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To cite this version:

Corentin Briat, Jean-François Lafay, Olivier Sename. Delay-Scheduled State-Feedback Design for Time-Delay Systems with Time-Varying Delays - A LPV Approach. Systems and Control Letters, 2009, 58, pp.664-671. 10.1016/j.sysconle.2009.06.001 . hal-00402238

HAL Id: hal-00402238

https://hal.science/hal-00402238v1

Submitted on 28 Oct 2010

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Delay-Scheduled State-Feedback Design for Time-Delay Systems with Time-Varying Delays - A LPV Approach

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Abstract

This paper is concerned with the synthesis of delay-scheduled state-feedback controllers which stabilize linear systems with time-varying delays. In this framework, it is assumed that the delay is approximately known in real-time and used in the controller in a scheduling fashion. First, a new model transformation turning a time-delay system into an uncertain LPV system is introduced. Using this transformation, a new delay-dependent stability test based on the so-called full block $\mathcal{S}$-procedure is developed and from this result, a new delay-dependent stabilization result is derived. Since the resulting LMI conditions depend polynomially on the parameters, a relaxation result is then applied in order to obtain a tractable finite set of finite-dimensional LMIs. The interests of the approach resides in 1) the synthesis of a new type of controllers scheduled by the delay value which has a lower memory consumption than controllers with memory (since it is not necessary to store past values of the state), and 2) an easy consideration of uncertainties on the delay knowledge.

Key words: Time-delay systems, Linear parameter varying systems, Robust linear matrix inequalities, Relaxation, Convex optimization

1. Introduction

Since several years, time-delay systems (TDS) have been intensively studied (Niculescu, 2001; Fridman and Shaked, 2002; Gu et al., 2003; Jiang and Han, 2005; Fridman, 2006; Gouaisbaut and Peaucelle, 2006a; Suplin et al., 2006; Xu et al., 2006; Kao and Rantzer, 2007). It has been shown that delays are often responsible of instability and poor performances and thus, they should not be neglected when analyzing system stability or synthesizing control laws. Since the advent of communication networks and embedded electronics, systems with time-varying delays have attracted more and more interest. Indeed, a communication network can be viewed as a communication channel inducing delays depending on the load of the network which varies in time.

In some applications, it may be possible to measure or compute the delay from a mathematical model and in this case, it could be interesting to use this information in the controller. In (Witrant et al., 2005), a predictive approach to control Network Controlled Systems is given but a network model is necessary to compute the prediction horizon. In (Sename et al., 1995) a state feedback with internal delay is designed but the robustness issue w.r.t delay measurement uncertainties is not considered. The authors proposed in (Briat et al., 2007) some preliminary results on a new control design technique for TDS. Using a new model transformation, the time-delay system is transformed into an uncertain Linear Parameter Varying (LPV) system where the delay acts as a parameter. However the results stand for interval delay (with non zero delay) only. Some interesting results on TDS based on Linear Fractional Transformation (LFT) may also be found in (Zhang et al., 2001; Roozbehani and Knospe, 2005; Gouaisbaut and Peaucelle, 2006b; Kao and Rantzer, 2007).

Following the preliminary results in (Briat et al., 2007), the LPV/uncertain system stability analysis and control synthesis tools are used to prove stability and stabilize time-delay systems (See (Apkarian and Adams, 1998; Scherer, 1999)). The main contributions of the paper are:

- Following the idea of (Briat et al., 2007), a new model transformation which corrects some weaknesses of the previous one is introduced.
- Using these results, a delay-dependent stability with guaranteed $\mathcal{L}_2$ performances test is provided. It is obtained using the so-called full-block $\mathcal{S}$-procedure and is expressed through parameter dependent LMI conditions. Computational approximations are then used in order

to make the problem tractable by numerical procedures.
– From this stability test a stabilization result is derived. The obtained controller is smoothly scheduled by an approximate delay value and the error on the delay knowledge is taken into account as a robustness constraint.

In this paper, the following systems are considered

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h(t)) + B_u u(t) + Ew(t) \\ z(t) &= Cx(t) + C_h x(t-h(t)) + D_u u(t) + Fw(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are respectively the system state, the control input, the exogenous input and the controlled output.

The time-varying delay $h(t)$ is assumed to belong to the set

$$\mathscr{H} := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, H),\ \dot{h} : \mathbb{R}_+ \to U \right\} \tag{2}$$

where $\mathcal{C}^1(I, J)$ denotes the set of continuous functions with continuous derivative mapping I to J, $H := [h_{min}, h_{max}]$ and $U := [\mu_{min}, \mu_{max}]$. The delay takes then bounded values and has a bounded derivative.

It is convenient to introduce the following set of vertices

$$\begin{aligned} \mathcal{V}_h &:= \{h_{min}, h_{max}\} \\ \mathcal{V}_\mu &:= \{\mu_{min}, \mu_{max}\} \qquad \mu_{max} < 1 \end{aligned} \tag{3}$$

The aim of the paper is to find delay-scheduled state-feedback controllers of the form

$$u(t) = K(\hat{h}(t))x(t) \tag{4}$$

which stabilize system (1) and where $K(\cdot)$ may be a linear, polynomial or rational function of the known value of the delay $\hat{h}(t) = h(t) + \delta_h(t)$, with knowledge error $\delta_h(t)$ belonging to

$$\Delta := \left\{ \delta_h : \mathbb{R}_+ \to \Delta,\ \dot{\delta}_h : \mathbb{R}_+ \to \Delta_\nu \right\} \tag{5}$$

where $\Delta := [-\delta, \delta]$ and $\Delta_\nu := [\nu_{min}, \nu_{max}]$.

Finally we define the set of known delay values:

$$\hat{\mathscr{H}} := \left\{ \hat{h} \in \mathcal{C}^1(\mathbb{R}_+, \hat{H}),\ \dot{\hat{h}} : \mathbb{R}_+ \to \hat{U} \right\} \tag{6}$$

with $\hat{H} := H + \Delta$ and $\hat{U} := U + \Delta_\nu$.

The notations are as follows, for symmetric matrices A, B, $A > B$ means $A - B$ is positive definite (i.e. $A - B > 0$). For a square matrix A we have $A^{\mathcal{S}} = A + A^T$ where A^T is the transpose of A. $Ker(A)$ is a basis of the null-space of A. $A_\perp$ is any basis of the orthogonal complement of $\Im(A)$ (i.e. $A^T A_\perp = 0$) and $Im(A)$ is the image set of A. $\oplus$ is the direct sum of matrices: $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. $\mathcal{L}_2$ is the space of signals with finite energy (finite $\mathcal{L}_2$-norm): $||f||^2_{\mathcal{L}_2} := \int_0^{+\infty} |f(t)|^2 dt < +\infty$.

The paper is organized as follows, in Section 2 we introduce some preliminary results. In Section 3, the new model transformation and the associated comparison system are presented. The first main result of the paper: a new delay-dependent stability test based on the use of the model transformation is developed in Section 4. Finally, the second main result of the paper: the delay-scheduled state-feedback design is detailed in Section 5.

2. Converting Polynomial into Linear Dependence

This section is devoted to the presentation of a relaxation technique for polynomially parameter dependent LMI. The key idea is to use the Finsler's lemma (Skelton et al., 1997) to linearize the dependence on the parameters. The following definition will be useful in the sequel

Definition 2.1 *A square matrix S is said to be $\mathcal{S}^2$-structured if it writes $S = [S_{ij}]_{i,j}$ with blocks $S_{ij} \in \mathbb{R}^{k \times k}$, $k > 1$ such that*

$$S_{ij} := \begin{cases} 0_{k \times k} & \text{if } i = j \\ S_{ij} = S_{ji}^T \in \mathbb{K}^k & \text{if } i \neq j \end{cases}$$

It is now possible to express the linearization lemma which has also been provided in (Sato, 2006; Sato and Peaucelle, 2007).

Lemma 2.1 *Suppose that a $\mathcal{M}(\delta)$ is a polynomially parametrized symmetric matrix in $\delta \in \Delta$ admitting a spectral factorization $\mathcal{M}(\delta) = U^T(\delta)\mathcal{N}U(\delta) < 0$ where $U(\delta)$ is a basis of polynomials. Then $\mathcal{M}(\delta) < 0$ for every $\delta \in \Delta$ if there exists a matrix $\mathcal{P}$ and a $\mathcal{S}^2$-structured matrix $\mathcal{R}$ such that*

$$\mathcal{N} + \mathcal{R} + \mathcal{P}V(\delta) + V^T(\delta)\mathcal{P}^T < 0 \quad \delta \in \Delta \tag{7}$$

where $U(\delta) = Ker(V(\delta))$ and $\mathcal{R}$ is $\mathcal{S}^2$-structured (i.e. $U^T(\delta)\mathcal{R}U(\delta) = 0$ with $\mathcal{R} = \mathcal{R}^T \neq 0$).

Proof: The proof is given in Appendix A and relies on the Finsler's Lemma (Skelton et al., 1997). □

Remark 2.1 *If the matrix $V(\delta)$ is affine in δ then lemma 2.1 can be used as a linearization procedure to linearize polynomially parameter dependent LMIs. It can be shown that every polynomially parametrized LMI can be expressed through a basis $U(\delta)$ which leads to an affine matrix $V(\delta)$. Indeed, the trivial case $U(\delta) = col(I, \delta_1, \ldots, \delta_N, \delta_1^2, \ldots)$ leads to an affine $V(\delta)$. Hence, assuming that a 'good' basis is used, then (7) is affine and thus can be treated as in the polytopic/affine framework (using multi-convexity).*

This approach is well dedicated to small and medium size problems since the size and the number of LMIs grows very quickly depending on the degree of polynomials, the number of parameters and the size of the initial LMI. In the case of large size problems, it may be interesting to use the relaxation proposed in (Ben-Tal and Nemirovski, 2002; Scherer, 2006). It is worth mentioning that this result is highly related to the so-called Sum-of-Squares (SoS) relaxation of parameter dependent LMIs.

An important remark concerns the relaxation of LMIs with rational parameter dependence. This type of LMIs

can indeed be turned into LMIs with polynomial dependence using the full-block $\mathcal{S}$-procedure (Scherer and Weiland, 2005) which can be, in turn, transformed into LMIs with linear dependence using lemma 2.1.

3. A new model transformation

A new model transformation allowing to turn a TDS with time-varying delays into an uncertain LPV system is provided in this section. The main advantages of this new operator compared to (Briat et al., 2007) is the consideration of zero delay values and a tighter computation of the $\mathcal{L}_2$ induced norm of the operator.

Define then the operator

$$\begin{aligned} \mathcal{D}_h : \mathcal{L}_2 &\to \mathcal{L}_2 \\ \eta(t) &\to \frac{1}{\sqrt{h(t)h_{max}}}\int_{t-h(t)}^{t}\eta(s)ds \end{aligned} \tag{9}$$

Proposition 3.1 *This operator enjoys the following properties:*

(i) $\mathcal{D}_h$ is $\mathcal{L}_2$ input/output stable.

(ii) $\mathcal{D}_h$ has a $\mathcal{L}_2$ induced norm lower than 1.

Proof: The proof is given in Appendix B and is based on a similar method as of (Gu et al., 2003). □

The remaining of the section explains the use of the operator $\mathcal{D}_h$ in order to turn a time-delay system into an uncertain LPV system. Let us consider system (1) and note that $x_h(t) = x(t) - h(t)\mathcal{D}_h(\dot{x}(t))$. Thus, substituting this equality into system (1) yields

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) - \alpha(t)A_h w_0(t) + B_u u(t) + Ew(t) \\ z_0(t) &= \dot{x}(t) \\ z(t) &= \bar{C}x(t) - \alpha(t)C_h w_0(t) + D_u u(t) + Fw(t) \\ w_0(t) &= \mathcal{D}_h(z_0(t)) \\ \alpha(t) &= \sqrt{h(t)h_{max}} \\ \bar{A} &= A + A_h \quad \bar{C} = C + C_h \end{aligned} \tag{10}$$

which is an uncertain LPV system with state $x(t)$ expressed in 'LFT' form. Indeed, the system is

- uncertain due to the presence of the 'unknown' structured norm bounded LTV dynamic operator $\mathcal{D}_h$ and
- parameter varying due to the presence $\sqrt{h(t)}$ whose dependence is even affine.

In order to deal with the uncertain operator $\mathcal{D}_h$, robust stability and robust control theories based on the full-block $\mathcal{S}$-procedure approach will be considered (Scherer and Weiland, 2005). The parameter dependence will be tackled using parameter dependent Lyapunov functions.

Remark 3.1 *According to (Gu et al., 2003), system (10) is not equivalent to (1) due to the use of the model transformation induced by the use of the operator $\mathcal{D}_h$. Without entering too much in details, the use of model transformation often introduces conservatism in the approach by adding additional dynamics to the comparison (the transformed) model which may be unstable even if the original system is stable. Hence the stability test performed on the comparison model (10) may fail even if the initial system (1) is stable. However, in the stabilization problem, this is less critical since it is aimed to stabilize the system and thus the additional dynamics will be implicitly stabilized.*

4. Delay-dependent stability

In this section, we develop the first main result of the paper: a new delay dependent stability test based on the comparison model (10) obtained from system (1) used along with the model transformation presented in section 3.

Lemma 4.1 *System (10) without control input (i.e. $u(t) = 0$) is asymptotically stable for $h \in \mathscr{H}$ and satisfies the $\mathcal{L}_2$ performance constraint $||z||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$ if there exist a smooth matrix function $P : H \to \mathbb{S}^n_{++}$, matrix functions $D : H \times U \to \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that the LMI (8) holds for all $\alpha \in [\sqrt{h_{min}h_{max}},\ h_{max}]$ and $\dot{h} \in \mathcal{V}_\mu$.*

Proof: The proof is given in Appendix C. It is based on an application of the full-block $\mathcal{S}$-procedure to deal with the uncertain operator $\mathcal{D}_h$ while the parameters are considered through the use a parameter dependent Lyapunov. □

A direct corollary can be obtained immediately by choosing all the parameter dependent matrices to be constant.

Corollary 4.1 *System (10) without control input (i.e. $u(t) = 0$) is quadratically asymptotically stable for $h \in \mathscr{H}$ and satisfies the $\mathcal{L}_2$ performance constraint $||z||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$ if there exist matrices $P, D \in \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} [\bar{A}^T P]^{\mathcal{S}} & -\alpha P A_h & PE & \bar{C}^T & \bar{A}^T D \\ \star & -D & 0 & -\alpha C_h^T & -\alpha A_h^T D \\ \star & \star & -\gamma I_p & F^T & E^T D \\ \star & \star & \star & -\gamma I_q & 0 \\ \star & \star & \star & \star & -D \end{bmatrix} < 0$$

holds for all $\alpha \in \{\sqrt{h_{min}h_{max}},\ h_{max}\}$.

Proof: The proof is straightforward application of lemma 4.1. Since the LMI is linear in $\sqrt{h}$ then using a convex argument it suffices to check the feasibility of the LMI at the vertices of H (i.e. $\mathcal{V}_h$) to conclude on the stability over the whole set H. □

The next corollary is obtained by fixing all parameter dependent matrices such that the resulting parameter dependent LMI is of maximum degree 2. It is then relaxed using Lemma 2.1.

Corollary 4.2 *System (10) without control input (i.e. $u(t) = 0$) is asymptotically stable for $h \in \mathscr{H}$ and satisfies the $\mathcal{L}_2$ performance constraint $||z||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$ if there exist matrices $P_0, P_1, D_0, D_1 \in \mathbb{S}^n$, a full matrix Θ, a $\mathcal{S}^2$-*

$$\begin{bmatrix} [\bar{A}^T P(h)]^{\mathcal{S}} + \frac{dP}{dh}\dot{h} & -\alpha P(h) A_h & P(h)E & \bar{C}^T & \bar{A}^T D(h,\dot{h}) \\ \star & -D(h,\dot{h}) & 0 & -\alpha C_h^T & -\alpha A_h^T D(h,\dot{h}) \\ \star & \star & -\gamma I_p & F^T & E^T D(h,\dot{h}) \\ \star & \star & \star & -\gamma I_q & 0 \\ \star & \star & \star & \star & -D(h,\dot{h}) \end{bmatrix} < 0 \tag{8}$$

structured matrix Φ and a scalar $\gamma > 0$ such that for all $(h_d, \mu_d) \in \mathcal{V}_h \times \mathcal{V}_\mu$ the LMIs

$$\Psi + \Phi + V_s(h_d)^T \Theta^T + \Theta V_s(h_d) < 0 \tag{11}$$

$$P_0 + h_d P_1 > 0 \tag{12}$$

$$D_0 + \sqrt{h_d} D_1 > 0 \tag{13}$$

holds where $\beta = \sqrt{h_{max}}$, $V_s(h_d) = \begin{bmatrix} \sqrt{h_d} I & -I & 0 \\ 0 & -\sqrt{h_d} I & -I \end{bmatrix}$

and

$$\Psi = \begin{bmatrix} \Psi_0 & \Psi_{1/2}/2 & \Psi_1/3 \\ \star & \Psi_1/3 & \Psi_{3/2}/2 \\ \star & \star & 0 \end{bmatrix}$$

$$\Psi_0 = \begin{bmatrix} [\bar{A}^T P_0]^{\mathcal{S}} + P_1 \dot{h} & 0 & P_0 E & C^T & \bar{A}^T D_0 \\ \star & -D_0 & 0 & 0 & 0 \\ \star & \star & -\gamma I_p & F^T & E^T D_0 \\ \star & \star & \star & -\gamma I_q & 0 \\ \star & \star & \star & \star & -D_0 \end{bmatrix}$$

$$\Psi_{1/2} = \begin{bmatrix} 0 & -\beta P_0 A_h & 0 & 0 & \bar{A}^T D_1 \\ \star & -D_1 & 0 & -\beta C_h^T & -\beta A_h^T D_0 \\ \star & \star & 0 & 0 & E^T D_1 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & -D_1 \end{bmatrix}$$

$$\Psi_1 = \begin{bmatrix} \bar{A}^T P_1 + P_1 \bar{A} & 0 & P_1 E & 0 & 0 \\ \star & 0 & 0 & 0 & -\beta A_h^T D_1 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix}$$

$$\Psi_{3/2} = \begin{bmatrix} 0 & -\beta P_1 A_h & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix}$$

Proof: Since system (10) is linear in $\sqrt{h(t)}$ then matrices of the form $P(h) = P_0 + P_1 h > 0$ and $D = D_0 + D_1\sqrt{h}$ are chosen. The reason why P does not involve a term in $\sqrt{h}$ is the non-differentiability of $\sqrt{h}$ for $h = 0$. However such a term can be used when the delay is not assumed to take 0 values.

Note that $P(h) > 0$ if and only if (12) is satisfied and $D(h) > 0$ if and only if (13) using standard convex arguments. Then substitute the explicit expressions of $D(h)$ and $P(h)$ into (8) leads to the expression

$$\Psi_0 + \Psi_{1/2}\sqrt{h} + \Psi_1 h + \Psi_{3/2} h^{3/2} < 0 \tag{17}$$

Since the LMI is affine in $\dot{h}$ (only Ψ_0 depends on $\dot{h}$) it is not necessary to include it into the spectral factor $U(\cdot)$. Then computing the spectral factorization of (17) with $U_s(h) = \begin{bmatrix} I \\ \sqrt{h} I \\ hI \end{bmatrix}$ we get

$$U_s^T(h) \begin{bmatrix} \Psi_0 & \Psi_{1/2}/2 & \Psi_1/3 \\ \star & \Psi_1/3 & \Psi_{3/2}/2 \\ \star & \star & 0 \end{bmatrix} U_s(h) < 0 \tag{18}$$

Applying lemma 2.1 on (18) linearizes the parameter dependence which becomes affine in $\sqrt{h}$. Indeed, we have

$$V_s(h) = \begin{bmatrix} \sqrt{h} I & -I & 0 \\ 0 & \sqrt{h} I & -I \end{bmatrix}$$

and $V_s(h) U_s(h) = 0$. Finally, using a convexity argument it suffices to check the negative definiteness of the LMI at the vertices of $H \times U$ (i.e. for every element of $\mathcal{V}_h \times \mathcal{V}_\mu$) to check the feasibility of the LMI over the whole space $H \times U$ and delay and we obtain (11). □

5. Delay-Scheduled state-feedback design

This section is devoted to the determination of a delay-scheduled controller of the form (4) which stabilizes system (1) and ensures $\mathcal{L}_2$ performance of the closed-loop system. The closed-loop system obtained from the interconnection of (1) and (4) is governed by the equations

$$\begin{bmatrix} -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial \hat{h}} + [X(\hat{h})\bar{A}^T + Y^T(\hat{h})B_u^T]^{\mathcal{S}} & X(\hat{h})\bar{C}^T + Y^T(\hat{h})D_u^T & -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial \hat{h}} + \bar{A}X(\hat{h}) + BY(\hat{h}) & \alpha h A_h \tilde{D}(\xi) & E \\ \star & -\gamma I_q & \bar{C}X(\hat{h}) + DY(\hat{h}) & h\alpha C_h \tilde{D}(\xi) & F \\ \star & \star & -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial \hat{h}} - \tilde{D}(\xi) & 0 & 0 \\ \star & \star & \star & -\tilde{D}(\xi) & 0 \\ \star & \star & \star & \star & -\gamma I_p \end{bmatrix} < 0 \tag{14}$$

$$\Psi_0 = \begin{bmatrix} -(\mu_d + \delta_\nu)X_1 + [X(\delta_d)\bar{A}^T + Y^T(\delta_d)B_u^T]^{\mathcal{S}} & X(\delta_d)\bar{C}^T + Y^T(\delta_d)D_u^T & -\dot{\delta}_d X_1 + \bar{A}X(\delta_d) & 0 & E \\ \star & -\gamma I_q & \bar{C}X(\delta_d) & 0 & F \\ \star & \star & -(\mu_d + \delta_\nu)X_1 - \tilde{D}_0 & 0 & 0 \\ \star & \star & \star & -\tilde{D}_0 & 0 \\ \star & \star & \star & \star & -\gamma I_p \end{bmatrix} \tag{15}$$

$$\Psi_1 = \begin{bmatrix} [\bar{A}X_1 + B_u Y_1]^{\mathcal{S}} & X_1\bar{C}^T + Y_1^T D_u^T & \bar{A}X_1 & \beta A_h \tilde{D}_1 & 0 \\ \star & 0 & \bar{C}X_1 & \beta C_h \tilde{D}_1 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix} \tag{16}$$

$$\begin{aligned} \dot{x}(t) &= \bar{A}_{cl}(h, \delta_h)x(t) - \alpha(t)A_h w_0(t) + Ew(t) \\ z(t) &= \bar{C}_{cl}(h, \delta_h)x(t) - \alpha(t)C_h w_0(t) + Fw(t) \\ z_0(t) &= \dot{x}(t) \\ w_0(t) &= \mathcal{D}_h(z_0(t)) \end{aligned} \tag{19}$$

with $\hat{h} = h + \delta_h \in \hat{H}$. Since the state feedback is of the form $K(\hat{h})$ then the closed-loop system matrices are given by $\bar{A}_{cl}(\hat{h}) = A + A_h + B_u K(\hat{h})$, $\bar{C}_{cl}(\hat{h}) = C + C_h + D_u K(\hat{h})$.

At this point, several techniques can be employed to compute the controller (4):

(i) either use a change of variable and in this case it is possible to fix a desired form to the controller; or

(ii) elaborate a stabilizability test and deduce a suitable controller either by explicit formulae or implicitly by solving a SDP.

In the present paper we propose a solution based on a change of variable which has several benefits compared to the second one. The main one is the the possibility of choosing the structure of the controller (constant, affine, polynomial or rational). The resulting controller is then implicitly implementable. This is not the case of controllers computed using the second technique (with explicit construction formulae) which may depend on the derivative of the delay, supposed to be unknown. Moreover, the first approach allows for an easy computation of rational controllers through an appropriate choice of the decision matrices.

Theorem 5.1 *The system (10) is stabilizable with a delay-scheduled state feedback gain $K(\hat{h}) = Y(\hat{h})X^{-1}(\hat{h})$ if there exists a smooth matrix function $X : \hat{H} \to \mathbb{S}^n_{++}$, matrix functions $\tilde{D} : H \times U \times \hat{H} \times \hat{U} \to \mathbb{S}^n_{++}$, $K : \hat{H} \to \mathbb{R}^{m\times n}$ and a scalar $\gamma > 0$ such that the LMI (14) holds for all $h \in H$, $\dot{h} \in U$, $\hat{h} \in \hat{H}$ and $\dot{\hat{h}} \in \hat{\Delta}$, where $\xi = col(h, \delta_h, \dot{h}, \dot{\delta}_h)$ and $\alpha = \sqrt{hh_{max}}$.*

Moreover the closed-loop system satisfies $||z||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$.

Proof: The proof is given in Appendix D and is mainly based on the dualization lemma (Scherer and Weiland, 2005). □

We develop immediately the following corollary where all the parameter dependent matrices are chosen to be constant:

Corollary 5.1 *The system (10) is stabilizable by a constant state feedback $K = YX^{-1}$ if there exists constant matrices $X, \tilde{D} \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$ and a scalar $\gamma > 0$ such that the LMI (20)*

$$\begin{bmatrix} [X\bar{A}^T + Y^T B_u^T]^{\mathcal{S}} & X\bar{C}^T + Y^T D_u^T & \bar{A}X & \alpha A_h \tilde{D} & E \\ \star & -\gamma I_q & \bar{C}X & \alpha C_h \tilde{D} & F \\ \star & \star & -\tilde{D} & 0 & 0 \\ \star & \star & \star & -\tilde{D} & 0 \\ \star & \star & \star & \star & -\gamma I_p \end{bmatrix} < 0 \tag{20}$$

holds for all $h \in \mathcal{V}_h$ *and* $\alpha = \sqrt{h h_{max}}$.

Moreover the closed-loop system satisfies $||z||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$.

Proof: It is a straightforward application of theorem 5.1 when using constant decision matrices. Then with a convexity argument it is possible to conclude on the feasibility of the LMI over the whole parameter space while checking the feasibility at the vertices of the set H only (i.e. $\mathcal{V}_h$). □

Finally the corollary below stands when the parameter dependent matrices are chosen such that rational controllers of degree 1 in $\hat{h}$ can be designed:

Corollary 5.2 *The system (10) is stabilizable with a delay-scheduled state feedback* $K(\hat{h}) = Y(\hat{h})X(\hat{h})^{-1}$ *with* $Y(\hat{h}) = Y_0 + Y_1\hat{h}$ *and* $X(\hat{h}) = X_0 + \hat{h}X_1$ *if there exist matrices* $X_0, X_1, \tilde{D}_0, \tilde{D}_1 \in \mathbb{S}^n$, $Y_0, Y_1 \in \mathbb{R}^{m\times n}$, *a full matrix* Π, *a* $\mathcal{S}^2$*-structured matrix* Φ *and a scalar* $\gamma > 0$ *such that the following LMIs*

$$\Psi + \Phi + \mathcal{K}_1^T(\xi)\Pi^T + \Pi\mathcal{K}_1(\xi) < 0 \tag{21}$$

$$\tilde{D}_0 + \tilde{D}_1\sqrt{h_d} > 0 \tag{22}$$

$$X_0 + X_1(h_d + \delta_d) > 0 \tag{23}$$

hold for all $\xi \in \mathcal{V}_h \times \mathcal{V}_\mu \times \mathcal{V}_\delta \times \mathcal{V}_\nu$ *with* $\mathcal{V}_\delta := \{-\delta, \delta\}$, $\mathcal{V}_\nu := \{\nu_{min}, \nu_{max}\}$, $\xi = (h_d, \mu_d, \delta_d, \delta_\nu)$, $\Psi = \begin{bmatrix} \Psi_0 & \Psi_{1/2}/2 \\ \star & \Psi_1 \end{bmatrix}$ *where* Ψ_0 *is given by (15),* Ψ_1 *by (16) and*

$$\Psi_{1/2} = \begin{bmatrix} 0 & 0 & 0 & \beta A_h \tilde{D}_0 & 0 \\ \star & 0 & 0 & \beta C_h \tilde{D}_0 & 0 \\ \star & \star & -\tilde{D}_1 & 0 & 0 \\ \star & \star & \star & -\tilde{D}_1 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix}$$

Proof: The proof is similar as for Corollary 4.2. Affine parameter dependent matrices $Y(\hat{h}) = Y_0 + Y_1\hat{h}$ and $X(\hat{h}) = X_0 + X_1\hat{h}$ are chosen. The matrix $\tilde{D}$ is fixed to be affine in $\sqrt{h(t)}$. Then (14) may be rewritten as

$$\begin{bmatrix} I \\ \sqrt{h}I \end{bmatrix}^T \underbrace{\begin{bmatrix} \Psi_0 & \Psi_{1/2}/2 \\ \star & \Psi_1 \end{bmatrix}}_{\Psi} \begin{bmatrix} I \\ \sqrt{h}I \end{bmatrix}^T < 0 \tag{24}$$

The linearization procedure is then applied and yields (21). The negative definiteness of $\tilde{D}$ and positive definiteness of X are defined by (22) and (23) respectively. □

6. Example

We aim to stabilize the following time delay system with time-varying delay

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} x_h(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \end{aligned} \tag{25}$$

with $h(t) \in [0.1, 0.9]$, $|\dot{h}| \leq 0.2$, $\dot{\delta}_h, \delta_h \in [-0.1, 0.1]$. Constant scaling (i.e. constant $\tilde{D}$) is chosen and we compute an affine state-feedback of the form $K(\hat{h}) = K_0 + K_1\hat{h}$ with

$$\begin{aligned} K_0 &= \begin{bmatrix} -27.1322 & -17.8448 \end{bmatrix} \\ K_1 &= \begin{bmatrix} -4.6802 & -3.1368 \end{bmatrix} \end{aligned} \tag{26}$$

Using this controller, the minimal $\mathcal{L}_2$ performance gain is $\gamma = 9.89$. It is important to note that, for this example, computing a rational gain does not improve the result and should not be used.

7. Conclusion

We have presented in this paper a new model transformation which refines the transformation proposed in (Briat et al., 2007) by allowing the consideration of a larger class of delay values (especially including 0). The methodology is similar and is based on the transformation of a time-delay system into an uncertain LPV system where the delay acts as a time-varying parameter. Based on that description, it is possible to propose a new delay-dependent stability lemma based on the full-block $\mathcal{S}$-procedure and derive a constructive approach for the computation of stabilizing state-feedback controllers. Both constant and delay-scheduled controllers are considered and uncertainties on the knowledge of the delay are taken into account in the synthesis problem. All the results are given in terms of parameter dependent LMIs depending polynomially on the parameters. These parametrized LMIs are then relaxed into a set of constant LMIs using a linearization result.

Appendix A. Proof of Lemma 2.1

The proof is a simple application of the Finsler's lemma (Skelton et al., 1997) recalleds hereunder:

Lemma A.1 *Let M by a symmetric matrix and B a matrix of appropriate dimensions, then the following statements are equivalent:*

(i) Inequality $Ker[B]^T M Ker[B] < 0$ holds.

(ii) There exists a matrix N of appropriate dimensions such that $M + B^T N + N^T B < 0$

First note that if the quadratic form $U^T(\delta)\mathcal{R}U(\delta)$ is added to $U^T(\delta)\mathcal{N}U(\delta)$, the expression of the quadratic form remains unchanged (since $\mathcal{R}$ is $\mathcal{S}^2$-structured and hence $U^T(\delta)\mathcal{R}U(\delta) = 0$). However this additional term modifies the eigenvalues the resulting LMI and thus provides extra degrees of freedom. Apply the Finsler's lemma on $U^T(\delta)\mathcal{N}U(\delta) < 0$ leads to the existence of $\mathcal{Q}(\delta)$ such that

$$\mathcal{N} + \mathcal{R} + \mathcal{Q}(\delta)V(\delta) + V^T(\delta)\mathcal{Q}(\delta)^T < 0$$

These parameter dependent LMIs are fully equivalent. Then fixing $\mathcal{Q}$ to be parameter independent (loosing then equivalence) yields the proposed result (7).

Appendix B. Proof of Proposition 3.1

Let us prove first that for a $\mathcal{L}_2$ input signal we get a $\mathcal{L}_2$ output signal. Assume that $\eta(t)$ is continuous and denote by $\eta_p(t)$ all the signals satisfying $d\eta_p(t)/dt = \eta(t)$ then we have

$$\mathcal{D}_h(\eta(t)) = \frac{\eta_p(t) - \eta_p(t - h(t))}{\sqrt{h(t)h_{max}}} \tag{B.1}$$

If $h(t)$ is positive then (B.1) is bounded since $\eta(t)$ is continuous and belongs to $\mathcal{L}_2$. Now, let us show that when the delay reaches 0, the output signal remains bounded. Let us suppose that there exist a (possibly infinite) family of time instants $0 \le t_0 < \ldots < t_i < t_{i+1} < \ldots$ such that $h(t_i) = 0$. Since $\eta_p(t)$ is continuously differentiable and hence we have

$$\lim_{t \to t_i} \frac{\eta_p(t) - \eta_p(t - h(t))}{\sqrt{h(t)h_{max}}} = \sqrt{\frac{h(t_i)}{h_{max}}}\eta(t_i)$$

Since $\eta(t)$ is continuous and belongs to $\mathcal{L}_2$, we can state that $\eta(t_i)$ is always finite and then the output signal satisfies

$$\lim_{t \to t_i} \frac{\eta_p(t) - \eta_p(t - h(t))}{\sqrt{h(t)h_{max}}} = 0$$

This shows that the output signal is bounded if the delay reaches zero. We have shown that the output signal remains bounded for any value of $h(t)$, let us prove now that it has a finite induced $\mathcal{L}_2$-norm using a similar method as in (Gu et al., 2003). We have the following definition

$$||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} := \int_0^{+\infty} \frac{dt}{h(t)h_{max}} \int_{t-h(t)}^{t} \eta^T(\theta)d\theta \cdot \int_{t-h(t)}^{t} \eta(\theta)d\theta$$

Then using the Jensen's inequality (see (Gu et al., 2003)) we obtain

$$||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} \le \int_0^{+\infty} \frac{dt}{h_{max}} \int_{t-h(t)}^{t} \eta^T(\theta)\eta(\theta)d\theta \tag{B.2}$$

In order to exhibit the norm of the input signal into the expression, the idea is to exchange the order of integration.

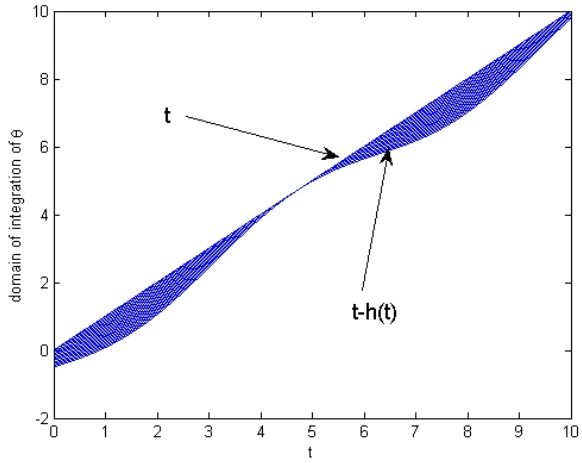

Fig. B.1. Evolution of the domain of integration of (B.2)

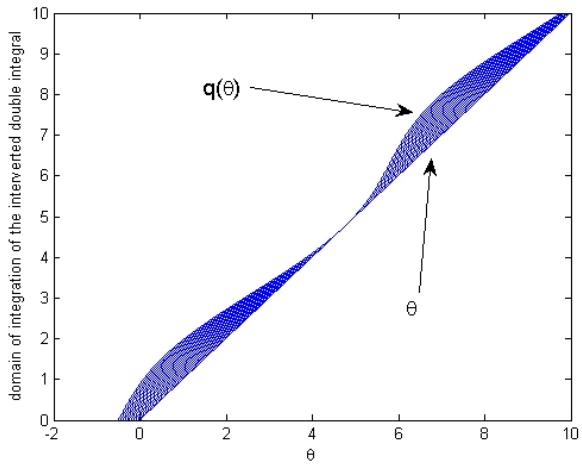

Fig. B.2. Evolution of the domain of integration of (B.3)

The exchange is possible if the dependence between t and θ is reversed. In the integral equation (B.2), θ depends on t, in the 'exchanged' one t should depend on θ. Note that the integration domain for (B.2) is defined by

$$\{(t, \theta) : \ t \in \mathbb{R}^+, \ \theta \in [t - h(t), t]\}$$

and is represented by the blue/dark surface on Figure B.1.

The exchange is based on the inverse function of $p(t) = t - h(t)$ which is denoted by $q := p^{-1}$. This inverse function exists since $p(t)$ is strictly increasing according to the assumption $\dot{h} < 1$. The inverse function can be seen as the symmetric of $p(t)$ with respect to the axis $\theta = t$ on Figure B.1 and after rotation/flipping of the plane we get the integral domain depicted on Figure B.2 which is formally defined by

$$\{(t, \theta) : \ \theta \in \mathbb{R}^+, \ t \in [\theta, q(\theta)]\}$$

It is easily seen that t depends on θ and hence the order of integration can be exchanged.

Hence we get

$$||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} \leq \frac{1}{h_{max}} \int_{-h(0)}^{+\infty} \eta^T(\theta)\eta(\theta)d\theta \int_{\theta}^{q(\theta)} dt \tag{B.3}$$
$$= \frac{1}{h_{max}} \int_{-h(0)}^{+\infty} \eta^T(\theta)\eta(\theta)(q(\theta)-\theta)d\theta$$

Moreover since the symmetry preserves distances, we have $|q(\theta)-\theta| \leq h_{max}$ and considering zero initial conditions (i.e. $\eta(s)=0$ for all $s \leq 0$) we get

$$||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} \leq ||\eta||^2_{\mathcal{L}_2}$$

stating that $\mathcal{D}_h$defines a $\mathcal{L}_2$ input/output stable operator with an $\mathcal{L}_2$-induced norm lower than 1.

Appendix C. Proof of Lemma 4.1

It is convenient to introduce first a simplified version of the full-block $\mathcal{S}$-procedure (Scherer, 2001):

Lemma C.1 *Let us consider system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_0(t)w_0(t) \\ z_0(t) &= C_0(t)x(t) + D_{00}(t)w_0(t) \\ w_0(t) &= \Delta(t)z_0(t) \end{aligned} \tag{C.1}$$

and such that the uncertainty $\Delta(t)$ satisfies the integral quadratic constraint (IQC)

$$\int_0^t z_0(s)^T \begin{bmatrix} \Delta(s) \\ I \end{bmatrix}^T M_\Delta(s) \begin{bmatrix} \Delta(s) \\ I \end{bmatrix} z_0(s)ds \geq 0 \tag{C.2}$$

where $M_\Delta(s)$ is a structured symmetric matrix (not necessarily positive semidefinite) and $z_0 \in \mathcal{L}_2$. Then if there exists a matrix $P = P^T > 0$ and $M_\Delta(s)$ such that the following LMI holds (where we omit the dependence on time):

$$\begin{bmatrix} \dot{P} + A^TP + PA & PB_0 \\ B_0^TP & 0 \end{bmatrix} + \begin{bmatrix} 0 & C_0^T \\ I & D_{00}^T \end{bmatrix} M_\Delta(s) \begin{bmatrix} 0 & I \\ C_0 & D_{00} \end{bmatrix} < 0 \tag{C.3}$$

then the system (C.1) is asymptotically stable.□

The main difficulty is the complete characterization of $\Delta(t)$ in finding a 'good' matrix $M_\Delta(s)$ such that (C.2) holds. Indeed, the integral quadratic constraint must be satisfied for every $z_0(t) \in \mathcal{L}_2$ and all 'trajectories' of the uncertain matrix $\Delta(t)$, meaning that the problem is truly infinite dimensional. Due to this fact, it is not possible (or extremely difficult) to find a matrix $M_\Delta(s)$ which totally describes the uncertain operator $\Delta(s)$ especially when $\Delta(s)$ contains dynamic LTV operators. In the remaining of the proof the matrix M_Δ will be chosen as constant.

Let us consider system

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) - \alpha(t)A_h\mathcal{D}_h(\dot{x}(t)) + Ew(t) \\ z(t) &= \bar{C}x(t) - \alpha(t)C_h\mathcal{D}_h(\dot{x}(t)) + Fw(t) \end{aligned} \tag{C.4}$$

with $\bar{A} = A + A_h$, $\bar{C} = C + C_h$ and $\alpha(t) = \sqrt{h(t)h_{max}}$. Denoting $w_0(t) := \mathcal{D}_h(\dot{x}(t))$, $z_0(t) = \dot{x}(t)$ and $\Delta(\cdot) := \mathcal{D}_h(\cdot)$. This system exactly falls into the framework of Lemma C.1.

It is possible to extend the approach to deal with robust $\mathcal{H}_\infty$ performances by adding the input/output constraint:

$$\int_0^t \begin{bmatrix} w(s) \\ z(s) \end{bmatrix}^T \begin{bmatrix} -\gamma I & 0 \\ 0 & \gamma^{-1}I \end{bmatrix} \begin{bmatrix} w(s) \\ z(s) \end{bmatrix} ds > 0 \tag{C.5}$$

where γ is a positive scalar and we get the LMI

$$\begin{aligned} &\begin{bmatrix} \frac{\partial P}{\partial h}\dot{h} + \bar{A}^TP(h) + P(h)\bar{A} & -\alpha PA_h & PE \\ \star & 0 & 0 \\ \star & \star & -\gamma(h,\dot{h})I \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \bar{A}^T \\ I & -\alpha A_h^T \\ 0 & E^T \end{bmatrix} \mho(h,\dot{h}) \begin{bmatrix} 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \end{bmatrix} \\ &+\gamma^{-1}(h,\dot{h}) \begin{bmatrix} \bar{C}^T \\ -\alpha A_h^T \\ F^T \end{bmatrix} \begin{bmatrix} \bar{C}^T \\ -\alpha A_h^T \\ F^T \end{bmatrix}^T < 0 \end{aligned} \tag{C.6}$$

where $\mho(h,\dot{h})$ verifies for all $\eta \in \mathcal{L}_2$ the IQC

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix}^T \mho(h,\dot{h}) \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix} ds > 0 \tag{C.7}$$

The separator $\mho(h,\dot{h}) = \mho^*(h,\dot{h})$ is chosen such that it characterizes the $\mathcal{L}_2$ induced norm of $\mathcal{D}_h$ that is

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix}^T \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mho_1} \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix} ds > 0 \tag{C.8}$$

for all $\eta \in \mathcal{L}_2$. Hence a set of separators can be parametrized by $\mho = \mho_1 \otimes D(h,\dot{h})$ where $D(h,\dot{h}) = D(h,\dot{h})^T > 0$ for all $(h,\dot{h}) \in H \times U$. Hence we have

$$\mho(h,\dot{h}) := \begin{bmatrix} -D(h,\dot{h}) & 0 \\ \star & D(h,\dot{h}) \end{bmatrix} \tag{C.9}$$

where $D : H \times U \to \mathbb{S}^n_{++}$. Then expanding (C.6) and performing a Schur complement on quadratic term

$$-\begin{bmatrix} \bar{C}^T & \bar{A}^TD(h,\dot{h}) \\ -\alpha C_h & -\alpha A_hD(h,\dot{h}) \\ F & E^TD(h,\dot{h}) \end{bmatrix} \begin{bmatrix} -\gamma^{-1}(h,\dot{h})I_q & 0 \\ 0 & -D^{-1}(h,\dot{h}) \end{bmatrix} (\star)^T$$

yields inequality (8).

Appendix D. Proof of Theorem 5.1

First note that the real unknown delay is denoted by $h(t)$ and the estimated one by $\hat{h}(t) = h(t) + \delta_h(t)$. Since the controller depends on X^{-1} and Y then both matrices

should depend on $\hat{h}$ only, which is the only known parameter. On the other hand, since $\tilde{D}$ is not involved in the controller expression, it may depend on any parameter (i.e. $h(t), \delta_h(t), \dot{h}(t), \dot{\delta}_h(t)$). In what follows, we define the compact notation $\xi = col(h, \delta_h, \dot{h}, \dot{\delta}_h)$ for simplicity.

Note now that LMI (C.6) can be rewritten in the following form

$$(\star)^T M(h,\dot{h}) \underbrace{\left[\begin{array}{ccc} I & 0 & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & 0 & I \\ \bar{C} & -\alpha C_h & F \end{array}\right]}_{S^-} < 0 \qquad \text{(D.1)}$$

where $M(h,\dot{h}) = \begin{bmatrix} \dot{h}\dfrac{dP(h)}{dh} & P(h) \\ P(h) & 0 \end{bmatrix} \oplus \mho(\xi) \oplus [-\gamma I_p] \oplus [\gamma^{-1} I_q]$.

In order to provide convex synthesis conditions for the controllers, the multiple products between the closed-system matrices $\bar{A}_{cl}(\hat{h}), \bar{C}_{cl}(\hat{h})$ and decision matrices P, D must be avoided. It is possible to modify the LMI condition (into an equivalent version) in order to keep one product between system matrices and decision variables only. This is performed by the mean of the dualization lemma (Scherer et al., 1997; Scherer and Weiland, 2004) which is recalled below for completeness:

Lemma D.1 *Let M ($n_M = dim(M)$) be a nonsingular symmetric matrix with respectively $n^+(M)$ and $n^-(M)$ positive and negative eigenvalues respectively. Let $\mathcal{S}^- = Im(S^-)$ a subspace of dimension $n^-(M)$ of $\mathbb{R}^{n_M}$ and $\mathcal{S}^+ = Im(S^+)$ such that $\mathcal{S}^+$ is the orthogonal complement of $\mathcal{S}^-$ (i.e. $(S^-)^T S^+ = 0$).*

The following statements are equivalent:

(i) The matrix inequality $(S^-)^T M S^- < 0$ holds.

(ii) The matrix inequality $(S^+)^T M^{-1} S^+ > 0$ holds.

□

First substitute the closed-loop system matrices into (D.1). Note that $\dim(M) = 4n + p + q$, $n^-(M) = 2n + p$ and $\text{rank}[S^-] = 2n+p$ (S^- is defined in (D.1)), where $n^-(M)$ is the number of strictly negative eigenvalues of the symmetric matrix M, $n = \dim(x)$, $p = \dim(w)$ and $q = \dim(z)$. Since $n^-(M) = \text{rank}[S^-]$ then it is possible to apply the dualization lemma and we get

$$(\star)^T M^{-1}(\xi) \underbrace{\left[\begin{array}{ccc} -\bar{A}_{cl}^T(\hat{h}) & -\bar{C}_{cl}^T(\hat{h}) & 0 \\ I_n & 0 & I_n \\ \hline \alpha A_h^T & \alpha C_h^T & 0 \\ 0 & 0 & -I_n \\ \hline -E^T & -F^T & 0 \\ 0 & I_q & 0 \end{array}\right]}_{S^+} > 0 \qquad \text{(D.2)}$$

where $M^{-1}(\xi) = \begin{bmatrix} \dfrac{dP(\hat{h})}{dt} & P(\hat{h}) \\ \star & 0 \end{bmatrix}^{-1} \oplus \mho^{-1}(\xi) \oplus [-\gamma^{-1}(\xi) I_p] \oplus [\gamma(\xi) I_q]$. Let $X = P^{-1}$, then $\dfrac{dX(\hat{h})}{dt} = -X\dfrac{dP(\hat{h})}{dt}X$ and $\begin{bmatrix} \dfrac{dP(\hat{h})}{dt} & P(\hat{h}) \\ \star & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & X(\hat{h}) \\ \star & \dfrac{dX(\hat{h})}{dt} \end{bmatrix}$. Denote also $\mho^{-1}(\xi) = \begin{bmatrix} -\tilde{D}(\xi) & 0 \\ \star & \tilde{D}(\xi) \end{bmatrix}$ with $\tilde{D}(\xi) := D(\xi)^{-1} \in \mathbb{S}^n_{++}$. Then expand (D.2) and noticing that $\tilde{D}(\xi) > 0$, the Schur complement can be used on the quadratic term:

$$-\begin{bmatrix} \alpha A_h \tilde{D} & E \\ \alpha C_h \tilde{D} & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{D}(\xi)^{-1} & 0 \\ 0 & \gamma^{-1}(\xi) I_p \end{bmatrix} (\star)^T \qquad \text{(D.3)}$$

Finally multiplying the LMI by -1 (to get a negative definite inequality) we obtain inequality (14) in which $Y(\hat{h}) = K(\hat{h})X(\hat{h})$ is a linearizing change of variable. This concludes the proof.

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