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# THE SPACES $H^n(\mathfrak{osp}(1|2), M)$ FOR SOME WEIGHT MODULES $M$

DIDIER ARNAL, MABROUK BEN AMMAR AND BECHIR DALI

ABSTRACT. We entirely compute the cohomology for a natural and large class of  $\mathfrak{osp}(1|2)$  modules  $M$ . We study the restriction to the  $\mathfrak{sl}(2)$  cohomology of  $M$  and apply our results to the module  $M = \mathfrak{D}_{\lambda, \mu}$  of differential operators on the super circle, acting on densities.

## 1. INTRODUCTION

The simplest Lie superalgebra is the algebra  $\mathfrak{osp}(1|2)$ . For such an algebra, the notion of Cartan subalgebra and weight module is well known (see section 2 for definitions and notations). In this paper, we consider such a weight module  $M$ , with moreover the assumption that one of the odd element (noted here  $A$ ) acts through a surjective map.

This generalizes the notion of  $\ell \downarrow$  modules for  $\mathfrak{sl}(2)$  [8], a class of modules admitting a finite dimensional and nontrivial extension, but our main motivation is the study of deformations of some actions of vector fields on the supercircle or the superspace  $\mathbb{R}^{1|1}$ , this theory was developed by Ovsienko and many other authors and some conjectures about the cohomology of natural modules coming from the action of  $\mathfrak{osp}(1|2)$  on differential operators on densities were presented (see [4, 3, 5]). The first cohomology group for this module was computed by Basdouri and Ben Ammar [2], it was conjectured that the second cohomology group would be generated by cup-product of nontrivial 1 cocycles, that the 2 cocycles whose  $\mathfrak{sl}(2)$  restriction is trivial are trivial, and so one.

In this paper, we first entirely determine the cohomology for our  $\mathfrak{osp}(1|2)$  module  $M$  and prove that the restriction map is one to one from  $H^n(\mathfrak{osp}(1|2), M)$  to  $H^n(\mathfrak{sl}(2), M)$ . Then we apply this to the module of the differential operators on densities, computing completely their cohomologies and explicitly describing the cocycles.

## 2. DEFINITIONS AND NOTATIONS

First, we define the Lie superalgebra  $\mathfrak{osp}(1|2)$  and the module  $M$ . We define the superalgebra  $\mathfrak{g} = \mathfrak{osp}(1|2)$  as the real algebra whose basis is  $(H, X, Y, A, B)$ . The elements  $H, X$  and  $Y$  are even (with parity 0, or in  $\mathfrak{g}_0$ ) and the elements  $A, B$  are odd (with parity 1, or in  $\mathfrak{g}_1$ ), the bracket is graded antisymmetric, we denote this property by

$$[U, V] = -(-1)^{UV}[V, U].$$

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The commutation relations are :

$$\begin{aligned} [H, X] &= X, & [H, Y] &= -Y, & [X, Y] &= 2H, \\ [H, A] &= \frac{1}{2}A, & [X, A] &= 0, & [Y, A] &= -B, \\ [H, B] &= -\frac{1}{2}B, & [X, B] &= A, & [Y, B] &= 0, \\ [A, A] &= 2X, & [A, B] &= 2H, & [B, B] &= -2Y. \end{aligned}$$

The bracket satisfies the graded Jacobi identity

$$(-1)^{UV} [[U, V], W] + (-1)^{VW} [[V, W], U] + (-1)^{WU} [[W, V], U] = 0.$$

We consider the subalgebra  $\mathbb{R}H$  as the Cartan subalgebra of  $\mathfrak{osp}(1|2)$ , its adjoint action is trivially split, with roots  $0, \pm\frac{1}{2}, \pm 1$ .

The even subalgebra  $\mathfrak{g}_0$  of  $\mathfrak{osp}(1|2)$  is of course the simple Lie algebra  $\mathfrak{sl}(2)$ . From the relations, it is clear that, as a graded Lie algebra,  $\mathfrak{osp}(1|2)$  is generated by its odd part  $\mathfrak{g}_1 = \text{Span}(A, B)$ .

We consider here a special class of  $\mathfrak{osp}(1|2)$  modules  $M$ . We first suppose  $M$  is a complex  $\mathbb{Z}_2$  graded vector space  $M_0 \oplus M_1$  (the elements of  $M_i$  are said homogenous with parity  $i$ ) and the  $H$  action is diagonalized on  $M$ , that is we decompose  $M$  (and thus  $M_0$  and  $M_1$ ) into weight spaces  $M^\alpha$  (resp.  $M_i^\alpha$ ) :

$$M = \bigoplus_{\alpha \in \Sigma} M^\alpha, \quad Hv_\alpha = \alpha v_\alpha, \quad \forall v_\alpha \in M^\alpha.$$

( $\Sigma \subset \mathbb{C}$  is the set of weights).

If  $V$  is a  $H$ -invariant vector subspace, then  $V$  itself can be decomposed in  $V = \bigoplus_{\alpha \in \Sigma} V^\alpha$  with  $V^\alpha = M^\alpha \cap V$ . For instance, each  $M_i$  can be decomposed.

The commutation relations imply directly

$$\begin{aligned} AM_i^\alpha &\subset M_{i+1}^{\alpha+\frac{1}{2}}, & XM_i^\alpha &\subset M_i^{\alpha+1}, \\ BM_i^\alpha &\subset M_{i+1}^{\alpha-\frac{1}{2}}, & YM_i^\alpha &\subset M_i^{\alpha-1}. \end{aligned}$$

Then we add the condition that the action of  $A$  is onto (or equivalently  $X$  is onto). This conditions implies that  $M$  does not have any minimal weight vector  $v$ , with weight  $\alpha_0$ . Indeed, if such a vector exists, the relation  $v = Aw = A \sum_{\beta \in \Sigma} w_\beta$  ( $w_\beta \in M^\beta$ ) implies

$$Hv = \alpha_0 v = \sum_{\beta} HAw_\beta = \sum_{\beta} (\beta + \frac{1}{2}) Aw_\beta = \sum_{\beta} \alpha_0 Aw_\beta,$$

or  $Aw_\beta = 0$  if  $\beta \neq \alpha_0 - \frac{1}{2}$ , and  $0 \neq v = Aw_{\alpha_0 - \frac{1}{2}}$ , therefore  $\alpha_0 - \frac{1}{2} \in \Sigma$ , which is impossible. Then our modules  $M$  are infinite dimensional.

For  $\mathfrak{sl}(2)$ , the simple modules for which  $X$  are onto are the modules  $\ell \downarrow$ . It is well known that these modules are the only (with the 'symmetric' case  $\ell \uparrow$ )  $\mathfrak{sl}(2)$ -modules admitting finite dimensional nontrivial extensions for some values of  $\ell$  (see [8]).

We now consider the cohomology groups  $H^n(\mathfrak{osp}(1|2), M)$  of these modules. A  $n$  cochain is a mapping  $f$  from  $\mathfrak{osp}(1|2)^n$  to  $M$  which is  $n$  linear and graded antisymmetric:

$$f(U_1, \dots, U_i, \dots, U_j, \dots, U_n) = -(-1)^{U_i U_j} f(U_1, \dots, U_j, \dots, U_i, \dots, U_n).$$

Defining the graded sign  $\varepsilon_U(\sigma)$  for a permutation  $\sigma \in \mathfrak{S}_n$  acting on the elements  $U_i$  as the product  $\varepsilon(\sigma)\varepsilon(\tau)$  of the usual sign  $\varepsilon(\sigma)$  of  $\sigma$  by the sign of the induced permutation  $\tau$  on

the set of indices  $i$  for odd elements  $U_i$ , we have :

$$f(U_{\sigma(1)}, \dots, U_{\sigma(n)}) = \varepsilon_U(\sigma) f(U_1, \dots, U_n).$$

Due to this property, we use the following notation:

$$U_1 \cdots U_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon_U(\sigma) (U_{\sigma^{-1}(1)} \otimes \cdots \otimes U_{\sigma^{-1}(n)})$$

and for any  $\sigma \in \mathfrak{S}_n$ ,

$$f(U_1 \cdots U_n) = \varepsilon_U(\sigma) f(U_{\sigma^{-1}(1)} \otimes \cdots \otimes U_{\sigma^{-1}(n)}) = \varepsilon_U(\sigma) f(U_{\sigma^{-1}(1)}, \dots, U_{\sigma^{-1}(n)}).$$

The cochain  $f$  is homogeneous with parity  $f$  if  $f(\mathfrak{g}_{i_1} \otimes \cdots \otimes \mathfrak{g}_{i_n}) \subset M_{f+\sum i_j}$ . The space of  $n$  cochain is denoted  $C^n(\mathfrak{osp}(1|2), M)$ , or  $C^n$  if no confusion is possible.

On such a cochain  $f$ , the coboundary operator is defined, using the Koszul rule for signs, by the relation (see [7] for instance):

$$\begin{aligned} (\partial f)(U_0, \dots, U_n) &= \sum_{i=0}^n (-1)^i (-1)^{U_i(f+U_0+\dots+U_{i-1})} U_i f(U_0, \dots, \hat{i}, \dots, U_n) + \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} (-1)^{U_i(U_0+\dots+U_{i-1})} (-1)^{U_j(U_0+\dots+\hat{i}+\dots+U_{j-1})} f([U_i, U_j], U_0, \dots, \hat{i}, \dots, \hat{j}, \dots, U_n). \end{aligned}$$

If  $f$  is a  $n$  cochain,  $\partial f$  is a  $n+1$  cochain with the same parity  $f$ , we can verify directly that  $\partial \circ \partial = 0$  (or we can use a shift on degree and usual cohomology computations). The  $n$  cocycles are the  $n$  cochains such that  $\partial f = 0$ , the  $n$  coboundaries are the cochains in the image of  $\partial$ , we put as usual

$$Z^n = \ker(\partial : C^n \longrightarrow C^{n+1}), \quad B^n = \partial(C^{n-1}), \quad H^n(\mathfrak{osp}(1|2), M) = Z^n / B^n.$$

$H^n(\mathfrak{osp}(1|2), M)$  is the  $n^{\text{th}}$  cohomology group for the module  $M$ .

### 3. THE COHOMOLOGY

**3.1.  $\mathfrak{osp}(1|2)$  cohomology.** The cohomology is described by the following

**Theorem 3.1.** (The groups  $H^n(\mathfrak{osp}(1|2), M)$ )

Let us denote by  $\ker A$  (respectively  $\ker B$ ) the subspaces of  $M$ , kernel of the morphism  $v \mapsto Av$  (respectively  $v \mapsto Bv$ ) in  $M$ . Then we have the following linear isomorphisms:

- (i)  $H^0(\mathfrak{osp}(1|2), M) = \ker A \cap \ker B$ .
- (ii)  $H^1(\mathfrak{osp}(1|2), M) \simeq (\ker A \cap \ker B) \oplus ((\ker A)^{-\frac{1}{2}} / B((\ker A)^0))$ .
- (iii)  $H^2(\mathfrak{osp}(1|2), M) \simeq (\ker A)^{-\frac{1}{2}} / B((\ker A)^0)$ .
- (iv)  $H^n(\mathfrak{osp}(1|2), M) = 0$  if  $n > 2$ .

The realization of these isomorphisms will be explicitly detailed in the proof. Before to prove this theorem, we shall give some preliminary results.

First we say that a  $n$  cochain  $f$  is reduced if  $f(AU_2 \cdots U_n) = 0$  for any  $U_2, \dots, U_n$  in  $\{A, H, B, Y\}$ . Observe that if  $f$  is reduced then we have also  $f(XU_2 \cdots U_n) = 0$  for any  $U_2, \dots, U_n$  in  $\{A, H, B, Y\}$ , since

$$0 = (\partial f)(A^2 U_2 \cdots U_n) = -f([A, A]U_2 \cdots U_n) = -2f(XU_2 \cdots U_n).$$

**Proposition 3.2.** (Each cochain is cohomologous to a reduced one)

*Let  $f$  be a  $n$  cochain. Then there exists a  $n - 1$  cochain  $g$  such that  $f - \partial g$  is reduced.*

**Proof.** If  $n = 0$ , any cochain is reduced and there is nothing to do. Suppose now  $n > 0$ , Recall that the vectors  $X, A, H, B, Y$  are root vectors with respective weight  $1, \frac{1}{2}, 0, -\frac{1}{2}, -1$ . Define the weight of  $U_1 \otimes \cdots \otimes U_n$  as the sum of the weights of the vectors  $U_i$ .

First, we kill  $f(A^n)$ . Indeed, if  $g_0$  is the  $n - 1$  cochain such that  $g_0(U_1 \cdots U_{n-1}) = 0$  except if  $U_1 = \cdots = U_{n-1} = A$  and  $g_0(A^{n-1}) = v$  where  $v$  is such that  $nAv = (-1)^f f(A^n)$  then  $(\partial g_0)(A^n) = f(A^n)$  and  $f_0 = f - \partial g_0$  vanishes on  $A^n$ . If  $n = 1$ , the proposition is proved.

Now, by induction, we suppose there is  $g_k$  such that  $f_k = f - \partial g_k$  vanishes on any product of the form  $A^{n-k}U_{n-k+1} \cdots U_n$  with  $U_j \in \{A, H, B, Y\}$ .

Suppose  $n - k > 1$  and consider a  $k + 1$  product of the form  $U_{n-k} \cdots U_n$ . If one of the  $U_i$  is  $A$ ,  $f_k$  vanishes on  $A^{n-k-1}U_{n-k} \cdots U_n$ , if there is no such  $U_i$ , but if  $U_i = U_j = H$ , then  $f_k$  vanishes on  $A^{n-k-1}U_{n-k} \cdots U_n$ . The monomial  $T$  with maximal weight for which  $f_k(A^{n-k-1}T)$  could be not zero is thus  $HB^k$  and its weight is  $w(T) = -\frac{k}{2}$ .

By induction, we can suppose  $f'_k$  vanishes for any monomial of the form  $A^{n-k}S$  and any monomial of the form  $A^{n-k-1}T$  with  $w(T) > \ell$  ( $\ell \leq -\frac{k}{2} + \frac{1}{2}$ ). We choose now  $g_{k+1}^\ell(U_1 \cdots U_{n-1}) = 0$  except if  $U_1 \cdots U_{n-1} = A^{n-k-2}U_{n-k} \cdots U_n$  and  $w(U_{n-k} \cdots U_n) = \ell$  and  $U_j \in \{H, B, Y\}$ . Then for such a monomial,

$$\begin{aligned} 0 &= g_{k+1}^\ell([A, A]A^{n-k-3}U_{n-k} \cdots U_n) = g_{k+1}^\ell([A, U_j]A^{n-k-2}U_{n-k} \cdots \hat{j} \cdots U_n) \\ &= g_{k+1}^\ell([U_i, U_j]A^{n-k-1}U_{n-k} \cdots \hat{i} \cdots \hat{j} \cdots U_n) \end{aligned}$$

and

$$(\partial g_{k+1}^\ell)(A^{n-k-1}U_{n-k} \cdots U_n) = (-1)^{g_{k+1}^\ell} (n - k - 1) A g_{k+1}^\ell(A^{n-k-2}U_{n-k} \cdots U_n).$$

We can then choose the value of  $g_{k+1}^\ell$  such that  $f'_k - \partial g_{k+1}^\ell$  vanishes on any monomial of the form  $A^{n-k-1}T$  with  $w(T) \geq \ell$ .

By induction, we prove there is  $g'$  such that  $f' = f - \partial g'$  is reduced.

**Proposition 3.3.** (Localization for cocycles)

*Suppose than  $f$  is a  $n$  reduced cocycle. Then*

- (i) *If  $n > 0$ ,  $f = 0$  if and only if  $f(B^n) = 0$ .*
- (ii) *If  $n > 1$ , any reduced cocycle vanishing on  $HB^{n-1}$  is a coboundary.*

**Proof.** (i) With the antisymmetry condition on  $f$ , the only possibly non vanishing terms for  $f$  are monomials containing  $B$  (as odd vector) and  $H$  and  $Y$  as even vector, but each of them at most one time.  $f(U_1 \cdots U_n) = 0$  except if  $U_1 \cdots U_n$  is  $HB^{n-1}$  or  $B^n$  or  $YB^{n-1}$  and, if  $n > 1$ ,  $HYB^{n-2}$ .

Now, the cocycle relation allows us to compute these vectors with the only knowledge of  $f(B^n)$  :

$$\begin{aligned} (\partial f)(AB^n) &= (-1)^f Af(B^n) + \sum_{i=1}^n (-1)^i (-1)^{(i-1)} f([A, B]B^{n-1}) \\ &= (-1)^f Af(B^n) - 2nf(HB^{n-1}) = 0, \end{aligned}$$

and

$$\begin{aligned}
(\partial f)(B^{n+1}) &= \sum_{i=0}^n (-1)^i (-1)^{f+i} Bf(B^n) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} (-1)^{i+j-1} f([B, B]B^{n-1}) \\
&= (n+1) [(-1)^f Bf(B^n) + nf(YB^{n-1})] = 0, \\
(\partial f)(HB^n) &= Hf(B^n) + \sum_{i=1}^n (-1)^i (-1)^{f+i-1} Bf(HB^{n-1}) + \\
&\quad + \sum_{j=1}^n (-1)^j (-1)^{j-1} f([H, B]B^{n-1}) + \\
&\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} (-1)^{i+j} f([B, B]HB^{n-2}) \\
&= (H + \frac{n}{2}id)f(B^n) - n [(-1)^f Bf(HB^{n-1}) + (n-1)f(HYB^{n-2})] = 0.
\end{aligned}$$

Thus a reduced cocycle  $f$  is completely determined by the vector  $f(B^n)$ , especially  $f = 0$  if and only if  $f(B^n) = 0$ .

(ii) Suppose now  $f(HB^{n-1}) = 0$  and  $n > 1$ . Then our computation proves that  $f(B^n)$  is in the kernel of  $A$ . We define a  $n-1$  cochain  $g$  by putting  $g(U_1 \cdots U_{n-1}) = 0$  except for

$$g(YB^{n-2}) = \frac{1}{n(n-1)} f(B^n).$$

Then

$$(\partial g)(B^n) = n(n-1)g(YB^{n-2}) = f(B^n).$$

and

$$\begin{aligned}
(\partial g)(AU_1 \cdots U_{n-1}) &= (-1)^g Ag(U_1 \cdots U_{n-1}) + \\
&\quad + \sum_{j=1}^{n-1} (-1)^j (-1)^{U_j(g+1+U_2+\cdots+U_{j-1})} U_j g(AU_1 \cdots \hat{j} \cdots U_{n-1}) + \\
&\quad + \sum_{j=1}^{n-1} (-1)^j (-1)^{U_j(U_1+\cdots+U_{j-1})} g([A, U_j]U_1 \cdots \hat{j} \cdots U_{n-1}) + \\
&\quad + \sum_{1 \leq i < j \leq n-1} (-1)^{i+j} (-1)^{U_i(1+U_1+\cdots+U_{i-1})+U_j(1+U_1+\cdots+U_{j-1})} \\
&\quad \quad g([U_i, U_j]AU_1 \cdots \hat{i} \cdots \hat{j} \cdots U_{n-1}) \\
&= \sum_{j=1}^{n-1} (-1)^j (-1)^{U_j(U_1+\cdots+U_{j-1})} g([A, U_j]U_1 \cdots \hat{j} \cdots U_{n-1}).
\end{aligned}$$

This is non vanishing only if  $[A, U_j] = B$  and  $U_1 \cdots \hat{j} \cdots U_{n-1} = YB^{n-3}$  or  $[A, U_j] = Y$  and  $U_1 \cdots \hat{j} \cdots U_{n-1} = B^{n-2}$ . In the first case, we are computing  $\partial g(AY^2B^{n-3})$ , but, the antisymmetry condition on  $\partial g$  gives  $\partial g(AY^2B^{n-3}) = 0$ . In the second case there is no such  $U_j$ . Thus  $\partial g$  vanishes on any monomial  $AU_1 \cdots U_{n-1}$ .

Now  $f - \partial g$  is a cocycle vanishing on any  $AU_2 \cdots U_n$  and on  $B^n$ , thus  $f = \partial g$ .

### Proof of Theorem 3.1.

(i) If  $n = 0$ , there is no coboundaries, the cocycles are the vector  $f \in M$  such that  $(\partial f)(U) = (-1)^f Uf = 0$  for any  $U$  in  $\mathfrak{osp}(1|2)$  these vectors are in  $\ker A \cap \ker B$ . Conversely, since  $A$  and  $B$  generate  $\mathfrak{osp}(1|2)$  as an algebra, each vector in  $\ker A \cap \ker B$  is 0 cocycle.

(ii) Suppose  $n = 1$ . We saw that up to a coboundary,  $f$  is vanishing on  $A$  and  $X$ . Thus  $f(H)$  belongs to  $\ker A$  since

$$(\partial f)(AH) = (-1)^f Af(H) = 0.$$

Let us now decompose  $f(H)$  on weight vectors :

$$f(H) = \sum_{\alpha \in \Sigma} v_\alpha, \quad Hv_\alpha = \alpha v_\alpha, \quad Av_\alpha = 0.$$

Put  $g = \sum_{\alpha \neq 0} \frac{1}{\alpha} v_\alpha$ . Then  $(\partial g)(A) = (-1)^g Ag = 0$  and  $(\partial g)(H) = Hg = \sum_{\alpha \neq 0} v_\alpha$ . The 1 cocycle  $f' = f - \partial g$  is reduced and satisfies  $f'(H) \in \ker A \cap \ker H$ . Now

$$0 = (\partial f')(HB) = Hf'(B) - (-1)^f Bf'(H) + \frac{1}{2}f'(B) = (H + \frac{1}{2}id)f'(B) - (-1)^f Bf'(H).$$

The first term is in  $\bigoplus_{\alpha \neq -\frac{1}{2}} M^\alpha$ , the second one in  $B(\ker H) \subset M^{-\frac{1}{2}}$ . Thus these two terms vanish. Therefore  $f'(H)$  is in  $\ker A \cap \ker B$ . We now suppose  $f(H) \in \ker A \cap \ker B$ . Then  $(H + \frac{1}{2}id)f(B) = 0$ .

On the other hand, we have  $Af(B) = 2(-1)^f f(H)$ . Thus  $f(B)$  is in the affine space of solutions for these two last equations. The corresponding linear space is  $(\ker A)^{-\frac{1}{2}}$ . But we can still add a coboundary  $\partial g$  to  $f$  with  $Ag = Hg = 0$ , then  $f(B)$  becomes  $f(B) + (-1)^g Bg$ . That means, we can impose to look for solution in an affine space parallel to  $(\ker A)^{-\frac{1}{2}}/B(\ker A \cap \ker H)$ .

To be more precise, let us choose a supplementary space  $V$  to  $(\ker A)^{-\frac{1}{2}}$  in  $M^{-\frac{1}{2}}$  and a supplementary space  $W$  to  $B((\ker A)^0)$  in  $(\ker A)^{-\frac{1}{2}}$ :

$$M^{-\frac{1}{2}} = (\ker A)^{-\frac{1}{2}} \oplus V = B((\ker A)^0) \oplus W \oplus V.$$

Up to a coboundary,  $f(H)$  belongs to  $\ker A \cap \ker B$  and  $f(B)$  to  $W \oplus V$ . Write  $f(B) = w + v$ , we get  $Af(B) = Av = 2(-1)^f f(H)$ . This relation characterizes  $v$  since  $A|_V$  is one-to-one. We associate to  $f$  the vector  $(f(H), w)$  in  $(\ker A \cap \ker B) \oplus W$ .

Conversely, let  $u$  be in  $\ker A \cap \ker B$ , homogeneous with parity  $u$  and  $v$  the unique vector in  $V_{u+1}$  such that  $Av = 2(-1)^u u$ . Choose any  $w$  in  $W$  and define a map  $f : \mathfrak{osp}(1|2) \rightarrow M$  by putting  $f(A) = f(X) = 0$ ,  $f(H) = u$ ,  $f(B) = v + w$ , and  $f(Y) = -(-1)^f B(v + w)$ . Then we verify directly that

$$\begin{aligned} (\partial f)(AX) &= (\partial f)(AA) = (\partial f)(AH) = 0 \\ (\partial f)(AB) &= (-1)^f A(v + w) - 2u = (-1)^f Av - 2u = 0, \\ (\partial f)(AY) &= -AB(v + w) - (v + w) = -(AB + BA)(v + w) - (v + w) \\ &= -(2H + id)(v + w) = 0. \end{aligned}$$

The map  $\partial f$  is then a reduced 2 cocycle and moreover, we have

$$\partial f(B^2) = (-1)^f 2Bw + 2(-(-1)^f Bw) = 0.$$

Thus  $\partial f = 0$ , that is,  $f$  is a 1 cocycle.

Now, suppose  $f$  is a coboundary, then there is  $g$  such that

$$Ag = 0 \quad \text{and} \quad f(H) = u = Hg.$$

This implies  $H^2g = Hu = 0$ , thus  $g \in (\ker A)^0$  and  $u = 0$ , thus  $v = 0$  and  $f(B) = w = (-1)^g Bg \in B((\ker A)^0) \cap W$ , thus  $w = 0$ . Conversely, if  $v = 0$  and  $w = (-1)^g Bg$  with  $g \in (\ker A)^0$ , then  $f' = f - \partial g$  is a reduced 1 cocycle such that  $f'(B) = 0$ , thus  $f' = 0$  and  $f$  is a coboundary. Thus, the map  $f \mapsto (u, w)$  realizes an isomorphism between  $H^1(\mathfrak{osp}(1|2), M)$  and  $(\ker A \cap \ker B) \oplus W$ .

We proved (ii) since  $W \simeq (\ker A)^{-\frac{1}{2}}/B((\ker A)^0)$ .

(iii) Suppose  $n \geq 2$  and  $f$  is a reduced  $n$  cocycle. Since

$$0 = (\partial f)(AHB^{n-1}) = (-1)^f Af(HB^{n-1}),$$

we get as above:  $f(HB^{n-1})$  is in  $\ker A$ .

We decompose  $f(HB^{n-1}) = \sum v_\alpha$  with  $(H - \alpha id)v_\alpha = Av_\alpha = 0$ . Define the  $n-1$  cochain  $g$  by  $g(U_1 \dots U_{n-1}) = 0$  except for  $g(B^{n-1})$  and  $g(YB^{n-2})$  and

$$g(B^{n-1}) = \sum_{\alpha \neq -\frac{n-1}{2}} \frac{1}{\alpha + \frac{n-1}{2}} v_\alpha, \quad (-1)^g Ag(YB^{n-2}) = g(B^{n-1}).$$

Then  $\partial g$  is a  $n$  cocycle, the only non vanishing terms in  $\partial g(AU_2 \dots U_n)$  are  $Ag(YB^{n-2})$  and  $g([A, Y]B^{n-2})$ . Both happen only if  $U_2 \dots U_n = YB^{n-2}$  and

$$(\partial g)(AYB^{n-2}) = (-1)^g Ag(YB^{n-2}) - g(B^{n-1}) = 0.$$

Thus  $f' = f - \partial g$  is a reduced  $n$  cocycle and  $f'(HB^{n-1}) = v_{-\frac{n-1}{2}} \in (\ker A)^{-\frac{n-1}{2}}$ . From now on, we suppose  $f$  is a reduced  $n$  cocycle such that  $f(HB^{n-1}) \in (\ker A)^{-\frac{n-1}{2}}$ .

Suppose now  $n = 2$ .

If  $f(HB)$  is in  $B(\ker A \cap \ker H)$ , we put  $g(X) = g(A) = 0$  and  $(-1)^g Bg(H) = f(HB)$  with  $Ag(H) = Hg(H) = 0$ , then we choose  $g(B)$  such that  $Ag(B) = (-1)^g 2g(H)$  and  $g(Y)$  such that  $Ag(Y) = (-1)^g g(B)$ . Then  $f - \partial g$  is a 2 cocycle vanishing on  $AX$ ,  $AA$ ,  $AB$  and  $AY$  and on  $HB$ . We saw that  $f - \partial g$  is then a coboundary. Thus,  $f$  is a coboundary.

Conversely, let  $w$  be a vector in  $(\ker A)^{-\frac{1}{2}}/B((\ker A)^0)$  (or in the supplementary space  $W$  for  $B((\ker A)^0)$  in  $(\ker A)^{-\frac{1}{2}}$ ). Then

$$\begin{aligned} ABw &= -w = (AB + BA)w, \\ -2Bw &= 2HBw = (AB + BA)Bw, \\ AB^2w &= -2Bw - BABw = -Bw. \end{aligned}$$

We put  $f(XU) = f(AU) = 0$ , for any  $U$ ,  $f(HB) = w$ , put  $f(B^2) = -4(-1)^f Bw$ ,  $f(HY) = -(-1)^f Bw$ , and  $f(YB) = 2B^2w$ . The 3 cocycle  $\partial f$  vanishes on  $A^2U$  for any  $U$ , we consider it on  $AHB$ ,  $AB^2$ ,  $AYB$  and  $AYH$ .

$$\begin{aligned} (\partial f)(AHB) &= (-1)^f Aw = 0, \\ (\partial f)(AB^2) &= -4ABw - 2f([A, B]B) = 4w - 4w = 0, \\ (\partial f)(AHY) &= -ABw + f([A, Y]H) = w - w = 0, \\ (\partial f)(AYB) &= (-1)^f Af(YB) - f([A, Y]B) + f([A, B]Y) \\ &= (-1)^f [2AB^2w + 4Bw - 2Bw] = 0. \end{aligned}$$

$\partial f$  is a reduced 3 cocycle, we moreover have

$$(\partial f)(B^3) = (-1)^f 3Bf(B^2) - 3f([B, B]B) = -12B^2w + 6f(YB) = 0.$$

Thus  $\partial f = 0$ ,  $f$  is then a reduced 2 cocycle. Now if  $f = \partial g$ , then

$$w = f(HB) = \partial g(HB) = (H + \frac{1}{2}id)g(B) - Bg(H).$$

Let  $g(B) = \sum_\alpha u_\alpha$ ,  $g(H) = \sum_\alpha x_\alpha$  and  $g(A) = \sum_\alpha y_\alpha$  where  $u_\alpha$ ,  $x_\alpha$  and  $y_\alpha$  are in  $M^\alpha$ , then we get

$$w = \sum_{\alpha \neq -\frac{1}{2}} \left( (\alpha + \frac{1}{2})u_\alpha - Bx_{\alpha+\frac{1}{2}} \right) - Bx_0.$$



But  $w$  is in  $W$ , thus,  $(\alpha + \frac{1}{2})u_\alpha - Bx_{\alpha+\frac{1}{2}} = 0$  if  $\alpha \neq -\frac{1}{2}$  and then  $w = -Bx_0$ . Moreover, we have

$$0 = f(HA) = (H - \frac{1}{2}id)g(A) - Ag(H) = \sum_{\alpha \neq \frac{1}{2}} \left( (\alpha - \frac{1}{2})y_\alpha - Ax_{\alpha-\frac{1}{2}} \right) - Ax_0.$$

Thus,  $Ax_0 = 0$ , therefore  $x_0 \in (\ker A)^0$  and  $w = -Bx_0 \in W \cap B((\ker A)^0) = \{0\}$ , this implies  $f = 0$ .

We proved the point (iii).

(iv) Suppose  $n > 2$ . We saw that any  $n$  cocycle  $f$  can be chosen such that  $f$  is reduced and  $f(HB^{n-1}) \in (\ker A)^{-\frac{n-1}{2}}$ . We define  $g$  by  $g(U_1 \dots U_{n-1}) = 0$  except

$$g(HYB^{n-3}) = -\frac{1}{(n-1)(n-2)}f(HB^{n-1})$$

and  $g(YB^{n-2})$ , chosen such that

$$Ag(YB^{n-2}) - 2(n-2)g(HYB^{n-3}) = 0.$$

Then  $(\partial g)(AHB^{n-2}) = 0$  and if  $U_2 \dots U_n \neq HB^{n-2}$ , then the only non vanishing terms in  $(\partial g)(AU_2 \dots U_n)$  have the form  $\pm g([A, U_j]U_2 \dots \hat{j} \dots U_n)$  with  $[A, U_j] = Y$ , which is impossible, or  $[A, U_j] = B$ , this means  $U_j = Y$ , but there is another index  $i \neq j$  with  $U_i = Y$  and this is still impossible or  $[A, U_j] = H$ , this means  $U_j = B$  and  $U_2 \dots U_n = HB^{n-2}$ , which is impossible. Thus  $f - \partial g$  is reduced and vanishes on  $HB^{n-1}$ , it is a coboundary,  $f$  is a coboundary,  $H^n(\mathfrak{osp}(1|2), M) = 0$ .

**3.2. Restriction to  $\mathfrak{sl}(2)$ .** We keep our notations.

**Lemma 3.4.** (Characterization for  $B((\ker A)^0)$ )

Let  $w$  be a vector in  $M$  such that  $w \in (\ker A)^{-\frac{1}{2}}$  and  $Bw \in Y((\ker X)^0)$ . Then  $w$  is in  $B((\ker A)^0)$ .

**Proof.** We suppose  $Bw = B^2v$ , with  $Hv = A^2v = 0$ . Thus  $ABv + BAv = 0$  and

$$2HBv = -Bv = AB^2v + BABv, \quad ABw = AB^2v = -Bv - BABv.$$

But

$$2Hw = (AB + BA)w = ABw = -w.$$

Or  $w = B(v + ABv)$ . But

$$2HA v = (AB + BA)Av = ABAv = Av.$$

Finally:

$$A(v + ABv) = Av + A^2Bv = Av - ABAv = 0.$$

This proves our lemma.

Let  $f$  be a  $n$  cochain for  $\mathfrak{osp}(1|2)$ . Its restriction  $f|_{\mathfrak{sl}(2)}$  to  $\mathfrak{sl}(2)^n$  is a  $n$  cochain for the  $\mathfrak{sl}(2)$  module  $M$ . If  $f$  is a cocycle (resp. a coboundary),  $f|_{\mathfrak{sl}(2)}$  is a cocycle (resp. a coboundary). The map  $f \mapsto f|_{\mathfrak{sl}(2)}$  defines a map  $\varphi$  from  $H^n(\mathfrak{osp}(1|2), M)$  to  $H^n(\mathfrak{sl}(2), M)$ .

**Proposition 3.5.** (Restriction of  $\mathfrak{osp}(1|2)$  cocycle and triviality)

A  $n$  cocycle  $f$  for  $\mathfrak{osp}(1|2)$  is a coboundary (a trivial cocycle) if and only if its restriction  $f|_{\mathfrak{sl}(2)}$  is a  $\mathfrak{sl}(2)$  coboundary. Or:  $\varphi$  is one to one.

**Proof.** We just consider  $n \leq 2$  and  $f$  choosen as in Theorem 3.1.

A 0 cocycle is a vector  $f$  in  $\ker A \cap \ker B$ , it is trivial if and only if  $f = 0$ .

A 1 cocycle is cohomologous to a cocycle  $f$  such that:

$$\begin{aligned} f(A) = f(X) = 0, \quad f(H) = u \in (\ker A)^0, \\ f(B) = v + w \in V \oplus W, \quad f(Y) = -(-1)^f B(v + w). \end{aligned}$$

Here  $V$  is a supplementary space for  $(\ker A)^{-\frac{1}{2}}$  in  $M^{-\frac{1}{2}}$ ,  $W$  a supplementary space for  $B((\ker A)^0)$  in  $(\ker A)^{-\frac{1}{2}}$ , and  $v$  is the only vector in  $V$  such that  $Av = 2(-1)^f u$ . We saw that  $f$  is characterized by  $u$  and  $w$ .

Suppose there is  $g$  in  $M$  such that  $(f - \partial g)|_{\mathfrak{sl}(2)}$  vanishes, thus  $Hg = f(H) = u$ , since  $u$  is in  $M^0$ , this relation forces  $u = 0$ , therefore  $v = 0$ . Now  $Xg = f(X) = 0$ ,  $Hg = f(H) = 0$  and  $Yg = f(Y) = -(-1)^f Bw$ . Our lemma says that  $w$  is in  $B((\ker A)^0)$ , thus  $w = 0$ ,  $f = 0$ .

A 2 cocycle is cohomologous to a cocycle  $f$  such that:

$$\begin{aligned} f(AU) = f(XU) = 0, \quad f(HB) = w \in W, \quad f(BB) = -4(-1)^f Bw, \\ f(HY) = -(-1)^f Bw, \quad f(YB) = 2B^2w. \end{aligned}$$

And  $f$  is characterized by  $w$ .

Suppose there is  $g$  in  $C^1(\mathfrak{sl}(2), M)$  such that  $(f - \partial g)|_{\mathfrak{sl}(2)}$  vanishes, put:

$$g(X) = \sum_{\alpha} x_{\alpha}, \quad g(H) = \sum_{\alpha} h_{\alpha}, \quad g(Y) = \sum_{\alpha} y_{\alpha}, \quad (x_{\alpha}, h_{\alpha}, y_{\alpha} \in M^{\alpha}).$$

We get

$$\begin{aligned} f(XH) &= Xg(H) - Hg(X) - g([X, H]) = \sum_{\alpha \neq 1} ((-\alpha + 1)x_{\alpha} + Xh_{\alpha-1}) + Xh_0 = 0, \\ f(HY) &= Hg(Y) - Yg(H) - g([H, Y]) = \sum_{\gamma \neq -1} ((\gamma + 1)y_{\gamma} - Yh_{\gamma+1}) - Yh_0 = -(-1)^f Bw. \end{aligned}$$

Since  $Bw$  is in  $M^{-1}$ , this implies  $Hh_0 = Xh_0 = 0$  and  $Yh_0 = (-1)^f Bw$ . Our lemma says that  $w$  is in  $B((\ker A)^0)$ , therefore  $w = 0$  and  $f = 0$ .

**Remark 3.6.** *In the same way as for Theorem 3.1, it is easy to compute the cohomology for the  $\mathfrak{sl}(2)$  module  $M$ . Here it is:*

$$\begin{aligned} H^0(\mathfrak{sl}(2), M) &= \ker X \cap \ker Y, \quad H^1(\mathfrak{sl}(2), M) \simeq (\ker X \cap \ker Y) \oplus (\ker X)^{-1}/Y((\ker X)^0), \\ H^2(\mathfrak{sl}(2), M) &\simeq (\ker X)^{-1}/Y((\ker X)^0), \quad H^{>2}(\mathfrak{sl}(2), M) = 0. \end{aligned}$$

#### 4. APPLICATION TO $\mathfrak{D}_{\lambda, \mu}$

##### 4.1. Differential operators on weighted densities.

We define the superspace  $\mathbb{R}^{1|1}$  in terms of its superalgebra of functions, denoted by  $C^{\infty}(\mathbb{R}^{1|1})$  and consisting of elements of the form:

$$F(x, \theta) = f_0(x) + f_1(x)\theta,$$

where  $x$  is the even variable,  $\theta$  is the odd variable ( $\theta^2 = 0$ ) and  $f_0(x), f_1(x) \in C^{\infty}(\mathbb{R})$ . We consider the contact bracket on  $C^{\infty}(\mathbb{R}^{1|1})$  defined on  $C^{\infty}(\mathbb{R}^{1|1})$  by:

$$\{F, G\} = FG' - F'G + \frac{1}{2}\eta(F)\bar{\eta}(G),$$

where  $\eta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$  and  $\bar{\eta} = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x}$ . Let  $\text{Vect}(\mathbb{R}^{1|1})$  be the superspace of vector fields on  $\mathbb{R}^{1|1}$ :

$$\text{Vect}(\mathbb{R}^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_i \in C^\infty(\mathbb{R}^{1|1}) \right\},$$

where  $\partial_\theta$  stands for  $\frac{\partial}{\partial \theta}$  and  $\partial_x$  stands for  $\frac{\partial}{\partial x}$ . We can realize the algebra  $\mathfrak{osp}(1|2)$  as a subalgebra of  $\text{Vect}(\mathbb{R}^{1|1})$ :

$$\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta}, X_\theta).$$

where, the vector field  $X_G$  is defined for any  $G \in C^\infty(\mathbb{R}^{1|1})$  by

$$X_G = G \partial_x + \frac{1}{2} \eta(G) \bar{\eta}.$$

Here, we have  $(-X_x, X_1, -X_{x^2}, 2X_\theta, X_{x\theta}) = (H, X, Y, A, B)$ . The bracket on  $\mathfrak{osp}(1|2)$  is then given by  $[X_F, X_G] = X_{\{F, G\}}$ .

We denote by  $\mathfrak{F}_\lambda$  the space of all weighted densities on  $\mathbb{R}^{1|1}$  of weight  $\lambda$ :

$$\mathfrak{F}_\lambda = \left\{ F(x, \theta) \alpha^\lambda \mid F(x, \theta) \in C^\infty(\mathbb{R}^{1|1}) \right\} \quad (\alpha = dx + \theta d_\theta).$$

The action of  $\mathfrak{osp}(1|2)$  on  $\mathfrak{F}_\lambda$  is given by

$$X_G(F \alpha^\lambda) = ((G \partial_x + \frac{1}{2} \eta(G) \bar{\eta})(F) + \lambda G' F) \alpha^\lambda.$$

Any differential operator  $A$  on  $\mathbb{R}^{1|1}$  defines a linear mapping from  $\mathfrak{F}_\lambda$  to  $\mathfrak{F}_\mu$  for any  $\lambda$  by:  $A : F \alpha^\lambda \mapsto A(F) \alpha^\mu$ ,  $\mu \in \mathbb{R}$ , thus, the space of differential operators becomes a family of  $\mathfrak{osp}(1|2)$  modules denoted  $\mathfrak{D}_{\lambda, \mu}$ , for the natural action:

$$X_G \cdot A = X_G \circ A - (-1)^{AG} A \circ X_G.$$

For more details see, for instance [1, 2, 3, 5]

## 4.2. Cohomology.

Let us consider the  $\mathfrak{osp}(1|2)$ -module  $\mathfrak{D}_{\lambda, \mu}$  of differential operators on densities on  $\mathbb{R}^{1|1}$ .

We put here  $p = \mu - \lambda$  and choose the following basis for  $\mathfrak{D}_{\lambda, \mu}$ :

$$a_{m,k} = x^m \partial_x^k, \quad b_{m,k} = x^m \theta \partial_\theta \partial_x^k, \quad c_{m,k} = x^m \theta \partial_x^k, \quad d_{m,k} = x^m \partial_\theta \partial_x^k - x^m \theta \partial_x^{k+1}.$$

(Here,  $m$  and  $k$  are natural integral numbers), we say that  $a_{m,k}$  and  $b_{m,k}$  are even vectors (see below) and  $c_{m,k}$  and  $d_{m,k}$  are odd vectors.

In fact they are weight vectors for the action of  $H$ :

$$\begin{aligned} H a_{m,k} &= (k - m - p) a_{m,k} & H b_{m,k} &= (k - m - p) b_{m,k} \\ H c_{m,k} &= (k - m - p - \frac{1}{2}) c_{m,k} & H d_{m,k} &= (k - m - p + \frac{1}{2}) d_{m,k}. \end{aligned}$$

Similarly, a direct computation give the following relations for the  $A$  and  $B$  actions on these vectors :

$$\begin{aligned} A a_{m,k} &= m c_{m-1,k}, & A b_{m,k} &= d_{m,k}, \\ A c_{m,k} &= a_{m,k}, & A d_{m,k} &= m b_{m-1,k} \end{aligned}$$

and

$$\begin{aligned} B a_{m,k} &= (m - 2k + 2p) c_{m,k} - k d_{m,k-1}, & B b_{m,k} &= d_{m+1,k} - (2\lambda + k) c_{m,k}, \\ B c_{m,k} &= a_{m+1,k} + k b_{m,k-1}, & B d_{m,k} &= (m - 2k + 2p - 1) b_{m,k} + (2\lambda + k) a_{m,k}. \end{aligned}$$

From these formulas (or directly), we can compute the  $X$  and  $Y$  actions, getting:

$$\begin{aligned} Xa_{m,k} &= ma_{m-1,k}, & Xb_{m,k} &= mb_{m-1,k}, \\ Xc_{m,k} &= mc_{m-1,k}, & Xd_{m,k} &= md_{m-1,k}, \end{aligned}$$

and

$$\begin{aligned} Ya_{m,k} &= (2k - 2p - m)a_{m+1,k} + k(2\lambda + k - 1)a_{m,k-1} + kb_{m,k-1}, \\ Yb_{m,k} &= (2k - 2p - m)b_{m+1,k} + k(2\lambda + k)b_{m,k-1}, \\ Yc_{m,k} &= (2k - 2p - m - 1)c_{m+1,k} + k(2\lambda + k - 1)c_{m,k-1}, \\ Yd_{m,k} &= (2k - 2p - m + 1)d_{m+1,k} + k(2\lambda + k)d_{m,k-1} - (2\lambda + 2k + 1)c_{m,k}. \end{aligned}$$

From these formulas, we immediately get

$$\ker A \cap \ker B = \begin{cases} \text{Span}(a_{0,0}) & \text{if } p = 0, \\ \text{Span}(d_{0,k}) & \text{if } p = k + \frac{1}{2}, k \in \{0, 1, 2, \dots\} \text{ and } 2\lambda + k = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

and

$$(\ker A)^{-\frac{1}{2}} = \begin{cases} \text{Span}(a_{0,k}) & \text{if } p = k + \frac{1}{2}, k \in \{0, 1, 2, \dots\}, \\ \text{Span}(d_{0,k}) & \text{if } p = k + 1, k \in \{0, 1, 2, \dots\}, \\ 0 & \text{elsewhere.} \end{cases}$$

Moreover, if  $p = k + \frac{1}{2}$ ,  $(\ker A)^0 = \text{Span}(d_{0,k})$ ,  $B((\ker A)^0) = \text{Span}(a_{0,k})$  if  $2\lambda + k \neq 0$ , 0 if it is not the case. Similarly, if  $p = k + 1$ , then  $B((\ker A)^0) = B(\text{Span}(a_{0,k+1})) = \text{Span}(d_{0,k})$ .

Now we deduce :

**Proposition 4.1.** (The cohomology for  $\mathfrak{D}_{\lambda,\mu}$ )

The dimensionalities for the cohomology groups  $H^n(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$  are:

$$\begin{aligned} (i) \quad \dim(H^0(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) &= \begin{cases} 1 & \text{if } \lambda = \mu, \\ 1 & \text{if } \lambda = -\frac{k}{2} \text{ and } \mu = \frac{k+1}{2}, k \in \{0, 1, 2, \dots\}, \\ 0 & \text{in the other cases.} \end{cases} \\ (ii) \quad \dim(H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) &= \begin{cases} 1 & \text{if } \lambda = \mu, \\ 2 & \text{if } \lambda = -\frac{k}{2} \text{ and } \mu = \frac{k+1}{2}, k \in \{0, 1, 2, \dots\}, \\ 0 & \text{in the other cases.} \end{cases} \\ (iii) \quad \dim(H^2(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) &= \begin{cases} 1 & \text{if } \lambda = -\frac{k}{2} \text{ and } \mu = \frac{k+1}{2}, k \in \{0, 1, 2, \dots\}, \\ 0 & \text{in the other cases.} \end{cases} \\ (iv) \quad \dim(H^n(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})) &= 0. \end{aligned}$$

We remind here the results of [2] for the  $H^1$ .

To be more precisely, in the following, we give explicit basis for these cohomology groups

- (i)  $H^0(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\lambda}) = \text{Span}(id)$  and  $H^0(\mathfrak{osp}(1|2), \mathfrak{D}_{-\frac{k}{2}, \frac{k+1}{2}}) = \text{Span}(\partial_\theta \partial_x^k - \theta \partial_x^{k+1})$ .
- (ii) The space  $H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\lambda})$  is spanned by the cohomology class of the reduced 1 cocycle  $h_\lambda$  defined by:

$$h_\lambda(X) = h_\lambda(A) = 0, \quad h_\lambda(H) = -id, \quad h_\lambda(B) = \theta \cdot \quad \text{and} \quad h_\lambda(Y) = -2x \cdot.$$

While the space  $H^1\left(\mathfrak{osp}(1|2), \mathfrak{D}_{-\frac{k}{2}, \frac{k+1}{2}}\right)$  is spanned by the cohomology classes of the reduced 1 cocycles  $f_k$  and  $\tilde{f}_k$  defined respectively by:

$$f_k(X) = f_k(A) = 0, \quad f_k(H) = \partial_\theta \partial_x^k - \theta \partial_x^{k+1}, \quad f_k(B) = \theta \partial_\theta \partial_x^k \text{ and } f_k(Y) = 2x f_k(H),$$

$$\tilde{f}_k(X) = \tilde{f}_k(A) = \tilde{f}_k(H) = 0, \quad \tilde{f}_k(B) = \partial_x^k \text{ and } \tilde{f}_k(Y) = -2k \partial_\theta \partial_x^{k-1} + 2\theta(k+1) \partial_x^k.$$

(iii) A similar realization of  $H^2\left(\mathfrak{osp}(1|2), \mathfrak{D}_{-\frac{k}{2}, \frac{k+1}{2}}\right)$  is easy, we prefer to give an explicit, nontrivial, reduced 2 cocycle as a cup product. Let

$$\Omega_k(U, V) = (f_k \vee h_{-\frac{k}{2}})(U, V) := f_k(U) \circ h_{-\frac{k}{2}}(V) - (-1)^{UV} f_k(V) \circ h_{-\frac{k}{2}}(U).$$

Since  $f_k$  and  $h_{-\frac{k}{2}}$  are cocycles, a direct computation shows that  $\Omega_k$  is a 2 cocycle, it is nontrivial since its restriction to  $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$  is nontrivial:

$$\Omega_k(X_f, X_g) = -(-1)^k \omega(f, g)(k \partial_\theta \partial_x^{k-1} - (k+1) \theta \partial_x^k)$$

where  $\omega$  is the Gelfand-Fuchs cocycle defined by  $\omega(f, g) = f'g'' - g'f''$ .

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