Planar graphs without adjacent cycles of length at most seven are 3-colorable

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Abstract

We prove that every planar graph in which no i-cycle is adjacent to a j-cycle whenever $3 \leq i \leq j \leq 7$ is 3-colorable and pose some related problems on the 3-colorability of planar graphs.

1 Introduction

In 1976, Appel and Haken proved that every planar graph is 4-colorable [3, 4], and as early as 1959, Grötzsch [15] proved that every planar graph without 3-cycles is 3-colorable. As proved by Garey, Johnson and Stockmeyer [14], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, some sufficient conditions for planar graphs to be 3-colorable were stated. In 1976, Steinberg [19] raised the following:

Steinberg's Conjecture '76 *Every planar graph without 4- and 5-cycles is 3-colorable.*

In 1969, Havel [16] posed the following problem:

Havel's Problem '69 *Does there exist a constant C such that every planar graph with the minimum distance between triangles at least C is 3-colorable?*

Havel [12, 13] proved that if C exists, then $C \geq 2$, which was improved to $C \geq 4$ by Aksionov and Mel'nikov [2] and, independently, by Steinberg (see [2]).

These two challenging problems remain open. In 1991, Erdös suggested the following *relaxation of Steinberg's conjecture*: Determine the smallest value of k, if it exists, such that every planar graph without cycles of length from 4 to k is 3-colorable. Abbott and Zhou [1] proved that such a k does exist, with $k \leq 11$. This result was later on improved to $k \leq 10$ by Borodin [5] and to $k \leq 9$ by Borodin [6] and Sanders and Zhao [18]. The best known bound for such a k is 7, and it was proved by Borodin, Glebov, Raspaud, and Salavatipour [10].

At the crossroad of Havel's and Steinberg's problems, Borodin and Raspaud [11] proved that every planar graph without 3-cycles at distance less than four and without 5-cycles is 3-colorable. (The distance here was improved to three by Borodin and Glebov [7] and Xu [21], and recently it was decreased to two by Borodin and Glebov [8].) Furthermore, Borodin and Raspaud [11] proposed the following conjecture:

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Strong Bordeaux Conjecture '03 *Every planar graph without 5-cycles and without adjacent triangles is 3 colorable.*

By adjacent cycles we mean those with at least one edge in common.

Obviously, this conjecture implies *Steinberg's Conjecture*. In [9], Borodin, Glebov, Jensen and Raspaud considered the adjacency between cycles in planar graphs, where all lengths of cycles are authorized; in a sense, this kind of problems is related to *Havel's problem*. More specifically, they proved that every planar graph without triangles adjacent to cycles of length from 3 to 9 is 3-colorable and proposed the following conjecture:

Novosibirsk 3-Color Conjecture '06 *Every planar graph without 3-cycles adjacent to cycles of length 3 or 5 is 3-colorable.*

Clearly, this one implies both the *Strong Bordeaux Conjecture* and *Steinberg's Conjecture*.

Many other sufficient conditions for the 3-colorability of planar graphs were proposed in which cycles with lengths from specific sets are forbidden (for example, see [20]). In this note we consider an approach based on the adjacencies of cycles. Let us start with some definitions:

GA - Graph of Non-Adjacencies A *graph of non-adjacencies* is one whose vertices are labelled by integers greater than two and each integer appears at most once. Given a graph $G_{\mathcal{A}}$ of non-adjacencies, we say that a graph G *respects* $G_{\mathcal{A}}$ if no two cycles of lengths i and j are adjacent in G if the vertices labelled with i and j are adjacent in $G_{\mathcal{A}}$.

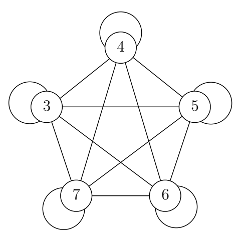

Figure 1: A graph of non-adjacencies.

Example. Let $G_{\mathcal{A}}$ be the graph depicted by Figure 1. A graph G respecting $G_{\mathcal{A}}$ is a graph in which there is no i-cycle adjacent to a j-cycle for $3 \le i \le j \le 7$.

We propose the following natural general question:

Problem 1 *Under which conditions of adjacencies is a planar graph 3-colorable?*

Our main result in this note (proved in Section 2) is that each planar graph respecting the graph $G_{\mathcal{A}}$ depicted by Figure 1 is 3-colorable.

Theorem 1 *Every planar graph in which no i-cycle is adjacent to a j-cycle whenever $3 \le i \le j \le 7$ is 3-colorable.*

Clearly, Theorem 1 is an extension of the above mentioned result by Borodin, Glebov, Raspaud, and Salavatipour [10].

The model of non-adjacencies can be made more precise. Define a function f on the edges of $G_{\mathcal{A}}$ by putting:

- $f(ij) = -1$ if the cycles of lengths i and j should not be adjacent in G,
- $f(ij) = 0$ if the cycles of lengths i and j should not be intersecting in G,
- $f(ij) = k$ if the distance between cycles of lengths i and j in G should be greater than k (the distance between two cycles C_1 and C_2 is defined as the length of a shortest path between two vertices of C_1 and C_2).

Montassier, Raspaud, Wang and Wang [17] suggested a *relaxation of Havel's problem* and proved

Theorem 2 *1. Every planar graph in which the cycles of length 3, 4, 5, and 6 are at distance at least 3 from each other is 3-colorable.*

2. Every planar graph in which the cycles of length 3, 4, and 5 are at distance at least 4 from each other is 3-colorable.

Note that the graphs studied in Theorem 2 respect the graphs of non-adjacencies depicted by Figure 2.

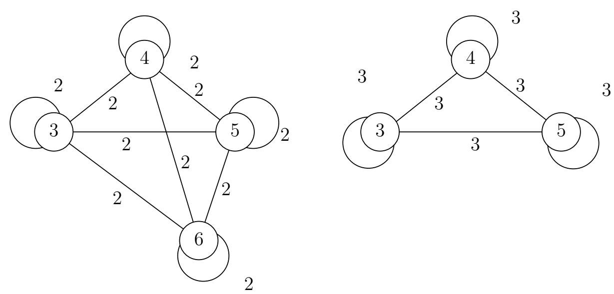

Figure 2: Some $G_{\mathcal{A}}$'s.

We conclude with some specific problems; see Figure 3.

Problem 2 *Let G be a planar graph respecting $G_{(A)}$ depicted by Figure 3. Let f_0 be an i-face with $3 \leq i \leq 11$. Prove that every proper 3-coloring of $G[V(f_0)]$ can be extended to the whole graph.*

Problem 3 *Let G be a planar graph respecting $G_{(B)}$ (resp. $G_{(C)}$, $G_{(D)}$, $G_{(E)}$) depicted by Figure 3. Prove that G is 3-colorable.*

The result on planar graphs respecting $G_{(C)}$ would imply *Steinberg's Conjecture*. The problem on planar graphs respecting $G_{(D)}$ is the *Novosibirsk 3-Color Conjecture*. Finally, the problem on planar graphs respecting $G_{(E)}$ for any finite k would provide the answer to *Havel's Problem*. The first attempt could be to study planar graphs respecting $G_{(B)}$ or subgraphs of $G_{\mathcal{A}}$ in Figure 1.

2 Proof of Theorem 1

Our proof is based on the following coloring extension lemma:

Lemma 1 *Suppose G is a connected planar graph respecting $G_{\mathcal{A}}$ depicted by Figure 1 and f_0 is an i-face with $3 \leq i \leq 11$; then every proper 3-coloring of $G[V(f_0)]$ can be extended to the whole G.*

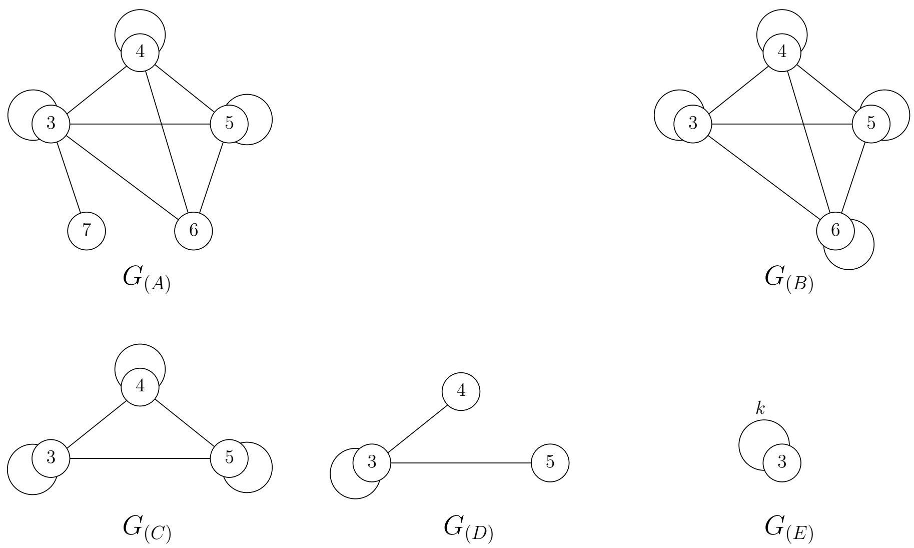

Figure 3: Some $G_{\mathcal{A}}$'s.

It is easy to see that Lemma 1 implies Theorem 1. Indeed, let G be a minimal counterexample to Theorem 1; clearly, G is connected. If G contains a triangle C_3, we fix the colors of the vertices of C_3 and apply Lemma 1 to $G \setminus int(C_3)$ and to $G \setminus out(C_3)$. If G does not contain triangles, then G is 3-colorable by Grötzsch's Theorem [15].

So, it suffices to prove Lemma 1. Note that our proof of Lemma 1 is built on the following result by Borodin, Glebov, Raspaud, and Salavatipour [10]:

Theorem 3 *Every proper 3-coloring of the vertices of any face of length 8 to 11 in a connected planar graph without cycles of length 4 to 7 can be extended to a proper 3-coloring of the whole graph.*

Let $G = (V(G), E(G), F(G))$ be a plane graph, where $V(G)$, $E(G)$ and $F(G)$ denote the sets of vertices, edges and faces of G, respectively. The neighbour set and the degree of a vertex v are denoted by $N(v)$ and $d(v)$, respectively. Let f be a face of G. We use $b(f)$, $V(f)$ to denote the boundary of f, the set of vertices on $b(f)$, respectively. A k-vertex (resp. ${}^{\geq}k$-vertex, ${}^{\leq}k$-vertex) is a vertex of degree k (resp. $\geq k$, $\leq k$). The same notation is used for faces and cycles: k-face, ${}^{\geq}k$-face, ${}^{\leq}k$-faces are faces of length k, $\geq k$, $\leq k$. Let C be a cycle of G. By $int(C)$ and $ext(C)$ denote the sets of vertices located inside and outside C, respectively. C is said to be *a separating cycle* if both $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$. Let $c_i(G)$ be the number of cycles of length i in G. Let C be a cycle of G, and let u and v be two vertices on C. We use $C[u, v]$ to denote the path of C clockwisely from u to v , and let $C(u, v) = C[u, v] \setminus \{u, v\}$.

By $\mathcal{G}$ denote the set of plane graphs that respects $G_{\mathcal{A}}$ depicted in Figure 1.

Assume that G is a counterexample to Lemma 1 with:

1. $c(G) = c_4(G) + c_5(G) + c_6(G) + c_7(G)$ as small as possible, and
2. $\sigma(G) = |V(G)| + |E(G)|$ minimum under the previous condition.

Without loss of generality, assume that the unbounded face f_0 is an i-face with $3 \leq i \leq 11$ such that a 3-coloring ϕ of $G[V(f_0)]$ cannot be extended to G. Let $C_0 = b(f_0)$. All face different from f_0 are called *internal*.

Claim 1 *G is 2-connected; hence, the boundary of every face is a cycle.*

Proof
Observe first that, by the minimality of G, there is no cut-vertex in $V(f_0)$. Now assume that B is a pendant block with the cut-vertex $v \in V(G) \setminus V(f_0)$. We first extend ϕ to $G \setminus (B \setminus v)$, then we color B with 3 colors by the minimality of G or Grötzsch's Theorem, permute the colors if necessary, and finally get an extension of ϕ to G. $\square$

Claim 2 $\forall v \in int(C_0), d(v) \geq 3$.

Proof
Let v be a $^{\leq}2$-vertex with $v \in int(C_0)$. We can first extend ϕ to $G \setminus v$ and then color v. $\square$

Claim 3 *G contains no separating k-cycles with $3 \leq k \leq 11$.*

Proof
Let C be a separating cycle of length from 3 to 11. By the minimality of G, we can extend ϕ to $G \setminus int(C)$. Then we extend the 3-coloring of $G[V(C)]$ to $G \setminus out(C)$ using the minimality of G. $\square$

Claim 4 *$G[V(f_0)]$ is a chordless cycle.*

Proof
Let uv be a chord of C_0. Then by the minimality of G, we can extend ϕ to $G \setminus uv$ and so to G. $\square$

Claim 5 $|f_0| \neq 4, 5, 6, 7$.

Proof
Let $C_0 = x_1x_2 \ldots x_k$ with $4 \leq k \leq 7$. Let G' be the graph obtained from G by adding $8 - k$ 2-vertices on the edge x_1x_2. Then observe that $c(G') < c(G)$ and $G' \in \mathcal{G}$. By choosing some good colors to the added vertices, we can extend the coloring of the outer face of G' to the whole graph G' by the minimality of G. This yields a proper 3-coloring of G, a contradiction. $\square$

Now we show that G contains no internal k-faces with $4 \leq k \leq 7$. Due to Claim 3 and the cycles adjacencies conditions, every k-cycle with $4 \leq k \leq 7$ bounds a face. This will show that G contains no k-cycles with $4 \leq k \leq 7$. Finally, Theorem 3 will complete the proof of Lemma 1.

Claim 6 *G contains no internal 7-faces.*

Proof
Let $f = x_1x_2x_3x_4x_5x_6x_7$ be an internal 7-face and $C_f = b(f)$.

Observation 1 *Let u, v two vertices of $V(f)$. Let $P_{u,v}$ be a path linking u and v such that $P_{u,v} \cap V(f) = \{u, v\}$ and $C_f(u, v) \in int(P_{u,v} \cup C_f[v, u])$. Let $P_{v,u}$ be a path linking u and v such that $P_{v,u} \cap V(f) = \{u, v\}$ and $C_f(v, u) \in int(P_{v,u} \cup C_f[u, v])$ (see Figure 4). It may happen that $P_{u,v}$ or/and $P_{v,u}$ does not exist.*

By the cycles adjacencies conditions or by Claim 3, we are sure that:

- *In Case (1) depicted by Figure 4, the path $P_{u,v}$ (resp. $P_{v,u}$) has at least 8 vertices (resp. 8 vertices) since there is no 7-cycle adjacent to $^{\leq}7$-cycles.*
- *In Case (2) depicted by Figure 4, the path $P_{u,v}$ (resp. $P_{v,u}$) has at least 8 vertices (resp. 11 vertices) since otherwise $P_{u,v} \cup C_f[v, u]$ (resp. $C_f[u, v] \cup P_{v,u}$) is a separating $^{\leq}11$-cycle.*
- *In Case (3) depicted by Figure 4, the path $P_{u,v}$ (resp. $P_{v,u}$) has at least 9 vertices (resp. 10 vertices) since otherwise $P_{u,v} \cup C_f[v, u]$ (resp. $C_f[u, v] \cup P_{v,u}$) is a separating $^{\leq}11$-cycle.*

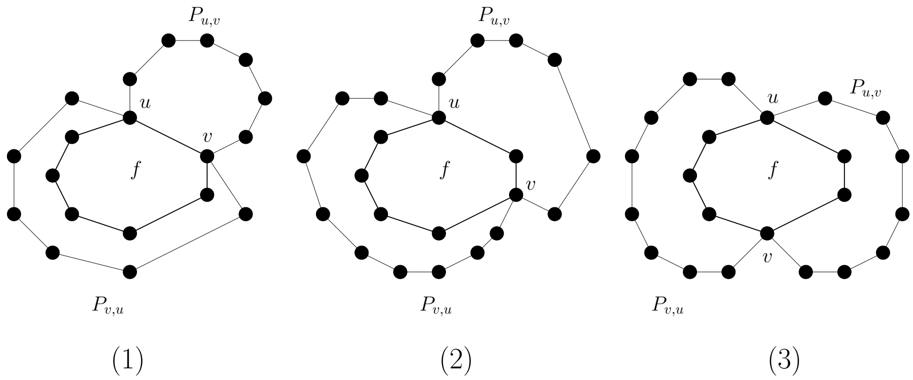

Figure 4: The paths $P_{u,v}$ and $P_{v,u}$.

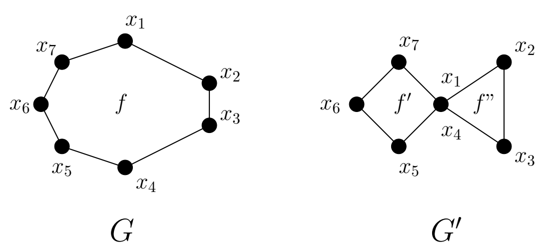

Figure 5: The identification of x_1 with x_4.

Let G' be the graph obtained from G by identifying x_1 with x_4, see Figure 5.

We will show that this identification does not create $^{\leq}$7-cycles, except $C_{f'} = x_1x_5x_6x_7$ and $C_{f''} = x_1x_2x_3$, which are a 4-cycle and a 3-cycle, respectively.

Suppose to the contrary that C^* is a $^{\leq}$7-cycle in G' created by the identification of x_1 and x_4 in G, different from $C_{f'}$ and $C_{f''}$.

By $l(x, y)$ denote the distance between the vertices x and y in $(V(G), E(G) \setminus \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_1\})$. The cycle C^* must go through at least two vertices of $x_1, \ldots, x_7$ (otherwise, its length cannot decrease by the identification). By Observation 1, the following table gives the length of C^* going through the vertices x and y of C_f:

$x, y \in C_f$	$l(x,y)$	$\|C^*\|$
x_1, x_2	7	8
x_1, x_3	7	8
x_1, x_4	8	8
x_1, x_5	8	9
x_1, x_6	7	9
x_1, x_7	7	8
x_2, x_3	7	8
x_2, x_4	7	8
x_2, x_5	8	10
x_2, x_6	8	11
x_2, x_7	7	9
x_3, x_4	7	8
x_3, x_5	7	9
x_3, x_6	8	11
x_3, x_7	8	10
x_4, x_5	7	8
x_4, x_6	7	9
x_4, x_7	8	9
x_5, x_6	7	8
x_5, x_7	7	9
x_6, x_7	7	8

Hence, such a cycle C^* cannot exist. The identification does not create $^{\leq}$7-cycles; moreover, by the cycles adjacencies conditions, f is not adjacent to $^{\leq}$7-cycles ; so it is for f' and f". It follows that the identification does not create a $^{\leq}$7-cycle adjacent to a $^{\leq}$7-cycle. This implies that $G' \in \mathcal{G}$.

We now show that x_1 and x_4 can be choosen so that the identification does not damage ϕ, i.e. we can choose x_1 and x_4 such that $|N(x_1) \cap C_0| + |N(x_4) \cap C_0| \leq 2$ (otherwise, the pre-coloring ϕ in G' might be not proper, or not defined at all).

Observation 2 *If u is an inner vertex, then $|N(u) \cap C_0| \leq 1$.*

Proof

Let u be an inner vertex; then $|N(u) \cap C_0| \leq 2$ by the cycles adjacencies conditions. Suppose that $|N(u) \cap C_0| = 2$ and assume that $N(u) \cap C_0 = \{x, y\}$. Then $C_0[u, v] \cup vxu$ or $C_0[v, u] \cup uxv$ is a separating $^{\leq}$11-cycle since $d(u) \geq 3$ and u has a neighbor not in C_0, a contradiction.

□

Hence, if $|C_0 \cap C_f| \leq 3$, we can choose x_1 and x_4 such that $|N(x_1) \cap C_0| + |N(x_4) \cap C_0| \leq 2$.

Since C_0 has no chord and $|f_0| \neq 7$, it follows that $|C_0 \cap C_f| \leq 5$ by the previous observation.

Consider the case $|C_0 \cap C_f| = 5$; now $C_0 \cap C_f$ is a set of consecutive vertices on C_0. Assume that $C_0 \cap C_f = \{x_1, x_4, x_5, x_6, x_7\}$; then $C_0[x_1, x_4] \cup x_1x_2x_3x_4$ is a separating $^{\leq}$11-cycle, a contradiction.

Now consider the case $|C_0 \cap C_f| = 4$. Again, $C_0 \cap C_f$ is a set of consecutive vertices on C_0. Assume that $C_0 \cap C_f = \{x_1, x_2, x_6, x_7\}$; then by the cycles adjacencies conditions, x_4 has no neighbor on C_0. Hence $|N(x_1) \cap C_0| + |N(x_4) \cap C_0| \leq 2$.

So we can choose x_1 and x_4 such that ϕ is not damaged. Finally, observe that $c(G') = c(G)$ and $\sigma(G') < \sigma(G)$. Hence, using the minimality of G, we can extend ϕ to the whole graph G' and so to G.

□

Claim 7 *G contains no internal k-faces, with $4 \leq k \leq 6$.*

The proof of Claim 7 is similar to that of Claim 6 but easier and is left to the reader.

This completes the proof of Theorem 1.

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