Every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable

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Abstract

An acyclic coloring of a graph G is a coloring of its vertices such that : (i) no two adjacent vertices in G receive the same color and (ii) no bicolored cycles exist in G. A list assignment of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Let G be a graph and L be a list assignment of G. The graph G is acyclically L-list colorable if there exists an acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G is acyclically L-list colorable for any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is acyclically k-choosable. In this paper, we prove that every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable.

1 Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that two adjacent vertices do not use the same color. A *k-coloring* of G is a proper coloring of G using k colors ; a graph admitting a k-coloring is said to be *k-colorable*. An *acyclic coloring* of a graph G is a proper coloring of G such that G contains no bicolored cycles ; in other words, the graph induced by every two color classes is a forest. A list assignment of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Let G be a graph and L be a list assignment of G. The graph G is *acyclically L-list colorable* if there is an acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G is acyclically L-list colorable for any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is *acyclically k-choosable*. The *acyclic choice number* of G, $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable. Borodin *et al.* [1] first investigated the acyclic choosability of planar graphs proving that:

Theorem 1 *[1] Every planar graph is acyclically 7-choosable.*

and put forward to the following challenging conjecture:

Conjecture 1 *[1] Every planar graph is acyclically 5-choosable.*

This conjecture if true strengthens Borodin's Theorem [3] on the acyclic 5-colorability of planar graphs and Thomassen's Theorem [11] on the 5-choosability of planar graphs.

In 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [7]). This problem remains open. In 1990, Erdős suggested the following relaxation of Steinberg's Conjecture: what is the smallest integer i such that every planar graph without cycles of lengths 4 to i is 3-colorable? The best known result is $i = 7$ [2]. This question is also studied in the choosability case: what is the smallest integer i such that every planar graph without cycles of lengths 4 to i is 3-choosable? In [12], Voigt proved that Steinberg's Conjecture can not be extended to list coloring ; hence, $i \geq 6$. Nevertheless, in 1996, Borodin [4] proved that every planar graph without cycles of lengths 4 to 9 is 3-colorable ; in fact, 3-choosable. So, $i \leq 9$.

In this paper, we study the question of Erdős in the acyclic choosability case:

Problem 1 *What is the smallest integer i such that every planar graph without cycles of lengths 4 to i is acyclically 3-choosable?*

Note that it is proved that every planar graph without cycles of lengths 4 to 6 is acyclically 4-choosable [10]. Also, the relationship between the maximum average degree of G (or the girth of G) and its acyclic choice number was studied (see for example [9, 8, 5]).

Our main result is the following:

Theorem 2 *Every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable.*

Hence, in Problem 1, $6 \leq i \leq 12$.

Section 2 is dedicated to the proof of Theorem 2. Follow some notations we will use:

Notations Let G be a planar graph. We use $V(G)$, $E(G)$ and $F(G)$ to denote the set of vertices, edges and faces of G respectively. Let $d(v)$ denote the degree of a vertex v in G and $r(f)$ the length of a face f in G. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. *$\geq k$-vertex, $\leq k$-vertex*). We use the same notations for faces : a *k-face* (resp. *$\geq k$-face, $\leq k$-face*) is a face of length k (resp. at least k, at most k). A k-face having the boundary vertices x_1, x_2, ..., x_k in the cyclic order is denoted by $[x_1x_2...x_k]$. For a vertex $v \in V(G)$, let $n_i(v)$ denote the number of i-vertices adjacent to v for $i \geq 1$, and $m_3(v)$ the number of 3-faces incident to v. A 3-vertex is called *3^*-vertex* if it is incident to a 3-face and adjacent to a 2-vertex (for example in Figure 1, the vertex t is a 3^*-vertex). A 3-face $[rst]$ with $d(r) = d(s) = d(t) = 3$ and with a 3^*-vertex on its boundary is called a *3^*-face*. *Two 3-faces $[rst]$ and $[uvw]$ are called linked* if there exists an edge tv which connects these two 3-faces such that $d(t) = d(v) = 3$ (see Figure 2). *A vertex v is linked to a 3-face $[rst]$* if there exists an edge between v and one vertex of the boundary of $[rst]$, say t, such that $d(t) = 3$ (for example in Figure 1, the vertex v is linked to the 3-face $[rst]$). Let $n^*(v)$ be the number of 3^*-face linked to v.

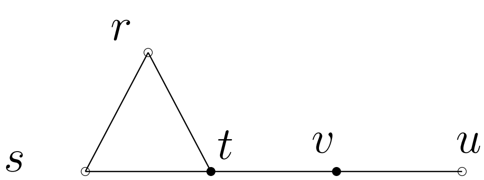

Figure 1: The vertex t is a 3^*-vertex and the vertex v is linked to the 3-face $[rst]$

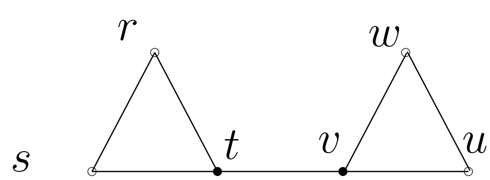

Figure 2: The two 3-faces [rst] and [uvw] are linked

2 Proof of Theorem 2

To prove Theorem 2 we proceed by contradiction. Suppose that H is a counterexample with the minimum order to Theorem 2 which is embedded in the plane. Let L be a list assignment with $|L(v)| = 3$ for all $v \in V(H)$ such that there does not exist an acyclic coloring c of H with for all $v \in V(H), c(v) \in L(v)$.

Without loss of generality we can suppose that H is connected. We will first investigate the structural properties of H (Section 2.1), then using Euler's formula and the discharging technique we will derive a contradiction (Section 2.2).

2.1 Structural properties of H

Lemma 1 *The minimal counterexample H to Theorem 2 has the following properties:*

(C1) H contains no 1-vertices.

(C2) A 3-face has no 2-vertex on its boundary.

(C3) A 2-vertex is not adjacent to a 2-vertex.

(C4) A 3-face has at most one 3^*-vertex on its boundary.

(C5) A 3-face $[rst]$ with $d(r) = d(s) = d(t) = 3$ is linked to at most one 3^*-face.

(C6) Two 3^*-faces cannot be linked.

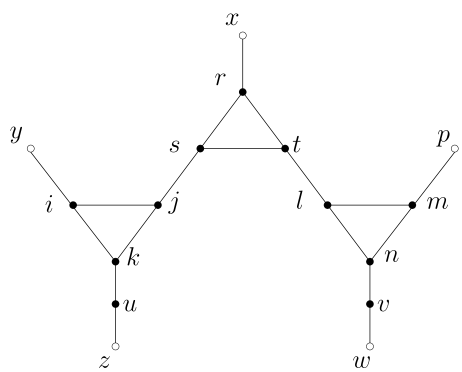

Figure 3: $[rst]$ is linked to two 3^*-faces $[ijk]$ and $[lmn]$

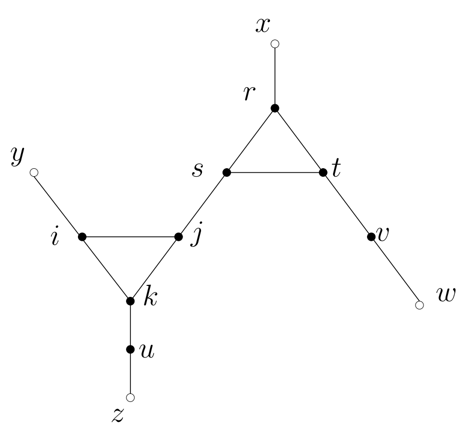

Figure 4: The two 3^*-faces $[rst]$ and $[ijk]$ are linked

Proof

(C1) Suppose H contains a 1-vertex u adjacent to a vertex v. By minimality of H, the graph $H' = H\backslash\{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of H'. To extend this coloring to H we just color u with $c(u) \in L(u)\backslash\{c(v)\}$. The obtained coloring is acyclic, a contradiction.

(C2) Suppose H contains a 2-vertex u incident to a 3-face $[uvw]$. By minimality of H, the graph $H' = H\backslash\{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of H'. We show that we can extend this coloring to H by coloring u with $c(u) \in L(u)\backslash\{c(v), c(w)\}$.

(C3) Suppose H contains a 2-vertex u adjacent to a 2-vertex v. Let t and w be the other neighbors of u and v respectively. By minimality of H, the graph $H' = H\backslash\{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of H'. We show that we can extend this coloring to H. Assume first that $c(t) \neq c(v)$. Then we just color u with $c(u) \in L(u)\backslash\{c(t), c(v)\}$. Now, if $c(t) = c(v)$, we color u with $c(u) \in L(u)\backslash\{c(v), c(w)\}$. In the two cases, the obtained coloring is acyclic, a contradiction.

(C4) Suppose H contains a 3-face $[rst]$ with two 3^*-vertices s and t. Suppose that t (resp. s) is adjacent to a 2-vertex v (resp. x) with $v \neq r, s$ by (C2) (resp. $x \neq r, t$). Let u (resp. y) be the other neighbor of v (resp. x) with $u \neq r, s$ (resp. $y \neq r, t$). By the minimality of H, $H' = H\backslash\{v\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of H'. We show now that we can extend c to H. If $c(u) \neq c(t)$, we color then v with a color different from $c(u)$ and $c(t)$ and the coloring obtained is acyclic. Otherwise, $c(u) = c(t)$. If we cannot color v, this implies without loss of generality $L(v) = \{1, 2, 3\}$, $c(u) = c(t) = c(x) = 1$, $c(r) = 2$ and $c(s) = c(y) = 3$. Observe that necessarily $L(t) = \{1, 2, 3\}$ (otherwise we can recolor t with $\alpha \in L(t)\backslash\{1, 2, 3\}$ and color v properly *i.e* v receives a color distinct of those of these neighbors). For a same reason $L(s) = \{1, 2, 3\}$ and $L(x) = \{1, 2, 3\}$. Now, we recolor t with the color 3, s with the color 1 and x with the color 2, then we can color v with the color 2. It is easy to see that the coloring obtained is acyclic.

(C5) Suppose H contains a 3-face $[rst]$ incident to three 3-vertices such that two of them are linked to two 3^*-faces $[ijk]$ and $[lmn]$. Suppose $[ijk]$ and $[lmn]$ are linked to $[rst]$ respectively by the edges sj and tl. Call y the third neighbor of i, x the third neighbor of r, and p the third neighbor of m. Suppose that the 2-vertex u (resp. v) is adjacent to k and z (resp. n and w). For example, H contains the graph depicted by Figure 3. By the minimality of H, $H' = H\backslash\{v\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of H'. We show now that we can extend c to H. If $c(w) \neq c(n)$, we color then v with a color different from $c(w)$ and $c(n)$ and the coloring obtained is acyclic. Otherwise, $c(w) = c(n)$. If we cannot color v, this implies without loss of generality $L(v) = \{1, 2, 3\} = L(l) = L(m)$, $c(w) = c(n) = c(t) = c(p) = 1$, and by permuting the colors of l and m, we are sure that $L(r) = \{1, 2, 3\} = L(s)$ and $c(x) = c(j) = 1$, then by permuting the colors of r and s, we are sure that $L(i) = \{1, 2, 3\} = L(k)$, $c(y) = c(u) = 1$, and $c(z) \in \{2, 3\}$. Let $\alpha = \{2, 3\} \setminus \{c(z)\}$. We recolor k, s, l, v with α and m, r, i with $c(z)$. The coloring obtained is acyclic.

(C6) Suppose H contains a 3-face $[rst]$ incident to three 3-vertices such that one vertex is linked to a 3^*-face, say s is linked by the edge sj to the 3^*-face $[ijk]$ and one vertex is a 3^*-vertex, say t. Call y the third neighbor of i, x the third neighbor of r. Suppose that the 2-vertex u (resp. v) is adjacent to k and z (resp. t and w). For example, H contains the graph depicted by Figure 4. By the minimality of H, $H' = H\backslash\{v\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of H'. We show now that we can extend c to H. If $c(w) \neq c(t)$, we color then v with a color different from $c(w)$ and $c(t)$ and the coloring obtained is acyclic. Otherwise, $c(w) = c(t)$. If we cannot color v, this implies without loss of generality $L(v) = \{1, 2, 3\} = L(r) = L(s)$, $c(w) = c(t) = c(x) = c(j) = 1$, and by permuting the colors of r and s, we are sure that $L(i) = \{1, 2, 3\} = L(k)$, $c(y) = c(u) = 1$, and $c(z) \in \{2, 3\}$. Let $\alpha = \{2, 3\} \setminus \{c(z)\}$. We recolor k, s, v with α and r, i with $c(z)$. The coloring obtained is acyclic.

□

Lemma 2 *Let H be a connected plane graph with n vertices, m edges and r faces. Then, we have* *the following:*

$$\sum_{v\in V(H)} (11d(v) - 26) + \sum_{f\in F(H)} (2r(f) - 26) = -52 \tag{1}$$

Proof

Euler's formula $n - m + f = 2$ can be rewritten as $(22m - 26n) + (4m - 26f)) = -52$. The relation $\sum_{v\in V(H)} d(v) = \sum_{f\in F(H)} r(f) = 2m$ completes the proof. □

2.2 Discharging procedure

Let H be a counterexample to Theorem 2 with the minimum order. Then, H satisfies Lemma 1.

We define the weight function $\omega : V(H) \cup F(H) \longrightarrow \mathbb{R}$ by $\omega(x) = 11d(x) - 26$ if $x \in V(H)$ and $\omega(x) = 2r(x) - 26$ if $x \in F(H)$. It follows from Equation (1) that the total sum of weights is equal to -52. In what follows, we will define discharging rules (R1) and (R2) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -52 < 0$$

and hence demonstrates that no such counterexample can exist.

The discharging rules are defined as follows:

(R1.1) Every ≥ 3-vertex v gives 2 to each adjacent 2-vertex.

(R1.2) Every ≥ 4-vertex v gives 9 to each incident 3-face and 1 to each linked 3^*-face.

(R2.1) Every 3^*-vertex v gives 5 to its incident 3-face.

(R2.2) Every 3-vertex v, different from a 3^*-vertex, which is not linked to a 3^*-face, gives 7 to its incident 3-face (if any).

(R2.3) Every 3-vertex v, different from a 3^*-vertex, linked to a 3^*-face gives 1 to each linked 3^*-face and gives 6 to its incident 3-face (if any).

In order to complete the proof, it suffices to prove that the new weight $\omega^*(x)$ is non-negative for all $x \in V(H) \cup F(H)$.

Let $v \in V(H)$ be a k-vertex. Then, $k \geq 2$ by (C1).

- If $k = 2$, then $\omega(v) = -4$ and v is adjacent to two ≥ 3-vertices by (C3). By (R1.1), $\omega^*(v) = -4 + 2 \cdot 2 = 0$.
- If $k = 3$, then $\omega(v) = 7$. Since H contains no 4-cycles, v is incident to at most one 3-face. Assume first that v is not incident to a 3-face. Then by (R1.1) and (R2.3), v gives at most 3 times 2. Hence, $\omega^*(v) \geq 7 - 3 \cdot 2 \geq 1$. Assume now that v is incident to a 3-face. If v is a 3^*-vertex, then $\omega^*(v) = 7 - 5 - 2 = 0$ by (R1.1) and (R2.1). If v is linked to a 3^*-face then $\omega^*(v) \geq 7 - 6 - 1 = 0$ by (R2.3). If v is not adjacent to a 2-vertex and not linked to a 3^*-face then $\omega^*(v) = 7 - 7 = 0$ by (R2.2).
- If $k \geq 4$, then $\omega(v) = 11k - 26$. Observe by (C1), (C2) and definitions of $n^*(v)$ and of linked vertices that:

$$m_3(v) \leq \left\lfloor \frac{k}{2} \right\rfloor \quad \text{and} \quad k - 2m_3(v) \geq n_2(v) + n^*(v)$$

$$k \geq 2m_3(v) + n_2(v) + n^*(v) \tag{2}$$

It follows by (R1.1), (R1.2) and Equation (2) that:

$$\begin{aligned}
\omega^*(v) &= 11k - 26 - 9m_3(v) - n^*(v) - 2n_2(v) \\
&\geq 11k - 26 - 9m_3(v) - \frac{9}{2}n^*(v) - \frac{9}{2}n_2(v) \\
&\geq 11k - 26 - \frac{9}{2}k \\
&\geq \frac{13}{2}k - 26 \\
&\geq 0
\end{aligned}$$

Suppose that f is a k-face. Then, $k = 3$ or $k \geq 13$ by hypothesis.

- If $k \geq 13$, then $\omega^*(f) = \omega(f) = 2k - 26 \geq 0$.
- If $k = 3$, then $\omega(f) = -20$. Suppose $f = [rst]$. By (C2), f is not incident to a 2-vertex ; hence, $d(r) \geq 3, d(s) \geq 3, d(t) \geq 3$. By (C4) f is incident to at most one 3^*-vertex. Now, observe that if one of the vertices r, s, t is a ≥ 4-vertex, then by (R1.2) (R2.1) (R2.2) (R2.3) $\omega^*(f) \geq -20 + 9 + 5 + 6 = 0$. So assume $d(r) = d(s) = d(t) = 3$ and let r_0, s_0, t_0 be the other neighbors of r, s, t, respectively. Suppose that f is a 3^*-face and let r be its unique 3^*-vertex. By (C6) none of s and t are linked to a 3^*-face. Moreover s_0 and t_0 give 1 to f by (R1.2) and (R2.3). Hence $\omega^*(f) = -20+5+2\cdot 7+2\cdot 1 = 1$. Finally assume that f is not a 3^*-face. By (C5) at most one of r, s, t is linked to a 3^*-face. Hence $\omega^*(f) \geq -20+6+2\cdot 7 = 0$, by (R1.2), (R2.2) and (R2.3).

We proved that, for all $x \in V(H) \cup F(H),\ \omega^*(x) \geq 0$. This completes the proof of Theorem 2.

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