tau_m N \\ * & -\tau_m R \end{bmatrix} < 0,$$

where M_i , for $i = 1, 2$ are given in Theorem 1 and

$$\begin{aligned} \Pi_1^i &= M_1^T P M_3^i + M_3^{iT} P M_1 - M_2^T S M_2 \\ &\quad - N M_2 - M_2^T N^T, \\ \Pi_2^i &= M_2^T S M_3^i + M_3^{iT} S M_2 \end{aligned}$$

and $M_3^i = [A^i \ A_d^i]$. The system (1) is thus asymptotically stable for the sampling period less than τ_m .

Proof: First note that the second condition of Theorem 1 is linear with respect to the system parameters A and A_d . The application to polytopic systems is straightforward. However the first inequality is not linear. Noting that

$M_3^T R M_3$ can be rewritten as $(M_3^T R)R^{-1}(R M_3)$, the Schur complement allows obtaining the first condition Theorem 3. As both conditions become linear with respect to the matrices A and A_d , one has to solve simultaneously the LMIs for all the Ω vertices. ■

Remark 1: Theorem 3 can be extended to the case of constant sampling period by considering:

$$\begin{aligned} \bar{\Pi}_1^i &= \Pi_1^i - (M_1 - M_2)^T U M_2 - M_2^T U^T (M_1 - M_2) \\ \bar{\Pi}_2^i &= \Pi_2^i + (M_1 - M_2)^T U M_3 - M_3^T U^T (M_1 - M_2) \end{aligned}$$

IV. EXPONENTIAL STABILITY OF SYSTEMS WITH SAMPLED INPUTS

In this section, a study of the convergence rate of the solutions of sampled-data systems is provided. The objective is to ensure that the solutions are bounded by a decreasing exponential function and to estimate the exponential rate α of convergence. Consider the definition of exponential convergence dedicated to time-delay systems:

Definition 1: [10] For given $\alpha > 0$ and $\beta > 1$, the closed-loop system (1) is said to be α -stable, or 'exponentially stable with the rate α ', if its solution $x(t; t_0, \phi)$ satisfies:

$$|x(t, t_0, \phi)| \leq K |\phi| e^{-\alpha(t-t_0)}. \quad (9)$$

As a comment, this definition can be extended to the case of negative α . In this situation, α corresponds to the rate of divergence of the solutions of system (1). The following theorem holds:

Theorem 4: For a given $\alpha > 0$, assume that there exist symmetric positive definite matrices P , R and $S \in \mathbb{R}^{n \times n}$ and a matrix $N \in \mathbb{R}^{2n \times n}$ that satisfy:

$$\begin{aligned} \Pi_1^\alpha + \tau_m \Pi_2^\alpha &< 0, \\ \begin{bmatrix} \Pi_1^\alpha & \tau_m N \\ * & -\tau_m(1 + 2\alpha\tau_m)R \end{bmatrix} &< 0, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \Pi_1^\alpha &= \Pi_1 + 2\alpha(M_1^T P M_1 - M_2^T R M_2), \\ \Pi_2^\alpha &= \Pi_2 + 2\alpha M_2^T S M_2 \end{aligned}$$

and where the matrices M_i , for $i = 1, 2, 3$ are given in Theorem 1. The system (1) is thus exponentially stable with a decay rate α for any time-varying period less than τ_m .

Proof: Consider V as in (6). To prove the exponential stability, the requirement on V becomes $\dot{V}(x_t) + 2\alpha V(x_t) < 0$, where α is a scalar. Note that if α is a positive scalar, then the solutions of the systems which satisfies the previous inequality are exponentially stable with the decay rate α . If α is negative, the system is not necessarily unstable, but the solutions of the systems are bounded by an exponential function. Denoting $W^\alpha(x_t) = \dot{V}(x_t) + 2\alpha V(x_t)$, the following equality is provided:

$$\begin{aligned} W^\alpha(x_t) &= \xi^T(t) [2M_1^T P M_3 - M_2^T S M_2 + 2\alpha M_1^T P M_1 \\ &\quad + \tau_m(2M_3^T S M_2 + M_3^T R M_3 + 2\alpha M_2^T S M_2) \\ &\quad - \tau(t)(M_3^T R M_3 + 2M_3^T S M_2 + 2\alpha M_2^T S M_2)] \xi(t) \\ &\quad - \int_{t_k}^t \dot{x}^T(s)(1 - 2\alpha(\tau_m - \tau(t)))R\dot{x}(s)ds \end{aligned}$$

Consider the integral term of the equation above. It is split into two terms. The first one is

$-(1 - 2\alpha\tau_m) \int_{t_k}^t \dot{x}^T(s)R\dot{x}(s)ds$. The second one is $-2\alpha\tau(t) \int_{t_k}^t \dot{x}^T(s)R\dot{x}(s)ds$. Applying the Jensen's inequality, this last term is bounded by:

$$-\tau(t) \int_{t_k}^t \dot{x}^T(s)R\dot{x}(s)ds \leq -\int_{t_k}^t \dot{x}^T(s)ds R \int_{t_k}^t \dot{x}(s)ds \leq -\xi(t)M_2^T R M_2 \xi(t)$$

The term $2\xi^T(t)N M_2 \xi(t) = 2\xi^T(t)N \int_{t_k}^t \dot{x}(s)ds$ is introduced. Using a classical bounding ensuring that

$$2\xi^T(t)N M_2 \xi(t) \leq \tau(t)\xi^T(t)N\{(1 - 2\alpha\tau_m)R\}^{-1}N^T\xi(t) + (1 - 2\alpha\tau_m) \int_{t-\tau(t)}^t \dot{x}^T(s)R\dot{x}(s)ds,$$

Combining the previous inequalities, the following inequality is obtained

$$W^\alpha(x_t) \leq \xi^T(t) [\Pi_1^\alpha + (\tau_m - \tau(t))\Pi_2^\alpha + \tau(t)N\{(1 - 2\alpha\tau_m)R\}^{-1}N^T] \xi(t)$$

This inequality has to be satisfied for all values of the delay $\tau(t) \in [0, \tau_m[$. As this equation depends linearly on the delay function, it is necessary and sufficient to ensure the negativity of the matrix at $\tau(t) = 0$ and τ_m . Applying the Schur complement on the vertice $\tau(t) = \tau_m$ leads to the stability conditions of Theorem 4. Integrating the differential inequality over a sampling interval leads to

$$V^\alpha(x_t) < x^T(t_k)P x(t_k)e^{2\alpha(t-t_k)}$$

The previous expression implies the definition of the exponential stability (9). ■

Remark 2: This theorem can be also extended to the cases of constant sampling period and polytopic systems. For a space limitation, the details of the theorems are not presented here but are straight forward.

V. SYSTEM WITH TWO SAMPLING PERIODS

In this section, the system is assumed to have to sampling periods T_1 and T_2 , i.e. the sampling input is sampled first with T_1 then with T_2 and so on. Figure 2 presents an example of sampling delay $\tau(t)$ corresponding to this problem. This systems has been already exposes in [17] and dealt in [6]. The objective is to prove that system (1) for these two sampling periods even if one of them is greater than the maximum allowable sampling period. Exponential stability conditions from Theorem 4 allows to quantify the convergence and divergence of the solutions within each sampling periods. As suggested in [11], combine this convergence and divergence rates, stability conditions care derived. The following theorem is proposed:

Theorem 5: Consider system (1) subject to the two sampling periods T_1 and T_2 . If there exist $\lambda_1, \lambda_2, \alpha_1$ and $\alpha_2 \in \mathbb{R}$, symmetric positive definite matrices P_i, R_i and $S_i \in \mathbb{R}^{n \times n}$ and a matrix $N_i \in \mathbb{R}^{2n \times n}$ that satisfy for $i = 1, 2$:

$$\Pi_{1i}^{\alpha_i} + T_i \Pi_{2i}^{\alpha_i} < 0,$$

$$\begin{bmatrix} \Pi_{1i}^{\alpha_i} & T_i N_i \\ * & -T_i(1 + 2\alpha_i T_i)R \end{bmatrix} < 0, \quad (11)$$

$$P_1 < \lambda_2 P_2, \quad P_2 < \lambda_1 P_1, \quad (12)$$

and such that

$$c = \lambda_1 \lambda_2 e^{-2(\alpha_1 T_1 + \alpha_2 T_2)} < 1, \quad (13)$$

where

$$\begin{aligned} \Pi_{1i}^{\alpha_i} &= M_1^T P_i M_3 + M_3^T P_i M_1 - M_2^T S_i M_2 - N_i M_2 \\ &\quad - M_2^T N_i^T - (M_1 - M_2)^T U_i M_2 - M_2^T U_i^T (M_1 - M_2) \\ &\quad + 2\alpha_i (M_1^T P_i M_1 - M_2^T R_i M_2), \\ \Pi_{2i}^{\alpha_i} &= M_2^T S_i M_3 + M_3^T S_i M_2 + M_3^T R_i M_3 \\ &\quad + (M_1 - M_2)^T U_i M_3 - M_3^T U_i^T (M_1 - M_2) \\ &\quad + 2\alpha_i (M_2^T S_i M_2 + 2(M_1 - M_2)^T U_i M_3) \end{aligned}$$

and where the matrices M_j , for $j = 1, 2, 3$ are given in Theorem 1. The system (1) with the two sampling periods T_1 and T_2 is thus asymptotically stable.

Proof: The proof is based on the discrete-time Lyapunov theory together with Theorem 4 applied to the case of constant sampling period. The notations V^i , for $i = 1, 2$, denote the functional derived by P_i, R_i, S_i and U_i . Each functional V_i corresponds to a sampling period. Consider the k^{th} sampling instant. Without loss of generality, it is assumed that $t_{k+1} - t_k = T_1$ and $t_{k+2} - t_{k+1} = T_2$. According to Theorem 4, the conditions of Theorem 5 implies that:

$$\begin{aligned} \forall t \in [t_k, t_{k+1}[, \quad \dot{V}_1(x_t) + 2\alpha_1 V_1(x_t) < 0 \\ \forall t \in [t_{k+1}, t_{k+2}[, \quad \dot{V}_2(x_t) + 2\alpha_2 V_2(x_t) < 0 \end{aligned}$$

Integrating this differential inequality and noting that $V_1(x_{t_k}) = x^T(t_k)P_1 x(t_k)$ and $V_2(x_{t_{k+1}}) = x^T(t_{k+1})P_2 x(t_{k+1})$, the following bounds are obtained:

$$\begin{aligned} V_1(x_{t_{k+1}}) &\leq x^T(t_k)P_1 x(t_k)e^{-2\alpha_1 T_1} \\ V_2(x_{t_{k+2}}) &\leq x^T(t_{k+1})P_2 x(t_{k+1})e^{-2\alpha_2 T_2} \end{aligned} \quad (14)$$

Consider now the sampling instant $k = 2k'$. Without loss of generality, after this sampling instant, the input is sampled with the period T_1 . So at time t_{k+2} , we consider $V_1(x_{t_{k+2}}) = x^T(t_{k+2})P_1 x(t_{k+2})$. By vertu of (12) and of the definition of V_2 , one has:

$$V_1(x_{t_{k+2}}) \leq \lambda_2 x^T(t_{k+2})P_2 x(t_{k+2}) \leq V_2(x_{t_{k+2}})$$

Using (14), the following inequality is satisfied:

$$\begin{aligned} V_1(x_{t_{k+2}}) &\leq \lambda_2 x^T(t_{k+1})P_2 x(t_{k+1})e^{-2\alpha_2 T_2} \\ &\leq \lambda_2 V_2(x_{t_{k+1}})e^{-2\alpha_2 T_2} \end{aligned}$$

Applying the same bounded method yields:

$$\begin{aligned} V_2(x_{t_{k+1}}) &\leq \lambda_1 x^T(t_k)P_1 x(t_k)e^{-2\alpha_1 T_1} \\ &\leq \lambda_1 V_1(x_{t_k})e^{-2\alpha_1 T_1} \end{aligned}$$

Combining the two previous inequality leads to:

$$\begin{aligned} V_1(x_{t_{k+2}}) &\leq (\lambda_1 \lambda_2 e^{-2(\alpha_1 T_1 + \alpha_2 T_2)})^{k'+1} V_1(x_0) \\ \text{and } V_2(x_{t_{k+1}}) &\leq (\lambda_1 \lambda_2 e^{-2(\alpha_1 T_1 + \alpha_2 T_2)})^{k'} V_2(x_{t_1}) \end{aligned}$$

Then if condition (13) is satisfied, the terms $V_1(x_{t_{k+2}})$ tends to 0 as k is going to infinity. From the conditions from Theorem 5, the variation of V_1 are bounded by an exponential function between the sampling t_k and t_{k+2} . The same property also holds for V_2 . Finally, V_1 and V_2 converges asymptotically to zero and the solutions of system (1) are stable. ■

Time-varying period		
Theorems	τ_m	NDV
[3]	0.8696	$5n^2 + 2n$
[14]	0.8696	$7n^2 + n$
[12]	0.8871	$16n^2 + 3n$
[1]	1.009	$8n^2 + 4n$
[9]	1.1137	$3.5n^2 + 1.5n$
[7]	1.3659	$0.5(n^2 + n) + 1$
Th.1	1.6894	$3.5n^2 + 1.5n$
Constant period		
Theorems	τ_m	NDV
[9]	1.3277	$5n^2 + n$
[7]	1.3659	$0.5(n^2 + n) + 1$
Th.2	1.7198	$5n^2 + n$
Theoretical bound	1.72	-

TABLE I
MAXIMAL ALLOWABLE SAMPLING PERIOD τ_m

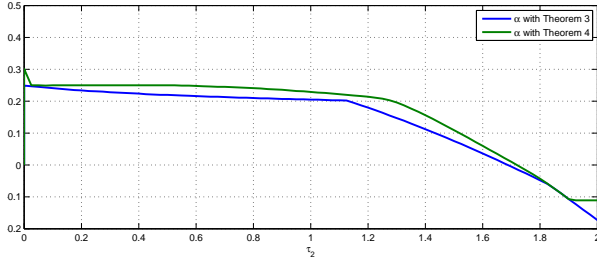


Fig. 1. Relation between the exponential decay rate α and the sampling period τ_m with the cases of constant and time-varying periods for Example 1

Remark 3: Condition (13) can be easily adapted to other sequences of sampling. For example, if the repeated sampling sequence is T_1, T_2 and T_1 once more, the conditions to ensure stability becomes:

$$c' = \lambda_1 \lambda_2 e^{-2(2\alpha_1 T_1 + \alpha_2 T_2)} < 1$$

Remark 4: By taking $U = 0$ in Theorem 5, it is possible to consider time varying sampling periods for T_1 and T_2 . However the condition should be rewritten differently:

$$c'' = \lambda_1 \lambda_2 e^{-2(2\alpha_1 T_{1min} + \alpha_2 T_{2max})} < 1$$

where T_{1min} is the smallest stabilizing sampling period and T_{2max} is the largest unstable sampling periods.

VI. EXAMPLES

A. Example 1

Consider system (1) from [3], [9] with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 0.375 & -1.15 \end{bmatrix}$$

The results are summarized in Table I for time-varying sampling and constant samplings periods. It can be seen that the results from Theorem 1 and 2 are less conservative than the one from the literature. Figure 1 shows the relation between the maximal convergence rate α and the sampling period in the constant and time-varying case. It shows that the conditions from Theorem 4 also holds for negative values of α .

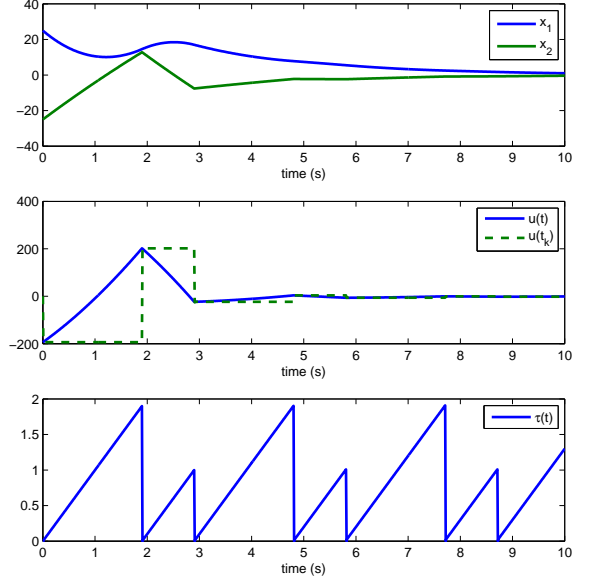


Fig. 2. Simulation of the states, input (continuous and sampled) and the sampling delay $\tau(t)$ for Example 2 with two sampling periods

Consider now the same system with two different sampling periods $T_1 = 1$ and $T_2 = 1.9$. From Theorem 2 (see Table I), the system is not stable if only the period T_2 is employed in the sampler. However, based on the conditions of Theorem 5 with $\alpha_1 = 0.21$ and $\alpha_2 = -0.11$, choosing $\lambda_1 = 1$ and using the principle of the generalized eigenvalues to minimize λ_2 under LMI constraints, we obtain $\lambda_2 = 1.0004$, (which finally means that P_1 and P_2 are approximately the same matrices). The last condition (13) holds since $c = 0.998$, which ensure the stability of the system with one stabilizing sampling period and an unstable one. Figure 2 shows the simulation of the states, input (continuous and sampled) and the sampling delay $\tau(t)$. In [6], the authors obtain less conservative result based on a discrete-time approach. They prove the system can be stable for instance when $T_1 = 1$ and $T_2 = 2.5$. Even if Theorem 5 do not ensure stability for such sampling periods it is still interesting since it can deal with system with parameter uncertainties.

B. Example 2

Consider the process model from [3] with

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 + g^2 \\ -1 \end{bmatrix}$$

where $|g_1| \leq 0.1$, and $|g_2| \leq 0.3$. With the state feedback gain $K = -[2.6884 \quad 0.6649]$, in [3] and in [9], it was respectively proven that the system is stable for any sampling interval smaller than 0.35, 0.4476. In this article, Theorem 3 ensures that the system is stable for all samplings sequence whose period is less than 0.602. Theorem 3 adapted to the case of constant period ensures that the system is stable all periods less than 0.703. It is clear that the stability conditions

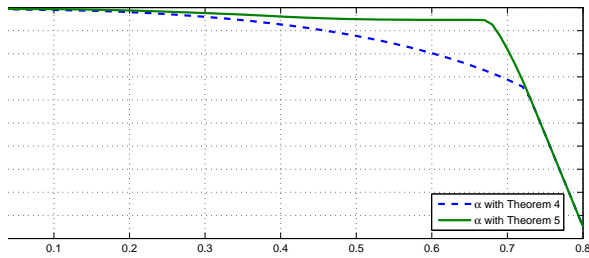


Fig. 3. Relation between the exponential decay rate α and the sampling period τ_m with the cases of constant and time-varying periods for Example 2

Theorems	τ_m	NDV
[3]	0.8696	$5n^2 + 2n$
[14]	0.8696	$7n^2 + n$
[12]	0.8871	$16n^2 + 3n$
[9]	1.9999	$3.5n^2 + 1.5n$
[1]	2.034	$8n^2 + 4n$
Th.1&2	1.9999	$3.5n^2 + 1.5n$

TABLE II

MAXIMAL ALLOWABLE SAMPLING PERIOD τ_m FOR EXAMPLE 3

are less conservative than the two others. Figure 3 shows the evolution of the convergence rate with respect to the maximum allowable sampling period τ_m for the cases of constant and time-varying periods.

C. Example 3

Consider system (1) from [3], [9] with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

The results on asymptotic stability are summarized in Table II. It can be seen that the maximal allowable sampling periods provided by Theorem 1 and 2 are the same as the ones from [9]. In [1], the authors derive less conservative result. This comes from the consideration of particular tools dedicated to time-delay systems. Nevertheless, this example shows the limits of the approaches in such case since the systems remain stable for some τ_m greater than 3.

VII. CONCLUSION

In this article, an analysis of linear invariant and time-varying systems with constant and time-varying sampling periods is provided. Tractable conditions are derived to ensure asymptotic stability and also to obtain an estimate of the convergence rate of the solutions. The examples show the efficiency of the method and the reduction of the conservatism compared to other results from the literature. Moreover, the article copes with the stability analysis of systems under several sampling periods. One of the periods can be greater than the allowable sampling delay. This has been treated by a continuous-time approach and allows considering uncertain or time-varying systems. Future works would focus on a reduction of the conservatism.

REFERENCES

- [1] Y. Ariba and F. Gouaisbaut, *Delay-dependent stability analysis of linear systems with time-varying delay*, Proc. of the 46th IEEE Conference on Decision and Control (New Orleans, LA, USA), 2007.
- [2] T. Chen and B.A. Francis, *Optimal sampled-data control systems*, Springer-Verlag, Berlin, Germany, 1995.
- [3] E. Fridman, A. Seuret, and J.-P. Richard, *Robust sampled-data stabilization of linear systems: An input delay approach*, *Automatica* **40** (2004), no. 8, 1141–1446.
- [4] H. Fujioka, *Stability analysis of systems with aperiodic sample-and-hold devices*, *Automatica* **45** (2009), no. 3, 771–775.
- [5] J.P. Hespanha, P. Naghshtabrizi, and Y. Xu, *A survey of recent results in networked control systems*, *Proceedings of the IEEE* **95** (2007), no. 1, 138–162.
- [6] X.G. Li, A. Cella, S.-I. Niculescu, and A. Reama, *Some remarks on the stability of networked control systems with periodic scheduling*, Submitted to the European Control Conference, 2009.
- [7] L. Mirkin, *Exponential stability of impulsive systems with application to uncertain sampled-data systems*, *IEEE Trans. on Automatic Control* **52** (2007), no. 6, 1109–1112.
- [8] ———, *Some remarks on the use of time-varying delay to model sample-and-hold circuits*, *IEEE Trans. on Automatic Control* **52** (2007), no. 6, 1009–1112.
- [9] P. Naghshtabrizi, J.P. Hespanha, and A.R. Teel, *Exponential stability of impulsive systems with application to uncertain sampled-data systems*, *Systems and Control Letters* **57** (2008), no. 5, 378–385.
- [10] S.-I. Niculescu, C.-E. de Souza, L. Dugard, and J.-M. Dion, *Robust exponential stability of uncertain systems with time-varying delays*, *Proceedings of the 33rd IEEE CDC*, 1994, p. 431.
- [11] X.-M. Sun, G.-P. Liu, D. Rees, and W. Wang, *Stability of systems with controller failure and time-varying delay*, *IEEE Trans. on Automatic Control* **53** (2008), no. 10, 2391–2396.
- [12] D. Yue, Q.-L. Han, and J. Lam, *Network-based robust H^∞ control of systems with uncertainty*, *Automatica* **41** (2005), no. 6, 640–644.
- [13] ———, *Stability and stabilization of nonuniform sampling systems*, *Automatica* **44** (2008), no. 12, 3222–3226.
- [14] D. Yue, Q.-L. Han, and C. Peng, *State feedback controller design for networked control systems*, *IEEE Trans. on Automatic Control* **51** (2004), no. 11, 640–644.
- [15] Sandro Zampieri, *A survey of recent results in Networked Control Systems*, Proc. 17th IFAC World Congress (Seoul, Korea), July 2008.
- [16] W. Zhang and M.S. Branicky, *Stability of networked control systems with time-varying transmission period*, Allerton Conf. Communication, Control, and Computing, October 2001.
- [17] W. Zhang, M.S. Branicky, and S.M. Phillips, *Stability of networked control systems*, *IEEE Control Systems Magazine* (2001), no. 21.