

# On $sl(3)$ KZ equations and $\mathcal{W}_3$ null-vector equations

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Based mostly on the article [arxiv:0811.4587](https://arxiv.org/abs/0811.4587) [[hep-th](#)]

# Families of non-rational 2d CFTs

Algebra	Target Dim.	Theory
$\widehat{sl(2)}$	3d	$SL(2, \mathbb{R})$ WZNW $\sim$ Strings in $AdS_3$
$\widehat{sl(2)}$	3d	$H_3^+$ model
$\widehat{sl(2)/u(1)}$	2d	Strings in 2d black hole
$Vir = \mathcal{W}_2$	1d	Liouville theory $L = \partial\phi\bar{\partial}\phi + e^{b\phi}$

Algebra	Target Dim.	Theory
$\widehat{sl(3)}$	8d	$SL(3, \mathbb{R})$ WZNW
...	$2 \leq d \leq 8$	...
$\mathcal{W}_3$	2d	Conformal $sl(3)$ Toda theory $L = (\partial\phi, \bar{\partial}\phi) + e^{b(e_1, \phi)} + e^{b(e_2, \phi)}$

# Plan of the talk

- Conjectured relation between correlation functions in the  $SL(N)$  WZNW model and  $sl(N)$  conformal Toda theory
- Case  $N = 2$  and an application
- Case  $N = 3$

# Correlation functions

$$\Omega_m = \left\langle \prod_{i=1}^m \Phi^{j_i}(x_i|z_i) \right\rangle^{SL(N)_k \text{ WZNW}} \quad (1)$$

- $z \in \mathbb{C}$ : position on Riemann sphere

- $j \in \mathbb{C}^{N-1} = \widehat{sl(N)}$  spin;  $x \in \mathbb{C}^{\frac{N(N-1)}{2}} =$  isospin

→ order- $N$  Knizhnik–Zamolodchikov equations

$$\tilde{\Omega}_m = \left\langle \prod_{i=1}^m V_{\alpha(j_i)}(z_i) \prod_{a=1}^{\frac{N(N-1)}{2}(m-2)} V_{\alpha_d}(y_a) \right\rangle^{sl(N) \text{ Toda } b=\frac{1}{\sqrt{k-N}}} \quad (2)$$

- $y \in \mathbb{C}$ : position on Riemann sphere (Sklyanin variable)

- $\alpha(j) \in \mathbb{C}^{N-1} = \mathcal{W}_N$  momentum;  $V_{\alpha_d} =$  degenerate field

# The relation

Sklyanin's separation of variables = integral transformation

$$\Psi(\{x_i\}) = S \cdot \tilde{\Psi}(\{y_a\}) = \int \prod_a dy_a S(\{x_i\}|\{y_a\}) \tilde{\Psi}(\{y_a\}) \quad (3)$$

**Twist**  $\Theta_m = \prod_{a < b} (y_a - y_b)^\lambda \prod_{i,a} (y_a - z_i)^\mu \prod_{i < j} (z_i - z_j)^\nu$

Conjectured relation:

$$\Omega_m = S \cdot \Theta_m \tilde{\Omega}_m$$

Status of relation	$sl(2)$ case	$sl(3)$ case
KZ-Compatible	Yes [Stoyanovsky + Feigin + Frenkel]	Only if $k = 3$
Proved in a model	Yes [SR + Teschner]	

# KZ equations and the Gaudin model

Knizhnik–Zamolodchikov equations

$$\left[ (k - N) \frac{\delta}{\delta z_i} + H_i \right] \Omega_m^{\text{WZNW}} = 0, \quad H_i = \sum_{\ell \neq i} \frac{\vec{D}_{(i)} \cdot \vec{D}_{(\ell)}}{z_i - z_\ell} \quad (4)$$

$H_i$  – Gaudin Hamiltonians

$$sl(2) \text{ differential operators } \begin{cases} D_{(i)}^- = \frac{\partial}{\partial x_i} \\ D_{(i)}^3 = x_i \frac{\partial}{\partial x_i} - j_i \\ D_{(i)}^+ = x_i^2 \frac{\partial}{\partial x_i} - 2j_i x_i \end{cases}$$

$$\text{Lax matrix } \boxed{L(u) = \sum_{i=1}^m \frac{\vec{t} \cdot \vec{D}_{(i)}}{u - z_i}} \begin{cases} u = \text{spectral parameter} \\ L(u) \in Mat_{N \times N} \end{cases}$$

# Sklyanin's Separation of Variables

$sl(2)$  Lax matrix

$$L(u) = \begin{bmatrix} \sum_{i=1}^m \frac{1}{u-z_i} \left( x_i \frac{\partial}{\partial x_i} - j_i \right) & \sum_{i=1}^m \frac{1}{u-z_i} \frac{\partial}{\partial x_i} \\ \sum_{i=1}^m \frac{1}{u-z_i} \left( x_i^2 \frac{\partial}{\partial x_i} - 2j_i x_i \right) & - \sum_{i=1}^m \frac{1}{u-z_i} \left( x_i \frac{\partial}{\partial x_i} - j_i \right) \end{bmatrix}$$

Building separated variables  $y_a$  from  $L(u)$ :

- Define  $y_a$  as the zeroes of  $B(u) = L_1^2(u)$
- Find associated momenta  $p_a = A(y_a)$  where  $A(u) = L_1^1(u)$
- Compute the kernel  $S(\{x_i\}|\{y_a\}) = \prod_{i=1}^m e^{x_i \frac{\prod_a (z_i - y_a)}{\prod_{l \neq i} (z_i - z_l)}}$  of the integral transformation

# $sl(2)$ KZ equations in Sklyanin variables

Characteristic equation [Sklyanin]

$$p_a^2 = \frac{1}{2}(L_\alpha^\beta L_\beta^\alpha)(y_a) = \sum_i \frac{1}{y_a - z_i} \left( H_i - \frac{1}{2} \frac{j_i(j_i + 1)}{y_a - z_i} \right) \quad (5)$$

Apply to  $S^{-1}\Omega_m^{\text{WZNW}}$  and inject KZ equations:

$$\left[ \frac{\partial^2}{\partial y_a^2} + \sum_{i=1}^m \frac{k-2}{y_a - z_i} \left( S^{-1} \frac{\delta}{\delta z_i} S + \frac{\Delta j_i}{y_a - z_i} \right) \right] S^{-1}\Omega_m = 0 \quad (6)$$

$$\sum_i \frac{1}{y_a - z_i} S^{-1} \frac{\delta}{\delta z_i} S = \sum_i \frac{1}{y_a - z_i} \left[ \frac{\partial}{\partial z_i} + \frac{\partial}{\partial y_a} \right] + \sum_b \frac{1}{y_a - y_b} \left[ \frac{\partial}{\partial y_b} - \frac{\partial}{\partial y_a} \right]$$

→ Second order BPZ equations for  $\Theta_m \tilde{\Omega}_m^{\text{Liouville}}$   
with  $V_{\alpha_d} = V_{-\frac{1}{2b}}$  [Stoyanovsky+Feigin+Frenkel]



# A family of solvable non-rational CFTs

$$H_3^+ \text{-Liouville} \quad S \cdot \Theta_m \tilde{\Omega}_m^{\text{Liouville}} = \Omega_m^{H_3^+}$$

$$\text{Now if} \quad \tilde{\Omega}_m^{(r)} \equiv \left\langle \prod_{i=1}^m V_{\alpha_i}(z_i) \prod_{a=1}^{m-2} V_{-\frac{r}{2b}}(y_a) \right\rangle \quad (r \neq 1)$$

$$\text{then} \quad \boxed{S \cdot \Theta_m^{(r)} \tilde{\Omega}_m^{(r)} = ?} \quad [\text{SR, 2008}]$$

Answer: Lagrangian  $L^{(r)} = \partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + (-\beta\bar{\beta})^r e^{2b\phi}$   
 with  $\phi, \beta, \gamma$  bosons,  $\Delta(\phi) = \Delta(\beta) = 0$ ,  $\Delta(\gamma) = 1$

- Correlation functions of descendants?
- Differential equation if  $r \in \mathbb{N} + b^2\mathbb{N}$ ?

→ extend  $sl(2)$  family by a subfamily with 2 parameters  $(r, b)$   
 and  $c(r, b) = 3 + 6(b + b^{-1}(1 - r))^2$

# $sl(3)$ Separation of Variables

$L(u) = \sum_{i=1}^m \frac{\vec{t} \cdot \vec{D}_{(i)}}{u - z_i}$ : the  $3 \times 3$  Lax matrix

- $B(u) = L_2^1 L_3^2 L_3^2 - L_3^2 L_3^1 L_2^2 + L_3^1 L_3^2 L_1^1 - L_1^2 L_3^1 L_3^1$
- $A(u) = -L_1^1 + \frac{L_3^1 L_1^2}{L_3^2}$
- Kernel  $S(\{x_i\}, \{y_a\})$ : no explicit expression

As before  $B(y_a) = 0$  and  $p_a = A(y_a)$

Characteristic equation:

$$p_a^3 - p_a \cdot \frac{1}{2} (L_\alpha^\beta L_\beta^\alpha)(y_a) + \frac{1}{4} (L_\alpha^\beta L_\beta^\alpha)'(y_a) + \frac{1}{6} \left( L_\alpha^\beta L_\beta^\gamma L_\gamma^\alpha + L_\beta^\alpha L_\gamma^\beta L_\alpha^\gamma \right) (y_a) = 0 \quad (7)$$

# $sl(3)$ KZ equations in Sklyanin variables

$$\left[ \frac{\partial^3}{\partial y_a^3} + \frac{\partial}{\partial y_a} \cdot \sum_{i=1}^m \frac{k-3}{y_a - z_i} S^{-1} \frac{\delta}{\delta z_i} S + \sum_{i=1}^m \frac{(k-3)\Delta_{j_i}}{(y_a - z_i)^2} \frac{\partial}{\partial y_a} + \sum_{i=1}^m \frac{(k-3)^{\frac{3}{2}} q_{j_i} - (k-3)\Delta_{j_i}}{(y_a - z_i)^3} \right] S^{-1} \Omega_m^{\text{WZNW}} \sim 0 \quad (8)$$

Null-vector equations in  $sl(3)$  Toda theory with  $\alpha_d = -b\omega_1$ :

$$\left[ \frac{\partial^3}{\partial y_a^3} + \frac{1}{b^2} D_2 + \frac{1}{b^4} D_1 + \frac{1}{b^2} \sum_{i=1}^m \frac{\Delta_{j_i}}{(y_a - z_i)^2} \frac{\partial}{\partial y_a} + \sum_{i=1}^m \frac{\frac{1}{b^3} q_{j_i} - \frac{1}{b^2} \Delta_{j_i}}{(y_a - z_i)^3} \right] \Theta_m \tilde{\Omega}_m^{\text{Toda}} \sim 0 \quad (9)$$

$$\frac{1}{b^2} = k - 3, \quad \alpha(j) = -bj + b^{-1}(e_1 + e_2), \quad j \in \mathbb{C}^2 \Leftrightarrow (\Delta_j, q_j)$$

# Comparison of differential equations

- $D_2 \stackrel{?}{\sim} \frac{\partial}{\partial y_a} \cdot \sum_{i=1}^m \frac{1}{y_a - z_i} S^{-1} \frac{\delta}{\delta z_i} S$
- Problem with  $D_1$  term where

$$\begin{aligned}
 D_1 = & - \sum_i \frac{1}{(y_a - z_i)^2} \frac{\partial}{\partial y_a} + 3 \sum_i \frac{1}{y_a - z_i} \sum_b \frac{1}{y_a - y_b} \left( \frac{\partial}{\partial y_b} - \frac{\partial}{\partial y_a} \right) \\
 & + 2 \left( \sum_i \frac{1}{y_a - z_i} \right)^2 \frac{\partial}{\partial y_a} - 2 \sum_{b \neq c} \frac{1}{y_a - y_b} \frac{1}{y_b - y_c} \left( \frac{\partial}{\partial y_b} - \frac{\partial}{\partial y_a} \right) \quad (10)
 \end{aligned}$$

A solution:  $\boxed{k \rightarrow 3 \Leftrightarrow b \rightarrow \infty}$  (critical level)

An explanation: Langlands correspondence?

A speculation:  $S \cdot \Theta_m \tilde{\Omega}_m^{\text{Toda}} \neq \Omega_m^{WZNW}$   
 but  $S \cdot \Theta_m \tilde{\Omega}_m^{\text{Toda}} = \Omega_m^{\text{Some new CFT}}$