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▶ To cite this version:

Bernhard Hermann Haak, Birgit Jacob. Observation of Volterra systems with scalar kernels. Journal of Integral Equations and Applications, 2010, pp.0. hal-00402810

HAL Id: hal-00402810

https://hal.science/hal-00402810

Submitted on 8 Jul 2009

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CONTROL OF VOLTERRA SYSTEMS WITH SCALAR KERNELS

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ABSTRACT. Volterra observations systems with scalar kernels are studied. New sufficient conditions for admissibility of observation operators are developed. The obtained results are applied to time-fractional diffusion equations of distributed order.

1. Introduction

Consider the following scalar abstract Volterra system

(1)
$$x(t) = f(t) + \int_0^t a(t-s) Ax(s) ds.$$

Here, the operator A is supposed to be a closed operator with dense domain on a Banach space X having its spectrum contained in some open sectorial region of the complex plane, symmetric to the real axis and open to the left:

$$\sigma(A) \subseteq -\Sigma_{\omega}$$
 where $\Sigma_{\omega} = \{z \in \mathbb{C} : |\arg(z)| < \omega\}$

for some $\omega \in (0,\pi)$. Moreover, the resolvent of A is supposed to satisfy a growth condition of the type $\|\lambda R(\lambda,A)\| \leq M$ uniformly on each sector $\Sigma_{\pi-\omega-\varepsilon}$. Typical examples of such operators are generators of bounded strongly continuous semi-groups, where $\omega \leq \pi/2$. We call -A a sectorial operator of type $\omega \in (0,\pi)$, but we mention that 'sectoriality' may have different meanings for different authors in the literature.

The kernel function $a \in L^1_{loc}$ is supposed to be of sub-exponential growth so that its Laplace transform $\widehat{a}(\lambda)$ exists for all λ with positive real part. The kernel is called *sectorial of angle* $\theta \in (0,\pi)$ if

$$\widehat{a}(\lambda) \in \Sigma_{\theta}$$
 for all λ with positive real part.

We will consider only parabolic equations (1) in the sense of Pruess [26]. In the case that -A is sectorial of some angle $\omega \in (0, \pi)$ this is equivalent to require $\widehat{a}(\lambda) \neq 0$ and $\frac{1}{\widehat{a}(\lambda)} \in \varrho(A)$ for all λ with positive real part.

In particular, when -A and a are both sectorial in the respective sense with angles that sum up to a constant strictly inferior to π , the Volterra equation is parabolic.

The kernel function is said to be k-regular if there is a constant K > 0 such that

$$|\lambda^n \widehat{a}^{(n)}(\lambda)| \le K|\widehat{a}(\lambda)|$$

for all n = 0, 1 ... k and all λ with positive real part. In Pruess [26, Theorem I.3.1] it is shown that parabolic equations with a k-regular kernel for $k \geq 1$ admit a unique solution family, i.e. a family of bounded linear operators $(S(t))_{t \geq 0}$ on X, such that

- (a) S(0) = I and $S(\cdot)$ is strongly continuous on \mathbb{R}_+ .
- (b) S(t) commutes with A, which means $S(t)(D(A)) \subset D(A)$ for all $t \geq 0$, and AS(t)x = S(t)Ax for all $x \in D(A)$ and $t \geq 0$.

Date: July 8, 2009.

(c) For all $x \in D(A)$ and all $t \ge 0$ the resolvent equations hold:

(2)
$$S(t)x = x + \int_0^t a(t-s)AS(s)x \, ds.$$

Moreover, $S \in C^{k-1}((0,\infty),\mathcal{B}(X))$ and $||t^n S^{(n)}(t)|| \leq K$ for all $n = 0,\ldots,k-1$.

The purpose of this article is to present conditions for the admissibility of observation operators to parabolic Volterra equations, that is, we consider the 'observed' system

(V)
$$\begin{cases} x(t) = f(t) + \int_0^t a(t-s) Ax(s) ds \\ y(t) = Cx(t) \end{cases}$$

Additionally to the sectoriality condition on A and the parabolicity condition on the Volterra equation, the operator C in the second line is supposed to be an operator from X into another Banach space Y that acts as a bounded operator from $X_1 \to Y$ where $X_1 = \mathcal{D}(A)$ endowed by the graph norm of A. In order to guarantee that the output function lies locally in L_2 we impose the following condition.

Definition 1.1. A bounded linear operator $C: X_1 \to Y$ is called *finite-time admissible* for the Volterra equation (1) if there are constants $\eta, K > 0$ such that

$$\left(\int_0^t \|CS(r)x\|^2 \, dr \right)^{\frac{1}{2}} \le Ke^{\eta t} \|x\|$$

for all $t \geq 0$ and all $x \in \mathcal{D}(A)$.

The notion of admissible observation operators is well studied in the literature for Cauchy systems, that is, $a \equiv 1$, see for example [17], [27], and [28]. Admissible observation operators for Volterra systems are studied in [12], [18], [19] and [22]. The Laplace transform of S, denoted by H, is given by

$$H(\lambda)x = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1}x, \quad \text{Re } \lambda > 0.$$

The following necessary condition for admissibility was shown in [19].

Proposition 1.2. If C is a finite-time admissible observation operator for the Volterra equation (1), then there is a constant M > 0 such that

(3)
$$\|\sqrt{\operatorname{Re}\lambda}CH(\lambda)\| \le M$$
, $\operatorname{Re}\lambda > 0$.

In [19] it is shown that (3) is also sufficient for admissibility if X is a Hilbert space, Y is finite-dimensional and A generates a contraction semigroup. However, in general this condition is not sufficient (see e.g. [17]).

We show that the slightly stronger growth condition on the resolvent

$$\sup_{r>0} \| (1 + \log^+ r)^{\alpha} r^{\frac{1}{2}} CH(r) \| < \infty,$$

is sufficient for admissibility if $\alpha > \frac{1}{2}$ (see Theorem 3.6). This result generalizes the sufficient condition of Zwart [29] for Cauchy systems to general Volterra systems (1).

Our second main result, Theorem 3.1 provides a perturbation argument to obtain admissibility for the controlled Volterra equation from the admissibility of the control operator for the underlying Cauchy equation. In the particular case of diagonal semigroups and one-dimensional output spaces Y this improves a direct Carleson measure criterion from Haak, Jacob, Partington and Pott [12].

We proceed as follows. In Section 2 we obtain an integral representation for the solution family $(S(t))_{t\geq 0}$ and several regularity results of the corresponding kernel.

Section 3 is devoted to sufficient condition for admissibility of observation operators. A perturbation result as well as a general sufficient condition is obtain. Several examples are included as well.

To enhance readability of the calculations, for rest of this article, K denotes some positive constant that may change from one line to the other unless explicitly quantified.

2. Regularity transfer

The main result of this section is formulated in the following proposition. Let $s(t, \mu)$ denote the solution of the scalar equation

$$s(t,\mu) + \mu \int_0^t a(t-r)s(r,\mu) dr = 1$$
 $t > 0, \mu \in \mathbb{C}$.

Proposition 2.1. Suppose that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular, and sectorial of angle $\theta < \pi/2$. Then there exists a family of functions v_t such that

$$\mathcal{L}(v_t)(\mu) = s(t,\mu)$$
 and $S(t) = \int_0^\infty v_t(s)T(s) ds$

satisfying

- (a) $\sup_{t>0} \|v_t\|_{L^1(\mathbb{R}_+)} < \infty$
- (b) $||v_t||_{L^2(\mathbb{R}_+)} \leq K(t^{-\theta/\pi} + t^{+\theta/\pi})$ where K depends only on θ and $C_{q,1}^{reg}$.
- (c) $||v_t||_{W^{1,1}} \le K(1 + t^{-\frac{2\theta}{\pi}} + t^{+\frac{2\theta}{\pi}}).$

Moreover, for $\gamma \in [0,1]$, $|\mu^{\gamma}s(t,\mu)| \leq Kt^{-\frac{2\gamma\theta}{\pi}}$

For the proof of this proposition the following two lemmas are needed.

Lemma 2.2. Suppose $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular and sectorial of angle $\theta \leq \pi$. Let $\rho_0 := 2\theta/\pi$. Then there exists a constant c > 0 such that

$$|\widehat{a}(\lambda)| \ge \begin{cases} c|\lambda|^{-\rho_0} & |\lambda| \ge 1\\ c|\lambda|^{\rho_0} & |\lambda| \le 1 \end{cases}$$

for all $\lambda \in \mathbb{C}_+$.

Proof. We borrow the argument from the proof of [25, Proposition 1]: we start with the analytic completion of the Poisson formula for the harmonic function $H(\lambda) = \arg \widehat{a}(\lambda)$, that is,

$$\log \widehat{a}(\lambda) = \kappa_0 + \frac{i}{\pi} \int_{-\infty}^{\infty} \left[\frac{1 - i\rho\lambda}{\lambda - i\rho} \right] h(i\rho) \frac{d\rho}{1 + \rho^2},$$

where $\kappa_0 \in \mathbb{R}$ is a constant. An easy calculation shows

$$|\operatorname{Re} \log \widehat{a}(\lambda)| \leq \kappa_0 + \rho_0 |\log \lambda|$$

for real $\lambda > 0$, and thus

$$|\widehat{a}(\lambda)| = e^{\log(|\widehat{a}(\lambda)|)} = e^{\operatorname{Re}\,\log\widehat{a}(\lambda)} \ge \begin{cases} c\lambda^{-\rho_0} & \lambda \ge 1\\ c\lambda^{\rho_0} & 0 \le \lambda \le 1 \end{cases},$$

where $c:=e^{-\kappa_0}>0$. This estimate, together with [26, Lemma 8.1] stating the existence of a constant c>0 such that $c^{-1}\leq \left|\widehat{a}(|\lambda|)/\widehat{a}(\lambda)\right|\leq c$ for all $\lambda\in\mathbb{C}_+$ completes the proof.

Lemma 2.3. Let $\theta \in (0, \pi)$. Then there exists $c_{\theta} > 0$ such that

$$(4) 1 + |\lambda| < c_{\theta} |1 + \lambda|$$

for all $\lambda \in \Sigma_{\pi-\theta}$.

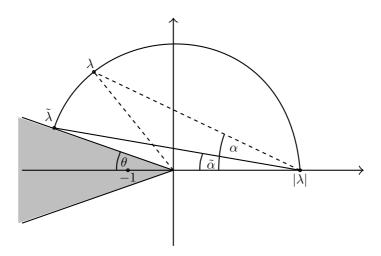


Figure 1. Illustration of (4)

Proof. Clearly, $\alpha > \widetilde{\alpha}$, see Figure 1. Since $\widetilde{\alpha} = \frac{\theta}{2}$, the assertion follows then from the fact that $\frac{|1+\lambda|}{1+|\lambda|} = \frac{\sin(\alpha)}{\sin(\theta-\alpha)} \ge \sin(\alpha) \ge \sin(\theta/2)$.

Proof of Proposition 2.1. (a) is [26, Proposition I.3.5]. This latter result is also the principal inspiration of the next part:

(b) Let $\sigma(\lambda, \mu) = (\hat{\mathcal{L}}s(\cdot, \mu))(\lambda)$, i.e. $\sigma(\lambda, \mu) = \frac{1}{\lambda(1+\mu\widehat{a}(\lambda))}$. Fix t > 0 and $\varepsilon > 0$. Then

$$s(t,\mu) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda t} \sigma(\lambda,\mu) \, d\lambda.$$

Then, by partial integration

$$\begin{split} s(t,\mu) &= \lim_{R \to \infty} \frac{1}{2\pi i} \bigg[\frac{1}{t} e^{\lambda t} \sigma(\lambda,\mu) \bigg]_{\lambda=\varepsilon-iR}^{\lambda=\varepsilon+iR} - \frac{1}{2\pi i} \int_{\varepsilon-iR}^{\varepsilon+iR} \frac{1}{t} e^{\lambda t} \frac{d}{d\lambda} \sigma(\lambda,\mu) \, d\lambda \\ &= -\frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{1}{t} e^{\lambda t} \frac{d}{d\lambda} \sigma(\lambda,\mu) \, d\lambda \end{split}$$

An elementary calculation gives

$$\frac{d}{d\lambda} \frac{1}{\lambda(1+\mu \widehat{a}(\lambda))} = -\frac{1+\mu \widehat{a}(\lambda) \left(1+\left(\frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right)\right)}{\lambda^2(1+\mu \widehat{a}(\lambda))^2}$$

By 1-sectoriality of the kernel, $\left|\frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right| \leq C = C_{a,1}^{\text{reg}}$ and so Lemma 2.3 yields for any $\delta > 0$,

$$\left(\int_{-\infty}^{\infty} \left| s(t, \delta + iy) \right|^{2} dy \right)^{\frac{1}{2}} \\
\leq C_{\theta} (1 + C) \frac{e^{\varepsilon t}}{2\pi t} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2} + x^{2})(1 + |\delta + iy||\widehat{a}(\varepsilon + ix)|)} dx \right)^{2} dy \right)^{\frac{1}{2}} \\
\leq \sqrt{2} C_{\theta} (1 + C) \frac{e^{\varepsilon t}}{2\pi t} \left(\int_{0}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2} + x^{2})(1 + |y||\widehat{a}(\varepsilon + ix)|)} dx \right)^{2} dy \right)^{\frac{1}{2}}$$

$$\leq \sqrt{2}C_{\theta}(1+C)\frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} \frac{1}{(\varepsilon^{2}+x^{2})^{2}(1+|y||\widehat{a}(\varepsilon+ix)|)^{2}} \, dy \right)^{\frac{1}{2}} dx$$

$$= \sqrt{2}C_{\theta}(1+C)\frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2}+x^{2})|\widehat{a}(\varepsilon+ix)|^{\frac{1}{2}}} \left(\int_{0}^{\infty} \frac{1}{(1+u)^{2}} \, du \right)^{\frac{1}{2}} dx$$

$$= \sqrt{2}C_{\theta}(1+C)\frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2}+x^{2})|\widehat{a}(\varepsilon+ix)|^{\frac{1}{2}}} \, dx$$

Now we split the integral into two parts, by considering the cases $\varepsilon^2 + x^2 \ge 1$ and $\varepsilon^2 + x^2 < 1$ to apply Lemma 2.2 which is controlling $|1/\widehat{a}|$. Substituting $x = \varepsilon t$ in both parts easily gives

$$||s(t,\cdot)||_{H^2} \le \widetilde{C}_{\theta} \frac{e^{\varepsilon t}}{t} (\varepsilon^{-1-\theta/\pi} + \varepsilon^{-1+\theta/\pi}),$$

which yields the assertion by letting $\varepsilon = \frac{1}{t}$.

(c) We argue in the same spirit as above: by partial integration

$$\label{eq:definition} \frac{d}{d\mu} \Big(\mu s(t,\mu) \Big) \quad = \quad \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{1}{t} e^{\lambda t} \frac{d^2}{d\mu \, d\lambda} \Big(\mu \sigma(\lambda,\mu) \Big) \, d\lambda$$

An elementary calculation gives

$$\frac{d^2}{d\lambda \, d\mu} \frac{\mu \widehat{a}(\lambda)}{(\lambda(1 + \mu \widehat{a}(\lambda)))^2} = \frac{1 + \mu \widehat{a}(\lambda) \left(1 + 2\left(\frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right)\right)}{\lambda^2 (1 + \mu \widehat{a}(\lambda))^3}$$

By 1-sectoriality of the kernel, $\left|\frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right| \leq C$ and so the Lemma yields any $\delta > 0$,

$$\int_{-\infty}^{\infty} \left| \frac{d}{d\mu} \left(\mu s(t, \delta + iy) \right) \right| dy$$

$$\leq C_{\theta} (1 + 2C) \frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2} + x^{2})(1 + |\delta + iy||\widehat{a}(\varepsilon + ix)|)^{2}} dx dy$$

$$\leq C_{\theta} (1 + 2C) \frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2} + x^{2})(1 + |y||\widehat{a}(\varepsilon + ix)|)^{2}} dx dy$$

$$= 2C_{\theta} (1 + 2C) \frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2} + x^{2})} \frac{1}{|\widehat{a}(\varepsilon + ix)|} \int_{0}^{\infty} \frac{1}{(1 + u)^{2}} du dx$$

$$= C_{\theta} (1 + 2C) \frac{e^{\varepsilon t}}{2\pi t} \int_{-\infty}^{\infty} \frac{1}{(\varepsilon^{2} + x^{2})} \frac{1}{|\widehat{a}(\varepsilon + ix)|} dx$$

$$\leq K(t^{-\frac{2\theta}{\pi}} + t^{+\frac{2\theta}{\pi}})$$

by choosing $\varepsilon = \frac{1}{t}$. This shows that $f_t(\mu) = \frac{d}{d\mu} \Big(\mu s(t, \mu) \Big) \in H^1(\mathbb{C}_+)$. We may apply Hardy's inequality (see e.g. [8, p.198], [14, Theorem 4.2]),

$$\int_0^\infty \frac{|\check{f}_t(r)|}{r} dr \le \frac{1}{2} \int_{-\infty}^\infty |f_t(i\omega)| d\omega$$

so that $\frac{\check{f}_t(r)}{r} \in L^1(\mathbb{R}_+)$ is Laplace transformable for every t > 0. Since

$$\mathcal{L}\left(\frac{\check{f}_t(r)}{r}\right)(\sigma) = \int_{\sigma}^{\infty} f_t(\mu) \, d\mu = \sigma s(t,\sigma),$$

we find that $\mu \mapsto \mu s(t,\mu) \in H^{\infty}(\mathbb{C}_+)$ with a norm controlled by a multiple of $(t^{\frac{+2\theta}{\pi}} + t^{-\frac{2\theta}{\pi}})$. This implies that $v'_t \in L^1(\mathbb{R}_+)$. Together with (a) the claim follows.

Finally, the same technique gives an estimate for the growth of $s(t, \mu)$:

$$\begin{split} \mu^{\gamma}s(t,\mu) & \leq & K\frac{|\mu|^{\gamma}e^{\varepsilon t}}{t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^{2}+r^{2})(1+|\mu||\widehat{a}(\varepsilon+ir)|)}\,dr\\ & \leq & K\frac{e^{\varepsilon t}}{t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^{2}+r^{2})|\widehat{a}(\varepsilon+ir)|^{\gamma}}\,\frac{|\mu|^{\gamma}|\widehat{a}(\varepsilon+ir)|^{\gamma}}{(1+|\mu||\widehat{a}(\varepsilon+ir)|)}\,dr\\ & \leq & K\frac{e^{\varepsilon t}}{t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^{2}+r^{2})|\widehat{a}(\varepsilon+ir)|^{\gamma}}\,dr\\ & \stackrel{\varepsilon=\frac{1}{t}}{\leq} & K(t^{-\frac{2\gamma\theta}{\pi}}+t^{+\frac{2\gamma\theta}{\pi}}). \end{split}$$

3. Sufficient conditions for finite-time admissibility

In this section we present the two main results of this paper.

Theorem 3.1. Let A generate an exponentially stable strongly continuous semi-group $(T(t))_{t\geq 0}$ and let $C: X_1 \to Y$ be bounded. Further we assume that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular and sectorial of angle $\theta < \pi/2$. Then finite-time admissibility of C for the semigroup $(T(t))_{t\geq 0}$ implies that of C for the solution family $(S(t))_{t\geq 0}$.

Proof. By Proposition 2.1 there exists a family of functions v_t such that $||v_t||_{L^2(\mathbb{R}_+)} \le Kt^{-\theta/\pi}$ for some constant K > 0 independent of $t \ge 0$ and

$$S(t) = \int_0^\infty v_t(r) T(r) dr, \qquad t \ge 0.$$

For $x \in \mathcal{D}(A)$ we have thus

$$CS(t)x = \int_0^\infty v_t(r) \, CT(r)x \, dr.$$

Note that finite-time admissibility of C for $(T(t))_{t\geq 0}$ implies the existence of a constant M>0 such that

$$||CT(\cdot)x||_{L^2(\mathbb{R}_+)} \le M||x||, \qquad x \in \mathcal{D}(A),$$

thanks to the exponential stability of $(T(t))_{t\geq 0}$. Thus the result follows from Cauchy-Schwarz inequality.

By replacing the Cauchy-Schwarz inequality by Hölder's inequality, similar arguments can be used to obtain sufficient conditions for L^p -admissibility.

Corollary 3.2. Assume in addition to the hypotheses of the theorem that one of the following conditions is satisfied:

- (a) Y is finite-dimensional, X is a Hilbert space and A generates a contraction semigroup;
- (b) X is a Hilbert space and A generates a normal, analytic semigroup;
- (c) A generates an analytic semigroup and $(-A)^{1/2}$ is an finite-time admissible observation operator for $(T(t))_{t\geq 0}$.

If there exists a constant M > 0 such that

(5)
$$||C(\lambda - A)^{-1}|| \le \frac{M}{\sqrt{\operatorname{Re}\lambda}}, \qquad \operatorname{Re}\lambda > 0,$$

then C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$.

Proof. Under the assumption of the corollary, the inequality (5) implies that C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$, see [16], [13], [23]. Thus the result follows from Theorem 3.1.

The following corollary is an immediate consequence of the Carleson measure criterion of Ho and Russell [15].

Corollary 3.3. Assume in addition to the hypotheses of the theorem that A admits a Riesz basis of eigenfunctions (e_n) on a Hilbert space X with corresponding eigenvalues λ_n . If $Y = \mathbb{C}$ and if

$$\mu = \sum_{n} |Ce_n|^2 \delta_{-\lambda_n}$$

is a Carleson measure on \mathbb{C}_+ , then C is finite-time admissible for the solution family $(S(t))_{t\geq 0}$.

A nice sufficient condition for admissibility for Cauchy problems is given by Zwart [29]. For the convenience of the reader we reproduce it here:

Theorem 3.4 (Zwart). Let A be the infinitesimal generator of an exponentially stable C_0 -semigroup $T(t)_{t\geq 0}$ on the Hilbert space H and let $C: X_1 \to Y$ be bounded, where Y is another Hilbert space. If for some $\alpha > \frac{1}{2}$,

(6)
$$\sup_{\operatorname{Re} \lambda > 0} \| (1 + \log^{+} \operatorname{Re} \lambda)^{\alpha} (\operatorname{Re} (\lambda))^{\frac{1}{2}} CR(\lambda, A) \| < \infty,$$

then C is a finite-time admissible observation operator.

Notice that the condition (6) can be reformulated by saying that in the sense of Evans, Opic and Pick (see [10, 9, 11])

$$\forall x \in X: \qquad \|CR(\cdot, A)x\|_{\frac{1}{2}, \infty, \mathbb{A}} < \infty$$

where $\mathbb{A} = (0, \alpha)$, see also Cobos, Frenandez-Cabrera and Triebel [7] for logarithmic type interpolation functors.

Corollary 3.5. Let in addition to the assumptions of Theorem 3.4, a be a 1-regular and sectorial kernel of type $< \pi/2$. Then C is finite-time admissible for the solution family $(S(t))_{t>0}$.

In some situations, the condition of sectoriality of angle $< \pi/2$ in the above corollary may be inconvenient. Under merely 1-regularity one can also obtain admissibility by the following direct argument.

Theorem 3.6. Let $S(\cdot)$ be a bounded solution family to (V) with a 1-regular kernel function a. Let $C: X_1 \to Y$ be bounded and assume that for some $\alpha > \frac{1}{2}$,

(7)
$$\sup_{r>0} \| (1 + \log^+ r)^\alpha r^{\frac{1}{2}} CH(r) \| < \infty.$$

Then C is finite-time admissible for $(S(t))_{t\geq 0}$.

Note that the exponent $\alpha > \frac{1}{2}$ is optimal in the sense that for $\alpha < \frac{1}{2}$ it is even wrong in the case $a \equiv 1$, see [20]. About the case $\alpha = \frac{1}{2}$ nothing is known at the moment.

Proof. Let $\lambda \in \mathbb{C}_+$ and let φ such that $\lambda = |\lambda| e^{2i\varphi}$. Then, by resolvent identity,

$$(1 + (\log^{+}(\operatorname{Re}\lambda))^{\alpha}\lambda^{\frac{1}{2}}CH(\lambda)$$

$$= (1 + (\log^{+}(\operatorname{Re}\lambda))^{\alpha}\lambda^{-\frac{1}{2}}C\frac{1}{\widehat{a}(\lambda)}R(\frac{1}{\widehat{a}(\lambda)}, A)$$

$$= (1 + \log^{+}|\lambda|)^{\alpha}|\lambda|^{\frac{1}{2}}CH(|\lambda|) e^{-i\varphi}\frac{\widehat{a}(|\lambda|)}{\widehat{a}(\lambda)}\Big[I + \Big(\frac{1}{\widehat{a}(|\lambda|)} - \frac{1}{\widehat{a}(\lambda)}\Big)R(\frac{1}{\widehat{a}(\lambda)}, A)\Big]$$

$$= (1 + \log^{+}|\lambda|)^{\alpha}|\lambda|^{\frac{1}{2}}CH(|\lambda|) e^{-i\varphi} \left[I + (1 - \frac{\widehat{a}(|\lambda|)}{\widehat{a}(\lambda)})AR(\frac{1}{\widehat{a}(\lambda)}, A) \right].$$

By [26, Lemma 8.1], $c^{-1} \leq |\widehat{a}(|\lambda|)/\widehat{a}(\lambda)| \leq c$ for some c > 0. This yields uniform boundedness of expression in brackets and so the assumed estimate (7) gives

(8)
$$\|\lambda \mapsto CH(r+\lambda)\|_{H^{\infty}(\mathbb{C}_+)} \le K(1+\log^+ r)^{-\alpha} r^{-\frac{1}{2}}.$$

Since $(S(t))_{t\geq 0}$ is bounded,

$$\|\lambda \mapsto H(r+\lambda)x\|_{H^{2}(\mathbb{C}_{+})} = \|e^{-rt}S(t)x\|_{H^{2}(\mathbb{C}_{+})} \le Kr^{-\frac{1}{2}} \|x\| \qquad \forall r > 0$$

and together with (8), we infer

(9)
$$\|\lambda \mapsto CH(r+\lambda)^2 x\|_{H^2(\mathbb{C}_+)} \le \frac{K}{(1+\log^+ r)^\alpha r} \|x\| \quad \forall r > 0.$$

Moreover, the estimate

$$\left\|\lambda \mapsto \frac{1}{r+\lambda} CH(r+\lambda)x\right\|_{H^2(\mathbb{C}_+)} \le \left\|\lambda \mapsto CH(r+\lambda)x\right\|_{H^\infty(\mathbb{C}_+)} \left\|\lambda \mapsto \frac{1}{r+\lambda}\right\|_{H^2(\mathbb{C}_+)}$$

implies

(10)
$$\|\lambda \mapsto \frac{1}{r+\lambda} CH(r+\lambda)x\|_{H^2(\mathbb{C}_+)} \le \frac{K}{(1+\log^+ r)^{\alpha} r} \|x\| \qquad \forall r > 0.$$

Since $\frac{d}{d\lambda}H(\lambda) = \left(\frac{\lambda \hat{a}'(\lambda)}{\hat{a}(\lambda)}\right)H(\lambda)^2 - \frac{1}{\lambda}\left(1 + \frac{\lambda \hat{a}'(\lambda)}{\hat{a}(\lambda)}\right)H(\lambda)$ we infer from (9) and (10) that

$$\left\| \mu \mapsto \frac{d}{d\mu} C H(r+\mu) x \right\|_{H^2(\mathbb{C}_+)} \le \frac{K}{(1+\log^+ r)^{\alpha} r} \|x\| \qquad \forall r > 0.$$

Finally, (inverse) Laplace transform yields

$$\left\| t \mapsto rte^{-rt}CS(t)x \right\|_{L^2(\mathbb{R}_+)} \le \frac{K}{(1 + \log^+ r)^{\alpha}} \|x\| \qquad \forall r > 0$$

and that is the estimate we need in the following dyadic decomposition argument: notice that $xe^{-x} \ge 2e^{-2}$ for $x \in [1, 2]$. Fix some $t_0 > 0$. Then,

$$\int_{0}^{t_{0}} \left\| CS(t)x \right\|^{2} dt = \sum_{n=1}^{\infty} \int_{t_{0}2^{-n}}^{t_{0}2^{-n+1}} \left\| CS(t)x \right\|^{2} dt$$

$$\leq \frac{e^{4}}{4} \sum_{n=1}^{\infty} \int_{t_{0}2^{-n}}^{t_{0}2^{-n+1}} \left\| t2^{n}t_{0}^{-1}e^{t2^{n}t_{0}^{-1}}CS(t)x \right\|^{2} dt$$

$$\leq K \sum_{n=1}^{\infty} \frac{1}{(1 + \log^{+}(2^{n}t_{0}^{-1}))^{2\alpha}} \|x\|^{2} \leq K \|x\|^{2}.$$

4. Example

In this section we apply the obtained results to time-fractional diffusion equation of distributed order.

Let A generate an exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$. For $\omega > 0$, $0 < 2\alpha < \beta \leq 1$ we study a time-fractional diffusion equation of distributed order of the form

(11)
$$\omega D_t^{\alpha} x(t) + D_t^{\beta} x(t) = Ax(t), \quad t \ge 0,$$

$$x(0) = x_0.$$

where $D_t^{\alpha}x = \left(-\frac{\partial}{\partial t}\right)^{\alpha}x$ denotes the Caputo derivative of x, given by the Phillips functional calculus of the right shift semigroup, that is,

$$D_t^{\gamma} x(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} x'(s) \, ds.$$

for $\gamma \in (0,1)$. Time-fractional diffusion equations of distributed order have attracted attention as a possible tool for the description of anomalous diffusion and relaxation phenomena in many areas such as turbulence, disordered medium, intermittent chaotic systems, mathematical finance and stochastic mechanics. For further information on time-fractional diffusion equations of distributed order we refer the reader to [1, 2, 3, 4, 5, 6, 21, 24].

Using the Laplace transform equation (11) is equivalent to

$$x(t) = x_0 + \int_0^t a(t-s) Ax(s) ds,$$

where

$$a(t) = t^{\beta - 2\alpha - 1} E_{\beta - \alpha, \beta - 2\alpha} (-\omega t^{\beta - \alpha}).$$

Here $E_{\gamma,\delta}$, where $\gamma,\delta>0$, denotes the Mittag-Leffler function

$$E_{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \delta)}$$

The Laplace transformation of the kernel a is given by

$$\hat{a}(\lambda) = \frac{\lambda^{-\alpha}}{\omega + \lambda^{\beta - \alpha}}.$$

Thus the kernel a satisfies the assumption of Theorem 3.1.

We note that this example does e.g. not satisfy the assumption of [12, Theorem 3.10] due to the 'mixed' growth conditions near infinity and the origin, such that, even when A is the Dirichlet Laplacian on a bounded domain, the latter result cannot be used to guarantee admissibility whereas a 'standard' Carlseon measure reduction using Corollary 3.3 still applies.

ACKNOWLEDGEMENT

This research was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from June 15 to June 28, 2008. We would like to thank the MFO for excellent working conditions.

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