



HAL
open science

Notes on Austin's multiple ergodic theorem

Thierry de La Rue

► **To cite this version:**

| Thierry de La Rue. Notes on Austin's multiple ergodic theorem. 2009. hal-00400975v4

HAL Id: hal-00400975

<https://hal.science/hal-00400975v4>

Preprint submitted on 16 Oct 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Notes on Austin's multiple ergodic theorem

Thierry de la Rue

Abstract

The purpose of this note is to present my understanding of Tim Austin's proof of the multiple ergodic theorem for commuting transformations, emphasizing on the use of joinings, extensions and factors. The existence of a sated extension, which is a key argument in the proof, is presented in a general context.

1 Introduction

The norm convergence of multiple ergodic averages for commuting transformations (Theorem 1.1 below) was first proved in 2008 by Terence Tao [8]. We intend to present here the quite different proof proposed by Tim Austin [1] using the machinery of joinings, extensions and factors. This text is written after Austin's talk at the conference *Dynamical Systems and Randomness*, (held in Paris, Institut Henri Poincaré, May 2009), and a short conversation with him following his talk. The major part is, as far as I understand it, quite faithful to Austin's original proof. The only slightly original contribution is the proof of the existence of a sated extension, which is presented in a general context.

Theorem 1.1. *Let $d \geq 1$, and T_1, \dots, T_d be d commuting, measure-preserving invertible transformations of the standard Borel probability space (X, \mathcal{A}, μ) . Then for any choice of $f_1, \dots, f_d \in L^\infty(\mu)$, the multiple ergodic averages*

$$\frac{1}{N} \sum_{n=1}^N f_1 \circ T_1^n \cdots f_d \circ T_d^n$$

converge in $L^2(\mu)$ as $N \rightarrow \infty$.

The strategy

The case $d = 1$ corresponds to the standard ergodic theorem of Von Neumann, and in this case the limit is clearly identified, as the orthogonal projection of the function f_1 on the subspace of L^2 -functions which are measurable with respect to the factor σ -algebra

$$\mathcal{I}^{T_1} := \left\{ A \in \mathcal{A} : \mu(A \Delta T_1^{-1} A) = 0 \right\}.$$

(This factor σ -algebra is called the *isotropy factor* by Austin; isotropy factors play a crucial role here and we will use the above notation for several transformations in the sequel.)

The proof for $d \geq 2$ is presented by induction on d : We assume that $d \geq 2$ is such that Theorem 1.1 has already been proved up to the case of $d - 1$ commuting transformations.

Then we identify a simple class \mathcal{C} of systems $(X, \mathcal{A}, \mu, T_1, \dots, T_d)$ with d commuting transformations for which the desired result is easily deduced from the $(d - 1)$ -case. Next step consists in the introduction of a larger class of systems, the so-called \mathcal{C} -sated systems. (Note that the notion of *satedness* is not explicit in [1]: It has been formalized in a subsequent work by Austin [2] dealing also with some polynomial sequences.) The \mathcal{C} -sated systems are characterized by a quite simple structure of their joinings with any \mathcal{C} -system, and this enables us to prove for them the theorem, using the induction hypothesis and a version of Van der Corput lemma. Finally, and this is the point where the machinery of joinings plays its crucial role, we show that any system possesses an extension which is \mathcal{C} -sated. Since we know that the theorem holds for \mathcal{C} -sated systems, it obviously holds for all their factors, hence for all systems.

For the sake of simplicity, we first present the induction step passing from one to two commuting transformations, admitting the existence of a \mathcal{C} -sated extension for any system. Then we will see how the same argument can be generalized to pass from $d - 1$ to d transformations. Finally, in a completely independent section, we prove a general result on joinings showing why any system admits a \mathcal{C} -sated extension, achieving the proof of Theorem 1.1.

2 The case of two commuting transformations

In all this section, we assume $d = 2$ and we present the argument showing how the theorem for two commuting transformations can be proved, using the well-known result in the case of a single transformation.

2.1 \mathcal{C} -systems

We first observe that there are two very simple cases in which the convergence in L^2 of the ergodic averages

$$\frac{1}{N} \sum_{n=1}^N f_1 \circ T_1^n f_2 \circ T_2^n \quad (1)$$

is a trivial consequence of the single-transformation case:

- If $T_1 = \text{Id}$, which amounts to saying that the isotropy factor \mathcal{I}^{T_1} is the whole σ -algebra \mathcal{A} : The above ergodic average reduces to f_1 times an ergodic average for the single transformation T_2 . Obviously, it is enough that f_1 be \mathcal{I}^{T_1} -measurable to get this reduction.
- If $T_1 = T_2$, in other words if the isotropy factor $\mathcal{I}^{T_2 T_1^{-1}}$ is the whole σ -algebra \mathcal{A} : Then (1) reduces to an ergodic average for the product $f_1 f_2$ and the single transformation $T_1 = T_2$. Note that this reduction holds as soon as f_1 is measurable with respect to $\mathcal{I}^{T_2 T_1^{-1}}$.

Now, we introduce the class of \mathcal{C} -systems as the class of systems $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, T_2)$ for which

$$\mathcal{A} = \mathcal{I}^{T_1} \vee \mathcal{I}^{T_2 T_1^{-1}}.$$

In other words, \mathbf{X} is a \mathcal{C} -system if it is isomorphic to a joining of two systems of the form $\mathbf{X}_1 = (X_1, \mathcal{A}_1, \mu_1, \text{Id}, S)$ and $\mathbf{X}_2 = (X_2, \mathcal{A}_2, \mu_2, T, T)$. In a system of the class \mathcal{C} , any bounded measurable function f_1 can be arbitrarily well approximated in L^2 by a finite sum of products

of the form gh , where g is \mathcal{G}^{T_1} -measurable and h is $\mathcal{G}^{T_2 T_1^{-1}}$ -measurable. For each such term gh , we can simultaneously apply the two reductions explained above, and we get that the L^2 -convergence of the ergodic averages (1) holds in any \mathcal{C} -system.

2.2 \mathcal{C} -sated systems

2.2.1 Looking for characteristic factors

In any system $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, T_2)$, it is now natural to consider the factor σ -algebra

$$\mathbf{X}_{\mathcal{C}} := \mathcal{G}^{T_1} \vee \mathcal{G}^{T_2 T_1^{-1}}.$$

Considering the action of T_1, T_2 on this factor, we obviously get a \mathcal{C} -system which we also denote by $\mathbf{X}_{\mathcal{C}}$. It is straightforward to check that, in fact, this factor is the largest factor in \mathbf{X} (in the sense of inclusion of σ -algebras) on which the action of the transformation gives rise to a \mathcal{C} -system: We call it the *largest \mathcal{C} -factor of \mathbf{X}* .

Observe that if f_1 is measurable with respect to $\mathbf{X}_{\mathcal{C}}$, the same argument as above proves the convergence in L^2 of the ergodic averages (1). Now, we are looking for simple conditions on \mathbf{X} ensuring that, when studying the convergence of these ergodic averages, we can replace f_1 by its projection $\mathbb{E}[f_1 | \mathbf{X}_{\mathcal{C}}]$, which would immediately lead to the desired conclusion. In other words, we are looking for conditions implying

$$\left\| \frac{1}{N} \sum_{n=1}^N f_1 \circ T_1^n f_2 \circ T_2^n \right\|_{L^2} \xrightarrow{N \rightarrow \infty} 0 \quad \text{as soon as } \mathbb{E}[f_1 | \mathbf{X}_{\mathcal{C}}] = 0. \quad (2)$$

(Although we shall not explicitly use here the notion of *characteristic factors*, we can note that the above condition is equivalent to “ $(\mathbf{X}_{\mathcal{C}}, \mathcal{A})$ is a pair of characteristic factors for the convergence of the ergodic averages (1)”, see Definition 4.1 in [1].)

2.2.2 Van der Corput lemma

To obtain the convergence to 0 in (2), we will make use of the following lemma.

Lemma 2.1 (Van der Corput). *Let (u_n) be a bounded sequence in a Hilbert space. If*

$$\lim_{H \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \frac{1}{N} \sum_{n=1}^N \langle u_n u_{n+h} \rangle = 0,$$

then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N u_n \right\| = 0.$$

For the sake of completeness, a proof of this lemma is included in Annex A.

2.2.3 A sufficient condition for the convergence

In view of Lemma 2.1, we are led to study the expression

$$\frac{1}{H} \sum_{h=1}^H \frac{1}{N} \sum_{n=1}^N \int_X f_1 \circ T_1^n f_2 \circ T_2^n \overline{f_1} \circ T_1^{n+h} \overline{f_2} \circ T_2^{n+h} d\mu.$$

Using the invariance of μ with respect to T_1 , we can rewrite each integral in the form

$$\int_X f_1 \overline{f_1} \circ T_1^h (f_2 \overline{f_2} \circ T_2^h) \circ (T_2 T_1^{-1})^n d\mu.$$

For each fixed h , the usual ergodic theorem for the single transformation $T_2 T_1^{-1}$ gives

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int_X f_1 \overline{f_1} \circ T_1^h (f_2 \overline{f_2} \circ T_2^h) \circ (T_2 T_1^{-1})^n d\mu \\ = \int_X f_1 \overline{f_1} \circ T_1^h \mathbb{E} \left[f_2 \overline{f_2} \circ T_2^h \mid \mathcal{J}^{T_2 T_1^{-1}} \right] d\mu. \end{aligned}$$

Now, it is convenient to view the latter integral as

$$\int_{X \times X} f_1(x_1) \overline{f_1(T_1^h x_1)} f_2(x_2) \overline{f_2(T_2^h x_2)} d(\mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu)(x_1, x_2), \quad (3)$$

where $\mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu$ denotes the relatively independent self-joining of \mathbf{X} over its factor σ -algebra $\mathcal{J}^{T_2 T_1^{-1}}$. Remark that this probability distribution is invariant under the action of the transformation $\widetilde{T}_1 := T_1 \otimes T_2 : (x_1, x_2) \mapsto (T_1 x_1, T_2 x_2)$. Indeed, if ϕ_1 and ϕ_2 are bounded measurable functions on X , we have

$$\begin{aligned} & \int_{X \times X} \phi_1(T_1 x_1) \phi_2(T_2 x_2) d(\mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu)(x_1, x_2) \\ &= \int_X \mathbb{E} \left[\phi_1 \circ T_1 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] \mathbb{E} \left[\phi_2 \circ T_2 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] d\mu \\ &= \int_X \mathbb{E} \left[\phi_1 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] \circ T_1 \mathbb{E} \left[\phi_2 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] \circ T_2 d\mu(x) \quad (\text{because } \mathcal{J}^{T_2 T_1^{-1}} \text{ is invariant} \\ & \quad \text{by both transformations}) \\ &= \int_X \mathbb{E} \left[\phi_1 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] \circ T_1 \mathbb{E} \left[\phi_2 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] \circ T_1 d\mu(x) \quad (\text{since on } \mathcal{J}^{T_2 T_1^{-1}}, T_1 \text{ and } T_2 \\ & \quad \text{coincide}) \\ &= \int_X \mathbb{E} \left[\phi_1 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] \mathbb{E} \left[\phi_2 \mid \mathcal{J}^{T_2 T_1^{-1}} \right] d\mu(x) \\ &= \int_{X \times X} \phi_1(x_1) \phi_2(x_2) d(\mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu)(x_1, x_2) \end{aligned}$$

Of course, $\mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu$ is also invariant by the transformation $\widetilde{T}_2 := T_2 \otimes T_2$. This gives us a big system with two commuting transformations $\widetilde{\mathbf{X}} := \left(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu, \widetilde{T}_1, \widetilde{T}_2 \right)$. Observe that our original system \mathbf{X} is a factor of $\widetilde{\mathbf{X}}$, which is obtained by considering only the first coordinate.

We have to average (3) over h . Writing this average in $\widetilde{\mathbf{X}}$ gives

$$\int_{X \times X} f_1 \otimes f_2(x_1, x_2) \frac{1}{H} \sum_{h=1}^H \overline{f_1 \otimes f_2} \left(\widetilde{T}_1^h(x_1, x_2) \right) d(\mu \otimes_{\mathcal{J}^{T_2 T_1^{-1}}} \mu)(x_1, x_2).$$

Applying the usual ergodic theorem for $\widetilde{T}_1 = T_1 \otimes T_2$ in $\widetilde{\mathbf{X}}$, we see that the latter expression converges, as $H \rightarrow \infty$, to

$$\int_{X \times X} f_1 \otimes f_2 \mathbb{E}_{\widetilde{\mathbf{X}}} \left[\overline{f_1 \otimes f_2} \mid \mathcal{G}^{\widetilde{T}_1} \right] d(\mu \otimes_{\mathcal{G}^{T_2 T_1^{-1}}} \mu).$$

An obvious sufficient condition for this limit to vanish is $\mathbb{E}_{\widetilde{\mathbf{X}}} [f_1 \otimes f_2 \mid \mathcal{G}^{T_1 \otimes T_2}] = 0$. The following proposition links this condition with the largest \mathcal{C} -factor $\widetilde{\mathbf{X}}_{\mathcal{C}}$ of $\widetilde{\mathbf{X}}$.

Proposition 2.2. *If $\mathbb{E}_{\widetilde{\mathbf{X}}} [f_1(x_1) \mid \widetilde{\mathbf{X}}_{\mathcal{C}}] = 0$, then $\mathbb{E}_{\widetilde{\mathbf{X}}} [f_1 \otimes f_2 \mid \mathcal{G}^{T_1 \otimes T_2}] = 0$.*

Proof. Observe that $\mathcal{G}^{\widetilde{T}_1}$ is a \mathcal{C} -factor of $\widetilde{\mathbf{X}}$, hence we have $\mathcal{G}^{\widetilde{T}_1} \subset \widetilde{\mathbf{X}}_{\mathcal{C}}$ by definition of the maximal \mathcal{C} -factor. Therefore, a sufficient condition for the above conclusion to hold is $\mathbb{E}_{\widetilde{\mathbf{X}}} [f_1 \otimes f_2 \mid \widetilde{\mathbf{X}}_{\mathcal{C}}] = 0$. Moreover, we can also note that $(x_1, x_2) \mapsto f_2(x_2)$ is also $\widetilde{\mathbf{X}}_{\mathcal{C}}$ -measurable: Indeed, on the σ -algebra generated by the second coordinate, the transformations \widetilde{T}_1 and \widetilde{T}_2 coincide, hence this σ -algebra is itself a \mathcal{C} -factor of $\widetilde{\mathbf{X}}$. We can then write

$$\mathbb{E}_{\widetilde{\mathbf{X}}} [f_1 \otimes f_2 \mid \widetilde{\mathbf{X}}_{\mathcal{C}}] = f_2(x_2) \mathbb{E}_{\widetilde{\mathbf{X}}} [f_1(x_1) \mid \widetilde{\mathbf{X}}_{\mathcal{C}}],$$

from which the proposition follows immediately. \square

2.2.4 \mathcal{C} -sated systems

From the above reasoning, we see that if our system \mathbf{X} is such that

$$\mathbb{E}_{\mathbf{X}} [f_1 \mid \mathbf{X}_{\mathcal{C}}] = 0 \implies \mathbb{E}_{\widetilde{\mathbf{X}}} [f_1(x_1) \mid \widetilde{\mathbf{X}}_{\mathcal{C}}] = 0, \quad (4)$$

then the convergence in L^2 of the ergodic averages (1) holds. Observe that in the RHS of (4), we are considering together two important factors of the big system $\widetilde{\mathbf{X}}$: On the one hand the factor generated by the first coordinate (which, as already mentioned, is nothing but the original system \mathbf{X}), and on the other hand the largest \mathcal{C} -factor $\widetilde{\mathbf{X}}_{\mathcal{C}}$ of $\widetilde{\mathbf{X}}$. In other words, we are considering a joining of our system \mathbf{X} with the \mathcal{C} -system $\widetilde{\mathbf{X}}_{\mathcal{C}}$. This motivates the following fundamental definition.

Definition 2.3. *The system \mathbf{X} is said to be \mathcal{C} -sated if, for any joining λ of \mathbf{X} with a \mathcal{C} -system \mathbf{Y} and any bounded measurable function f on \mathbf{X} , we have*

$$\mathbb{E}_{\lambda} [f(x) \mid \mathbf{Y}] = \mathbb{E}_{\lambda} \left[\mathbb{E}_{\mathbf{X}} [f(x) \mid \mathbf{X}_{\mathcal{C}}] \mid \mathbf{Y} \right]. \quad (5)$$

In other words, the system is \mathcal{C} -sated if any joining of \mathbf{X} with a \mathcal{C} -system is relatively independent over the largest \mathcal{C} -factor $\mathbf{X}_{\mathcal{C}}$ of \mathbf{X} .

Of course, any \mathcal{C} -sated system satisfies (4). Hence, a partial conclusion up to this point can be stated as follows:

Proposition 2.4. *If $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, T_2)$ is a \mathcal{C} -sated system, then the ergodic averages (1) converge in L^2 for any choice of f_1 and f_2 in $L^\infty(\mu)$.*

3 The case of d commuting transformations

In this section, we assume that $d \geq 2$ is such that Theorem 1.1 has already been proved in the case of $d - 1$ commuting transformations, and we adapt the arguments of the preceding section to see how to prove the ergodic theorem in the case of \mathcal{C} -sated systems of d commuting transformations.

In this general case, we define the class of \mathcal{C} -systems as the class of systems $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, \dots, T_d)$ for which

$$\mathcal{A} = \mathcal{I}^{T_1} \vee \mathcal{I}^{T_2 T_1^{-1}} \vee \dots \vee \mathcal{I}^{T_d T_1^{-1}}.$$

A factor σ -algebra of a system $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, \dots, T_d)$ on which the action of T_1, \dots, T_d defines a \mathcal{C} -system will be called a \mathcal{C} -factor of \mathbf{X} . In any system \mathbf{X} there always exists a largest \mathcal{C} -factor

$$\mathbf{X}_{\mathcal{C}} := \mathcal{I}^{T_1} \vee \mathcal{I}^{T_2 T_1^{-1}} \vee \dots \vee \mathcal{I}^{T_d T_1^{-1}}.$$

($\mathbf{X}_{\mathcal{C}}$ is defined as a factor σ -algebra of \mathbf{X} , but we will use the same notation to denote the \mathcal{C} -system obtained by considering the action of T_1, \dots, T_d on this sub- σ -algebra.)

The same argument as in the case $d = 2$ proves that convergence in L^2 of ergodic averages

$$\frac{1}{N} \sum_{n=1}^N f_1 \circ T_1^n \cdots f_d \circ T_d^n \quad (6)$$

reduces in the class of \mathcal{C} -systems to the case of $d - 1$ commuting transformations, and in fact it is enough for this reduction to be valid that f_1 be measurable with respect to $\mathbf{X}_{\mathcal{C}}$. Hence we are just looking for conditions ensuring that we can replace f_1 in (6) by its projection $\mathbb{E}_{\mathbf{X}} [f_1 | \mathbf{X}_{\mathcal{C}}]$.

The class of \mathcal{C} -sated systems is defined word for word as in Definition 2.3. Assuming now that $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, \dots, T_d)$ is a \mathcal{C} -sated system, we have to show that for any choice of $f_1, \dots, f_d \in L^\infty(\mu)$,

$$\mathbb{E}_{\mathbf{X}} [f_1 | \mathbf{X}_{\mathcal{C}}] = 0 \implies$$

$$\lim_{H \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \frac{1}{N} \sum_{n=1}^N \int_X f_1 \circ T_1^n \cdots f_d \circ T_d^n \overline{f_1} \circ T_1^{n+h} \cdots \overline{f_d} \circ T_d^{n+h} d\mu = 0, \quad (7)$$

which in turn, by Lemma 2.1, implies

$$\left\| \frac{1}{N} \sum_{n=1}^N f_1 \circ T_1^n \cdots f_d \circ T_d^n \right\|_{L^2} \xrightarrow{N \rightarrow \infty} 0.$$

3.1 Furstenberg self-joining

Since we have assumed the validity of Theorem 1.1 for $d - 1$ commuting transformations, the averages

$$\frac{1}{N} \sum_{n=1}^N \int_X g_1 \circ T_1^n g_2 \circ T_2^n \cdots g_d \circ T_d^n d\mu = \frac{1}{N} \sum_{n=1}^N \int_X g_1 g_2 \circ (T_2 T_1^{-1})^n \cdots g_d \circ (T_d T_1^{-1})^n d\mu$$

converge for any choice of g_1, \dots, g_d in $L^\infty(\mu)$. Applying this convergence in the case of indicator functions $g_i = \mathbb{1}_{A_i}$, $i = 1, \dots, d$, it is standard to see that the limit defines a probability measure λ on X^d by the formula

$$\lambda(A_1 \times \dots \times A_d) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int_X \mathbb{1}_{A_1}(T_1^n x) \cdots \mathbb{1}_{A_d}(T_d^n x) d\mu(x).$$

Moreover, it is straightforward to check that λ enjoys the following properties:

- λ is invariant by the transformation $T_i \otimes \dots \otimes T_i$ for all $i = 1, \dots, d$;
- The d marginal distributions of λ are equal to μ ;
- λ is also invariant by the transformation $T_1 \otimes T_2 \otimes \dots \otimes T_d$.

The above first two properties together mean that λ is a d -fold self-joining of the system $\mathbf{X} = (X, \mathcal{A}, \mu, T_1, \dots, T_d)$. this self-joining was introduced by Furstenberg in [4], and therefore referred to as *Furstenberg self-joining* by Austin.

As in the case of two commuting transformations, we now define a big system

$$\widetilde{\mathbf{X}} := (X^d, \mathcal{A}^{\otimes d}, \lambda, \widetilde{T}_1, \widetilde{T}_2, \dots, \widetilde{T}_d),$$

where $\widetilde{T}_1 := T_1 \otimes T_2 \otimes \dots \otimes T_d$ and, for $2 \leq i \leq d$, $\widetilde{T}_i := T_i \otimes T_i \otimes \dots \otimes T_i$. Observe by considering the first coordinate that \mathbf{X} is a factor of $\widetilde{\mathbf{X}}$. Note also that, for $2 \leq i \leq d$, \widetilde{T}_1 and \widetilde{T}_i coincide on the sub- σ -algebra \mathcal{A}_i generated by the i -th coordinate, from which we can deduce

$$\mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_d \subset \widetilde{\mathbf{X}}_{\mathcal{C}}. \quad (8)$$

We now turn back to (7). For any fixed h , we have by definition of Furstenberg self-joining

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N \int_X f_1 \circ T_1^n \cdots f_d \circ T_d^n \overline{f_1} \circ T_1^{n+h} \cdots \overline{f_d} \circ T_d^{n+h} d\mu \\ \xrightarrow{N \rightarrow \infty} \int_{X^d} (f_1 \otimes \dots \otimes f_d) \overline{(f_1 \otimes \dots \otimes f_d) \circ \widetilde{T}_1^h} d\lambda. \end{aligned} \quad (9)$$

Averaging the latter expression over $h \in \{1, \dots, H\}$, and letting H go to infinity gives, using the mean ergodic theorem for the single transformation \widetilde{T}_1 ,

$$\int_{X^d} (f_1 \otimes \dots \otimes f_d) \mathbb{E}_{\widetilde{\mathbf{X}}} \left[\overline{(f_1 \otimes \dots \otimes f_d)} \mid \mathcal{I}^{\widetilde{T}_1} \right] d\lambda. \quad (10)$$

Assume now that $\mathbb{E}_{\mathbf{X}} [f_1 \mid \mathbf{X}_{\mathcal{C}}] = 0$. Then if \mathbf{X} is a \mathcal{C} -sated system, we have

$$\mathbb{E}_{\widetilde{\mathbf{X}}} \left[f_1(x_1) \mid \widetilde{\mathbf{X}}_{\mathcal{C}} \right] = \mathbb{E}_{\mathbf{X}} [f(x) \mid \mathbf{X}_{\mathcal{C}}] = 0.$$

Recalling (8), we get

$$\mathbb{E}_{\widetilde{\mathbf{X}}} \left[f_1 \otimes \dots \otimes f_d \mid \widetilde{\mathbf{X}}_{\mathcal{C}} \right] = f_2(x_2) \cdots f_d(x_d) \mathbb{E}_{\widetilde{\mathbf{X}}} \left[f_1(x_1) \mid \widetilde{\mathbf{X}}_{\mathcal{C}} \right] = 0,$$

and since $\mathcal{I}^{\widetilde{T}_1} \subset \widetilde{\mathbf{X}}_{\mathcal{C}}$,

$$\mathbb{E}_{\widetilde{\mathbf{X}}} \left[f_1 \otimes \dots \otimes f_d \mid \mathcal{I}^{\widetilde{T}_1} \right] = 0,$$

which proves (7).

We thus have proved the following partial result:

Proposition 3.1. *If the statement of Theorem 1.1 is valid for $d - 1$ commuting transformations, then it is also valid for any \mathcal{C} -sated system of d commuting transformations.*

4 Existence of \mathcal{C} -sated extensions

The purpose of this section is to prove the existence of a \mathcal{C} -sated extension for any dynamical system, in a general context including the case we need to finish the proof of Theorem 1.1. An important part of the arguments used below was developed in [7] for the study of another class of systems, namely the class of all factors of all countable self-joinings of a given system. But as mentioned in [3], they work in a quite general setting which we present here in details.

From now on, let \mathcal{C} denote a class of dynamical systems, which we always assume to be stable under taking isomorphisms. As before, we call \mathcal{C} -factor of a dynamical system $(X, \mathcal{A}, \mu, T_1, \dots, T_d)$ any factor sub- σ -algebra on which the action of T_1, \dots, T_d defines a system in the class \mathcal{C} . In the particular case of class \mathcal{C} used in the preceding sections, it was quite obvious to see that any system admits a largest \mathcal{C} -factor. This is in fact a general result provided a stability assumption on \mathcal{C} .

Lemma 4.1. *If the class \mathcal{C} is stable under taking countable joinings, then any system \mathbf{X} admits a largest \mathcal{C} -factor, which we denote by $\mathbf{X}_{\mathcal{C}}$.*

Proof. We just set

$$\mathbf{X}_{\mathcal{C}} := \{A \in \mathcal{A} : A \text{ belongs to some } \mathcal{C}\text{-factor of } \mathbf{X}\},$$

and we claim that it is a \mathcal{C} -factor. Since (X, \mathcal{A}, μ) is a standard Borel space, the σ -algebra \mathcal{A} equipped with the metric $d(A, B) := \mu(A \Delta B)$ is separable (where we naturally identify subsets A and A' of X when $\mu(A \Delta A') = 0$). Therefore there exists a countable family $(A_i)_{i \in I}$ dense in $\mathbf{X}_{\mathcal{C}}$, and for each i there is some \mathcal{C} -factor \mathcal{F}_i containing A_i . Since the class \mathcal{C} is stable under taking countable joinings, $\mathcal{F} := \bigvee_{i \in I} \mathcal{F}_i$ is itself a \mathcal{C} -factor. By density, we have $\mathbf{X}_{\mathcal{C}} \subset \mathcal{F}$ but, since $\mathbf{X}_{\mathcal{C}}$ contains every \mathcal{C} -factors, we have $\mathbf{X}_{\mathcal{C}} = \mathcal{F}$. \square

If \mathcal{C} is stable under taking countable joinings, we can thus repeat Definition 2.3 in this more general setting:

Definition 4.2. *The system \mathbf{X} is said to be \mathcal{C} -sated if any joining of \mathbf{X} with a \mathcal{C} -system is relatively independent over the largest \mathcal{C} -factor $\mathbf{X}_{\mathcal{C}}$ of \mathbf{X} .*

Proposition 4.3. *If the class \mathcal{C} is stable under taking countable joinings and under taking factors, then any system \mathbf{X} is \mathcal{C} -sated.*

The proof is based on a fundamental lemma, published simultaneously in two papers [5, 6], stating that if two systems \mathbf{X} and \mathbf{Y} are not disjoint, then \mathbf{X} possesses a non-trivial common factor with a joining of countably many copies of \mathbf{Y} . We slightly rephrase this lemma in order to make it more convenient for our purposes:

Lemma 4.4. *Let λ be a joining of two systems $\mathbf{X} = (X, \mathcal{A}, \mu, (T_j))$ and $\mathbf{Y} = (Y, \mathcal{B}, \nu, (S_j))$, and let g be a bounded measurable function defined in \mathbf{Y} . Then there exists a factor sub- σ -algebra \mathcal{F} in \mathbf{X} such that the action of T_1, \dots, T_d on \mathcal{F} is isomorphic to a factor of some joining of countably many copies of \mathbf{Y} , and satisfying*

$$\mathbb{E}_{\lambda} [g(y) \mid \mathbf{X}] = \mathbb{E}_{\lambda} [g(y) \mid \mathcal{F} \otimes \{\emptyset, Y\}]. \quad (11)$$

Proof. We consider a countable family of copies of the dynamical system defined by the joining λ , and consider their relatively independent joining λ_∞ over their common factor \mathbf{X} . Then λ_∞ is a probability measure on the space $X \times Y^\mathbb{N}$, which is easily seen to be invariant under the shift transformation on each x -fiber, $\sigma : (x, y_0, y_1, y_2, \dots) \mapsto (x, y_1, y_2, y_3, \dots)$. Moreover, λ_∞ conditioned on each such fiber is a product measure. A relative version of Kolmogorov 0-1 law (see *e.g.* [6], Lemma 9) gives that, modulo λ_∞ , the σ -algebra \mathcal{I}^σ of shift-invariant events coincides with the σ -algebra $\mathcal{A} \otimes \{\emptyset, Y^\mathbb{N}\}$ generated by the x coordinate. Consider now a bounded measurable function g on Y , and set $g_\infty(x, (y_n)) := g(y_1)$ for $x \in X$ and $(y_n) \in Y^\mathbb{N}$. Applying the ergodic theorem in the dynamical system $(X \times Y^\mathbb{N}, \lambda_\infty, \sigma)$ to the function g_∞ , we obtain

$$\frac{1}{N} \sum_{n=1}^N g(y_n) \xrightarrow[N \rightarrow \infty]{L^2(\lambda_\infty)} \mathbb{E}_{\lambda_\infty} [g_\infty \mid \mathcal{I}^\sigma] = \mathbb{E}_{\lambda_\infty} [g(y_1) \mid \mathbf{X}],$$

and by definition of λ_∞ the latter is equal to $\mathbb{E}_\lambda [g(y) \mid \mathbf{X}]$. Hence, $\mathbb{E}_\lambda [g(y) \mid \mathbf{X}]$ coincides modulo λ_∞ with a function which is measurable with respect to $(y_n)_{n \in \mathbb{N}}$. It follows that the factor \mathcal{F} of \mathbf{X} generated by $\mathbb{E}_\lambda [g(y) \mid \mathbf{X}]$ is isomorphic to a factor of the joining of countably many copies of \mathbf{Y} obtained by considering the (y_n) -coordinates in λ_∞ . Finally, with this definition of \mathcal{F} , we obviously have (11). \square

Proof of Proposition 4.3. Let \mathbf{X} be any dynamical system, and λ be a joining of \mathbf{X} with a \mathcal{C} -system \mathbf{Y} . For a given bounded measurable function g defined in \mathbf{Y} , let \mathcal{F} be the factor sub- σ -algebra given by Lemma 4.4. By stability of \mathcal{C} under taking countable joinings and factors, \mathcal{F} is a \mathcal{C} -factor of \mathbf{X} , and \mathcal{F} is therefore contained in the largest \mathcal{C} -factor $\mathbf{X}_\mathcal{C}$. Equation (11) then gives

$$\mathbb{E}_\lambda [g(y) \mid \mathbf{X}] = \mathbb{E}_\lambda [g(y) \mid \mathbf{X}_\mathcal{C} \otimes \{\emptyset, Y\}],$$

and this equation means that in the joining λ , \mathbf{X} and \mathbf{Y} are relatively independent over $\mathbf{X}_\mathcal{C}$. This proves that \mathbf{X} is \mathcal{C} -sated. \square

The class \mathcal{C} of dynamical systems which is used in Section 2, and its generalization in Section 3, are easily proved to be stable under taking countable joinings, but unfortunately they are not stable under taking factors (see Annex B).

This is why it is necessary in general to pass to extensions to get \mathcal{C} -sated systems. The remaining of the section is devoted to the proof of the following theorem.

Theorem 4.5. *Let \mathcal{C} be a class of dynamical systems which is stable under taking countable joinings. Then any system admits a \mathcal{C} -sated extension.*

For \mathcal{C} satisfying the hypothesis of the above theorem, we start by introducing the class $\overline{\mathcal{C}}$ consisting of dynamical systems which are factors of \mathcal{C} -systems. Obviously $\overline{\mathcal{C}}$ is stable by taking factors, and we can also check that $\overline{\mathcal{C}}$ is stable under taking countable joinings. Indeed, let $\overline{\mathbf{Z}}$ be a joining of a countable family $(\overline{\mathbf{X}}_i)_{i \in I}$ of $\overline{\mathcal{C}}$ -systems. For each i , let \mathbf{X}_i be a \mathcal{C} -extension of $\overline{\mathbf{X}}_i$, and define \mathbf{Y}_i as the relatively independent joining of \mathbf{X}_i and $\overline{\mathbf{Z}}$ over their common factor $\overline{\mathbf{X}}_i$. Then, consider the relatively independent joining \mathbf{Z} of the \mathbf{Y}_i 's over their common factor $\overline{\mathbf{Z}}$. In \mathbf{Z} , each factor $\overline{\mathbf{X}}_i$ of $\overline{\mathbf{Z}}$ is identified with a factor of \mathbf{X}_i , hence $\overline{\mathbf{Z}}$ itself, which is generated by all the $\overline{\mathbf{X}}_i$'s, is contained in the σ -algebra generated by the \mathbf{X}_i 's.

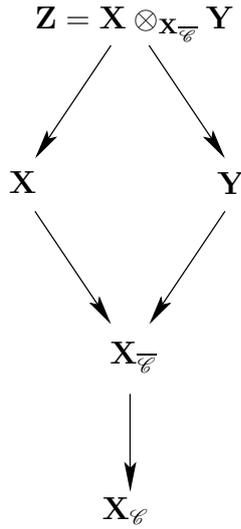
$\overline{\mathbf{Z}}$ is thus a factor of the joining of the \mathbf{X}_i 's defined by \mathbf{Z} , and since \mathcal{C} is stable under taking countable joinings, this joining is a \mathcal{C} -system.

In any system $\mathbf{X} = (X, \mathcal{A}, \mu, (T_j))$, there exist therefore a largest \mathcal{C} -factor $\mathbf{X}_{\mathcal{C}}$, and a largest $\overline{\mathcal{C}}$ -factor $\mathbf{X}_{\overline{\mathcal{C}}}$. Since any \mathcal{C} -system is obviously a $\overline{\mathcal{C}}$ -system, $\mathbf{X}_{\mathcal{C}} \subset \mathbf{X}_{\overline{\mathcal{C}}}$.

Proposition 4.6. \mathbf{X} is \mathcal{C} -sated if and only if $\mathbf{X}_{\mathcal{C}} = \mathbf{X}_{\overline{\mathcal{C}}}$.

Proof. The *if* part is a direct corollary of Proposition 4.3 applied to the class $\overline{\mathcal{C}}$. Conversely, let us assume that \mathbf{X} is \mathcal{C} -sated. Let \mathbf{Y} be a \mathcal{C} -extension of $\mathbf{X}_{\overline{\mathcal{C}}}$, and consider the relatively independent joining of \mathbf{X} and \mathbf{Y} over their common factor $\mathbf{X}_{\overline{\mathcal{C}}}$: Since \mathbf{X} is \mathcal{C} -sated, this joining is relatively independent over $\mathbf{X}_{\mathcal{C}}$. But this is only possible if $\mathbf{X}_{\overline{\mathcal{C}}} \subset \mathbf{X}_{\mathcal{C}}$. \square

Proof of Theorem 4.5. We use the same construction as above: Given a dynamical system \mathbf{X} , we consider its largest $\overline{\mathcal{C}}$ -factor $\mathbf{X}_{\overline{\mathcal{C}}}$, a \mathcal{C} -extension \mathbf{Y} of $\mathbf{X}_{\overline{\mathcal{C}}}$, and the relatively independent joining of \mathbf{X} and \mathbf{Y} over their common factor $\mathbf{X}_{\overline{\mathcal{C}}}$. Let us denote by \mathbf{Z} the latter system: \mathbf{Z} is the extension of \mathbf{X} which will be proved to be \mathcal{C} -sated. For this, by Proposition 4.6 it is enough to establish that the largest $\overline{\mathcal{C}}$ -factor of \mathbf{Z} is \mathbf{Y} : Since \mathbf{Y} is a \mathcal{C} -system, this will give $\mathbf{Z}_{\mathcal{C}} = \mathbf{Z}_{\overline{\mathcal{C}}}$.



Let us consider a joining λ of \mathbf{Z} with a \mathcal{C} -system \mathbf{W} . Since the joining of \mathbf{Y} and \mathbf{W} induced by λ is still a \mathcal{C} -system, Proposition 4.3 ensures that, inside λ , \mathbf{X} and $(\mathbf{W} \vee \mathbf{Y})$ are relatively independent over $\mathbf{X}_{\overline{\mathcal{C}}}$. Hence \mathbf{X} and \mathbf{W} are relatively independent over \mathbf{Y} , and finally \mathbf{Z} and \mathbf{W} are relatively independent over \mathbf{Y} (because \mathbf{Z} is generated by \mathbf{X} and \mathbf{Y}). We thus have proved that any joining of \mathbf{Z} with a \mathcal{C} -system is relatively independent over \mathbf{Y} . Taking in particular the relatively independent joining of \mathbf{Z} with a \mathcal{C} -extension of $\mathbf{Z}_{\overline{\mathcal{C}}}$, we see that this is only possible if $\mathbf{Z}_{\overline{\mathcal{C}}} \subset \mathbf{Y}$. Since the converse inclusion obviously holds, this concludes the proof. \square

Annex A. Proof of Van der Corput Lemma

Here is a proof of Lemma 2.1. First, observe that, since the sequence (u_n) is bounded, for any H we have

$$\left\| \frac{1}{N} \sum_{n=1}^N u_n \right\| = \left\| \frac{1}{N} \sum_{n=1}^N \frac{1}{H} \sum_{h=1}^H u_{n+h} \right\| + O(H/N).$$

Using the classical inequality $(a+b)^2 \leq 2(a^2+b^2)$, then the triangular inequality and finally the Cauchy-Schwartz inequality in the form $(1/N \sum_1^N a_n)^2 \leq 1/N \sum_1^N a_n^2$, we get

$$\begin{aligned} \left\| \frac{1}{N} \sum_{n=1}^N u_n \right\|^2 &\leq 2 \left\| \frac{1}{N} \sum_{n=1}^N \frac{1}{H} \sum_{h=1}^H u_{n+h} \right\|^2 + O(H^2/N^2) \\ &\leq 2 \left(\frac{1}{N} \sum_{n=1}^N \left\| \frac{1}{H} \sum_{h=1}^H u_{n+h} \right\| \right)^2 + O(H^2/N^2) \\ &\leq \frac{2}{N} \sum_{n=1}^N \left\| \frac{1}{H} \sum_{h=1}^H u_{n+h} \right\|^2 + O(H^2/N^2) \end{aligned}$$

We now have to estimate

$$\frac{1}{N} \sum_{n=1}^N \left\| \frac{1}{H} \sum_{h=1}^H u_{n+h} \right\|^2 = \frac{1}{N} \sum_{n=1}^N \frac{1}{H} \sum_{h=1}^H \frac{1}{H} \sum_{h'=1}^H \langle u_{n+h}, u_{n+h'} \rangle.$$

We split the RHS into three pieces $P_{h=h'}$, $P_{h<h'}$, and $P_{h>h'}$, corresponding respectively to the terms where $h = h'$, $h < h'$, and $h > h'$. The first piece is simply controlled by choosing H large enough:

$$P_{h=h'} = \frac{1}{N} \sum_{n=1}^N \frac{1}{H^2} \sum_{h=1}^H \langle u_{n+h}, u_{n+h} \rangle = O(1/H).$$

The second and third pieces are treated with the same computation, we only detail here the case ($h < h'$):

$$\begin{aligned} P_{h<h'} &= \frac{1}{H} \sum_{h=1}^H \frac{1}{H} \sum_{h'=h+1}^H \frac{1}{N} \sum_{n=1}^N \langle u_{n+h}, u_{n+h'} \rangle \\ &= \frac{1}{H} \sum_{h=1}^H \frac{1}{H} \sum_{h'=1}^{H-h} \frac{1}{N} \sum_{n=1}^N \langle u_n, u_{n+h'} \rangle + O(H/N) \\ &= \frac{1}{H} \sum_{h=1}^{H-1} \frac{1}{H} \sum_{h'=1}^h \frac{1}{N} \sum_{n=1}^N \langle u_n, u_{n+h'} \rangle + O(H/N) \end{aligned}$$

Fixing H large enough, the hypothesis then implies that $P_{h<h'}$ can be made arbitrarily close to zero when $N \rightarrow \infty$, which achieves the proof.

Annex B. A factor of a \mathcal{C} -system is not always a \mathcal{C} -system

Here is an example showing that the class \mathcal{C} defined in Section 2 is not stable under taking factors. For each $\alpha \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, let us denote by R_α the translation on \mathbb{T} : $x \mapsto x + \alpha \pmod{1}$, and by μ the Haar measure on \mathbb{T} . For some fixed irrational α , we consider the system $\mathbf{X} = (\mathbb{T} \times \mathbb{T}, \mu \otimes \mu, T_1, T_2)$ where $T_1 := R_\alpha \otimes R_{2\alpha}$, and $T_2 := R_{2\alpha} \otimes R_{2\alpha}$. Denoting by x (respectively y) the first (respectively second) coordinate on $\mathbb{T} \times \mathbb{T}$, we observe that any function of $2x - y \pmod{1}$ is invariant by T_1 , hence is measurable with respect to the \mathcal{C} -factor $\mathbf{X}_{\mathcal{C}}$. Observe also that on the σ -algebra generated by y , T_1 and T_2 define the same action, hence any function of y is also measurable with respect to the \mathcal{C} -factor $\mathbf{X}_{\mathcal{C}}$. It follows that the factor of \mathbf{X} generated by $2x \pmod{1} = (2x - y) + y \pmod{1}$ is contained in $\mathbf{X}_{\mathcal{C}}$, hence is a factor of a \mathcal{C} -system. However, the action of (T_1, T_2) restricted to this factor is isomorphic to the action of $(R_{2\alpha}, R_{4\alpha})$ on \mathbb{T} . The latter is certainly not a \mathcal{C} -system, since both $\mathcal{I}^{R_{2\alpha}}$ and $\mathcal{I}^{R_{4\alpha}R_{2\alpha}^{-1}}$ are trivial.

References

- [1] Tim Austin, *On the norm convergence of nonconventional ergodic averages*, Ergodic Theory Dynam. Systems, to appear.
- [2] ———, *Pleasant extensions subject to some algebraic constraints, and applications*, Preliminary notes available on arXiv:0905.0518, 2009.
- [3] Thierry de la Rue, *An introduction to joinings in ergodic theory*, Discrete Contin. Dyn. Syst. **15** (2006), no. 1, 121–142.
- [4] Harry Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Analyse Math. **31** (1977), 204–256.
- [5] E. Glasner, J.-P. Thouvenot, and B. Weiss, *Entropy theory without a past*, Ergodic Theory Dynam. Systems **20** (2000), no. 5, 1355–1370.
- [6] M. Lemańczyk, F. Parreau, and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fund. Math. **164** (2000), no. 3, 253–293.
- [7] E. Lesigne, B. Rittaud, and T. de la Rue, *Weak disjointness of measure-preserving dynamical systems*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 1173–1198.
- [8] T. Tao, *Norm convergence of multiple ergodic averages for commuting transformations*, Ergodic Theory Dynam. Systems **28** (2008), no. 2, 657–688.