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# DUALITY AND INTERTWINING FOR DISCRETE MARKOV KERNELS: A RELATION AND EXAMPLES

THIERRY HUILLET<sup>1</sup>, SERVET MARTINEZ<sup>2</sup>

ABSTRACT. We work out some relations between duality and intertwining in the context of discrete Markov chains, fixing up the background of previous relations first established for birth and death chains and their Siegmund duals. In view of the results, the monotone properties resulting from the Siegmund dual of birth and death chains are revisited in some detail, with emphasis on the non neutral Moran model. We also introduce an ultrametric type dual extending the Siegmund kernel. Finally we discuss the sharp dual, following closely the Diaconis-Fill study.

**Running title:** Duality and Intertwining.

**Keywords:** *Duality; Intertwining; Siegmund dual; generalized ultrametric matrices; birth and death chains; Moran models; sharp strong stationary time; sharp dual.*

**MSC 2000 Mathematics Subject Classification:** 60 J 10, 60 J 70, 60 J 80.

## 1. INTRODUCTION

Our work is devoted to the study of duality and intertwining relations between Markov chain kernels. Even if these concepts can be established only as relations between matrices, as we define them in the next section, our study is on its probabilistic consequences. For this purpose we need that the matrices are non negative and substochastic to be able to define a dual Markov chain. The fact that the intertwining kernel is stochastic allows a rich probabilistic interpretation that has been given in [2], [4], [7] and [8].

A main problem is the existence of a duality relationship between substochastic kernels. Indeed, once this fact is established, then several relations can be deduced when the starting chain is irreducible and positive recurrent. This is the statement of one of our main result, which is Theorem 2. The hypotheses of this theorem rely on a duality relationship between kernels.

In the following sections, we find additional examples where these duality relations between substochastic kernels can be established: for the well-known Siegmund kernel the hypotheses of Theorem 2 are verified for monotone chains, see Corollary 6; for a generalized ultrametric potential kernel some conditions for the existence of the dual are given in Proposition 11; for birth and death (BD) chains the properties

derived from monotonicity are summarized in Corollary 7; and in Proposition 10 we show that the non-neutral Moran model is monotone when its bias mechanism is nondecreasing.

For birth and death chains, we revisit the properties relating non negative spectrum and monotonicity (see Proposition 9) and for the Moran model we identify some cases with non negative spectra and also when stronger properties are satisfied.

The section 5 follows closely the ideas on sharp stationary times and duals developed in [1], [4] and [7]. In Proposition 13 we show a sharpness result alluded to in Remark 2.39 of [4] and in Theorem 2.1 in [7]. One of its corollaries is Proposition 14 where the condition for sharpness is written in terms of the dual function. This applies to the intertwining of a monotone chain under the Siegmund dual, in this case both chains can start from the state 0. In the BD case we also study some quantitative aspects of the absorption time.

We point out that even if duality and intertwining can be set for Markov chains acting on general state spaces and/or with continuous time, we restrict ourselves to the discrete time and space in order to be able to present quickly our main results and avoid to introduce additional overburden notations.

## 2. DUALITY AND INTERTWINING

**2.1. Notation.** Let  $I$  be a countable set. By  $\mathcal{F}(I)$  we denote the set of real functions, and by  $\mathcal{F}_b(I)$  and  $\mathcal{F}_+(I)$  we denote respectively its bounded and positive elements. Since  $I$  is countable the set  $\mathcal{F}(I)$  is identified with the set of vectors  $\mathbb{R}^I$ . Let  $\partial$  be a point that does not belong to  $I$ , and denote  $I^\partial := I \cup \{\partial\}$ . Every  $f \in \mathcal{F}(I)$  is extended canonically to a function  $f^\partial$  that satisfies  $f^\partial(\partial) = 0$ .

If  $A$  is any set we denote by  $\mathbf{1}_A$  or  $\mathbf{1}(A)$  its characteristic function. We denote by  $\mathbf{1}$  the unit function defined on  $I$  (or in other sets  $\hat{I}$  and  $\tilde{I}$  that we introduce further).

A non negative matrix  $P = (P(x, y) : x, y \in I)$  is called a kernel on  $I$ . (Sometimes we will emphasize the non negativity by saying a non negative kernel.) It obviously acts on the set  $\mathcal{F}_+(I)$ . A substochastic kernel is such that  $P\mathbf{1} \leq \mathbf{1}$ , it is stochastic when the equality  $P\mathbf{1} = \mathbf{1}$  holds, and strictly substochastic if it is substochastic and there exists some  $x \in I$  such that  $P\mathbf{1}(x) < 1$ . When  $P$  is substochastic, it obviously acts on  $\mathcal{F}_b(I)$ .

The kernel  $P$  is irreducible when for any pair  $x, y \in I$  there exists  $n > 0$  such that  $P^{(n)}(x, y) > 0$ .

A point  $x_0 \in I$  is an absorbing point of the kernel  $P$  when  $P(x_0, y) = \delta_{y, x_0}$  for all  $y \in I$ .

When  $P$  is a substochastic kernel there exists a uniquely defined (in distribution) Markov chain  $X = (X_n : n < \mathcal{T}^X)$  taking values in the countable set  $I$ , with

lifetime  $\mathcal{T}^X$  and with transition kernel  $P$ . We have the equality  $P = P_X$  where  $P_X$  is the kernel acting on the set of functions  $\mathcal{F}_b(I)$  (or  $\mathcal{F}_+(I)$ ) by

$$P_X f(x) = \mathbb{E}(f(X_1) \cdot \mathbf{1}(\mathcal{T}^X > 1)), \quad x \in I.$$

$P$  generates the semigroup  $(P^n : n \geq 1)$ , each matrix  $P^n$  acting on  $\mathcal{F}_b(I)$  or  $\mathcal{F}_+(I)$ , and it verifies

$$P^n f(x) = \mathbb{E}(f(X_n) \cdot \mathbf{1}(\mathcal{T}^X > n)), \quad x \in I, n \geq 1.$$

The lifetime  $\mathcal{T}^X$  is such that

- If  $P$  is stochastic then  $\mathcal{T}^X = +\infty$   $\mathbb{P}_x$ -a.e. for all  $x \in I$ ;
- If  $P$  is strictly substochastic then there exists some  $x \in I$  such that  $\mathbb{P}_x(\mathcal{T}^X < +\infty) > 0$ . When  $P$  is irreducible strictly substochastic then for all  $x \in I$  it holds  $\mathbb{P}_x(\mathcal{T}^X < +\infty) = 1$ .

The kernels will be denoted by  $P, \widehat{P}, \widetilde{P}$ , they will be defined on the countable sets  $I, \widehat{I}, \widetilde{I}$  respectively. When these kernels are substochastic the associated Markov chains will be respectively denoted by  $X, \widehat{X}, \widetilde{X}$ , and the lifetimes of these chains will be respectively  $\mathcal{T}, \widehat{\mathcal{T}}, \widetilde{\mathcal{T}}$ .

**2.2. Strictly substochastic kernel.** If  $P$  is strictly substochastic we can add a new state  $\partial$  to  $I$ , and  $X$  is extended to the Markov chain  $X^\partial = (X_t^\partial : t \geq 0)$  by

$$X_t^\partial = X_t, \quad t < \mathcal{T}; \quad X_t^\partial = \partial, \quad t \geq \mathcal{T},$$

so  $\partial$  is an absorbing state of the new chain. The transition kernel  $P_{X^\partial}$  of  $X^\partial$  is stochastic and it is given by

$$P_{X^\partial} g(x) = \mathbb{E}_x(g(X_1^\partial) \cdot \mathbf{1}(T_\partial^{X^\partial} > 1)) + g(\partial) \mathbb{P}_x(T_\partial^{X^\partial} \leq 1),$$

for all  $g \in \mathcal{F}_b(I^\partial)$  or  $g \in \mathcal{F}_+(I^\partial)$ . Then,

$$(1) \quad [g(\partial) = 0] \Rightarrow [(P_{X^\partial}^n g)|_I = P^n(g|_I), \quad \forall n \geq 1].$$

Therefore, since the canonical extension of  $f \in \mathcal{F}(I)$  to  $f^\partial \in \mathcal{F}(I^\partial)$  satisfies  $f^\partial(\partial) = 0$ , the right hand side of (1) is verified for  $g = f^\partial$ .

We recall that  $h \in \mathcal{F}_b(I)$  (or  $h \in \mathcal{F}_+(I)$ ) is a  $P$ -harmonic function if  $Ph = h$ , or equivalently if it verifies

$$\mathbb{E}_x(h(X_n) \cdot \mathbf{1}(\mathcal{T} > n)) = h(x), \quad \forall x \in I, \forall n \geq 1.$$

We have that its extension  $h^\partial \in \mathcal{F}_b(I^\partial)$  (or  $h^\partial \in \mathcal{F}_+(I^\partial)$ ) such that  $h^\partial(\partial) = 0$  is a  $P_{X^\partial}$ -harmonic function.

Let us denote by

$$T_J^X = \inf\{n \geq 0 : X_n \in J\}$$

the hitting time of  $J \subseteq I$  of the chain  $X$ , where as usual we put  $+\infty = \inf \emptyset$ . When  $J = \{a\}$  is a singleton we put  $T_a^X$  instead of  $T_{\{a\}}^X$ . Observe that with this notation we have

$$\mathcal{T}^X = T_\emptyset^{X^\partial}.$$

To simplify the notation, for the Markov chains  $X, \widehat{X}, \widetilde{X}$ , the hitting times are denoted respectively by  $T_J = T_J^X, \widehat{T}_J = T_J^{\widehat{X}}, \widetilde{T}_J = T_J^{\widetilde{X}}$  (when  $J$  is a subset of  $I, \widehat{I}, \widetilde{I}$ , respectively).

Let us recall the structure of a non irreducible substochastic kernel  $P$ . In this case, up to permutation, we can partition

$$I = \bigcup_{l=1}^{\ell} I_l$$

in such a way that (see [9], Section 8.3):

$$P_{I_l \times I_l} \text{ is irreducible } \forall l \in \{1, \dots, \ell\},$$

and

$$\forall x \in I_l, y \in I_{l'} : P(x, y) > 0 \Rightarrow l \leq l'.$$

If  $P$  is stochastic then the last of these submatrices  $P_{I_\ell \times I_\ell}$  is stochastic, that is  $P_{I_\ell \times I_\ell} \mathbf{1}_{I_\ell} = \mathbf{1}_{I_\ell}$  and there could be other stochastic submatrices. If  $P$  is strictly substochastic then none or some of these submatrices  $P_{I_l \times I_l}, l = 1, \dots, \ell$ , could be stochastic. We put

$$St(P) = \{I_l : P_{I_l \times I_l} \text{ is stochastic}, l \in \{1, \dots, \ell\}\}.$$

Then, when  $P$  is stochastic  $St(P) \neq \emptyset$ , and if  $P$  is strictly substochastic then  $St(P)$  could be empty or not. When  $St(P) \neq \emptyset$  then it could contain a unique class or not, and also by a simple permutation we can always assume that it contains  $I_\ell$  (this permutation is not needed when  $P$  is stochastic).

**2.3. Definitions.** We recall the duality and the intertwining relations. As usual  $M'$  denotes the transposed of matrix  $M$ , that is  $M'(x, y) = M(y, x)$  for all  $x, y \in I$ .

**Definition 1.** Let  $P$  and  $\widehat{P}$  be two kernels defined on the countable sets  $I$  and  $\widehat{I}$ , and let  $H = (H(x, y) : x \in I, y \in \widehat{I})$  be a non negative matrix. Then  $\widehat{P}$  is said to be a  $H$ -dual of  $P$  if it verifies

$$(2) \quad H\widehat{P}' = PH.$$

We call  $H$  a dual function between  $(P, \widehat{P})$ .  $\square$

Note that for a kernel  $P$  the  $H$ -dual  $\widehat{P}$  exists when (2) holds and  $\widehat{P} \geq 0$ .

When  $|I| = |\widehat{I}|$  is finite and  $H$  is nonsingular we get that

$$\widehat{P}' = H^{-1}PH,$$

and so  $\widehat{P}'$  and  $P$  are similar matrices and have the same spectrum.

Duality is a symmetric notion between kernels, because if  $\widehat{P}$  is a  $H$ -dual of  $P$ , then  $P$  is a  $H'$ -dual of  $\widehat{P}$ .

We will assume that the non negative dual matrix  $H$  is nontrivial, in the sense that no row and no column vanishes completely. On the other hand note that if  $H$  is a dual function between  $(P, \widehat{P})$  then for all  $c > 0$ ,  $cH$  is also a dual function

between these matrices. Then, when it is necessary, we can always multiply all the coefficients of  $H$  by a strictly positive constant.

This notion of duality (2) coincides with the one between Markov processes that can be found in references [15], [18] and [4] among others. Indeed, let  $P$  and  $\widehat{P}$  be substochastic and let  $X$  and  $\widehat{X}$  be Markov chains with kernels  $P$  and  $\widehat{P}$  respectively. Then, if  $\widehat{P}$  is a  $H$ -dual of  $P$ , we have that  $\widehat{X}$  is a  $H$ -dual of  $X$ , which means that

$$(3) \quad \forall x \in I, y \in \widehat{I}, \forall n \geq 0 : \quad \mathbb{E}_x(H(X_n, y)) = \mathbb{E}_y(H(x, \widehat{X}_n)),$$

where we have extended  $H$  to  $(I \cup \{\partial\}) \times (\widehat{I} \cup \{\partial\})$  by putting  $H(x, \partial) = H(\partial, y) = H(\partial, \partial) = 0$ , for all  $x \in I, y \in \widehat{I}$ .

Let us now introduce intertwining.

**Definition 2.** Let  $P$  and  $\widetilde{P}$  be two kernels defined on the countable sets  $I$  and  $\widetilde{I}$  and let  $\Lambda = (\Lambda(y, x) : y \in \widetilde{I}, x \in I)$  be a stochastic matrix. We say that  $\widetilde{P}$  is a  $\Lambda$ -intertwining of  $P$ , if it verifies

$$\widetilde{P}\Lambda = \Lambda P.$$

$\Lambda$  is called a link between  $(P, \widetilde{P})$ .  $\square$

When  $|I| = |\widetilde{I}|$  is finite and  $\Lambda$  is nonsingular we get

$$\widetilde{P} = \Lambda P \Lambda^{-1}.$$

and so  $P$  and  $\widetilde{P}$  are similar and have the same spectrum.

Let  $P$  and  $\widetilde{P}$  be substochastic and denote by  $X$  and  $\widetilde{X}$  the associated Markov chains, if  $\widetilde{P}$  is a  $\Lambda$ -intertwining of  $P$  we say that  $\widetilde{X}$  is a  $\Lambda$ -intertwining of  $X$ . Obviously the intertwining is not a symmetric relation because  $\Lambda'$  is not necessarily stochastic. But when  $\Lambda$  is doubly stochastic we have that  $\widetilde{P}$  is a  $\Lambda$ -intertwining of  $P$  implies that  $P$  is a  $\Lambda'$ -intertwining of  $\widetilde{P}$ .

The stochastic intertwining between Markov chains has been deeply studied in [2], [4], [7] and [8].

### 3. RELATIONS BETWEEN DUALITY AND INTERTWINING

Let us introduce additional notation:

- By  $\mathbf{e}_a$  we denote a column vector with 0 entries except for its  $a$ -th entry which is 1;
- When  $P$  is an irreducible positive recurrent stochastic kernel, we denote by  $\pi = (\pi(x) : x \in I)$  its stationary distribution and we write it as a column vector. So  $\pi'P = \pi'$ , where  $\pi'$  is the row vector transposed of  $\pi$ .

Now we give a result on intertwining that will be often used.

**Proposition 1.** *Let  $P$  be an irreducible positive recurrent stochastic kernel and  $\pi$  be its stationary distribution. Assume  $\tilde{P}$  is a kernel that is a  $\Lambda$ -intertwining of  $P$ ,  $\tilde{P}\Lambda = \Lambda P$ . If  $\tilde{a}$  is an absorbing state in  $\tilde{P}$  then,*

$$(4) \quad \pi' = \mathbf{e}'_{\tilde{a}}\Lambda.$$

*Proof.* Since the chain  $P$  is positive recurrent with stationary distribution  $\pi$  and  $\Lambda$  is stochastic we get  $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=0}^{k-1} (\Lambda P^n)(x, y) = \pi(y)$ , in particular

$$(5) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n=0}^{k-1} (\Lambda P^n)(\tilde{a}, y) = \pi(y).$$

On the other hand from the assumption we get  $\tilde{P}^n(\tilde{a}, y) = \delta_{y, \tilde{a}}$  and then

$$(6) \quad (\tilde{P}^n \Lambda)(\tilde{a}, y) = \Lambda(\tilde{a}, y) \quad \forall n \geq 0, y \in I.$$

From (8) we have  $\tilde{P}^n \Lambda = \Lambda P^n$  for all  $n \geq 1$ , and so from (5) and (6) we deduce  $\Lambda(\tilde{a}, y) = \pi(y)$  which is equivalent to  $(\mathbf{e}'_{\tilde{a}} \Lambda)(y) = \pi(y)$ . Then (4) is shown.  $\square$

For a vector  $\rho \in \mathbb{R}^I$  we will denote by  $D_\rho$  the diagonal matrix with terms  $(D_\rho)(x, x) = \rho(x)$ ,  $x \in I$ .

Let  $P$  be an irreducible positive recurrent stochastic kernel with stationary distribution  $\pi$ . By irreducibility we have  $\pi > 0$ . Denote by  $\overleftarrow{P}$  the transition kernel of the reversed chain of  $X$ , so  $\overleftarrow{P}(x, y) = \pi(x)^{-1} P(y, x) \pi(y)$  or equivalently

$$(7) \quad \overleftarrow{P}' = D_\pi P D_\pi^{-1}.$$

We have that  $\overleftarrow{P}$  is in duality with  $P$  via  $H = D_\pi^{-1}$ . Note that  $\overleftarrow{P}$  is also irreducible and positive recurrent with stationary distribution  $\pi$  and that  $P' = D_\pi \overleftarrow{P} D_\pi^{-1}$ , so we can exchange the roles of  $P$  and  $\overleftarrow{P}$ . In the reversible case  $\overleftarrow{P} = P$ , the relation (7) expresses a self duality.

Let us give one of our main results that can be viewed as the generalization of Theorem 5.5 in [4] devoted to birth and death chains.

**Theorem 2.** *Let  $P$  be an irreducible positive recurrent stochastic kernel and let  $\pi$  be its stationary distribution. Assume  $\hat{P}$  is a (non negative) kernel and that it is a  $H$ -dual of  $P$ ,  $H\hat{P}' = PH$ , where  $H$  is nontrivial. Then*

$$(i) \quad \hat{P}H'D_\pi = H'D_\pi\overleftarrow{P}.$$

(ii) *The vector  $\varphi := H'\pi$  is strictly positive and it verifies*

$$\hat{P}\varphi = \varphi.$$

(iii)  *$\tilde{P} = D_\varphi^{-1}\hat{P}D_\varphi$  is a stochastic kernel and it is a  $\Lambda$ -intertwining of  $\overleftarrow{P}$ , so  $\Lambda$  is a stochastic link  $\Lambda$ , more precisely*

$$(8) \quad \tilde{P}\Lambda = \Lambda\overleftarrow{P} \quad \text{with } \Lambda := D_\varphi^{-1}H'D_\pi \text{ and they verify } \tilde{P}\mathbf{1} = \mathbf{1} = \Lambda\mathbf{1}.$$

Moreover we have the duality relation

$$K\tilde{P}' = PK \text{ with } K := HD_\varphi^{-1}.$$

(iv) Let  $I$  and  $\hat{I}$  be finite and  $\hat{P}$  be substochastic. Then:

(iv1) When  $\hat{P}$  is stochastic and irreducible then  $\varphi = c\mathbf{1}$  for some  $c > 0$ , and  $\tilde{P} = \hat{P}$ .

(iv2) If  $\hat{P}$  is strictly substochastic then it is not irreducible.

(iv3) If  $\hat{P}$  is non irreducible then  $St(\hat{P}) \neq \emptyset$  and there exist some constants  $c_l > 0$  for  $\hat{I}_l \in St(\hat{P})$  such that

$$(9) \quad \varphi(x) = \sum_{\hat{I}_l \in St(\hat{P})} c_l \mathbb{P}_x(\lim_{n \rightarrow \infty} \hat{X}_n \in \hat{I}_l) = \sum_{\hat{I}_l \in St(\hat{P})} c_l \mathbb{P}_x(\hat{T}_{\hat{I}_l} < \hat{T}).$$

(iv4) If  $\hat{P}$  has a unique stochastic class  $\hat{I}_\ell$ , then,

$$(10) \quad \frac{\varphi(x)}{\varphi(y)} = \mathbb{P}_x(\hat{T}_{\hat{I}_\ell} < \hat{T}) \text{ for any } y \in \hat{I}_\ell,$$

and the intertwining Markov chain  $\tilde{X}$  is given by the Doob transform

$$(11) \quad \mathbb{P}_x(\tilde{X}_1 = y_1, \dots, \tilde{X}_k = y_k) = \mathbb{P}_x(\hat{X}_1 = y_1, \dots, \hat{X}_k = y_k | \hat{T}_{\hat{I}_\ell} < \hat{T}).$$

(v) If  $\hat{a}$  is an absorbing state in  $\hat{P}$  then  $\hat{a}$  is an absorbing state in  $\tilde{P}$  and (4)  $\pi' = \mathbf{e}'_{\hat{a}} \Lambda$  holds. Moreover the sets of absorbing points in  $\hat{P}$  and  $\tilde{P}$  coincide.

(vi) If  $|I| = |\hat{I}|$  is finite and  $H$  is nonsingular then:  $\hat{P} = H'D_\pi \overleftarrow{P} D_\pi^{-1} H'^{-1}$  and  $\tilde{P} = \Lambda \overleftarrow{P} \Lambda^{-1}$ . Hence  $\hat{P}$ ,  $\overleftarrow{P}$ ,  $\tilde{P}$ , are similar matrices and  $P$ ,  $\hat{P}$ ,  $\tilde{P}$  have the same spectrum.

*Proof.* From  $H\hat{P}' = PH$ , we find

$$\hat{P}H' = H'D_\pi \overleftarrow{P} D_\pi^{-1}.$$

By multiplying to the right by  $D_\pi$  we get (i). The part (vi) follows directly in the finite nonsingular case.

Since  $D_\pi \mathbf{1} = \pi$  we get that  $\hat{P}H'\pi = H'D_\pi \overleftarrow{P} \mathbf{1}$ . Since  $\overleftarrow{P}$  is stochastic we get  $\hat{P}H'\pi = H'D_\pi \mathbf{1} = H'\pi$ . Let  $\varphi = H'\pi$ . Since  $\pi > 0$  and at each row of  $H$  there exists a strictly positive element, then  $\varphi > 0$ . Then (ii) holds. Now define,

$$\tilde{P} = D_\varphi^{-1} \hat{P} D_\varphi.$$

By using (i) we get,

$$\tilde{P} D_\varphi^{-1} H' D_\pi = D_\varphi^{-1} \hat{P} H' D_\pi = D_\varphi^{-1} H' D_\pi \overleftarrow{P}$$

Then the relation  $\tilde{P} \Lambda = \Lambda \overleftarrow{P}$  holds in (iii), moreover

$$\begin{aligned} \tilde{P} \mathbf{1} &= D_\varphi^{-1} \hat{P} D_\varphi \mathbf{1} = D_\varphi^{-1} \hat{P} \varphi = D_\varphi^{-1} \varphi = \mathbf{1}, \\ \Lambda \mathbf{1} &= D_\varphi^{-1} H' D_\pi \mathbf{1} = D_\varphi^{-1} H' \pi = D_\varphi^{-1} \varphi = \mathbf{1}. \end{aligned}$$



Then  $\tilde{P}$  and  $\Lambda$  are Markov kernels. Finally from the equality

$$HD_\varphi^{-1}\tilde{P}'D_\varphi = H\hat{P}' = PH,$$

the relation  $K\tilde{P}' = PK$  is straightforward. Hence (iii) is verified.

Now assume  $I$  is finite. If  $\hat{P}$  is an irreducible strictly substochastic kernel then necessary its spectral radius is strictly smaller that 1, which contradicts the equality  $\hat{P}\varphi = \varphi$ , because  $\varphi > 0$ . In the case  $\hat{P}$  is stochastic and irreducible, the equation  $\hat{P}\varphi = \varphi$ ,  $\varphi > 0$ , implies  $\varphi = c\mathbf{1}$  for some constant  $c > 0$ . So (iv1) and (iv2) follow.

Now assume that the matrix  $\hat{P}$  is substochastic and non irreducible. Let  $\hat{I} = \bigcup_{l=1}^\ell \hat{I}_l$  be the partition in irreducible components  $\hat{P}_{\hat{I}_l \times \hat{I}_l}$  such that  $x \in \hat{I}_l$ ,  $y \in \hat{I}_{l'}$  and  $\hat{P}(x, y) > 0$  implies  $l' \geq l$ . The last submatrix  $\hat{P}_{\hat{I}_\ell \times \hat{I}_\ell}$  verifies,

$$\hat{P}_{\hat{I}_\ell \times \hat{I}_\ell} \varphi|_{\hat{I}_\ell} = \varphi|_{\hat{I}_\ell}.$$

Then  $\hat{P}_{\hat{I}_\ell \times \hat{I}_\ell}$  is an irreducible substochastic matrix whose Perron-Frobenius eigenvalue is 1, so we deduce that  $\hat{P}_{\hat{I}_\ell \times \hat{I}_\ell}$  is stochastic and  $\varphi|_{\hat{I}_\ell} = c_\ell \mathbf{1}_{\hat{I}_\ell}$  for some constant  $c_\ell > 0$ , so  $St(\hat{P}) \neq \emptyset$ . Then, if  $(\hat{I}_l \in St(\hat{P}))$  are the irreducible stochastic classes the same argument implies that  $\varphi|_{\hat{I}_l} = c_l \mathbf{1}_{\hat{I}_l}$  for some quantity  $c_l > 0$  and this happens for all  $\hat{I}_l \in St(\hat{P})$ .

Let  $\hat{X} = (\hat{X}_t : t < \hat{T})$  be the Markov chain with kernel  $\hat{P}$ . It is known that all the trajectories that are not killed are attracted by  $\bigcup_{\hat{I}_l \in St(\hat{P})} \hat{I}_l$ , that is

$$\mathbb{P}_x(\lim_{n \rightarrow \infty} \hat{X}_n \in \bigcup_{\hat{I}_l \in St(\hat{P})} \hat{I}_l \mid \hat{T} = \infty) = 1.$$

On the other hand the equality  $\hat{P}\varphi = \varphi$  expresses that  $\varphi$  is an harmonic function for the chain  $\hat{X}$ . Hence, for all  $n \geq 0$  it is verified,

$$\begin{aligned} \varphi(x) &= \mathbb{E}_x(\varphi(\hat{X}_n), \hat{T} > n) \\ &= \sum_{\hat{I}_l \in St(\hat{P})} \mathbb{E}_x(\varphi(\hat{X}_n), \hat{T} > n, \hat{T}_{\hat{I}_l} < \hat{T}) \\ &\quad + \mathbb{E}_x(\varphi(\hat{X}_n), \hat{T} > n, \hat{T} < \min\{\hat{T}_{\hat{I}_l} : \hat{I}_l \in St(\hat{P})\}). \end{aligned}$$

Then, by taking  $n \rightarrow \infty$  in above expression and since  $\lim_{n \rightarrow \infty} \mathbb{P}_x(\min\{\hat{T}_{\hat{I}_l} : \hat{I}_l \in St(\hat{P})\} > \hat{T} > n) = 0$ , we get the relation (9),

$$\varphi(x) = \sum_{\hat{I}_l \in St(\hat{P})} c_l \mathbb{P}_x(\hat{T}_{\hat{I}_l} < \hat{T}).$$

Let us prove part (iv4). Since there is a unique stochastic class the equality (10) follows straightforwardly. Then the transition probabilities of  $\tilde{P}$  are given by the Doob  $h$ -transform

$$\tilde{P}(x, y) = \mathbb{P}_x(\hat{T}_{\hat{I}_\ell} < \hat{T})^{-1} \hat{P}(x, y) \mathbb{P}_y(\hat{T}_{\hat{I}_\ell} < \hat{T}) = \mathbb{P}_x(\hat{X}_1 = y \mid \hat{T}_{\hat{I}_\ell} < \hat{T}), \quad \forall x, y \in \hat{I}.$$

The Markov property gives the formula for every cylinder.

Finally, let us show part (v). Since the chain  $\overleftarrow{P}$  is positive recurrent with stationary distribution  $\pi$  it suffices to show that  $\hat{a}$  is an absorbing state for  $\overleftarrow{P}$ . This follows straightforwardly from the equality  $\overleftarrow{P} = D_\varphi^{-1} \hat{P} D_\varphi$ , indeed it implies  $\overleftarrow{P}(\hat{a}, y) = \hat{P}(\hat{a}, y) \frac{\varphi(y)}{\varphi(\hat{a})} = \delta_{y, \hat{a}}$ . Also this proves the equality of the set of absorbing points for both kernels  $\hat{P}$  and  $\overleftarrow{P}$ .  $\square$

**Remark 1.** We can exchange the roles of  $P$  and  $\overleftarrow{P}$  in the irreducible and positive recurrent case. Thus, in the hypothesis of the Theorem we can take  $\overleftarrow{P}$  instead of  $P$ , so  $\hat{P}$  is  $H$ -dual of  $\overleftarrow{P}$ ,  $H\hat{P}' = \overleftarrow{P}H$ , and in all the statements of the Theorem we must change  $P$  by  $\overleftarrow{P}$ .  $\square$

**Remark 2.** A probabilistic explanation of how appears  $\varphi := H'\pi > 0$  can be done when  $\hat{P}$  is substochastic and  $H$  is bounded. In this case the dual relation  $H\hat{P}' = PH$  is expressed by the expression (3),

$$\forall x \in I, y \in \hat{I}, \forall n \geq 0 : \mathbb{E}_x(H(X_n, y)) = \mathbb{E}_y(H(x, \hat{X}_n)).$$

Since by hypothesis  $X$  is an irreducible and positive recurrent Markov chain then  $\varphi$  appears as the following limit on the left hand side,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{n \leq k} \mathbb{E}_x(H(X_n, y)) = \sum_{u \in I} \pi(u) H(u, y) = \varphi(y).$$

$\square$

**Remark 3.** We have

$$(12) \quad \Lambda(x, y) = \frac{1}{\varphi(x)} H(y, x) \pi(y),$$

in particular  $\Lambda(x, y) = 0$  if and only if  $H(y, x) = 0$ .  $\square$

**Remark 4.** The formulas in Theorem 2 state that  $\Lambda$ ,  $\hat{P}$  and  $\overleftarrow{P}$  are invariant when  $H$  is multiplied by a strictly positive constant. Then, we can fix  $c > 0$  and take  $cH$  in order to have  $\varphi(x) = 1$  for all  $x \in \hat{I}$ , or equivalently  $c_l = 1$ , for some fixed stochastic class  $\hat{I}_l \in St(\hat{P})$ .

**Remark 5.** When the starting equality between stochastic kernels is the intertwining relation  $\overleftarrow{P}\Lambda = \Lambda\overleftarrow{P}$ , then we have the duality relation  $H\hat{P}' = PH$  with  $H = D_\pi^{-1}\Lambda'$  and  $\hat{P} = \overleftarrow{P}$ . In this case  $\varphi = \mathbf{1}$ .

We note the equality  $\hat{I} = \tilde{I}$  of the sets where the kernels  $\hat{P}$  and  $\overleftarrow{P}$  are defined in Theorem 2. On the other hand we recall that in the finite case the positive recurrence property on  $P$  follows from irreducibility.

**Proposition 3.** Assume  $H$  is nonsingular and has a constant column that is strictly positive, that is

$$\exists \hat{a} \in \hat{I} : H\mathbf{e}_{\hat{a}} = c\mathbf{1} \text{ for some } c > 0.$$

Then:

(i)  $\hat{a}$  is an absorbing state for  $\hat{P}$  (so  $\{\hat{a}\}$  is a stochastic class).

(ii) Under the hypotheses of Theorem 2,  $\pi' = \mathbf{e}'_{\hat{a}}\Lambda$  holds and if  $\hat{P}$  is strictly substochastic and  $\{\hat{a}\}$  is the unique stochastic class then  $\mathbb{P}_y(\hat{T}_{\hat{a}} < \hat{T}) = \varphi(y)/\varphi(\hat{a})$  and the relation (11) is satisfied.

*Proof.* (i) From  $H\mathbf{e}_{\hat{a}} = c\mathbf{1}$  we get,

$$\begin{aligned} \mathbf{e}'_{\hat{a}}\hat{P} &= \mathbf{e}'_{\hat{a}}H'D_{\pi}\overleftarrow{P}D_{\pi}^{-1}H^{-1'} = (H\mathbf{e}_{\hat{a}})'D_{\pi}\overleftarrow{P}D_{\pi}^{-1}H^{-1'} \\ &= c\pi'\overleftarrow{P}D_{\pi}^{-1}H^{-1'} = (H^{-1}c\mathbf{1})' = \mathbf{e}'_{\hat{a}}. \end{aligned}$$

Then

$$\hat{P}(\hat{a}, y) = (\mathbf{e}'_{\hat{a}}\hat{P})(y) = \mathbf{e}'_{\hat{a}}(y) = \delta_{\hat{a}, y}.$$

So  $\hat{a}$  is an absorbing state for  $\hat{P}$ .

(ii) From Theorem 2 (v),  $\hat{a}$  is an absorbing state of  $\tilde{P}$  and  $\pi' = \mathbf{e}'_{\hat{a}}\Lambda$ . The rest of part (ii) follows straightforwardly.  $\square$

When  $P$  does not satisfy positive recurrence let us only consider the following special case.

**Proposition 4.** *Let  $x_0 \in I$  be an absorbing point of the kernel  $P$  and let  $\hat{P}$  be a substochastic kernel that is a  $H$ -dual of  $P$ :  $H\hat{P}' = PH$ . Then  $h(y) := H(x_0, y)$ ,  $y \in \hat{I}$ , is a non negative  $\hat{P}$ -harmonic function. When  $H$  is bounded and  $\hat{P}$  is a stochastic recurrent kernel, the  $x_0$ -row  $H(x_0, \cdot)$  is constant.*

*Proof.* It suffices to show that the function  $h$  is  $\hat{P}$ -harmonic. Since  $P(x_0, z) = \delta_{z, x_0}$   $\forall z \in I$ , we get  $(PH)(x_0, y) = H(x_0, y)$ . Therefore, if  $\hat{P}$  verifies the duality equality (2) we get,  $(H\hat{P}')(x_0, y) = H(x_0, y) = h(y)$ . Then

$$(\hat{P}h)(y) = \sum_{z \in \hat{I}} \hat{P}(y, z)H(x_0, z) = \sum_{z \in \hat{I}} H(x_0, z)\hat{P}'(z, y) = (H\hat{P}')(x_0, y) = h(y),$$

and the result is shown.  $\square$

#### 4. CLASSES OF DUAL MATRICES

We consider the finite set case. We assume  $I = \hat{I} = \tilde{I} = \{0, \dots, N\}$ , so the kernels are non negative  $I \times I$  matrices and when they are substochastic the associated Markov chains take values in  $I$ .

We will study some classes of non negative matrices  $H$  for which there exist substochastic kernels  $P$  and  $\hat{P}$  in duality relation (2). So, in these cases we would be able to apply the results established in Theorem 2, and Proposition 3.

**4.1. The potential case.** Let us see what happens with a quite general class of matrices, the finite potential kernels. Let  $R$  be a strictly substochastic kernel with no stochastic classes (this is the case if  $R$  is also irreducible). Then it has a well defined finite potential,

$$H = (\text{Id} - R)^{-1} = \sum_{n \geq 0} R^n \geq 0.$$

So  $H^{-1} = \text{Id} - R$ . (In particular no column nor row of  $H$  vanishes).

Let  $P$  be a substochastic kernel. Define

$$\widehat{P}' = H^{-1}PH = (\text{Id} - R)P(\text{Id} - R)^{-1}.$$

**Proposition 5.** *Assume that also the transposed matrix  $R'$  is substochastic. Then,  $\widehat{P}\mathbf{1} \geq 0$  and there exists a stochastic kernel  $P$  for which it is verified  $\widehat{P} \geq 0$ . Indeed, the constant stochastic kernel  $P = \frac{1}{N+1}\mathbf{1}\mathbf{1}'$  fulfills the property.*

*Proof.* Since  $R'$  is substochastic we have  $(\text{Id} - R')\mathbf{1} \geq 0$ . Then

$$\mathbf{1}'\widehat{P}' = \mathbf{1}'(\text{Id} - R)P(\text{Id} - R)^{-1} \geq 0.$$

Now, since  $(\text{Id} - R)^{-1} \geq 0$  and  $\widehat{P}' = (P - RP)(\text{Id} - R)^{-1}$ , we get that once the relation  $RP \leq P$  is verified then  $\widehat{P}' \geq 0$ . Since  $R$  is substochastic the matrix  $P = \frac{1}{N+1}\mathbf{1}\mathbf{1}'$  makes the job.  $\square$

**4.2. Siegmund kernel.** A well-known case of a kernel  $H$  arising as a potential of a strict substochastic kernel  $R$  as above, is the Siegmund kernel. Let  $R(x, y) = \mathbf{1}(x + 1 = y)$  so it is a strictly substochastic (because the  $N$ -th row vanishes) and it has no stochastic classes. Its transposed matrix  $R'(x, y) = \mathbf{1}(x = y + 1)$  is also substochastic.

By direct computation we get that  $H_S = (\text{Id} - R)^{-1}$  verifies

$$H_S(x, y) = \mathbf{1}(x \leq y)$$

so it is the Siegmund kernel. We have  $H_S^{-1} = \text{Id} - R$ , then  $H_S^{-1}(x, y) = \mathbf{1}(x = y) - \mathbf{1}(x + 1 = y)$ .

This case has been studied in detail, for instance see [4] Section 5. Let us summarize some well-known observations. We have  $(H_S\widehat{P}')(x, y) = \sum_{z \geq x} \widehat{P}(y, z)$  and

$(PH_S)(x, y) = \sum_{z \leq y} P(x, z)$ . Then, the equation  $H_S\widehat{P}' = PH_S$  gives

$$(13) \quad \widehat{P}(y, x) = \sum_{z \geq x} \widehat{P}(y, z) - \sum_{z > x} \widehat{P}(y, z) = \sum_{z \leq y} (P(x, z) - P(x + 1, z))$$

In particular  $\widehat{P} \geq 0$  requires the condition,

$$(14) \quad \forall y \in I : \sum_{z \leq y} P(x, z) \text{ decreases with } x \in I.$$

In this case  $P$  is called monotone.

Also, from  $\widehat{P}(N, x) = \sum_{z \leq N} (P(x, z) - P(x+1, z))$  we deduce that

$$P \text{ stochastic} \Rightarrow \widehat{P}(N, x) = \delta_{x, N},$$

so  $N$  is an absorbing state of  $\widehat{P}$ . Also from (13) we get that

$$(15) \quad \widehat{P}(N-1, N) = \sum_{z \leq N-1} P(N, z) = 1 - P(N, N).$$

We also observe that

$$\sum_{x \leq N} \widehat{P}(y, x) = \sum_{z \leq y} P(0, z),$$

in particular

$$(16) \quad \widehat{P}\mathbf{1} \leq \mathbf{1} \text{ so } \widehat{P} \text{ is substochastic,}$$

and also  $\sum_{x \leq N} \widehat{P}(0, x) = P(0, 0)$ . Then,

$$(17) \quad P(0, 0) = 1 \Rightarrow \widehat{P} \text{ is stochastic;}$$

$$(18) \quad P(0, 0) < 1 \Rightarrow \sum_{x \leq N} \widehat{P}(0, x) < 1 \text{ and } \widehat{P} \text{ loses mass through } 0.$$

This last case occurs for any irreducible stochastic kernel  $P$  with  $N \geq 1$ . Indeed, in this case  $P(0, 0) = 1$  cannot happen because it contradicts irreducibility.

Also we get,

$$P(0, 0) + P(0, 1) = 1 \Rightarrow \widehat{P} \text{ does not lose mass through } \{1, \dots, N\}.$$

When the finite matrix  $P$  is irreducible we can apply Theorem 2 and in this case

$$(19) \quad \varphi(x) = (H'_S \pi)(x) = \sum_{y \in I} \mathbf{1}(y \leq x) \pi(y) = \sum_{y \leq x} \pi(y) =: \pi^c(x),$$

is the cumulative distribution of  $\pi$ . We have that  $\pi^c$  is not constant because  $\pi > 0$ .

Let us show that

$$(20) \quad N \text{ is the unique absorbing state of } \widehat{P}.$$

Indeed, from (13) the unique absorption implies that  $x < N$  verifies  $\widehat{P}(x, y) = \delta_{y, x}$  if and only if  $\sum_{z \leq x} P(x, z) = 1$  and  $\sum_{z \leq x} P(x+1, z) = 0$ . Therefore, also from (13), we get

$$\sum_{z \leq x} P(y, z) = 1 \quad \forall y \leq x \text{ and } \sum_{z \leq x} P(y, z) = 0 \quad \forall y > x,$$

which contradicts the irreducibility of  $P$ .

We obtain the following result. In it we assume  $N \geq 1$  to avoid the trivial case when  $N = 0$  and  $P(0, 0) = 1$ .

**Corollary 6.** *Let  $H$  be the Siegmund kernel,  $P$  be a monotone finite irreducible stochastic kernel with stationary distribution  $\pi$ . Let  $H_S \widehat{P}' = PH_S$  with  $\widehat{P} \geq 0$ . Then:*

(i)  $\widehat{P}$  is a strictly substochastic kernel that loses mass through 0, and parts (iv2) and (iv3) of Theorem 2 hold.

(ii)  $\varphi = \pi^c$  and the stochastic intertwining kernel  $\Lambda$  verifies

$$(21) \quad \Lambda(x, y) = \mathbf{1}(x \geq y) \frac{\pi(y)}{\pi^c(x)}.$$

and the intertwining matrix  $\widetilde{P}$  of  $\overleftarrow{P}$  is given by

$$\widetilde{P}(x, y) = \widehat{P}(x, y) \frac{\pi^c(y)}{\pi^c(x)} \quad x, y \in I.$$

(iii)  $N$  is the unique absorbing state for  $\widehat{P}$  and Theorem 2 parts (iv4) and (v) are verified with  $\widehat{I}_\ell = \{N\}$  and  $\widehat{a} = N$ . In particular  $\pi' = \mathbf{e}'_N \Lambda$  holds.

(iv) The following relation holds:

$$(22) \quad \Lambda \mathbf{e}_N = \pi(N) \mathbf{e}_N$$

*Proof.* The first three parts are direct consequence of Theorem 2, relations (16), (18), (19), (20). Finally, (22) is a direct computation from (21) and  $\varphi(N) = 1$ .  $\square$

**4.3. Duality for finite state space birth and death chains.** Recall  $I = \widehat{I} = \widetilde{I} = \{0, \dots, N\}$ . Let  $X = (X_n : n \geq 0)$  be a discrete birth and death (BD) chain with transition Markov kernel  $P = (P(x, y) : x, y \in I)$ . Then  $P(x, y) = 0$  if  $|x - y| > 1$  and

$$P(x, x+1) = p_x, \quad P(x, x-1) = q_x, \quad P(x, x) = r_x, \quad x \in I,$$

with

$$q_x + r_x + p_x = 1 \quad \forall x \in I \quad \text{and boundary conditions } q_0 = p_N = 0.$$

We always take

$$q_x, p_x > 0, \quad x \in \{1, \dots, N-1\}.$$

We will assume the irreducible case, which in this case is equivalent to the condition

$$(23) \quad p_0 > 0, \quad q_N > 0.$$

(A unique exception will be done in Subsection 4.4 where we will explicitly assume that (23) is not satisfied.) The stationary distribution  $\pi = (\pi(x) : x \in I)$  verifies  $\pi(y) = \pi(0) \prod_{z < y} \frac{p_z}{q_{z+1}} > 0$ ,  $y \in \{1, \dots, N\}$ , where  $\pi(0)$  fulfills  $\sum_{y \in I} \pi(y) = 1$ .

The matrix  $P$  is self-adjoint with the inner product given by  $\pi$ , that is it verifies  $\pi(x)P(x, y) = \pi(y)P(y, x)$  for all  $x, y$ . So,  $P = \overleftarrow{P}$  where  $\overleftarrow{P} = D_\pi^{-1} P' D_\pi$  is the transition matrix of the time reversed process and  $P$  has real eigenvalues.

The unique constraint in (13) to get that  $\widehat{P} \geq 0$  is satisfied, is for  $y = x$ , that is we need that the condition  $\widehat{P}(x, x) \geq 0$  is verified and it reads

$$(24) \quad \forall x \in \{0, \dots, N-1\} : p_x + q_{x+1} \leq 1.$$

This is the equivalent of (14) for BD chains. So, when (24) is satisfied we say that  $P$  is monotone. In this case the Siegmund dual  $\widehat{P}$  exists and it is a BD kernel with

$$\widehat{P}(x, x-1) = p_x, \quad \widehat{P}(x, x) = 1 - (p_x + q_{x+1}), \quad \widehat{P}(x, x+1) = q_{x+1}.$$

The drift of  $X$  at  $x$  is  $f(x) := p_x - q_x$ , and the drift of  $\widehat{X}$  at  $x$  is  $\widehat{f}(x) = \widehat{p}_x - \widehat{q}(x) = -f(x+1) + (p_{x+1} - p_x)$ , so  $-f(x+1) - r_{x+1} \leq \widehat{f}(x) \leq -f(x) + r_x$ .

Note that  $\widehat{P}(0, 0) = 1 - (p_0 + q_1)$ ,  $\widehat{P}(0, 1) = q_1$ , then the Markov chain  $\widehat{X}$  loses mass through the state 0 if and only if  $p_0 = 0$  or equivalently  $r_0 < 1$ . On the other hand  $\widehat{P}(N, N) = 1 - (p_N + q_{N+1}) = 1$  (because  $p_N = q_{N+1} = 0$ ), so  $N$  is an absorbing state. When (24) holds we say that  $P$  is a monotone kernel. From this analysis, Corollary 6, and the reversibility relation  $P = \overleftarrow{P}$  we can state the following result.

**Corollary 7.** *Let  $H$  be the Siegmund kernel and  $P$  be a finite irreducible stochastic BD chain with monotone kernel  $P$  and whose parameters are  $p_x, q_x$ . Let  $\pi$  be the stationary distribution of  $P$ . Then, the dual matrix  $\widehat{P}$  defined by  $\widehat{P}' = H_S^{-1} P H_S$  is a strictly substochastic kernel that loses mass through the state 0. Moreover:*

(i)  $\varphi = \pi^c$  and parts (iv2) and (iv3) of Theorem 2 hold.

(ii)  $N$  is an absorbing state of  $\widehat{P}$  and  $\{N\}$  is the unique stochastic class of  $\widehat{P}$ , all the other states in  $I$  are transient, and Theorem 2 (iv4) is verified with  $\widehat{I}_\ell = \{N\}$ .

(iii) Let  $\Lambda$  be the stochastic kernel given by (21). Then, the  $\Lambda$ -intertwining matrix  $\widetilde{P}$  of  $P$  is given by

$$\widetilde{P}(x, x-1) = p_x \frac{\pi^c(x-1)}{\pi^c(x)}, \quad \widetilde{P}(x, x) = 1 - (p_x + q_{x+1}), \quad \widetilde{P}(x, x+1) = q_{x+1} \frac{\pi^c(x+1)}{\pi^c(x)}.$$

**4.4. Absorbing points for the BD kernels.** Let us modify the BD kernel  $P$  by taking 0 as an absorbing state. That is, instead of the irreducibility conditions (23) we take  $p_0 = 0$  and no restriction on  $q_N$ , it could be 0 or  $> 0$ . Assume  $P$  is monotone, so (24) holds. Then the BD kernel  $\widehat{P}$  is stochastic, see (17). In this case  $N$  is the unique absorbing state for  $\widehat{P}$ .

Let us describe what happens by exploiting the special form of the Siegmund dual. By evaluating (3) at  $y = 0$  we get

$$(25) \quad \mathbb{P}_x(X_n \leq 0) = \mathbb{P}_0(x \leq \widehat{X}_n),$$

and by evaluating (3) at  $x = N$  we obtain

$$(26) \quad \mathbb{P}_N(X_n > y) = \mathbb{P}_y(\widehat{X}_n < N).$$

Now there are two cases:

(i) If  $q_N > 0$  then 0 is the unique absorbing state for  $P$ . By (15) we get that  $\widehat{P}(N-1, N) = q_N > 0$ , so  $N$  is an absorbing state that attracts all the trajectories of the chain,  $\mathbb{P}_x(\lim_{n \rightarrow \infty} X_n = N) = 1$  for  $x \in I$ .

(ii) If  $q_N = 0$ , then 0 and  $N$  are absorbing states for  $P$ . By using (15) we get that  $\widehat{P}(N-1, N) = q_N = 0$ , so  $N$ , besides being an absorbing state for  $\widehat{P}$  is an isolated state for  $\widehat{P}$  (that is  $\widehat{P}(y, N) = 0$  for all  $y < N$ ). Therefore it does not attract any of the trajectories starting from a state different from  $N$ . Hence, the equation (26) is simply the equality  $1 = 1$  when  $y < N$ .

Let us summarize which is the picture for (ii):  $P$  has 0 and  $N$  as absorbing states that attract all the trajectories of its associated Markov chain  $X$ ,  $\widehat{P}$  is stochastic,  $N$  is a  $\widehat{P}$ -absorbing isolated state, and  $\widehat{P}|_{I \setminus \{N\} \times I \setminus \{N\}}$  is stochastic and irreducible. Let  $\widehat{\pi}_* = (\widehat{\pi}_*(z) : z \in I \setminus \{N\})$  be the stationary distribution of the submatrix  $\widehat{P}|_{I \setminus \{N\} \times I \setminus \{N\}}$ .

Let  $\phi(x) = \mathbb{P}_x(\lim_{n \rightarrow \infty} X_n = 0)$  be the absorption probability at 0 of the chain  $X$  starting from  $x$ . We have the following result.

**Proposition 8.** *If  $p_0 = 0$  and  $q_N = 0$  then 0 and  $N$  are absorbing states for  $P$  and  $\widehat{P}|_{I \setminus \{N\} \times I \setminus \{N\}}$  is stochastic and has 0 as an absorbing point. Let  $\phi(x) = \mathbb{P}_x(\lim_{n \rightarrow \infty} X_n = 0)$ , and  $\widehat{\pi}_* = (\widehat{\pi}_*(z) : z \in I \setminus \{N\})$  be the stationary distribution of  $\widehat{P}|_{I \setminus \{N\} \times I \setminus \{N\}}$ . Then*

$$\phi(x) = 1 - \frac{\eta(x)}{\eta(N)} = 1 - \widehat{\pi}_*^c(x+1)$$

where  $\widehat{\pi}_*^c$  is the cumulative distribution of  $\widehat{\pi}_*$  and  $\eta(x) := \sum_{y=0}^{x-1} \prod_{z=1}^y \frac{q_z}{p_z}$  is the scale function of  $P$ .

*Proof.* The first equality follows from the fact that  $\eta$  is a martingale and  $\eta(0) = 0$ . For the second relation we take  $x < N$  and let  $n \rightarrow \infty$  in the formula (25), which gives

$$\phi(x) = \sum_{z \geq x} \widehat{\pi}_*(z) = 1 - \widehat{\pi}_*^c(x+1).$$

□

**4.5. The spectral characterization.** Let us give a sufficient spectral property for the monotonicity of the kernel  $P$  for an irreducible BD chain taking values on  $I = \{0, \dots, N\}$ . Consider the polynomials  $(\mathbf{q}_y(t) : y \in I)$  with  $t \in [-1, 1]$ , determined by:  $\mathbf{q}_0(t) = 1$  for all  $t$  and the recurrence:

$$\begin{aligned} t\mathbf{q}_0(t) &= p_0\mathbf{q}_1(t) + r_0\mathbf{q}_0(t), \\ t\mathbf{q}_y(t) &= p_y\mathbf{q}_{y+1}(t) + r_y\mathbf{q}_y(t) + q_y\mathbf{q}_{y-1}(t), \quad y \in \{1, \dots, N-1\}. \end{aligned}$$

It holds  $\mathbf{q}_y(1) = 1$  for all  $y \geq 0$  and the polynomial  $\mathbf{q}_y(t)$  is of degree  $y$  in  $t$ .



Let  $Z := \{t_k : R_{N+1}(t_k) = 0\}$  be the zeros of the polynomial  $R_{N+1}(t) = tq_N(t) - r_Nq_N(t) - q_Nq_{N-1}(t)$ , which is of degree  $N + 1$ . The set  $Z$  constitutes the spectrum of  $P$  (see [12], p. 78). All the zeros are simple and we order them by  $1 = t_0 > t_1 > \dots > t_N \geq -1$ . The quantity  $1 - t_1$  is the spectral gap. The spectral probability measure on  $[-1, 1]$  is  $\mu(dt) := \sum_{k=0}^N \mu_k \delta_{t_k}$ , with respect to which  $(q_y(t) : y \geq 1)$  are orthogonal. It is known that  $\mu_0 = \pi_0$ .

Let  $N = 2N_0$  be even. Assume that the BD chain is given by  $r_x = 0$  for all  $x \in I$ , and that it is reflected at the boundaries  $\{0, N\}$ , so  $p_0 = q_N = 1$ . In this case the spectral measure is symmetric on  $[-1, 1]$ , in particular  $t_{N_0} = 0$  and  $t_{2N_0} = -1$ . When  $N = 2N_0 + 1$  is odd, the spectral measure is again symmetric, but  $\{0\}$  is no longer an eigenvalue and  $t_{N_0} > 0$ .

A spectral sufficient condition for the monotone property (24) is given below in part (i). This result can be found in Lemma 2.4 of [7], and here we give a different proof. On the other hand note that when  $r_x \geq 1/2 \forall x \in I$  then obviously the monotone condition (24) is satisfied. In part (ii) we reinforce this implication.

**Proposition 9.** (i) *If a BD chain is spectrally non negative, then it is monotone.*

(ii) *If  $r_x \geq 1/2$  for all  $x \in I$ , then the BD chain is spectrally positive.*

*Proof.* Let us show (i). For a BD chain  $X$  whose transition matrix  $P$  is spectrally non negative, there exists a BD chain  $Y$  taking values on  $\{0, \dots, 2N\}$  reflected at the boundary, started at an even integer and such that  $X \stackrel{d}{\sim} (Y_{2n}/2 : n \geq 0)$ . This follows simply from adapting [20], Th. 2.1 to the finite case. As noted just before, the spectral measure of  $Y$  is symmetric on  $[-1, 1]$  and by passing to  $X$  the spectrum is being folded: If  $\sum_{k=0}^{2N} \mu_k \delta_{t_k}$  is the symmetric spectral measure of  $Y$  with  $t_N = 0$  then  $2 \sum_{k=0}^N \mu_k \delta_{t_k^2}$  is the spectral measure of  $X$ . Let  $\alpha_y$  and  $\beta_y$  be the up and down probabilities that  $Y_m \rightarrow Y_{m+1} = Y_m \pm 1$  given that  $Y_m$  is in state  $y$  different from the endpoints. We have  $\alpha_y + \beta_y = 1$ , and then:

$$q_x = \beta_{2x} \beta_{2x-1}, \quad r_x = \beta_{2x} \alpha_{2x-1} + \alpha_{2x} \beta_{2x+1}, \quad p_x = \alpha_{2x} \alpha_{2x+1}.$$

This, together with  $p_0 = \alpha_1$  and  $q_N = \beta_{2N-1}$  allows to determine recursively the transition matrix of  $Y$  from the one of  $X$ . From these facts we deduce that our hypothesis implies

$$p_x + q_{x+1} = \alpha_{2x} \alpha_{2x+1} + \beta_{2x+2} \beta_{2x+1} < \alpha_{2x} \alpha_{2x+1} + \beta_{2x+1} < 1,$$

then the chain  $X$  is monotone.

For the proof of (ii) first note that  $P = D_\pi^{-\frac{1}{2}} Q D_\pi^{-\frac{1}{2}}$ , where  $Q$  is a symmetric matrix given by  $Q(x, y) = 0$  when  $|x - y| > 1$  and

$$Q(x, x+1) = \sqrt{p_x q_{x+1}} = Q(x+1, x), \quad Q(x, x) = r_x, \quad x \in I.$$

Now consider the superdiagonal matrix  $S$  such that  $S(x, y) = 0$  if  $y \notin \{x, x+1\}$  and

$$S(x, x) = \sqrt{p_x}, \quad x \in I; \quad S(x, x+1) = \sqrt{q_{x+1}}, \quad x \in I, \quad x \neq N.$$

Then  $S'S$  is a tridiagonal symmetric matrix, with  $S'S(x, y) = 0$  if  $|x - y| > 1$  and

$$S'S(x, x) = p_x + q_x, \quad S'S(x, x+1) = \sqrt{p_x q_{x+1}} = S'S(x+1, x), \quad x \in I.$$

Consider the diagonal matrix  $D_r$  with  $r = (r_0, \dots, r_N)$ . We have  $Q = 2D_r - I + S'S$ , so  $Q$  is the sum of a diagonal matrix and a symmetric positive definite matrix. We conclude that, if the holding probabilities  $r_x \geq 1/2$ , for all  $x \in I$ , then for all  $z \in \mathbb{R}^{N+1} \setminus \{0\}$ ,

$$z'Qz = \sum_{x=0}^N (2r_x - 1) |z_x|^2 + |Sz|^2 > 0$$

and so  $Q$  and  $P$  are positive definite.  $\square$

We emphasize that in part (ii) we show that  $r_x \geq 1/2 \forall x \in I$  implies that the spectrum is positive, and that this is a stronger property than monotonicity in view of (i). On the other hand the condition  $r_x \geq 1/2$  for all  $x \in I$  is sufficient to get a positive spectrum but, as it is easy to see, it is not necessary.

*Example:* An example showing that non negative spectrum is not necessary for the monotone property (24) is the BD chain given  $p_x = p$ ,  $q_x = q$ ,  $x = 1, \dots, N-1$  and boundary conditions  $r_0 = q$ ,  $p_0 = p$ ,  $q_N = q$ ,  $r_N = p$ , where  $p \in (0, 1)$  and  $q = 1 - p$ . Then the monotone property holds but the spectrum fail to be non negative. Indeed, from [6], p. 438 it follows that  $t_k = 2\sqrt{pq} \cos(\frac{k\pi}{N+1})$ ,  $k = 1, \dots, N-1$ ,  $t_0 = 1$ ,  $t_N = -1$ .  $\square$

**4.6. The Moran model.** Let us introduce the 2-allele Moran model with bias mechanism  $p$ . Let

$$p : [0, 1] \rightarrow [0, 1] \text{ be continuous with } 0 \leq p(0) \text{ and } p(1) \leq 1.$$

Denote  $q(u) := 1 - p(u)$ . The Moran model is a BD Markov chain  $X$  characterized by the quadratic transition probabilities  $p_x, r_x, q_x$ ,  $x \in I = \{0, \dots, N\}$ ,

$$q_x = \frac{x}{N} q\left(\frac{x}{N}\right), \quad r_x = \frac{x}{N} p\left(\frac{x}{N}\right) + \left(1 - \frac{x}{N}\right) q\left(\frac{x}{N}\right), \quad p_x = \left(1 - \frac{x}{N}\right) p\left(\frac{x}{N}\right).$$

Assuming  $p_0 = p(0) > 0$  and  $q_N = 1 - p(1) > 0$ , with  $y \in \{1, \dots, N\}$ , the BD chain is irreducible with invariant distribution

$$\frac{\pi(y)}{\pi(0)} = \prod_{x=1}^y \frac{p_{x-1}}{q_x} = \binom{N}{y} \prod_{x=1}^y \frac{p\left(\frac{x-1}{N}\right)}{q\left(\frac{x}{N}\right)} = \frac{p(0) \binom{N}{y}^{y-1}}{q\left(\frac{y}{N}\right)} \prod_{x=1}^{y-1} \frac{p\left(\frac{x}{N}\right)}{q\left(\frac{x}{N}\right)},$$

where  $\pi(0)$  is the normalizing constant.

If  $X$  is a Moran model defined by some bias  $p$ , then  $\bar{X}_n := N - X_n$  is also a Moran model with bias  $\bar{p}(u) = 1 - p(1 - u)$ , and so with parameters

$$\bar{q}_x = p_{N-x} = \frac{x}{N} \bar{q}\left(\frac{x}{N}\right), \quad \bar{p}_x = q_{N-x} = \left(1 - \frac{x}{N}\right) \bar{p}\left(\frac{x}{N}\right)$$

where  $\bar{q}(u) := 1 - \bar{p}(u)$ . The spectra of  $\bar{P}$  and  $P$  are the same.

**Proposition 10.** *Assume that in the Moran model the bias  $p$  is nondecreasing. Then the BD chain is monotone, that is condition (24)  $p_x + q_{x+1} \leq 1$  is fulfilled (and so the Siegmund dual exists).*

*Proof.* First, since  $p_N = q_{N+1} = 0$  we have nothing to verify for  $x = N$ . Let us see what happens with  $x = 0$ . We need to guarantee  $1 - p_0 - q_1 \geq 0$ , but this is true because  $p(1/N) \geq p(0) \geq Np(0) - (N-1)$ .

Let us consider the case  $x \in \{1, \dots, N-1\}$ . We have the following relations, where in the first inequality we use that  $p$  is nondecreasing,

$$\begin{aligned}
 p_x + q_{x+1} &= p\left(\frac{x}{N}\right) - \frac{x}{N}p\left(\frac{x}{N}\right) + \left(\frac{x+1}{N}\right) - \left(\frac{x+1}{N}\right)p\left(\frac{x+1}{N}\right) \\
 &\leq p\left(\frac{x}{N}\right)\left(1 - \frac{x}{N} - \left(\frac{x+1}{N}\right)\right) + \left(\frac{x+1}{N}\right) \\
 (27) \quad &= \frac{1}{N}\left(p\left(\frac{x}{N}\right)((N-1-2x) + (x+1))\right) \leq 1.
 \end{aligned}$$

Now, the last inequality  $\leq 1$  in (27) is fulfilled because:

If  $x = \frac{N-1}{2}$  it reduces to  $\frac{x+1}{N} \leq 1$ ;

If  $x < \frac{N-1}{2}$  it reduces to  $p\left(\frac{x}{N}\right) \leq \frac{N-x-1}{N-1-2x}$ , and this is satisfied because the right hand side of this expression is  $> 1$ ;

If  $\frac{N-1}{2} < x \leq N-1$  it is verified because  $N-1-x \geq 0$  and  $N-1-2x < 0$ .  $\square$

**Moran model with mutations.** A basic bias example is the mutation mechanism

$$(28) \quad p(u) = (1 - a_2)u + a_1(1 - u),$$

where  $(a_1, a_2)$  are mutation probabilities in  $(0, 1]$ . The drift is  $p(u) - u$ . When  $a_1 + a_2 \neq 1$ , the invariant probability measure satisfies  $\pi(x) = \binom{N}{x} \frac{(\alpha)_x (\beta)_{N-x}}{(\alpha+\beta)_N}$ ,  $x \in I$ , where  $(\alpha)_x := \Gamma(\alpha+x)/\Gamma(\alpha)$ .

When  $p$  is non-decreasing, we have  $a_1 + a_2 \leq 1$ . In  $\bar{p}(u)$  the roles of  $a_1$  and  $a_2$  are exchanged.

The case  $a_1 = a_2 = 1$ , that is  $p(u) = 1 - u$ , corresponds to the heat-exchange Bernoulli-Laplace model [6]. Here,  $\pi(x) = \binom{N}{x} \binom{N}{N-x} / \binom{2N}{N}$ . If  $a_1 = a_2 = 1/2$  then  $p(u) = 1/2$  which is amenable (through a suitable time substitution) to the Ehrenfest urn model provided  $N$  is even.

One-way mutations,  $(a_1, a_2) = (a_1, 0)$  or  $(0, a_2)$  lead to the choice  $p(1) = 1$  or  $p(0) = 0$  respectively, corresponding to the case in which  $N$  or  $0$  is an absorbing state respectively.

Except for some exceptional special cases, the spectral measure associated to the Moran model is not known. Let us supply some of these special cases.

**Spectral representation of the Moran model with mutations.** Assume  $a_1 + a_2 \neq 1$ , [13]. Here the eigenvalues are

$$(29) \quad t_k = 1 - \frac{k}{N} \left( a_1 + a_2 + \frac{k-1}{N} (1 - (a_1 + a_2)) \right).$$

which is non negative for all  $k \in I$ . The spectral gap is  $1 - t_1 = \frac{1}{N} (a_1 + a_2)$ .

When  $a_1 = a_2 = 1$ ,  $t_k = 1 - \frac{k}{N} (2N + 1 - k)$  and the spectral measure is given by  $\mu_k = \frac{2N+1-2k}{2N+1-k} \binom{N}{k} / \binom{2N-k}{N}$ . The expected return time to 0 is  $2^{2N} / \sqrt{\pi N}$  whereas the expected return time to  $N/2$  is of order  $\sqrt{\pi N}/2$ , much smaller.

When  $a_1 + a_2 = 1$ ,  $p(u) = a_1$  is constant and the transition probabilities become affine linear functions of the state. Here  $\pi(x) = \binom{N}{x} a_1^x \bar{a}_1^{N-x}$ ,  $\mu_k = \binom{N}{k} a_1^k \bar{a}_1^{N-k}$  and  $t_k = 1 - \frac{k}{N}$ . When  $a_1 = 1/2$ , the holding probabilities are  $r_x = 1/2$  and both  $\pi(x)$  and  $\mu_k$  are symmetric Binomial( $N, 1/2$ ) distributed.

**Cases with positive eigenvalues.** We may look for conditions on the mechanism  $p$  leading to  $r_x \geq \frac{1}{2}$  in which case the BD chain is spectrally positive. Assume  $p : [0, 1] \rightarrow (0, 1)$  is continuous, non-decreasing and so  $0 < p(0) \leq p(1) < 1$ .

Then, as can easily be checked when  $N$  is even:

$$r_x \geq 1/2 \forall x \in I \Leftrightarrow p(1/2) = 1/2 \text{ with } p(0) \leq \frac{1}{2} \leq p(1).$$

Indeed, imposing  $r_x \geq 1/2$  for all  $x$  leads to  $p(u) \geq 1/2$  if  $u \geq 1/2$ ,  $p(u) \leq 1/2$  if  $u \leq 1/2$  and  $p(1/2) = 1/2$ . Since  $p$  is non-decreasing these conditions are equivalent to  $p(1/2) = 1/2$ . The reciprocal also holds. When  $N$  is odd an analogous condition can be written. When the mutation mechanism satisfies  $0 < a_1 \leq 1 - a_2 < 1$ , the condition  $p(1/2) = 1/2$ , leads to  $a_1 = a_2$ . However, it is easy to see that the condition  $a_1 = a_2$  is not necessary for  $P$  to be spectrally positive.

**4.7. Generalized ultrametric case.** Let us examine another triangular matrix  $H$  that is also a potential matrix. It belongs to the class of generalized ultrametric matrices (see [19], [16]), a class that contains the ultrametric matrices introduced in [17].

Let  $C$  be a nonempty set strictly contained in  $I = \{0, \dots, N\}$ . Denote  $C' = I \setminus C$ . We put  $C(x) = C$  when  $x \in C$  and  $C(x) = C'$  otherwise. Take  $\alpha, \beta \geq 0$ , and put  $\gamma(x) = \alpha$  if  $x \in C$  and  $\gamma(x) = \beta$  otherwise. Now, define the matrix  $H_{\alpha, \beta}$  by

$$H_{\alpha, \beta}(x, y) = \mathbf{1}(x \leq y) + \gamma(x) \mathbf{1}(x \leq y) \mathbf{1}(C(x) = C(y)),$$

which is a clear generalization of the Siegmund dual because  $H_{0,0} = H_S$ . It is straightforward to check that  $H_{\alpha, \beta}$  belongs to the class of potential matrices introduced in Subsection 4.1, indeed  $H_{\alpha, \beta} = (\text{Id} - R)^{-1}$  with

$$R(x, y) = \mathbf{1}(x = y) - \frac{1}{1 + \gamma(x)} \mathbf{1}(x = y) + \frac{1}{1 + \gamma(x)} \mathbf{1}(x + 1 = y).$$

As it is easily checked  $R$  is an irreducible strictly substochastic matrix that loses mass through the state  $N$ . Then

$$H_{\alpha, \beta}^{-1} = \text{Id} - R, \text{ so } H_{\alpha, \beta}^{-1}(x, y) = \frac{1}{1 + \gamma(x)} \mathbf{1}(x = y) - \frac{1}{1 + \gamma(x)} \mathbf{1}(x + 1 = y).$$

In this case we are able to compute the inverse matrix  $H_{\alpha, \beta}^{-1}$ , the description of the inverse of any generalized ultrametric matrices can be found in [3]. We point out that  $R'$  is substochastic only when  $\alpha \geq \beta$  and in this case it is an irreducible strictly

substochastic that loses mass through the state 0. In the rest of this Subsection, we will put  $H = H_{\alpha, \beta}$  to avoid overburden notation.

We have,

$$\begin{aligned} (H\widehat{P}')(x, y) &= \sum_{z \geq x} H(x, z) \widehat{P}(y, z) = \sum_{z \geq x} \widehat{P}(y, z) + \gamma(x) \sum_{z \geq x, z \in C(x)} \widehat{P}(y, z), \\ (PH)(x, y) &= \sum_{z \leq y} P(x, z) H(z, y) = \sum_{z \leq y} P(x, z) + \gamma(y) \sum_{z \leq y, z \in C(y)} P(x, z). \end{aligned}$$

By permuting  $I$  we can always assume that  $C$  is an interval, that is  $C = \{1, \dots, k\}$  for some  $0 \leq k < N$ , and so  $C' = \{k+1, \dots, N\}$ . With this choice we have that each  $x \notin \{k, N\}$  verifies  $C(x) = C(x+1)$  and so  $\gamma(x) = \gamma(x+1)$ .

(i). Let  $x \neq k$ . From the above equalities we find

$$(H\widehat{P}')(x, y) - (H\widehat{P}')(x+1, y) = (1 + \gamma(x)) \widehat{P}(y, x).$$

(the case  $x = N$  follows from  $(H\widehat{P}')(N+1, y) = 0$ ). Then the equality  $H\widehat{P}' = PH$  implies,

$$(1 + \gamma(x)) \widehat{P}(y, x) = \sum_{z \leq y} (P(x, z) - P(x+1, z)) + \gamma(y) \sum_{z \leq y, z \in C(y)} (P(x, z) - P(x+1, z)).$$

(i1). Let  $x \neq k$  and  $y \leq k$ . In this case we have  $\gamma(y) = \alpha$  and  $z \leq y$  implies  $z \in C(y)$ . So, we find

$$(1 + \gamma(x)) \widehat{P}(y, x) = (1 + \alpha) \sum_{z \leq y} (P(x, z) - P(x+1, z)).$$

Then, a necessary and sufficient condition for  $\widehat{P}(y, x) \geq 0$  is that

$$(30) \quad \sum_{z \leq y} P(x+1, z) \leq \sum_{z \leq y} P(x, z),$$

and we get

$$(31) \quad \widehat{P}(y, x) = \left( \frac{1 + \alpha}{1 + \gamma(x)} \right) \sum_{z \leq y} (P(x, z) - P(x+1, z)).$$

(i2). Let  $x \neq k$  and  $y > k$ . In this case we have that  $\gamma(y) = \beta$ , and  $C(z) = C(y)$  if and only if  $z > k$ . Then,

$$(1 + \gamma(x)) \widehat{P}(y, x) = \sum_{z \leq y} (P(x, z) - P(x+1, z)) + \beta \sum_{k < z \leq y} (P(x, z) - P(x+1, z)),$$

and so

$$(32) \quad \widehat{P}(y, x) = \left( \frac{1}{1 + \gamma(x)} \right) \sum_{z \leq k} (P(x, z) - P(x+1, z)) + \left( \frac{1 + \beta}{1 + \gamma(x)} \right) \sum_{k < z \leq y} (P(x, z) - P(x+1, z)).$$

Then, a necessary and sufficient condition in order that  $\widehat{P}(y, x) \geq 0$  for  $x \neq k$  is that

$$(33) \quad \sum_{z \leq k} P(x+1, z) + (1 + \beta) \sum_{k < z \leq y} P(x+1, z) \leq \sum_{z \leq k} P(x, z) + (1 + \beta) \sum_{k < z \leq y} P(x, z).$$

We can summarize subcases (i1) and (i2) as for all  $x \neq k$

$$\widehat{P}(y, x) = \left( \frac{1 + \gamma(y)}{1 + \gamma(x)} \right) \sum_{z \leq y} (P(x, z) - P(x+1, z)) - \mathbf{1}(y > k) \frac{\beta}{1 + \gamma(x)} \sum_{z \leq k} (P(x, z) - P(x+1, z)).$$

The necessary and sufficient condition in order that  $\widehat{P}(y, x) \geq 0$  for  $x \neq k$  is constituted by (30) and (33).

(ii). Assume  $x = k$ . Recall that  $\gamma(k) = \alpha$  and  $\gamma(k+1) = \beta$ , so

$$(H\widehat{P}')(k, y) = \sum_{z \geq k} \widehat{P}(y, z) + \alpha \sum_{z \geq k, z \in C(k)} \widehat{P}(y, z) = \sum_{z > k} \widehat{P}(y, z) + (1 + \alpha) \widehat{P}(y, k).$$

$$(H\widehat{P}')(k+1, y) = \sum_{z > k} \widehat{P}(y, z) + \beta \sum_{z > k, z \in C(x)} \widehat{P}(y, z) = (1 + \beta) \sum_{z > k} \widehat{P}(y, z).$$

Then, by using  $H\widehat{P}' = PH$ ,

$$\begin{aligned} (1 + \alpha) \widehat{P}(y, k) &= (H\widehat{P}')(k, y) - (H\widehat{P}')(k+1, y) + \beta \sum_{z > k} \widehat{P}(y, z) \\ &= (PH)(k, y) - \left( \frac{1}{1 + \beta} \right) (PH)(k+1, y), \end{aligned}$$

and so

$$\begin{aligned} (1 + \alpha) \widehat{P}(y, k) &= \sum_{z \leq y} \left( P(k, z) - \left( \frac{1}{1 + \beta} \right) P(k+1, z) \right) \\ (34) \quad &+ \gamma(y) \sum_{z \leq y, z \in C(y)} \left( P(k, z) - \left( \frac{1}{1 + \beta} \right) P(k+1, z) \right). \end{aligned}$$

From (34) we deduce,

$$(35) \quad y \leq k : \widehat{P}(y, k) = \sum_{z \leq y} \left( P(k, z) - \left( \frac{1}{1 + \beta} \right) P(k+1, z) \right),$$

and

$$\begin{aligned} y > k : \widehat{P}(y, k) &= \left( \frac{1}{1 + \alpha} \right) \sum_{z \leq k} \left( P(k, z) - \left( \frac{1}{1 + \beta} \right) P(k+1, z) \right) \\ (36) \quad &+ \left( \frac{1 + \beta}{1 + \alpha} \right) \sum_{k < z \leq y} \left( P(k, z) - \left( \frac{1}{1 + \beta} \right) P(k+1, z) \right). \end{aligned}$$

Hence, the equations (30) and (33) imply that  $\widehat{P}(y, k) \geq 0$  for all  $y$ , and then they are necessary and sufficient for  $\widehat{P} \geq 0$ .

From (31) we find,

$$\forall y \leq k : \sum_{x < k} \widehat{P}(y, x) = \sum_{z \leq y} (P(0, z) - P(k, z)), \quad \sum_{x > k} \widehat{P}(y, x) = \left( \frac{1 + \alpha}{1 + \beta} \right) \sum_{z \leq y} P(k+1, z).$$

So, by using (35) we get

$$(37) \quad \forall y \leq k : \sum_{x \leq N} \widehat{P}(y, x) = \sum_{z \leq y} P(0, z) + \left( \frac{\alpha}{1 + \beta} \right) \sum_{z \leq y} P(k+1, z).$$

On the other hand, from (32) we obtain,

$$\begin{aligned} \forall y > k : \sum_{x < k} \widehat{P}(y, x) &= \left( \frac{1}{1+\alpha} \right) \sum_{z \leq k} (P(0, z) - P(k, z)) + \left( \frac{1+\beta}{1+\alpha} \right) \sum_{k < z \leq y} (P(0, z) - P(k, z)), \\ \sum_{x > k} \widehat{P}(y, x) &= \left( \frac{1}{1+\beta} \right) \sum_{z \leq k} P(k+1, z) + \sum_{k < z \leq y} P(k+1, z). \end{aligned}$$

By using (36) we get

$$\begin{aligned} \forall y > k : \sum_{x \leq N} \widehat{P}(y, x) &= \left( \frac{1}{1+\alpha} \right) \sum_{z \leq k} P(0, z) + \left( \frac{1+\beta}{1+\alpha} \right) \sum_{k < z \leq y} P(0, z), \\ (38) \qquad \qquad \qquad &+ \left( \frac{\alpha}{(1+\alpha)(1+\beta)} \right) \sum_{z \leq k} P(k+1, z) + \left( \frac{\alpha}{1+\alpha} \right) \sum_{k < z \leq y} P(k+1, z). \end{aligned}$$

**Proposition 11.** *Let  $P$  be a stochastic kernel and let  $\widehat{P}$  be a  $H_{\alpha, \beta}$ -dual of  $P$ . Then, a sufficient condition to have  $\widehat{P} \geq 0$  is the following one:*

$$(39) \qquad \exists \delta \in (0, 1) \quad \text{such that } \forall x \in \{0, \dots, N\} : \sum_{z \leq k} P(x, z) = \delta;$$

$$(40) \qquad \forall y \leq k : \sum_{z \leq y} P(x, z) \text{ decreases in } x \in \{1, \dots, k\};$$

$$(41) \qquad \forall y > k : \sum_{k < z \leq y} P(x, z) \text{ decreases in } x \in \{k+1, \dots, N\}.$$

Moreover, under the conditions (39), (40) and (41),  $\widehat{P}$  is substochastic if and only if  $\delta = \left( \frac{1+\beta}{1+\alpha+\beta} \right)$ . In this case  $\widehat{P}$  is conservative at sites  $k$  and  $N$ .

*Proof.* The relations (39), (40) and (41), are sufficient for  $\widehat{P} \geq 0$  because they imply the conditions (30) and (33). Now put

$$L(y) = \sum_{x \leq N} \widehat{P}(y, x).$$

From (37) we find that  $\{L(y) : y \leq k\}$  attains its maximum at  $y = k$  and by using (39) this maximum becomes  $L(k) = \delta + \left( \frac{\alpha\delta}{1+\beta} \right)$ . So, this last quantity must be at most 1 in order that  $\widehat{P}$  is substochastic. On the other hand, from (38) it follows that  $\{L(y) : y > k\}$  attains its maximum at  $y = N$  and that this maximum is

$$L(N) = \left( \frac{\delta}{1+\alpha} \right) + \left( \frac{(1+\beta)(1-\delta)}{1+\alpha} \right) + \left( \frac{\alpha\delta}{(1+\alpha)(1+\beta)} \right) + \left( \frac{\alpha(1-\delta)}{1+\alpha} \right).$$

By straightforward computations it follows that

$$L(N) = \frac{1}{1+\alpha} (1 + \alpha + \beta(1 - L(k))).$$

Then, by using  $L(k) \leq 1$  we deduce that  $L(N) \leq 1$  if and only if  $L(k) = 1$ , in which case  $L(N) = 1$ . The result is shown.  $\square$

If the ultrametric dual is seen as a perturbation of the Siegmund dual then there is a rigidity result for the BD chains.

**Proposition 12.** *Let  $P$  be the stochastic kernel of an irreducible BD chain on  $I = \{0, \dots, N\}$ . Assume that there exists a substochastic kernel  $\widehat{P}$  that is a  $H_{\alpha, \beta}$ -dual of  $P$ ,  $H_{\alpha, \beta} \widehat{P}' = P H_{\alpha, \beta}$ .*

*Then we necessarily have  $\beta = 0$  and the monotone property (24) is verified. Moreover, if  $k \geq 1$  then  $\alpha = \beta = 0$  and  $H_{\alpha, \beta} = H_{0,0} = H_S$  is the Siegmund dual.*

*If  $k = 0$  then  $\alpha \leq (1 - p_0)/q_1$ . If  $\alpha = (1 - p_0)/q_1$  the kernel  $\widehat{P}$  is stochastic, and when  $\alpha < (1 - p_0)/q_1$  the kernel  $\widehat{P}$  is substochastic and it only loses mass through  $\{0\}$ .*

*Proof.* From (32) we have

$$\begin{aligned} \widehat{P}(k+2, k-1) &= \left(\frac{1}{1+\alpha}\right) \sum_{z \leq k} (P(k-1, z) - P(k, z)) + \left(\frac{1+\beta}{1+\alpha}\right) \sum_{k < z \leq k+2} (P(k-1, z) - P(k, z)) \\ &= \left(\frac{1}{1+\alpha}\right) (1 - (1 - P(k, k+1))) - \left(\frac{1+\beta}{1+\alpha}\right) P(k, k+1). \end{aligned}$$

So  $\widehat{P}(k+2, k-1) = -P(k, k+1)(\beta/1+\alpha)$ , and we must necessarily have  $\beta = 0$ .

Since  $\beta = 0$ , from relations (30) and (33), it results that the conditions to have  $\widehat{P} \geq 0$  is that (24) is fulfilled, that is  $p_x + q_{x+1} \leq 1 \quad \forall x \in \{0, \dots, N-1\}$ .

On the other hand if  $k \geq 1$  we get from (37) that for  $y = k$ ,

$$\sum_{x \leq N} \widehat{P}(k, x) = \sum_{z \leq k} P(0, z) + \left(\frac{\alpha}{1+\beta}\right) \sum_{z \leq k} P(k+1, z) = 1 + \left(\frac{\alpha}{1+\beta}\right) P(k+1, k).$$

So, we must necessarily have  $\alpha = 0$ .

In the case  $k = 0$  from relation (38) it results that  $\sum_{y \in I} \widehat{P}(x, y) = 1$  for all  $y > 0$ .

The only case we must examine is (37) for  $k = 0$  and the condition  $\sum_{y \in I} \widehat{P}(0, y) = (1 - p - 0) + \alpha q_1 \leq 1$  implies  $\alpha \leq (1 - p_0)/q_1$ .  $\square$

## 5. STRONG STATIONARY TIMES

Let  $P$  be an irreducible positive recurrent stochastic kernel on the countable set  $I$  and  $X = (X_n : n \geq 0)$  be a Markov chains with kernel  $P$ . Let  $\pi$  be the stationary probability measure of  $X$ . We denote by  $\pi_0$  the initial distribution of  $X$  and in general  $\pi_n$  is the distribution of  $X_n$ ,  $\pi_n(\cdot) = \mathbb{P}_{\pi_0}(X_n = \cdot)$ . It verifies  $\pi'_n = \pi'_0 P^n$ .

A random time  $T$  is called a strong stationary time for  $X$ , if  $X_T$  has distribution  $\pi$  and it is independent of  $T$ , see [1]. The separation discrepancy is defined by,

$$\text{sep}(\pi_n, \pi) := \sup_{y \in I} \left[ 1 - \frac{\pi_n(y)}{\pi(y)} \right].$$



It satisfies  $\text{sep}(\pi_n, \pi) \geq \|\pi_n - \pi\|_{TV}$  where  $\|\pi_n - \pi\|_{TV} = \frac{1}{2} \sum_{y \in I} |\pi_n(y) - \pi(y)|$  is the total variation distance between  $\pi_n$  and  $\pi$ , see [1] and [4]. In Proposition 2.10 in [1] it was proven that every stationary time  $T$  verifies

$$(42) \quad \text{sep}(\pi_n, \pi) \leq \mathbb{P}_{\pi_0}(T > n) \quad n \geq 0.$$

Based upon this result the strong stationary time  $T$  is called sharp when there is equality in (42), that is

$$\text{sep}(\pi_n, \pi) = \mathbb{P}_{\pi_0}(T > n) \quad n \geq 0.$$

In Proposition 3.2 in [1] it was shown that a sharp strong stationary time always exists.

Let  $\tilde{P}$  be a stochastic kernel on the countable set  $\tilde{I}$  such that  $\tilde{P}$  is a  $\Lambda$ -intertwining of  $P$  where  $\Lambda$  is a nonsingular stochastic kernel, so  $\tilde{P}\Lambda = \Lambda P$ . Let  $\tilde{X} = (\tilde{X}_n : n \geq 0)$  be a Markov chain with kernel  $\tilde{P}$ .

Recall that when we are in the framework of Theorem 2, we have  $\tilde{P}\Lambda = \Lambda \overleftarrow{P}$ , so  $\tilde{P}$  is a  $\Lambda$ -intertwining of the reversal kernel  $\overleftarrow{P}$ . Hence, when the intertwining is constructed from a dual relation,  $\overleftarrow{P}$  and the reversed chain  $\overleftarrow{X}$  will play the role of  $P$  and  $X$  in the intertwining relation. In the reversible case  $\overleftarrow{P} = P$  both notations coincide, that is  $\overleftarrow{P} = P$  and we can take  $\overleftarrow{X} = X$ , this occurs for instance when  $P$  is the kernel of an irreducible BD chain.

The initial probability distributions of the chains  $X$  and  $\tilde{X}$  will be respectively  $\pi_0$  and  $\tilde{\pi}_0$ , that is  $X_0 \stackrel{d}{\sim} \pi_0$  and  $\tilde{X}_0 \stackrel{d}{\sim} \tilde{\pi}_0$ . We assume that the initial distributions are linked, this means:

$$(43) \quad \pi'_0 = \tilde{\pi}'_0 \Lambda.$$

When this relation is verified we say that  $\pi'_0$  and  $\pi_0$  is an admissible condition. Let  $\pi_n$  and  $\tilde{\pi}_n$  be the distributions of  $X_n$  and  $\tilde{X}_n$ . By the intertwining relation  $\tilde{P}^n \Lambda = \Lambda P^n$  for all  $n \geq 1$ , and the initial condition (43) we get

$$\pi'_n = \tilde{\pi}'_n \Lambda \quad \forall n \geq 0.$$

**5.1. The coupling.** Let us recall the coupling done in [4] between the intertwining Markov chains. Consider the kernel  $\overline{P}$  defined on  $I \times \tilde{I}$  by:

$$\overline{P}((x, \tilde{x}), (y, \tilde{y})) = \frac{P(x, y) \tilde{P}(\tilde{x}, \tilde{y}) \Lambda(\tilde{y}, y)}{(\Lambda P)(\tilde{x}, y)} \mathbf{1}((\Lambda P)(\tilde{x}, y) > 0).$$

The kernel  $\overline{P}$  is stochastic. Let  $\overline{X} = (\overline{X}_n : n \geq 0)$  be the chain taking values in  $I \times \tilde{I}$ , evolving with the kernel  $\overline{P}$  and having as initial distribution the vector  $(\pi_0, \tilde{\pi}_0)$  where  $\pi'_0 = \tilde{\pi}'_0 \Lambda$ . It can be checked that  $\overline{X}$  is a coupling of the chains  $X$  and  $\tilde{X}$ . Then, in the sequel we will write by  $X$  and  $\tilde{X}$  the components of  $\overline{X}$ , so  $\overline{X}_n = (X_n, \tilde{X}_n)$  for all  $n \geq 0$ . In the above construction it can be also checked that,

$$(44) \quad \Lambda(\tilde{x}, x) = \mathbb{P}(X_n = x \mid \tilde{X}_n = \tilde{x}) \quad \forall n \geq 0.$$

(For this equality also see [2]). In [4] this coupling was characterized as the unique one that verifies (44) and three other properties on conditional independence. These properties imply that the coupling also satisfies,

$$\Lambda(\tilde{x}_n, x_n) = \mathbb{P}\left(X_n = x_n \mid \tilde{X}_0 = \tilde{x}_0 \cdots \tilde{X}_n = \tilde{x}_n\right) \quad \forall n \geq 0.$$

In this process the original ergodic Markov chain  $X$  governed by  $P$ , may be viewed as a random output of the Markov process  $\tilde{X}$  governed by  $\tilde{P} = \Lambda P \Lambda^{-1}$ , when  $\Lambda$  is non singular. This is a setup reminiscent of filtering theory with  $\tilde{X}$  the hidden process and  $X$  the observable. The peculiarity of the intertwining construction is that the output  $X$  process is itself Markov.

The following concept was introduced in [4].

**Definition 3.** *The Markov chain  $\tilde{X}$  will be called a strong stationary dual of the Markov chain  $X$ , if  $\tilde{X}$  has an absorbing state  $\tilde{\partial}$  that verifies*

$$(45) \quad \pi(x) = \mathbb{P}\left(X_n = x \mid \tilde{X}_0 = \tilde{x}_0 \cdots \tilde{X}_{n-1} = \tilde{x}_{n-1}, \tilde{X}_n = \tilde{\partial}\right) \quad \forall x \in I, n \geq 0,$$

and where  $\tilde{x}_0 \cdots \tilde{x}_{n-1} \in \tilde{I}$  satisfy  $\mathbb{P}\left(\tilde{X}_0 = \tilde{x}_0 \cdots \tilde{X}_{n-1} = \tilde{x}_{n-1}, \tilde{X}_n = \tilde{\partial}\right) > 0$ .

In Theorem 2.4 in [1] it was shown that when the condition (45) holds then the absorption time  $\tilde{T}_{\tilde{\partial}}$  at  $\{\tilde{\partial}\}$  is a strong stationary time for  $X$ . Moreover in Remark 2.8 in [4] it is built a specific dual process  $\tilde{X}$  having an absorbing state  $\tilde{\partial}$  and whose absorption time  $\tilde{T}_{\tilde{\partial}}$  is sharp.

Assume that  $\tilde{\partial}$  is an absorbing state for  $\tilde{X}$ . From (4) we get  $\pi' = \mathbf{e}'_{\tilde{\partial}} \Lambda$ . When the initial conditions are linked by relation (43)  $\pi'_0 = \tilde{\pi}'_0 \Lambda$ , we get that  $\tilde{T}_{\tilde{\partial}}$  is a strong stationary time for  $X$ . Indeed, from  $\Lambda(\tilde{\partial}, x) = \mathbb{P}\left(X_n = x \mid \tilde{X}_0 = \tilde{x}_0 \cdots \tilde{X}_n = \tilde{\partial}\right)$ , it follows that  $\pi(x) = \mathbb{P}\left(X_n = x \mid \tilde{X}_0 = \tilde{x}_0 \cdots \tilde{X}_n = \tilde{\partial}\right)$  is verified because condition  $\pi' = \mathbf{e}'_{\tilde{\partial}} \Lambda$  holds. Observe that for the Siegmund dual and monotone kernels (that is verifying (14)) the absorbing state is  $\tilde{\partial} = N$ .

**5.2. Choice of the initial conditions.** Let  $\tilde{\partial}$  be an absorbing state of  $\tilde{X}$ . From (43) the initial conditions of the chains must verify  $\pi'_0 = \tilde{\pi}'_0 \Lambda$  to be able to perform the duality construction and to get that the absorption time  $\tilde{T}_{\tilde{\partial}}$  is a strong stationary time for  $X$ .

Assume that  $I = \tilde{I}$ . Since  $\Lambda$  is a stochastic matrix it has a left probability eigenvector  $\pi'_\Lambda$  satisfying  $\pi'_\Lambda = \pi'_\Lambda \Lambda$ . So, we can choose  $\tilde{X}_0 \stackrel{d}{\sim} X_0 \stackrel{d}{\sim} \pi_\Lambda$  because (43) is satisfied (we also use  $\stackrel{d}{\sim}$  to mean 'distributed as'). Then, when  $\tilde{X}$  is initially distributed as  $\pi_\Lambda$ ,  $\tilde{T}_{\tilde{\partial}}$  is a strong stationary time for the chain  $X$  starting from  $\pi_\Lambda$ .

If  $\Lambda$  is non irreducible then  $\pi_\Lambda$  could fail to be strictly positive. This is the case for the Siegmund kernel. In fact, from (21) it can be checked that  $\mathbf{e}_0$  is the unique

left eigenvector satisfying  $\mathbf{e}'_0 = \mathbf{e}'_0 \Lambda$  and so  $\pi_\Lambda = \mathbf{e}_0$ . Then, for the  $\Lambda$ -intertwining given by (21) the initial condition  $\tilde{X}_0 \stackrel{d}{\sim} \delta_0$  and  $X_0 \stackrel{d}{\sim} \delta_0$  is admissible.

When  $\tilde{b} \in \tilde{I}$ ,  $b \in I$ ,  $\tilde{b} \neq \tilde{\partial}$ , verify  $\mathbf{e}'_b \Lambda = \mathbf{e}'_b$  then  $\tilde{X}_0 \stackrel{d}{\sim} \delta_{\tilde{b}}$  and  $X_0 \stackrel{d}{\sim} \delta_b$  is an admissible initial condition (it verifies (43)). Then  $\tilde{T}_{\tilde{\partial}}$  starting from  $\tilde{b}$  is a strong stationary time for  $X$  starting from  $b$ , and it is strictly positive. In this case, both  $\tilde{X}_0$  and  $X_0$  start at a single point. We observe that the condition  $\mathbf{e}'_b \Lambda = \mathbf{e}'_b$  is equivalent to the following condition on the dual function:  $H\mathbf{e}_{\tilde{b}} = c\mathbf{e}_b$  for some  $c > 0$ . Indeed if  $H$  verifies this condition and since  $\Lambda = D_\varphi H' D_\pi$  (see (8)) we obtain  $\mathbf{e}'_b \Lambda = c'\mathbf{e}'_b$  with  $c' = c\pi(b)/\varphi(\tilde{b})$ . Since  $\Lambda$  is stochastic we get  $c' = 1$ , and so  $c = \varphi(\tilde{b})/\pi(b)$ . This gives  $H\mathbf{e}_{\tilde{b}} = \frac{\varphi(\tilde{b})}{\pi(b)}\mathbf{e}_b$ , which is exactly  $\mathbf{e}'_b \Lambda = \mathbf{e}'_b$ .

For the Siegmund kernel and  $P$  monotone,  $\Lambda$  is given by (21) and the equation (43) takes the form

$$\pi_0(x) = \sum_{z=x}^N \tilde{\pi}_0(z) \frac{\pi(x)}{\pi^c(z)} \quad \forall x \in I.$$

So we need that  $\pi_0(x)/\pi(x)$  decreases with  $x \in I$  and in this case  $\tilde{\pi}_0(x) = \pi^c(x) (\pi_0(x)/\pi(x) - \pi_0(x+1)/\pi(x+1))$ . These are, respectively, condition (4.7) and formula (4.10) in [4].

We recall that every monotone kernel  $P$  verifies condition  $\pi' = \mathbf{e}'_N \Lambda$  (see (4)). The  $\Lambda$ -intertwining  $\tilde{P}$  is the one of  $\overleftarrow{P}$ , and in this case  $\overleftarrow{X}$  and  $\tilde{X}$  denote the Markov chains associated to  $\overleftarrow{P}$  and  $\tilde{P}$ , respectively.

**5.3. Conditions for sharpness.** We now give a proof of the sharpness result alluded to in Remark 2.39 of [4] and in Theorem 2.1 in [7].

**Proposition 13.** *Let  $X$  be an irreducible positive recurrent Markov chain,  $\tilde{X}$  be a  $\Lambda$ -intertwining of  $X$  having  $\tilde{\partial}$  as an absorbing state. Assume that there exists  $d \in I$  such that*

$$(46) \quad \Lambda \mathbf{e}_d = \pi(d) \mathbf{e}_{\tilde{\partial}}.$$

*Then  $\tilde{X}$  is a sharp dual to  $X$ , that is for  $\tilde{X}_0 \stackrel{d}{\sim} \tilde{\pi}_0$  and  $X_0 \stackrel{d}{\sim} \pi_0$  with  $\pi'_0 = \tilde{\pi}'_0 \Lambda$ , we have:*

$$(47) \quad \text{sep}(\pi_n, \pi) = \mathbb{P}_{\tilde{\pi}_0}(\tilde{T}_{\tilde{\partial}} > n) \quad \forall n \geq 0.$$

*Proof.* From condition  $\Lambda \mathbf{e}_d = \pi(d) \mathbf{e}_{\tilde{\partial}}$  we get,

$$(48) \quad \pi_n(d) = \pi'_n \mathbf{e}_d = \tilde{\pi}'_n \Lambda \mathbf{e}_d = \pi(d) \tilde{\pi}_n(\tilde{\partial}).$$

Since  $\pi > 0$ , the last equalities imply that

$$(49) \quad \pi_n(d) > 0 \Leftrightarrow \tilde{\pi}_n(\tilde{\partial}) > 0.$$

On the other hand the condition  $\pi' = \mathbf{e}'_{\tilde{\partial}}\Lambda$  means that the  $\tilde{\partial}$ -row of  $\Lambda$  verifies  $\Lambda(\tilde{\partial}, \cdot) = \pi'(\cdot) > 0$ . Then, if for some  $n$  we have  $\tilde{\pi}_n(\tilde{\partial}) > 0$ , from  $\pi'_n = \tilde{\pi}'_n\Lambda$  we deduce  $\pi_n > 0$ . Moreover,

$$\pi_n(x) = \sum_{\tilde{x} \in \tilde{I}} \tilde{\pi}(\tilde{x})\Lambda(\tilde{x}, x) \geq \tilde{\pi}(\tilde{\partial})\Lambda(\tilde{\partial}, x) = \tilde{\pi}(\tilde{\partial})\pi(x)$$

Therefore, from (48) we get

$$\min_{x \in I} \frac{\pi_n(x)}{\pi(x)} = \tilde{\pi}(\tilde{\partial}) = \frac{\pi_n(d)}{\pi(d)}.$$

Then,  $\text{sep}(\pi_n, \pi) = 1 - \tilde{\pi}(\tilde{\partial})$ . Since  $\tilde{\partial}$  is an absorption state implies  $\tilde{\pi}_n(\tilde{\partial}) = \mathbb{P}_{\tilde{\pi}_0}(\tilde{T}_{\tilde{\partial}} \leq n)$ , we get the desired relation

$$\text{sep}(\pi_n, \pi) = \mathbb{P}_{\tilde{\pi}_0}(\tilde{T}_{\tilde{\partial}} > n) \quad \forall n \geq n_+, \text{ with } n_+ = \inf\{n \geq 0 : \tilde{\pi}_n(\tilde{\partial}) > 0\}.$$

Let us show that the relation (47) holds for  $n < n_+$ . First remark that in this case  $\tilde{\pi}_n(\tilde{\partial}) = 0$ , which by (49) implies  $\pi_n(d) = 0$ . Then  $\text{sep}(\pi_n, \pi) = 1$  and so the equality  $\text{sep}(\pi_n, \pi) = \mathbb{P}_{\tilde{\pi}_0}(\tilde{T}_{\tilde{\partial}} > n) = 1$  holds. We have proven that  $\tilde{X}$  is a sharp dual to  $X$ .  $\square$

**Proposition 14.** (i) *Assume the hypotheses of Theorem 2 are verified and that  $\hat{P}$  is a substochastic kernel having  $\hat{a}$  as an absorbing state in  $\hat{P}$ . Then, if there exists some  $d \in I$  such that*

$$(50) \quad \mathbf{e}'_d H = c \mathbf{e}'_{\hat{a}} \text{ for some } c > 0,$$

*then  $\hat{a}$  is an absorbing state for  $\tilde{X}$  and  $\tilde{X}$  is a sharp dual to  $\overleftarrow{X}$ . That is, when  $\pi'_0 = \tilde{\pi}'_0\Lambda$  the relation (47) is verified.*

(ii) *Assume the hypotheses of Theorem 2 are verified and that  $\hat{P}$  is a substochastic kernel verifying that there exist  $\hat{a} \in \hat{I}$ ,  $d \in I$  such that for some constants  $c' > 0$ ,  $c > 0$  we have*

$$(51) \quad H \mathbf{e}_{\hat{a}} = c' \mathbf{1} \text{ and } \mathbf{e}'_d H = c \mathbf{e}'_{\hat{a}}$$

*Then part (i) holds, and  $\tilde{X}$  is a sharp dual to  $\overleftarrow{X}$ .*

*Proof.* (i) From Theorem 2 (v) it follows that  $\hat{a}$  is an absorbing state for  $\tilde{P}$ . From Proposition 13 it suffices to show that  $d$  verifies (46):  $\Lambda \mathbf{e}_d = \pi(d) \mathbf{e}_{\hat{a}}$ . Since the hypothesis is  $H(d, y) = c \delta_{y, \hat{a}}$  for some  $c > 0$  and for all  $y \in \tilde{I}$ , the Remark 3 implies  $\Lambda(x, d) = c' \delta_{x, \hat{a}}$  for some  $c' > 0$ . Now, from Theorem 2 (v) and (4) we have  $\pi(d) = \Lambda(\hat{a}, d)$ , and we deduce  $c' = \pi(d)$ . Therefore  $\Lambda(x, d) = \pi(d) \delta_{x, \hat{a}}$  which is equivalent to (46).

(ii) From Proposition 3 we get that the first relation in (51) guarantees that  $\hat{a}$  is an absorbing state for  $\hat{P}$ . So, we are under the hypotheses of part (i) and the result follows.  $\square$

**Corollary 15.** (i) *For a monotone irreducible stochastic kernel  $P$ , the  $\Lambda$ -intertwining Markov chain  $\tilde{X}$  has  $N$  as absorbing state and it is a sharp dual of  $\overleftarrow{X}$ . Moreover, both chains  $\overleftarrow{X}$  and  $\tilde{X}$  can start at state 0.*

(ii) For a monotone irreducible stochastic BD kernel  $P$  we have that the BD chain  $\tilde{X}$  is a sharp dual to  $X$ .

*Proof.* For part (i), the properties required for sharpness for the Siegmund intertwining of BD chains follow straightforward because the  $N$ -th row of  $H_S$  verifies (50) with  $d = N$ . Also the relation (22) in Corollary 6 is exactly (46). The fact that the state 0 is admissible for both  $X$  and  $\tilde{X}$  is a consequence of  $\mathbf{e}'_0 = \mathbf{e}'_0 \Lambda$ . In part (ii) the only novelty is that for BD chains  $P = \overleftarrow{P}$ .  $\square$

We note that, by definition, for an absorbing point  $\hat{a}$  there is a unique state  $d$  verifying (50), as it occurs for the Siegmund kernel.

When  $d$  verifies the property (49) we say that  $d$  is a witness state in  $X$  that  $\tilde{X}$  hits  $\partial$ . It reflects the following more general situation. Assume that  $\Lambda$  fulfills  $\Lambda(x, y) > 0 \Leftrightarrow x \geq y$ . Then, from  $\pi' = \tilde{\pi}' \Lambda$  we get

$$\tilde{\pi}_0(x) > 0 \Rightarrow \pi_0(y) > 0 \quad \forall y \leq x.$$

Then if  $N$  is an absorbing state of  $\tilde{P}$  and  $P(y, y+1) > 0$  and  $\tilde{P}(y, y+1) > 0$  for all  $y \in \{0, \dots, N-1\}$  the equivalence  $\pi_n(N) > 0 \Leftrightarrow \tilde{\pi}_n(N) > 0$  is satisfied, and so  $N$  will be a witness state in  $X$  that  $\tilde{X}$  hits the state  $N$ .

**5.4. Times to absorption.** From Proposition 14, for the BD chains the random time  $\tilde{T}_N$  starting from the state 0 gives information on the speed of convergence to its invariant measure, of the original BD chain  $X$  starting from the state 0. In the sequel we denote by  $\tilde{T}_{N;0}$  a random variable distributed as the hitting time  $\tilde{T}_N$  when starting from 0, that is  $\mathbb{P}(\tilde{T}_{N;0} = n) = \mathbb{P}_0(\tilde{T}_N = n)$  for  $n \geq 1$ . We denote its variance by  $\text{Var}(\tilde{T}_{N;0})$ .

For BD chains absorbed at  $N$ , the probability generating function of  $\tilde{T}_N$  starting from 0 is, see [14] and [7],

$$(52) \quad \mathbb{E}\left(u^{\tilde{T}_{N;0}}\right) = \prod_{k=1}^N \frac{(1-t_k)u}{1-t_k u}, \quad u \in [0, 1],$$

where  $-1 < t_k < +1$ ,  $k = 1, \dots, N$  are the  $N$  distinct eigenvalues of both  $\tilde{P}$  and  $P$ , avoiding  $t_0 = 1$ . The formula (52) also reads

$$\mathbb{P}\left(\tilde{T}_{N;0} > n\right) = \sum_{l=1}^N \prod_{k \neq l} \frac{1-t_k}{t_l-t_k} t_l^n, \quad n \geq N-1.$$

Then,  $t_1^{-n} \mathbb{P}(\tilde{T}_{N;0} > n) \rightarrow \prod_{k=2}^N \frac{1-t_k}{t_1-t_k}$  as  $t \rightarrow \infty$ , and  $\tilde{T}_{N;0}$  has geometric tails with exponent  $t_1$ . Also,

$$\mathbb{E}(\tilde{T}_{N;0}) = \sum_{k=1}^N (1-t_k)^{-1} \quad \text{and} \quad \text{Var}(\tilde{T}_{N;0}) = \sum_{k=1}^N (1-t_k)^{-2} - \sum_{k=1}^N (1-t_k)^{-1}.$$

Since  $t_1$  is the dominant eigenvalue

$$(53) \quad \text{Var}\left(\tilde{T}_{N;0}\right) \leq \mathbb{E}(\tilde{T}_{N;0})/1-t_1.$$

When the eigenvalues  $t_k$  are non negative, then  $\tilde{T}_{N;0} \stackrel{d}{\sim} \sum_{k=1}^N \tau_k$  where the  $\tau_k$ s are independent and  $\tau_k \stackrel{d}{\sim} \text{Geometric}(1 - t_k)$ , the geometric distribution with success parameter  $1 - t_k$  on  $\{1, 2, \dots\}$ . Assume that the eigenvalues  $t_k$  are not all positive and put  $t_N < \dots < t_{l+1} < 0 \leq t_l < \dots < t_1 < t_0 = 1$ . Then (52) interprets as:

$$\tilde{T}_{N;0} - \sum_{k=l+1}^N B_k \stackrel{d}{\sim} \sum_{k=1}^l \tau_k,$$

where  $B_k \stackrel{d}{\sim} \text{Bernoulli}(1/(1 - t_k))$ ,  $\tau_k \stackrel{d}{\sim} \text{Geometric}(1 - t_k)$  and  $\tilde{T}_{N;0}$  are all mutually independent. All the previous results in this Subsection 5.4 can be found in [1], [4] and [7].

When  $t_k$  are known explicitly it is possible to compute  $\mathbb{E}(\tilde{T}_{N;0})$  and  $\text{Var}(\tilde{T}_{N;0})$ . So, we can search for conditions under which

$$\mathbb{E}(\tilde{T}_{N;0}) \rightarrow \infty \text{ and } \text{Var}(\tilde{T}_{N;0}) / (\mathbb{E}(\tilde{T}_{N;0})^2) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

If this is the case,  $\tilde{T}_{N;0} / \mathbb{E}(\tilde{T}_{N;0}) \rightarrow 1$  as  $N \rightarrow \infty$  in probability, and  $\lfloor \mathbb{E}(\tilde{T}_{N;0}) \rfloor$  is a cutoff time for  $X$  started at 0. In this goal, from (54) we get  $\text{Var}(\tilde{T}_{N;0} / \mathbb{E}(\tilde{T}_{N;0})) \leq 1 / ((1 - t_1) \mathbb{E}(\tilde{T}_{N;0}))$ . Then,  $(1 - t_1) \mathbb{E}(\tilde{T}_{N;0}) \rightarrow \infty$  as  $N \rightarrow \infty$  is a sufficient condition for  $\text{Var}(\tilde{T}_{N;0} / \mathbb{E}(\tilde{T}_{N;0})) \rightarrow 0$ . See [5] for recent developments and precisions.

*Example:* Consider the Moran model with mutations, and put  $a := a_1 + a_2$ ,  $\bar{a} := 1 - a$ . From (29) the eigenvalues  $t_k$  verify:  $1 - t_k = \frac{k}{N} (a + \bar{a} \frac{k-1}{N})$ . Using the approximation

$$\mathbb{E}(\tilde{T}_{N;0}) \sim N \int_0^1 \frac{dx}{(x + 1/N)(a + \bar{a}x)} = \frac{N^2}{Na - \bar{a}} \left( \int_0^1 \frac{dx}{x + 1/N} - \bar{a} \int_0^1 \frac{dx}{a + \bar{a}x} \right),$$

we get

$$\mathbb{E}(\tilde{T}_{N;0}) \sim N (\log N + \log a) / a \text{ and } \text{Var}(\tilde{T}_{N;0}) \sim (N/a)^2$$

showing that  $\text{Var}(\tilde{T}_{N;0} / \mathbb{E}(\tilde{T}_{N;0})) \sim (\log N)^{-2} \rightarrow 0$ . The expected mixing time is  $\mathbb{E}(\tilde{T}_{N;0}) \sim N \log N / a$  whereas the spectral gap is  $1 - t_1 = a/N$ .  $\square$

In general, the values  $t_k$  are not known. So it would be helpful to compute differently the mean and the variance of the absorption time  $\tilde{T}_{N;0}$ . This is the goal of our next paragraph in the BD chain context.

*The mean and the variance of the absorption time.* Let us compute  $\mathbb{E}(\tilde{T}_{N;0})$  and  $\text{Var}(\tilde{T}_{N;0})$  by the usual methods. We introduce the following sequences of independent random variables:

$$(S_y : y = 0, \dots, N - 1) \text{ with distribution } \mathbb{P}(S_y = n) = \mathbb{P}_y(\tilde{T}_{y+1} = n) \quad \forall n \geq 0,$$

so  $S_y$  is a copy of the time spent in hitting  $y + 1$  when  $\tilde{X}$  starts from  $y$ . We also assume that the sequence  $(S_y : y = 0, \dots, N - 1)$  is independent of the Markov

chain  $\tilde{X}$ . Observe that

$$\mathbb{P}\left(\sum_{y=0}^{N-1} S_y = n\right) = \mathbb{P}(\tilde{T}_{N;0} = n) \quad \forall n \geq 0.$$

When the initial condition is  $\tilde{X}_0 = y$ , we have the representation

$$(54) \quad S_y \stackrel{d}{\sim} \mathbf{1}(\tilde{X} = y+1) + \mathbf{1}(\tilde{X} = y)(1 + S'_y) + \mathbf{1}(\tilde{X} = y-1)(1 + S_{y-1} + S''_y),$$

where  $S'_y$  and  $S''_y$  are independent copies of  $S_y$ , which are independent from  $\tilde{X}$  and from the whole sequence  $(S_y : y = 0, \dots, N-1)$ . By taking expected values we find the recurrence relation

$$(55) \quad \mathbb{E}(S_y) = \frac{1}{\tilde{p}_y} + \frac{\tilde{q}_y}{\tilde{p}_y} \mathbb{E}(S_{y-1}).$$

Since  $\mathbb{E}(S_0) = 1/\tilde{p}_0$  we get by iteration,

$$(56) \quad \mathbb{E}(S_y) = \sum_{l=0}^y \frac{1}{\tilde{p}_y} \prod_{r=l+1}^y \frac{\tilde{q}_r}{\tilde{p}_r}$$

and so the mean of the absorption time at  $N$  starting from 0 is,

$$\mathbb{E}(\tilde{T}_{N;0}) = \sum_{y=0}^{N-1} \left( \sum_{l=0}^y \frac{1}{\tilde{p}_y} \prod_{r=l+1}^y \frac{\tilde{q}_r}{\tilde{p}_r} \right).$$

Also from (54) we obtain

$$\begin{aligned} S_y^2 &\stackrel{d}{\sim} \mathbf{1}(\tilde{X} = y+1) + \mathbf{1}(\tilde{X} = y)(1 + 2S'_y + S'^2_y) \\ &\quad + \mathbf{1}(\tilde{X} = y-1)(1 + S^2_{y-1} + S''^2_y + 2S_{y-1} + 2S''_y + 2S_{y-1}2S''_y). \end{aligned}$$

Therefore

$$\mathbb{E}(S_y^2) = \frac{1}{\tilde{p}_y} + \frac{2\tilde{r}_y}{\tilde{p}_y} \mathbb{E}(S_y) + \frac{\tilde{q}_y}{\tilde{p}_y} \mathbb{E}(S_{y-1}^2) + \frac{2\tilde{q}_y}{\tilde{p}_y} (\mathbb{E}(S_{y-1}) + \mathbb{E}(S_y) + \mathbb{E}(S_{y-1})\mathbb{E}(S_y)).$$

From (55) we find that  $\text{Var}(S_y) = \mathbb{E}(S_y^2) - \mathbb{E}(S_y)^2$  verifies

$$\begin{aligned} \text{Var}(S_y) &= \frac{1}{\tilde{p}_y} + \frac{2\tilde{r}_y}{\tilde{p}_y} \mathbb{E}(S_y) + \frac{\tilde{q}_y}{\tilde{p}_y} \mathbb{E}(S_{y-1}^2) + \frac{2\tilde{q}_y}{\tilde{p}_y} (\mathbb{E}(S_{y-1}) + \mathbb{E}(S_y) + \mathbb{E}(S_{y-1})\mathbb{E}(S_y)) \\ &\quad - \frac{1}{\tilde{p}_y^2} - \frac{\tilde{q}_y^2}{\tilde{p}_y^2} \mathbb{E}(S_{y-1})^2 - \frac{2\tilde{q}_y}{\tilde{p}_y^2} \mathbb{E}(S_{y-1}). \end{aligned}$$

Therefore

$$(57) \quad \text{Var}(S_y) = \frac{\tilde{q}_y}{\tilde{p}_y} \text{Var}(S_{y-1}) + A_y,$$

where

$$\begin{aligned} A_y &= \frac{\tilde{p}_y - 1}{\tilde{p}_y^2} + \frac{2(1 - \tilde{p}_y)}{\tilde{p}_y} \mathbb{E}(S_y) + \frac{2\tilde{q}_y(\tilde{p}_y - 1)}{\tilde{p}_y^2} \mathbb{E}(S_{y-1}) + \frac{2\tilde{q}_y}{\tilde{p}_y} \mathbb{E}(S_{y-1})\mathbb{E}(S_y) \\ &\quad - \frac{\tilde{q}_y(\tilde{q}_y - \tilde{p}_y)}{\tilde{p}_y^2} \mathbb{E}(S_{y-1})^2. \end{aligned}$$

Observe that from (56) the coefficient  $A_y$  can be computed in terms of the parameters of the BD chain  $\tilde{X}$ . In particular  $A_0 = \text{Var}(S_0) = (1 - \tilde{p}_0)/\tilde{p}_0^2$ . From the recurrence formula (57) and the value for  $\text{Var}(S_0)$ , we find the explicit expression,

$$\text{Var}(S_y) = \sum_{l=0}^y A_l \prod_{s=l+1}^y \frac{\tilde{q}_s}{\tilde{p}_s}.$$

Therefore, by using independence, the variance of the hitting time of  $N$  starting from 0 is,

$$(58) \quad \text{Var}(\tilde{T}_{N;0}) = \sum_{y=0}^{N-1} \text{Var}(S_y) = \sum_{y=0}^{N-1} \left( \sum_{l=0}^y A_l \prod_{s=l+1}^y \frac{\tilde{q}_s}{\tilde{p}_s} \right),$$

which can be explicitly computed simply in terms of the transition parameters of the BD chain  $\tilde{X}$ .

**Remark 6.** *Even if the expressions of the mean and the variance in (56) and (58) do not require the knowledge of the spectrum, they are difficult to handle in terms of the parameters, so in general we are not able to use them to describe the behavior of the mean and the variance when  $N$  is large.*

## 6. THE HYPERGEOMETRIC DUAL

For  $I = \{0, \dots, N\}$  let us suggest other potentially interesting examples of non-singular duality kernels  $H$  for which there exists a column of  $H$  which is constant so that Proposition 3 can be applied. For these examples,  $H^{-1}$  is known explicitly which turns out to be useful to decide whether for a given irreducible stochastic kernel the  $H$ -dual defines a substochastic matrix. If this occurs, the problem of interpreting the intertwining chain given by Theorem 2, remains a challenging problem for each specific case.

The Vandermonde dual and the hypergeometric duals that were first introduced in [18] in the context of neutral population genetics. In this context and also in nonneutral situations, the hypergeometric kernel plays a central role.

- **Vandermonde.**  $H(x, y) = (x/N)^y$ . In this case the column 0-th is constant.
- **Hypergeometric.**  $H(x, y) = \binom{N-x}{y} / \binom{N}{y}$ . In this case  $H = H'$ ,  $H$  is upper-left triangular, and (51) is verified with  $\hat{a} = 0$  and  $d = N$ ,

$$(59) \quad H\mathbf{e}_0 = \mathbf{1} \quad \text{and} \quad \mathbf{e}'_N H = \mathbf{e}'_0.$$

Let us comment on this choice of  $H$ .

When  $P$  is given by the reversible Moran model with completely monotone non-neutrality bias mechanism, the  $H$ -dual kernel  $\hat{P}$  can be interpreted in terms of a multi-sex backward process akin to the coalescent. As shown in [11], for the Moran model with bias  $p$  satisfying  $p(0) \in (0, 1)$  we have:  $\hat{P}\mathbf{1}(0) = 1$  and  $0 < \hat{P}\mathbf{1}(x) = 1 - \frac{x}{N}p(0) < 1$  for all  $x \neq 0$ . From  $p(0) \neq 0$ , all the states but  $a = 0$  of  $\hat{P}$  are mass-defective. The intertwining matrix  $\tilde{P}$  is the transition kernel of a skip-free to the left BD chain that can easily be obtained from [11], and 0 is the unique



absorbing state for  $\tilde{X}$ . The relation (59) fulfills the hypotheses of Proposition 14 with  $d = N$ , then in the above Moran model the sharpness property is satisfied.

On the other hand the link matrix  $\Lambda$  is upper-left triangular, stochastic and irreducible. Then, there exists a probability vector  $\pi_\Lambda$  that verifies  $\pi'_\Lambda \Lambda = \pi'_\Lambda$ , so  $\pi_0 = \tilde{\pi}_0 = \pi_\Lambda$  is an admissible initial condition for  $X$  and  $\tilde{X}$ . Also, from  $\mathbf{e}'_N \Lambda = \mathbf{e}'_0$  we get that another admissible initial condition is  $\pi_0 = \delta_0$  and  $\tilde{\pi}_0 = \delta_N$ . We can summarize this discussion in the following result.

**Corollary 16.** *Let  $X$  be the Moran chain with transition matrix  $P$  fulfilling the above monotonicity conditions on  $p$ . Then, the construction of the intertwining kernel  $\tilde{P}$  in Theorem 2 starting from the hypergeometric dual  $H$  can be done and the Markov chain  $\tilde{X}$  is well-defined. The absorbing state of  $\tilde{X}$  is 0, the process  $\tilde{X}$  is a sharp dual of  $X$  and  $\tilde{X}$  can be started at  $N$  while  $X$  starts at 0.*

Hence, the time  $\tilde{T}_{0;N}$  that  $\tilde{X}$  reaches 0 when it starts from  $N$ , is the stochastically smallest time at which  $X_{\tilde{T}_{0;N}} \stackrel{d}{\sim} \pi$  given  $X_0 = 0$  and  $\tilde{X}_0 = N$ . We point out that the time  $\tilde{T}_{0;N}$  that  $\tilde{X}$  reaches 0 when it starts from  $N$ , is distributed like the time  $\tilde{T}_{N;0}$  to reach  $N$  starting from 0 of the Siegmund intertwining BD chain to the Moran model, namely like (52). This is in accordance with Theorem 1.2 of [8], stating that for a skip-free to the right Markov chain absorbed at  $N$ , the law of the time it takes to hit  $N$  starting from 0 is given by (52). This result can be transferred to our skip-free to the left BD chain case, while exchanging the boundaries  $\{0, N\}$ .

Let us finally consider the Wright-Fisher transition matrix  $P$  given by

$$P(x, y) = \binom{N}{y} p \left( \frac{x}{N} \right)^y \left( 1 - p \left( \frac{x}{N} \right) \right)^{N-y}.$$

whose bias  $p(u)$  is again a completely monotone function, satisfying  $p(0) > 0$ . This process is not reversible, nor is it in the BD class. However, using the hypergeometric duality kernel it was shown in [10] that the  $H$ -dual  $\tilde{P}$  to  $P$  in (2) defines a substochastic matrix. From Theorem 2 we conclude that the corresponding  $\tilde{P}$  is  $\Lambda$ -linked to  $P$ .  $\square$ .

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