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# HYPERGEOMETRIC FUNCTIONS AND SINGULAR SOLUTIONS OF WAVE EQUATIONS WITH LORENTZ-INVARIANT POTENTIAL

ALI BENTRAD AND SATYANAD KICHENASSAMY

ABSTRACT. New solutions of equation  $u_{tt} - \Delta u + Au/(r^2 - t^2) = 0$ , in series of hypergeometric functions (HGF), are constructed. A sharpened estimate on hypergeometric functions is derived to prove the convergence of the series.

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## 1. INTRODUCTION

**1.1. Main results and motivation.** Hypergeometric solutions of linear PDEs have been studied almost since the beginning of the analytic study of PDEs, because they have built-in singularities and therefore permit a global representation of solutions near regular and singular points, unlike the representation in power series. Among the first examples are the Euler-Poisson-Darboux and Tricomi equations, see [10, book IV, chap. III] and [11]. These special equations have been studied from different perspectives in [4, 12].

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However, recent results show that many other PDEs also possess hypergeometric solutions. In particular, degenerate equations which fall beyond the scope of general existence results have been found to have solutions expressible in terms of HGF or series of HGF, see e.g. [21, 22, 23, 7, 5, 6]. The resulting information on singularities of solutions does not seem to be accessible via other methods.

These results open up the possibility that hypergeometric solutions are representative of the singular behavior of solutions of PDEs of rather general form; in a special case, this has been demonstrated in [7].

The present work gives another example of this phenomenon, for the equation:

$$(1) \quad Lu := u_{tt} - \Delta u + \frac{Au}{r^2 - t^2} = 0, \text{ for } 0 < t < r < R,$$

where  $u = u(x, t)$ ,  $r = |x|$ ,  $R$  is a positive constant,  $x \in \mathbb{R}^N$ , and  $N \geq 1$ . Note that the expression  $r^2 - t^2$  is Lorentz-invariant. The constant  $A$  is assumed to be real and less than  $\frac{1}{4}(N-1)^2$ , to make the various parameters defined below real; the expressions we obtain admit of complex extension. The solutions are considered outside the light-cone only for the same reason. The constant  $\frac{1}{4}(N-1)^2$  is the Hardy constant in dimension  $N+1$ .

The technical difficulties are twofold: (a) recognizing that the problem can be reduced to a hypergeometric equation (HGE) and (b) estimating the resulting solution.

Theorems 5 and 6 perform the reduction, and give solutions of the form:

$$(2) \quad u(x, t) = r^l \phi\left(\frac{x}{r}\right) f\left(\frac{t^2}{r^2}\right),$$

where  $l$  is an integer and  $\phi$  is a spherical harmonic:  $-\Delta_{S^{N-1}}\phi = \lambda\phi$ , where  $\Delta_{S^{N-1}}$  is the Laplace–Beltrami operator on the  $(N-1)$ -sphere. When  $N$  is odd and  $\phi$  is constant,  $f$  may be expressed in terms of elementary functions.

The estimate is proved in Lemmas 2 and 3, and gives the convergence of a series of HGFs representing the solution of the Cauchy problem with general analytic data proportional to  $\phi(x/r)$  (Theorem 8). The solutions are valid for  $0 < t < r < R-t$ , which is the domain of dependence predicted by finite speed of propagation. The usual estimates on the HGF, see Lemma 4, are not sufficient to obtain this result: they would only give convergence for  $r \leq R/2$  (Theorem 9).

*Remark 1. Solutions of inhomogeneous HGE with right-hand side  $z^{m-1}$  are also known [3]; they would enable one to obtain exact solutions to equation (1) with suitable right-hand sides. Recall that hypergeometric*

*solutions of the inhomogeneous wave equation with a constant right-hand side arise as a step in the construction of interacting waves for nonlinear equations with piecewise linear nonlinearities [15].*

Many of the HGF in this paper may be expressed in terms of Legendre functions  $P_\nu^\mu$  [2, formula 15.4.25]. However, general results on Legendre functions do not seem to give the estimates required for the convergence proof.

Replacing  $t$  by  $it$  in our solutions gives solutions of a Schrödinger equation with inverse-square, or Hardy potential, away from the origin in  $\mathbb{R}^{N+1}$ :

$$(3) \quad -u_{tt} - \Delta u + \frac{Au}{r^2 + t^2} = 0,$$

This equation arises by linearization about the “very singular solution” of  $-\Delta u = gu^p$  [8, 25]. Eigenfunctions of the Schrödinger equation with inverse-square potential on a bounded domain require Bessel functions [9], as in the case of zero potential. We do not consider the wave equation with potential  $A/r^2$ , which is not Lorentz-invariant.

Similarly, the potential  $A/(r^2 - t^2)$  arises naturally by linearizing the homogeneous or “self-similar” solution [13, 1]

$$(4) \quad U = (r^2 - t^2)^{-1/(p-1)} \text{ for } r > t,$$

of the nonlinear wave equation

$$(5) \quad u_{tt} - \Delta u = gu^p,$$

$p \neq 1$ , in  $N$  space dimensions, where  $p \neq (N+1)/(N-1)$ , and

$$g = \frac{2}{p-1} \left( N - 1 - \frac{2}{p-1} \right).$$

This solution is real for  $r > t$ . Linearization leads to (1) with

$$A = -gp = \frac{2p}{p-1} \left( \frac{2}{p-1} - N + 1 \right).$$

The constant  $A$  is positive if and only if  $0 < p < (N+1)/(N-1)$ . For  $p < 1$ , the self-similar solution  $U$  remains bounded on the light-cone; it is a polynomial if  $p = 1 - 1/k$ , with  $k = 2, 3, \dots$

Homogeneous solutions such as  $U$  have weak singularities for all time; they differ from solutions with blowup, for which Cauchy data in the remote past are not themselves singular—such solutions may be constructed systematically by Fuchsian Reduction [17, 16, 18]. The present paper shows that certain problems with characteristic singularities are also amenable to a reduction to the HGE, which is the prototype of a Fuchsian equation.

**1.2. Organization of the paper.** Background information on the hypergeometric equation is collected in Section 2, to make the paper self-contained. For further details, see e. g. [2, 14, 19, 20]. The comparison Lemmas 2 and 3 are perhaps new. Lemma 4 is a very simple form of estimates in [19, p. 283]. For details on the recent results on estimates for the HGF motivated by results found in Ramanujan's notebooks, see [19].

Section 3 constructs exact solutions for the Cauchy problem with initial data of special form (Theorem 6). In Section 4, these solutions are superposed to obtain more general solutions with controlled behavior (Theorems 8 and 9).

## 2. ESTIMATES OF HYPERGEOMETRIC FUNCTIONS

**2.1. Definitions.** The hypergeometric function (HGF)  $F(a, b ; c ; z)$  is defined for  $c \notin -\mathbb{N}$  by analytic continuation of the sum of the hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where  $(a)_n = a(a+1) \cdots (a+n-1)$  for  $n \geq 1$ , and  $(a)_0 = 1$ . It is symmetric in  $a$  and  $b$ . It solves the hypergeometric equation (HGE) with parameters  $(a, b, c)$ :

$$z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0.$$

After multiplication by  $z$ , and letting  $D = z d/dz$ , the HGE may be written

$$D(D+c-1)w - z(D+a)(D+b)w = 0.$$

A second solution of the HGE is

$$(6) \quad z^{1-c} F(a-c+1, b-c+1 ; 2-c ; z).$$

The only singularities of the HGE are 0, 1 and infinity, and they are regular.

**2.2. A simple reduction.** Any linear Fuchsian equation with three regular singularities is reducible to the HGE by a change of unknown of the form  $\tilde{w}(z) = z^\alpha (1-z)^\beta w(z)$ . We record a special case of this result, for later use.

**Lemma 1.** *Consider a solution  $f(z)$  of*

$$D(D+c-1)f - z(D+a)(D+b)f + \frac{Bzf}{1-z} = 0,$$

where  $B$  is a constant. If

$$(7) \quad \beta(\beta+a+b-c) + B = 0,$$

and  $f(z) = (1-z)^\beta g(z)$ , then

$$D(D+c-1)f - z(D+a+\beta)(D+b+\beta)f = 0.$$

*Proof.* The result may be proved either by appealing to general properties on the hypergeometric equation or in a self-contained manner. We give both arguments.

First, let  $\beta$  be one of the roots of equation (7). The equation solved by  $f$  has three regular singular points, at 0, 1, and infinity. The indices at 0, 1 and infinity are respectively  $(0, 1 - c)$ ,  $(\beta, \beta')$ , and  $(a, b)$ , where  $\beta$  and  $\beta'$  are the two roots of equation (7). It follows that  $g$  solves an equation with the same singular points, which are regular, and for which the indices at 0, 1 and infinity are respectively  $(0, 1 - c)$ ,  $(0, \beta' - \beta)$ , and  $(a + \beta, b + \beta)$ . Since zero is an index both at 0 and at 1, this equation must be equivalent to the hypergeometric equation for the function  $F(a + \beta, b + \beta; c; z)$ . This proves the result.

The self-contained proof runs as follows. Let  $y = z/(1 - z)$ , so that  $Dy = y + y^2$ . We find

$$\begin{aligned} Df &= (1 - z)^\beta (D - \beta y)g; \\ D^2 f &= (1 - z)^\beta (D^2 - \beta y(2D + 1) + \beta(\beta - 1)y^2)g. \end{aligned}$$

Substituting into the equation for  $f$ , and dividing through by  $(1 - z)^\beta$ , we find, after some algebra,

$$D(D + c - 1)g - z[(D + a)(D + b) + \beta(2D + 1)]g + Qg = 0,$$

where

$$\begin{aligned} Q &= \beta(\beta - 1)y^2(1 - z) - \beta(c - 1)y + \beta yz(a + b) + By \\ &= yz\beta(\beta + a + b - 1) + [B - \beta(c - 1)]y \\ &= \beta(\beta + a + b - 1)y(z - 1) + [B + \beta(\beta + a + b - c)]y. \end{aligned}$$

Since  $y(z - 1) = -z$ , we find

$$D(D + c - 1)g - z(D + a + \beta)(D + b + \beta)g + [B + \beta(\beta + a + b - c)]yg = 0,$$

from which the desired result follows.  $\square$

**2.3. Majorants for hypergeometric functions.** If the parameters  $a$ ,  $b$  and  $c$  are real and positive, the coefficients of  $F(a, b; c; z)$  increase as  $a$  and  $b$  are increased, and  $c$  is decreased. This makes it possible to compare different hypergeometric functions. Three results of this type are given next. Term-by-term domination of power series is expressed by the symbol  $\ll$ ; thus,

$$\sum_{n \geq 0} a_n z^n \ll \sum_{n \geq 0} b_n z^n$$

if  $|a_n| \leq b_n$  for every  $n \geq 0$ .

**Lemma 2.** *If  $c \geq \frac{1}{2}$  and  $0 \leq a \leq b - \frac{1}{2}$ , then*

$$(8) \quad F(a, b; c; z^2) \ll \frac{1}{2}[(1 + z)^{1-2b} + (1 - z)^{1-2b}].$$

If  $c \geq \frac{1}{2}$  and  $0 \leq a \leq b + \frac{1}{2}$ ,

$$(9) \quad F(a, b; c; z^2) \ll \frac{1}{2} [(1+z)^{-2b} + (1-z)^{-2b}].$$

*Proof.* Consider first the case  $a \leq b - \frac{1}{2}$ . The assumptions on the parameters ensure that  $F(a, b; c; z^2) \ll F(b - \frac{1}{2}, b; \frac{1}{2}; z^2)$ . Now

$$\begin{aligned} F(b - \frac{1}{2}, b; \frac{1}{2}; z^2) &= \sum_{n \geq 0} \frac{(b - \frac{1}{2})b \cdots (b + n - 1)}{\frac{1}{2} \cdot 1 \cdots (n - \frac{1}{2})n} z^{2n} \\ &= \sum_{n \geq 0} \frac{(2b - 1)_{2n}}{(1)_{2n}} z^{2n}. \end{aligned}$$

This expression is equal to the right-hand side of (8), thanks to the identity

$$(10) \quad (1 - z)^{-d} = \sum_{n \geq 0} \frac{(d)_n}{(1)_n} z^n.$$

If  $a \leq b + \frac{1}{2}$ , we observe that  $F(a, b; c; z^2) \ll F(b, b + \frac{1}{2}; \frac{1}{2}; z^2)$ , to which the previous considerations apply, with  $b$  replaced by  $b + \frac{1}{2}$ .  $\square$

**Lemma 3.** If  $c \geq \frac{3}{2}$  and  $0 \leq a \leq b - \frac{1}{2}$ , then

$$(11) \quad F(a, b; c; z^2) \ll \frac{1}{4z(1-b)} [(1+z)^{2-2b} - (1-z)^{2-2b}].$$

If  $b > 0$ ,  $c \geq \frac{3}{2}$ , and  $0 \leq a \leq b + \frac{1}{2}$ ,

$$(12) \quad F(a, b; c; z^2) \ll \frac{1}{2z(1-2b)} [(1+z)^{1-2b} - (1-z)^{1-2b}].$$

*Proof.* Assume first  $0 \leq a \leq b - \frac{1}{2}$ . We now have  $F(a, b; c; z^2) \ll F(b - \frac{1}{2}, b; \frac{3}{2}; z^2)$ . Since  $(\frac{3}{2})_n = 2(\frac{1}{2})_n(\frac{3}{2} + n - 1) = (\frac{1}{2})_n(2n + 1)$ ,

$$zF(b - \frac{1}{2}, b; \frac{3}{2}; z^2) = \sum_{n \geq 0} \frac{(b - \frac{1}{2})_n (b)_n}{(\frac{1}{2})_n (1)_n} \frac{z^{2n+1}}{2n + 1}.$$

This series represents the anti-derivative of  $F(b - \frac{1}{2}, b; \frac{1}{2}; z^2)$  that vanishes for  $z = 0$ . The first statement follows; the second is reduced to the first as before.  $\square$

We conclude with an estimate in terms of powers of  $(1 - z)$ .

**Lemma 4.** If  $a \geq b > c > 0$ , and  $d = a + b - c$ ,

$$(13) \quad F(a, b; c; z) \ll \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} (1 - z)^{-d}.$$

*Proof.* Since  $a + b = c + d$ , we have, for  $n \geq 1$ ,

$$\frac{(a+n)(b+n)}{(c+n)(d+n)} = 1 + \frac{ab - cd}{(c+n)(d+n)}.$$

Also,  $ab - cd = (c - a)(c - b) > 0$ . Therefore,

$$\frac{(a)_n(b)_n}{(c)_n(d)_n} \text{ increases with } n.$$

Let  $\Gamma_0 = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}$ . Since, for any  $x > 0$ ,

$$\Gamma(x) = \lim_{n \rightarrow \infty} n^{x-1} \frac{(1)_n}{(x)_n},$$

$\frac{(a)_n(b)_n}{(c)_n(d)_n}$  tends to  $\Gamma_0$  as  $n \rightarrow \infty$ . Therefore,

$$\frac{(a)_n(b)_n}{(c)_n(1)_n} \leq \Gamma_0 \frac{(d)_n}{(1)_n}.$$

The result follows, thanks to (10).  $\square$

### 3. HYPERGEOMETRIC SOLUTIONS

We now seek solutions of equation (1) of the form (2). We first obtain solutions of  $Lu = 0$ , and then select those with Cauchy data  $(r^l \phi_\lambda(x/r), 0)$ , or  $(0, r^l \phi_\lambda(x/r))$ , where  $\phi_\lambda$  is an eigenfunction of the Laplace-Beltrami operator on the sphere:  $-\Delta_{S^{N-1}} \phi_\lambda = \lambda \phi_\lambda$ , with  $\lambda$  constant and nonnegative.

**3.1. A family of solutions of  $Lu = 0$ .** Let  $u = \phi_\lambda(x/r)v(r, t)$ . We obtain

$$u_{tt} - \Delta u = \left( v_{tt} - v_{rr} - \frac{N-1}{r} v_r + \frac{\lambda}{r^2} v \right) \phi_\lambda.$$

Next, let  $z = t^2/r^2$  and  $v(r, t) = r^l(1-z)^\beta g(z)$ . We show that one may choose  $\beta$  so that  $g$  solves a hypergeometric equation:

**Theorem 5.** *Assume that*

$$(14) \quad \beta_l := \frac{1}{2} \left| l + \frac{N-1}{2} \right| \left\{ 1 + \sqrt{1 - \frac{A}{(l + \frac{N-1}{2})^2}} \right\}.$$

*Let*

$$(15) \quad a_\pm = -\frac{1}{2}(l-1 + \frac{N}{2}) \pm \frac{1}{2} \sqrt{(\frac{N}{2}-1)^2 + \lambda}.$$

*Then, equation  $Lu = 0$ , with*

$$u = r^l(1-z)^\beta g(z) \phi_\lambda(x/r),$$

*is equivalent to the HGE for  $g$ , with parameters*

$$(a_l, b_l, c) := (a_+ + \beta_l, a_- + \beta_l, 1/2).$$



*Remark 2.* The parameters  $a_l$  and  $b_l$  depend on  $l$ ,  $N$ ,  $\lambda$ ,  $A$ , but we single out the dependence in  $l$  since we will let it tend to infinity. They are real if and only if

$$(l + \frac{N-1}{2})^2 \geq A.$$

In that case,  $a_l \geq b_l$ .

*Proof.* Let  $v(r, t) = r^l f(z)$ . Recall that  $D = z \frac{d}{dz}$ . We shall first express  $Lu$  in terms of  $f$ , and then apply Lemma 1.

The derivatives of  $v$  with respect to  $t$  and  $x$  are as follows.

$$\begin{aligned} rv_t &= 2tr^{l-1}f', \\ r^2v_{tt} &= \frac{4}{z}r^lD(D - \frac{1}{2})f, \\ rv_r &= r^l(l - 2D)f, \\ r^2v_{rr} &= r^l(2D - l)(2D - l + 1)f, \\ r^2\frac{n-1}{r}v_r &= -(N-1)r^l(2D - l)f. \end{aligned}$$

It follows that

$$\begin{aligned} r^2(u_{tt} - \Delta u) &= \phi_\lambda \frac{4r^l}{z} \left\{ D(D - \frac{1}{2}) - z(D - \frac{l}{2})(D - \frac{l}{2} + \frac{1}{2}) \right. \\ &\quad \left. + z\frac{N-1}{2}(D - \frac{l}{2}) + \frac{1}{4}\lambda z \right\} f \\ &= \frac{4r^l}{z} \left\{ D(D - \frac{1}{2}) - z[(D - \frac{l}{2})(D - \frac{l+N}{2} + 1) - \frac{\lambda}{4}] \right\} f. \end{aligned}$$

Thanks to the choice of  $a_\pm$  in equation (15), we have

$$\begin{aligned} (D - \frac{l}{2})(D - \frac{l+N}{2} + 1) - \frac{\lambda}{4} \\ = D^2 - (l-1 + \frac{N}{2})D + \frac{l}{2}(\frac{l+N}{2} - 1) - \frac{\lambda}{4} = (D + a_+)(D + a_-). \end{aligned}$$

On the other hand,

$$r^2 \frac{Av}{r^2 - t^2} = \frac{r^l}{z} \frac{Azf}{1 - z}.$$

Adding this to the above expression for  $r^2(u_{tt} - \Delta u)$ , we obtain

$$\frac{z}{4}r^{2-l}Lu = \phi_\lambda \left[ D(D - \frac{1}{2}) - z(D + a_+)(D + a_-) + \frac{Az}{4(1-z)} \right] f.$$

Using Lemma 1 with  $(a, b, c) = (a_+, a_-, \frac{1}{2})$ , we conclude that, if  $\beta_l$  is given by (14), and

$$f(z) = (1-z)^{\beta_l}g(z),$$

then

$$\frac{z}{4}r^{2-l}Lu = (1-z)^{\beta_l}\phi_\lambda \left[ D(D - \frac{1}{2}) - z(D + a_+ + \beta_l)(D + a_- + \beta_l) \right] g.$$

This completes the proof.  $\square$

### 3.2. Solutions with special Cauchy data.

**Theorem 6.** *Let*

$$F_{l,\lambda} := r^l(1 - \frac{t^2}{r^2})^{\beta_l} F\left(a_l, b_l; \frac{1}{2}; \frac{t^2}{r^2}\right)$$

and

$$G_{l,\lambda} := tr^l(1 - \frac{t^2}{r^2})^{\beta_{l+1}} F\left(a_{l+1} + \frac{1}{2}, b_{l+1} + \frac{1}{2}; \frac{3}{2}; \frac{t^2}{r^2}\right).$$

The functions  $F_{l,\lambda}\phi_\lambda(\frac{x}{r})$  and  $G_{l,\lambda}\phi_\lambda(\frac{x}{r})$  both solve  $Lu = 0$ . For  $t = 0$ ,  $F_{l,\lambda} = \partial_t G_{l,\lambda} = r^l$  and  $\partial_t F_{l,\lambda} = G_{l,\lambda} = 0$ .

*Proof.* We now know that, if  $F$  is a HGF with parameters  $(a_l, b_l, \frac{1}{2})$ , then

$$r^l\phi_\lambda(\frac{x}{r})(1 - \frac{t^2}{r^2})^{\beta_l} F$$

solves  $Lu = 0$ . Taking  $F = F(a_l, b_l; \frac{1}{2}; \frac{t^2}{r^2})$ , the solution  $F_{l,\lambda}$  results. Taking for  $F$  the second solution (6) of the HGE and replacing  $l$  by  $l + 1$  leads, after rearranging, to the solution  $G_{l,\lambda}$ . The behavior for  $t = 0$  is checked by inspection.  $\square$

*Remark 3.* If  $\lambda = 0$  and  $N$  is odd,  $a_l - b_l$  is a half-integer, and the HGF may be expressed in terms of elementary functions. When  $a_l - b_l = \frac{1}{2}$ , we use (10). For  $a_l - b_l > \frac{1}{2}$ , we use the identity

$$\begin{aligned} & (c-a)_n z^{c-a-1} (1-z)^{a+b-c-n} F(a-n, b; c; z) \\ &= \frac{d^n}{dz^n} \left\{ z^{c-a+n-1} (1-z)^{a+b-c} F(a, b; c; z) \right\}. \end{aligned}$$

**3.3. Remarks on inhomogeneous problems.** Because the hypergeometric equation with polynomial right-hand side may also be solved in closed form [3], the method of §3.1 readily yields solutions with right-hand sides that are products of  $(1-z)^{\beta_l}$  by polynomials in  $r$  and  $z$ . We record next a different method that produces simple solutions of inhomogeneous problems of special form. As before, all computations are performed for  $r > t$ .

**Theorem 7.** *Assume that  $A, A'$  and  $\gamma$  are given constants. If  $h(x, t)$  is homogeneous of degree  $\gamma$  and satisfies*

$$h_{tt} - \Delta h + \frac{A'h}{r^2 - t^2} = 0,$$

then  $H := c(r^2 - t^2)^\alpha h$  solves

$$LH = (r^2 - t^2)^{\alpha-1} h$$

provided that

$$c[A - A' - 2\alpha(2\gamma + 2\alpha + N - 1)] = 1.$$

*Proof.* Let  $\square$  denote the wave operator. The assumptions imply that  $Lh = (A - A')h/(r^2 - t^2)$ . Since  $\square(r^2 - t^2)^\alpha = -2\alpha(2\alpha + N - 1)(r^2 - t^2)^{\alpha-1}$ ,

$$\begin{aligned} LH &= c(r^2 - t^2)^\alpha Lh - 4c\alpha(r^2 - t^2)^{\alpha-1}(th_t + rh_r) \\ &\quad - 2\alpha(2\alpha + N - 1)(r^2 - t^2)^{\alpha-1}ch. \end{aligned}$$

Since  $h$  is homogeneous of degree  $\gamma$ ,

$$th_t + rh_r = \gamma h.$$

Therefore,  $LH = (r^2 - t^2)^{\alpha-1}h$  if and only if

$$c[A - A' - 2\alpha(2\gamma + 2\alpha + N - 1)] = 1.$$

This completes the proof  $\square$

#### 4. SERIES SOLUTION OF THE CAUCHY PROBLEM

Let

$$v_0(r) = \sum_{l \geq 0} a_l r^l; \quad v_1(r) = \sum_{l \geq 0} b_l r^l$$

be two series with radius of convergence  $R > 0$ . Equation (1) has a unique solution such that  $u_\lambda = \phi_\lambda v_0$ ,  $\partial_t u_\lambda = \phi_\lambda v_1$  for  $t = 0$ ; it is equal to

$$(16) \quad \phi_\lambda \sum_{l \geq 0} a_l F_{l,\lambda}(r, t) + b_l G_{l,\lambda}(r, t).$$

Summation over  $\lambda$  yields the general solution. We focus here on estimations of  $u_\lambda$ . The subscript  $\lambda$  will be suppressed in the following.

**Theorem 8.** *If  $A < \frac{1}{4}(N - 1)^2$ , the series (16) is real and converges for  $t < r < R - t$ .*

*Remark 4.* Since we are dealing with a wave equation with speed one, the solution of the Cauchy problem with data for  $0 < r < R$  is uniquely determined on the domain of influence  $t < r < R - t$ . The domain of convergence in the theorem is therefore optimal.

*Proof.* The assumption on  $A$  ensures that  $\beta$ , hence  $a_l$  and  $b_l$ , are real. Also,  $a_l$  and  $b_l$  are positive if  $l$  is large. Since  $\lambda$  is an eigenvalue of the Laplacian on the sphere, is either 0 or  $\geq 1$ ; therefore,  $a_l - b_l \geq \frac{1}{2}$  unless  $N = 2$  and  $\lambda = 0$ , in which case  $a_l = b_l$ . In all cases, we may use Lemmas 2 and 3 if  $l$  is large.

If  $a_l - b_l \geq \frac{1}{2}$ , we appeal to (8) to estimate  $F_{l,\lambda}$  by

$$r^l \left( \frac{r^2 - t^2}{r^2} \right)^{\beta_l} \times \frac{1}{2} \left[ \left( 1 + \frac{t}{r} \right)^{1-2b_l} + \left( 1 - \frac{t}{r} \right)^{1-2b_l} \right].$$

Now,  $\beta_l = l + \eta_1$ ,  $b_l = \frac{1}{2}l + \eta_2$ , where  $\eta_1$  and  $\eta_2$  remain bounded as  $l \rightarrow \infty$ . Rearranging, and using the inequality  $(r - t)^l(r + t)^{-l} \leq 1$ , we obtain

$$\begin{aligned} 2|F_{l,\lambda}(r, t)| &\leq r^l \left( \frac{r^2 - t^2}{r^2} \right)^{l+\eta_1} \left[ \left( \frac{r+t}{r} \right)^{-l+1-2\eta_2} + \left( \frac{r-t}{r} \right)^{-l+1-2\eta_2} \right] \\ &\leq (r+t)^l r^{2\eta_2-2\eta_1-1} (r^2 - t^2)^{\eta_1} [(r+t)^{-2\eta_2} + (r-t)^{-2\eta_2}]. \end{aligned}$$

Therefore,  $\limsup_{l \rightarrow \infty} |F_{l,\lambda}|^{1/l} \leq r + t$ . It follows that  $\sum_{l \geq 0} a_l F_{l,\lambda}$  converges for  $r + t < R$ . The other series which makes up the solution is handled similarly. If  $a_l \leq b_l + \frac{1}{2}$ , we appeal to (12) instead of (8).  $\square$

We conclude with a different estimate, which gives better results near the light-cone, but is weaker than the previous one for  $r > R/2$ .

**Theorem 9.** *As  $l \rightarrow +\infty$ ,  $F_{l,\lambda} = \mathcal{O}(2^l(1 - t^2/r^2)^{\beta_l+\eta_3})$ , where  $\eta_3 = \mathcal{O}(1/l)$ .*

*Proof.* We estimate  $F_{l,\lambda}$  using Lemma 4. Since  $a_l + b_l - \frac{1}{2} = \mathcal{O}(1/l)$ , we conclude that

$$|F_{l,\lambda}(r, t)| \leq C'(1 - t^2/r^2)^{\beta_l+\eta_3},$$

where  $C' = \Gamma(\frac{1}{2})\Gamma(a_l + b_l - \frac{1}{2})/\Gamma(a_l)\Gamma(b_l)$ . Stirling's formula gives  $C' = \mathcal{O}(2^l)$ . The result follows.  $\square$

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