



HAL
open science

Ergodicity of self-attracting motion

Victor A. Kleptsyn, Aline Kurtzmann

► **To cite this version:**

| Victor A. Kleptsyn, Aline Kurtzmann. Ergodicity of self-attracting motion. 2009. hal-00400810v1

HAL Id: hal-00400810

<https://hal.science/hal-00400810v1>

Preprint submitted on 2 Jul 2009 (v1), last revised 31 Mar 2010 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ERGODICITY OF SELF-ATTRACTING MOTION

VICTOR KLEPTSYN, ALINE KURTZMANN

ABSTRACT. The aim of this paper is to study the asymptotic behaviour of a class of self-attracting motions on \mathbb{R}^d . Using stochastic approximation methods, these processes have already been studied by Benaïm, Ledoux and Raimond (2002) in a compact setting. We also relate the asymptotic behaviour of the self-attracting Brownian motion to the McKean-Vlasov process that was studied, via the decrease of the free energy, by Carrillo, McCann and Villani (2003). Mixing these methods, we manage to obtain sufficient conditions for the (limit-quotient) ergodicity of the self-attracting diffusion, together with a speed of convergence.

1. INTRODUCTION

1.1. Statement of the problem. This text is devoted to study the asymptotic behaviour of a Brownian motion, interacting with its own passed trajectory, so-called “self-interacting Brownian motion”. Namely, we fix an interaction potential function $W : \mathbb{R}^d \rightarrow \mathbb{R}$, and consider a stochastic differential equation

$$(1) \quad dX_t = \sqrt{2} dB_t - \left(\frac{1}{t} \int_0^t \nabla W(X_t - X_s) ds \right) dt,$$

where B_t is a standard Brownian motion, with an initial condition of given X_0 (with the condition of continuity at $t = 0$). This equation can be rewritten using the normalized occupation measure μ_t :

$$\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds,$$

where δ_x is the Dirac measure, concentrated at the point x . Using this convention, the equation (1) becomes

$$(2) \quad dX_t = \sqrt{2} dB_t - \nabla W * \mu_t(X_t) dt,$$

where $*$ stands for the convolution.

Note that the equations (1), (2) clearly have singularities at $t = 0$, which is the reason why sometimes these equations are considered only after some positive moment $r > 0$. We discuss the existence and uniqueness questions for the solution below, in Section 2.

Similar questions have already been studied since the 90's. The first time-continuous self-interacting processes have been introduced by Durrett and Rogers [9] under the name of “Brownian polymers”. They are solutions to SDEs of the form

$$(3) \quad dX_t = dB_t + \left(\int_0^t f(X_t - X_s) ds \right) dt$$

where $(B_t; t \geq 0)$ is a standard Brownian motion and f a given function. We remark that, in the latter equation, the drift term is given by the non-normalized measure $t\mu_t$ and not by μ_t as the process we will study here. As the process $(X_t; t \geq 0)$ evolves in an environment

The second author has been supported by the Swiss National Science Foundation grant PBNE2-119027.

changing with its past trajectory, this SDE defines a self-interacting diffusion, which can be either self-repelling or self-attracting, depending on the function f . In any dimension, Durrett & Rogers obtained that $|X_t|/t$ is bounded (by a deterministic constant) whenever f has a compact support. But, very few results are known as soon as f takes values of both signs.

Self-interacting diffusions, with dependence on the (convolved) normalized occupation measure $(\mu_t, t \geq 0)$ have been considered since the work of Benaïm, Ledoux & Raimond [3]. The main difference between these diffusions and Brownian polymers is that the drift term is divided by t . This implies that the long-time away interaction is less important than the near-time interaction (the interaction is not “uniform in time” anymore). The processes of Durrett & Rogers, and Benaïm, Ledoux & Raimond have been introduced to modelize the evolution of ants or polymers for instance.

Benaïm et al., in [3] and [4], have shown that the asymptotic behaviour of μ_t can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. Afterwards, one can go further in this study and give sufficient conditions for the a.s. convergence of the empirical measure. It happens that with a symmetric interaction, μ_t converges a.s. to a local minimum of a nonlinear free energy functional (each local minimum having a positive probability to be chosen). These results are valid for a compact manifold. They have recently been generalized to \mathbb{R}^d (see [10]) assuming a confinement potential satisfying some conditions –these hypotheses on the confinement potential are required since in general the process can be transient, and is thus very difficult to analyze). Most of these results are summarized in a recent survey of Pemantle [13], which also includes results concerning self-interacting random walks.

Coming back to the process introduced by Durrett and Rogers, all the results obtained have in common that the drift may overcome the noise, so that the randomness of the process is “controlled”. To illustrate that, let us mention, for the same model of Durrett & Rogers, the case of a repulsive and compactly supported function f , that was conjectured in 1992 in [9] and stays still unsolved:

Conjecture (Durrett & Rogers [9]). Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an odd function, of compact support. Then, for the process X defined by (3), the quotient X_t/t converges a.s. to 0.

In (1), the drift term is divided by t , and so it is bounded for a compactly supported interaction W . From that point of view, the study of (1) is as difficult as the process of the conjecture. Indeed, the interaction potential is in general not strong enough for the process to be recurrent, and the behaviour is then very difficult to analyze. In particular, it is hard to predict the relative importance of the drift term (in competition with the Brownian motion) in the evolution. Nevertheless, as the interaction potential is attractive enough, the diffusion (a bit modified) is comparable to an Ornstein-Uhlenbeck process and so, we have access to its ergodic behaviour.

Let us now point out a similarity between the self-attracting diffusion studied and the Markov diffusion corresponding to McKean & Vlasov’s PDE. Consider the Markov process defined by the SDE

$$(4) \quad dY_t = \sqrt{2} dB_t - \nabla W * \nu_t(Y_t) dt,$$

where ν_t stands for the *law* of Y_t , and W is a strictly uniformly convex function. When the random measure μ_t converges (for instance when W is symmetric), we emphasize that the asymptotic behaviour of ν_t is then close to the asymptotic behaviour of μ_t , corresponding to

the diffusion process (1). The Markov process (4) corresponds to a famous PDE introduced by MacKean and Vlasov. The asymptotic law of this process has been intensively studied these last years, using some mass transportation techniques and functional inequalities. Indeed, denote

$$\Pi(\nu)(dx) := \frac{1}{Z} e^{-W*\nu(x)} dx,$$

where Z is the normalization constant. Consider the process Y , starting from an initial law ν^* being a “fixed point” of Π (which is unique for a strictly convex W when the center of mass is fixed), that is a probability measure solution to $\nu = \Pi(\nu)$. Then the probability measure ν^* is invariant for the process Y . So, one wishes to prove that each law ν_t (starting from any absolutely continuous probability measure) converges to ν^* . In particular, Carrillo, McCann & Villani [6] have shown that the relative free energy corresponding to ν_t with respect to ν^* decreases exponentially fast to 0. Moreover, by Talagrand’s inequality, one compares the relative free energy and the Wasserstein distance in case of uniform convexity, and so they have obtained the decrease to 0 of the quadratic Wasserstein distance between ν_t and ν^* . Note that Bolley, Guillin & Villani [5], or Cattiaux, Guillin & Malrieu [8] have also studied the asymptotics of this Markov process.

We remark that a huge difference between the preceding Markov process and the (non-Markov) self-interacting diffusion is that the asymptotic σ -algebra is in general not trivial for the latter diffusion process.

1.2. Main result. Our results are analogous to those of Carrillo et al. [6]: under some assumptions imposed on the interaction potential W , we show that the normalized occupation measure μ_t almost surely converges to an equilibrium state, which is unique up to a translation:

Theorem 1 (Main result). *Suppose, that $W \in C^2(\mathbb{R}^d)$, and:*

- 1) *spherical symmetric: $W(x) = W(|x|)$;*
- 2) *uniformly convex:*

$$(5) \quad \exists C_W > 0 : \quad \forall x \in \mathbb{R}^d, \forall v \in \mathbb{S}^{d-1}, \quad \left. \frac{\partial^2 W}{\partial v^2} \right|_x \geq C_W;$$

- 3) *W has at most a polynomial growth: for some symmetric polynomial P , we have*

$$(6) \quad \forall x \in \mathbb{R}^d \quad |W(x)| + |\nabla W(x)| + \|\nabla^2 W(x)\| \leq P(|x|);$$

Then, there exists a unique symmetric density $\rho_\infty : \mathbb{R}^d \rightarrow \mathbb{R}_+$, such that almost surely, there exists c_∞ such that

$$\mu_t \xrightarrow[t \rightarrow \infty]{*-weakly} \rho_\infty(x - c_\infty) dx.$$

Remark 1. The assumption 1) corresponds to the physical assumption of the interaction force between two particles being directed along the line joining them.

The origin of the following remark will be clear after the discussion in §4

Remark 2. The density ρ_∞ is the same limit density as in the result of [6], uniquely defined (among the centered densities) by the following property: ρ_∞ is a positive function, proportional to $e^{-W*\rho_\infty}$.

We can also consider the same drifted motion in presence of an external potential V . For this, the following result is a generalization of Theorem 1:

Theorem 2. *Let X be the solution to the equation*

$$(7) \quad dX_t = \sqrt{2}dB_t - \left(\nabla V(X_t) - \frac{1}{t} \int_0^t \nabla W(X_t - X_s) ds \right) dt$$

Suppose, that $V \in C^2(\mathbb{R}^d)$ and $W \in C^2(\mathbb{R}^d)$, and:

- 1) *spherical symmetric: $W(x) = W(|x|)$;*
- 2) *V and W are convex, $\lim_{|x| \rightarrow \infty} V(x) = +\infty$, and one of them is uniformly convex:*

$$(8) \quad \exists C > 0 : \quad \forall x \in \mathbb{R}^d, \forall v \in \mathbb{S}^{d-1}, \quad \left. \frac{\partial^2 V}{\partial v^2} \right|_x \geq C \quad \text{or} \quad \left. \frac{\partial^2 W}{\partial v^2} \right|_x \geq C;$$

- 3) *V and W have at most a polynomial growth: for some symmetric polynomial P we have*

$$(9) \quad \forall x \in \mathbb{R}^d \quad |V(x)| + |W(x)| + |\nabla V(x)| + |\nabla W(x)| + \|\nabla^2 V(x)\| + \|\nabla^2 W(x)\| \leq P(|x|).$$

Then, there exists a unique density $\rho_\infty : \mathbb{R}^d \rightarrow \mathbb{R}_+$, such that almost surely

$$\mu_t \xrightarrow[t \rightarrow \infty]{*-weakly} \rho_\infty(x) dx.$$

As the proof of the latter Theorem coincides with the proof of Theorem 1 almost identically, we do not present it here. It suffices to add V in the arguments below. Moreover, if V admits a unique minimum, then the corresponding c_∞ is the minimum of V .

The proof of Theorem 1 is split into two parts. Consider a natural “reference point” for a measure μ :

Definition 1. For a measure μ on \mathbb{R}^d , its *center* is the point c_μ such that $W' * \mu(c_\mu) = 0$, or, equivalently, the point where the convolution $W * \mu$ (the potential generated by μ) takes its minimal value. Also, we define the *centered* measure μ^c as the translation of the measure μ , bringing c_μ to the origin:

$$(10) \quad \mu^c(A) = \mu(A + c_\mu).$$

Remark 3. This notion of center had previously been introduced by Raimond in [14]. Indeed, to study the linear d -dimensional case of Brownian polymers, Raimond defined the center and proved that the process X remains close to c_t (and that c_t converges a.s.).

The first part of the proof of Theorem 1 consists of proving the convergence of centered occupation measures:

Theorem 3. *Under the assumptions of Theorem 1, for some symmetric density function $\rho_\infty : \mathbb{R}^d \rightarrow \mathbb{R}_+$ almost surely*

$$\mu_t^c \xrightarrow[t \rightarrow \infty]{*-weakly} \rho_\infty(x) dx$$

The second is the convergence of centers:

Theorem 4. *Under the assumptions of Theorem 1 almost surely the centers $c_t := c_{\mu_t}$ converge to some (random) limit c_∞ .*

It is clear that the two latter theorems imply the main result. Let us sketch their proof.

1.3. Outline of the proof.

1.3.1. *Existence and uniqueness.* First, a standard remark is Markovianization: the behaviour of the pair (X_t, μ_t) is Markovian. We make it, together with some other standard remarks, in §2.1. Unfortunately, the Markov process (X_t, μ_t) is infinite-dimensional and, in general, we cannot reduce to a finite-dimensional process. So, we cannot use this information directly in order to obtain interesting properties on μ_t , because the state space is then too large.

After that, we discuss the global existence and uniqueness for the solutions of (2) in Section 2.

1.3.2. *Discretization.* A next step is discretization: we take a (well-chosen and deterministic) sequence of moments $T_n \rightarrow \infty$, with $T_n \gg T_{n+1} - T_n \gg 1$, and consider the behaviour of the measures μ_{T_n} . As $T_n \gg T_{n+1} - T_n$, it is natural to expect (and we will give the corresponding statement) that the occupation measures μ_t on the interval $[T_n, T_{n+1}]$ almost do not change and thus stay close to μ_{T_n} . So, on this interval we can approximate the solution X_t of equation (2) by the solution of the same equation with $\mu_t \equiv \mu_{T_n}$:

$$dY_t = \sqrt{2} dB_t - \nabla W * \mu_{T_n}(Y_t) dt, \quad t \in [T_n, T_{n+1}],$$

in other words, by a Brownian motion in a potential $W * \mu_{T_n}$ that does not depend on time.

On the other hand, the interval $T_{n+1} - T_n$ becomes longer and longer. So, using Birkhoff Ergodic Theorem¹, we see that the (normalized) distribution $\mu_{[T_n, T_{n+1}]}$ of values of X_t on these intervals becomes closer and closer to the equilibrium measures $\Pi(\mu_{T_n})$ for a Brownian motion in the potential $W * \mu_{T_n}$, where (see Section 5)

$$\Pi(\mu)(dx) := \frac{1}{Z(\mu)} e^{-(W * \mu)(x)} dx, \quad Z(\mu) := \int_{\mathbb{R}^d} e^{-(W * \mu)(x)} dx.$$

But

$$\mu_{T_{n+1}} = \frac{T_n}{T_{n+1}} \mu_{T_n} + \frac{T_{n+1} - T_n}{T_{n+1}} \mu_{[T_n, T_{n+1}]},$$

so we have

$$\mu_{T_{n+1}} \approx \frac{T_n}{T_{n+1}} \mu_{T_n} + \frac{T_{n+1} - T_n}{T_{n+1}} \Pi(\mu_{T_n}) = \mu_{T_n} + \frac{T_{n+1} - T_n}{T_{n+1}} (\Pi(\mu_{T_n}) - \mu_{T_n}),$$

and

$$\frac{\mu_{T_{n+1}} - \mu_{T_n}}{T_{n+1} - T_n} \approx \frac{1}{T_{n+1}} (\Pi(\mu_{T_n}) - \mu_{T_n}).$$

This motivates us to approximate the behaviour of measures μ_t by trajectories of the flow (on the infinite-dimensional space of measures!)

$$\dot{\mu} = \frac{1}{t} (\Pi(\mu) - \mu),$$

or after a logarithmic change of variables $\theta = \log t$,

$$\mu' = \Pi(\mu) - \mu.$$

These passages are rigorously done in Section 5.

¹see for instance [18], chap. XIII

1.3.3. *Flow $\dot{\mu} = \Pi(\mu) - \mu$.* In Section 3, we study the properties of the flow $\dot{\mu} = \Pi(\mu) - \mu$. First (as it is a flow on the infinite-dimensional space) we have to show local existence and uniqueness of its solutions. It is done by standard means (existence theorem for flows with Lipschitz right hand-side), together with some estimates for $\Pi(\mu)$, see §3.2. Then, using some estimates (provided in §2.2), we show global existence of the trajectories. Moreover, descending this flow on the space of centered measures (as a topological flow), we find a positively invariant compact set of measures: the probability measures with an exponential decrease.

Then, we pass to a physical interpretation of this flow in Section 4. This interpretation motivates to consider the “entropy” function for the measure μ (as was previously done by Carrillo et al. [6]), which turns out to be a Lyapunov function for this flow.

1.3.4. *Proof of Theorem 3.* We will proceed in two steps. The idea is to approximate our random object, that is the normalized occupation measure of the process, by a well-chosen deterministic flow.

First, we work with our main dynamical system: $\dot{\mu} = \Pi(\mu) - \mu$. We prove that every trajectory of this flow converges weakly to the unique centered-fixed point of Π . After that, it remains to show that the centered empirical measure itself converges. To this aim, we recall the notion of asymptotic pseudotrajectory. That means that the occupation measure is close (in a certain way we will explain later) to a deterministic flow and that the asymptotic properties of the random measure are the same as the deterministic flow. This will be done rigorously in §5.2.1.

1.4. **Centers convergence for $\dot{\mu} = \Pi(\mu) - \mu$.** Consider a deterministic trajectory μ_t of this flow. At this moment, it will already be shown in Section 3, that the centered measures μ_t^c converge to $\mu_\infty^c(dx) := \rho_\infty(x) dx$. Now, we are going to deduce that the centers c_t of μ_t converge, in other words, that the integral $\int_1^\infty \dot{c}_t dt$ converges.

But the drift velocity \dot{c}_t is translation-invariant, so it can be represented by a function $\varphi(\mu_t^c)$. The density ρ_∞ being symmetric, $\varphi(\mu_t^c)$ tends to 0. So, to show the convergence of the integral of the drift, one has to estimate, on one hand, the function φ in terms of distance between μ^c and ρ_∞ , and on the other hand, the speed of the convergence of μ_t^c to ρ_∞ . The first step is technical and is done in §6.1.

For the second step, we will use the estimates of exponential convergence given by Carrillo, McCann & Villani in [6]. Namely, to obtain a control on the speed of convergence of μ_t^c to ρ_∞ , it (more or less) suffices to estimate the decrease rate for the normalized *free energy* — the Lyapunov function \mathcal{F} given in Section 4, — that we will obtain using the *displacement convexity* of \mathcal{F} .

Now, the physical interpretation described in §4 and the estimates of Carrillo et al. motivate the inequalities, which provide the latter bound for $\mathcal{F}(\Pi(\mu))$ (up to some technical details, which we do not precise at the moment). We do this in §4.2.

Finally, the (deterministic) trajectories of $\dot{\mu} = \Pi(\mu) - \mu$ are not exactly the (random!) trajectories of the normalized occupation measure μ_t . The discretization procedure allows to interpret the latter ones as trajectories of the flow $\dot{\mu} = \Pi(\mu) - \mu$, suffering some random perturbations (smaller and smaller as the time tends to the infinity). So, to conclude the proof of Theorem 4, we improve the estimates, including the effects coming from these random perturbations.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

2.1. Markovian form; local existence and uniqueness. First step in studying the trajectories of SDE (2) is to pass to the couple (X_t, μ_t) . A standard remark is that the behaviour of this couple is (infinite-dimensional) Markovian. (Note that, in general, we cannot reduce the study to a finite-dimensional Markov process.) This reduction is easily implied by the identity

$$(11) \quad \mu_{t+s} = \frac{t}{t+s} \mu_t + \frac{1}{t+s} \int_t^{t+s} \delta_{X_u} du.$$

Note that the second term in the right hand-side of (11) can be written as $\frac{s}{t+s} \mu_{[t, t+s]}$, where $\mu_{[t, t+s]}$ is the normalized occupation measure during the time interval $[t, t+s]$:

$$\mu_{[t_1, t_2]} := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \delta_{X_u} du.$$

Now, passing μ_t to the left hand side of (11), dividing by s and passing to the limit as $s \rightarrow 0$, we obtain the following SDE for the couple (X_t, μ_t) :

$$(12) \quad \begin{cases} dX_t = \sqrt{2} dB_t - \nabla W * \mu_t(X_t) dt, \\ \dot{\mu}_t = \frac{1}{t} (-\mu_t + \delta_{X_t}). \end{cases}$$

For any $t_0 > 0$, the local existence and uniqueness of solutions to (12), in a neighborhood of t_0 , is implied by well-known arguments: see Theorem 11.2 of [15].

However, in order to study the asymptotic behaviour of solutions to (1), we first should show the global existence of these solutions, in other words, that they do not explode in a finite time. It will be done in §2.2.

Also, note that the equation (12) clearly has a singularity at $t = 0$. To avoid this singularity, sometimes the equation (12) is considered with an initial condition (X_r, μ_r) at some positive moment $r > 0$ (and thus for $t \in [r, \infty)$). After the time-shift $s = t - r$, the system (12) transforms to

$$(13) \quad \begin{cases} dX_s = \sqrt{2} dB_s - \nabla W * \mu_s(X_s) ds, \\ \dot{\mu}_s = \frac{1}{s+r} (-\mu_s + \delta_{X_s}). \end{cases}$$

In fact, we can restrict our consideration to such situations only (as, anyway, we are interested in the asymptotic behaviour of solutions at infinity), but it is interesting to show that the equation (1) has indeed existence and uniqueness of solutions for any initial value problem $X_0 = x_0$. It is done below, in §2.5.

2.2. Center-drift estimates. A natural “reference point” that one can associate to a measure μ is the equilibrium point $c = c_\mu$ of the potential it generates with W , defined by the equation $\nabla W * \mu(c) = 0$ (see Definition 1, §1.2), that we refer to as the *center* of the measure μ . Also, it will be convenient to consider the *centered measure* μ^c , obtained from μ by the translation that shifts the center to the origin.

Note that the Implicit Function Theorem allows to estimate (on an interval of existence of solution (X_t, μ_t) to (12)) the derivative \dot{c}_t of $c_t := c_{\mu_t}$. In particular, we will see that c_t is a C^1 -function on this interval.

Indeed, the function $(x, t) \mapsto \nabla W * \mu_t(x)$ is C^1 -smooth:

$$d(\nabla W * \mu_t)(x) = \nabla^2 W * \mu_t(x) dx + \nabla W * \dot{\mu}_t(x) dt = \nabla^2 W * \mu_t(x) dx + \frac{1}{t} \nabla W * (-\mu_t + \delta_{X_t})(x) dt,$$

and for any (x, t) we have $\nabla^2 W * \mu_t(x) \geq C_W > 0$. The Implicit Function Theorem thus implies that c_t is a C^1 -function of t (on the interval of existence of solution), and that

$$\begin{aligned} \dot{c}_t &= - \left(\frac{\partial}{\partial x} \nabla W * \mu_t(x) \Big|_{x=c_t} \right)^{-1} \frac{\partial}{\partial t} (\nabla W * \mu_t)(c_t) = -\frac{1}{t} (\nabla^2 W * \mu_t(c_t))^{-1} (\nabla W * \delta_{X_t})(c_t) \\ &= -\frac{1}{t} (\nabla^2 W * \mu_t(c_t))^{-1} \nabla W(-X_t + c_t) = \frac{1}{t} (\nabla^2 W * \mu_t(c_t))^{-1} \nabla W(X_t - c_t). \end{aligned}$$

This (rather naturally) justifies, that the center drifts towards X_t in dimension 1, and gives an upper bound on its speed in terms of distance between them:

$$(14) \quad |\dot{c}_t| \leq \frac{1}{t} \cdot \frac{P(|X_t - c_t|)}{C_W}.$$

2.3. Law of X -center distances: Ornstein-Uhlenbeck estimate. To continue our study, first we would like to obtain an estimate on the behaviour of the distance $|X_t - c_t|$. Namely, we are going to compare it with the (absolute value of) Ornstein-Uhlenbeck process, and to obtain exponential-decrease bounds on its occupation measure.

2.3.1. One-dimensional estimate. Note, that if $X_t \neq c_t$, we can write

$$d|X_t - c_t| = \sqrt{2} \operatorname{sign}(X_t - c_t) dB_t - \operatorname{sign}(X_t - c_t) \cdot \left(W' * \mu_t(X_t) + \frac{W'(X_t - c_t)}{t W'' * \mu_t(c_t)} \right) dt.$$

The Brownian term has constant intensity (and has the same law as the Brownian motion β_t); on the other hand, the drift is negative, and its absolute value admits a lower bound

$$\left| W' * \mu_t(X_t) + \frac{1}{t} \cdot \frac{W'(X_t - c_t)}{W'' * \mu_t(c_t)} \right| \geq |W' * \mu_t(X_t)| \geq \frac{C_W}{2} |X_t - c_t|,$$

because the second derivative $W'' * \mu_t$ of $W * \mu_t$ is estimated from below by C_W due to the same estimate for W .

Thus, the trajectories of $|X_t - c_t|$ drift towards zero at least as quickly as the trajectories of an Ornstein-Uhlenbeck process

$$(15) \quad dZ_t = \sqrt{2} d\beta_t - \frac{C_W}{2} Z_t dt, \quad Z_0 = X_0 - c_0.$$

This justifies the following, by the usual comparison result:

Lemma 1. *There is a probability-preserving measurable (with the natural filtrations) mapping from the trajectories of (1) to those of (15), such that for any pair of corresponding trajectories X_t and Z_t , we have almost surely*

$$|X_t - c_t| \leq |Z_t|.$$

2.3.2. d -dimensional case. In dimension $d > 1$, in case if the potential $W(x)$ depends not only on the norm $|x|$, the drift of c_t can be directed outwards X_t . Indeed, this component is given by $\frac{1}{t} (\nabla^2 W * \mu_t(c_t))^{-1} \nabla W(X_t - c_t)$, and both angles between $X_t - c_t$ and the corresponding gradient of W , and between the gradient of W and the matrix $(\nabla^2 W * \mu_t(c_t))^{-1}$ applied to it, can be big (almost 90 degrees). So, this is the place where the assumption 1) is used (see Remark 1), in order to get $(\dot{c}_t, X_t - c_t) \geq 0$. Thus, we show the following comparison result, which proof is similar to the proof of Lemma 1. Denote Z the d -dimensional Ornstein-Uhlenbeck process

$$dZ_t = \sqrt{2} dB_t - \frac{C_W}{2} Z_t dt, \quad Z_0 = X_0 - c_0.$$

Lemma 2. *We can compare the trajectories of (1) to those of (15), such that for any pair of corresponding trajectories X_t and Z_t , we have almost surely*

$$|X_t - c_t| \leq |Z_t|.$$

2.4. Global existence.

Proposition 1 (of global existence). *For any $r > 0$ and for any initial condition (X_r, μ_r) the solution to (12) exists (and is unique) on the whole interval $[r, +\infty)$.*

Proof. As we already have the local existence and uniqueness, it suffices to check that the solution X_t cannot explode in a finite time (this impossibility will imply that the measures μ_t , as the occupation measures of X_t , also stay in a compact domain for any bounded interval of time).

Let us introduce the increasing sequence of stopping times $\tau_0 = 0$ and

$$\tau_n := \inf \{t \geq \tau_{n-1}; |X_t| > n\}.$$

In order to show that the solution never explodes, we use the comparison of $X_t - c_t$ with the Ornstein-Uhlenbeck process (see §2.3). So, we have for the corresponding Ornstein-Uhlenbeck process Z , that

$$|X_{t \wedge \tau_n} - c_{t \wedge \tau_n}| \leq |Z_{t \wedge \tau_n}|.$$

As Z does not explode in a finite time, letting n go to infinity, we conclude that $X_t - c_t$ does not explode in a finite time. To conclude, one has to use the inequality (14):

$$|\dot{c}_t| \leq \frac{1}{t} \frac{P(|X_t - c_t|)}{C_W} \leq \frac{1}{t} \frac{P(|Z_t|)}{C_W}.$$

Any trajectory of Z being bounded on any finite interval of time, the integral $\int_r^t \frac{1}{s} \frac{P(|Z_s|)}{C_W} ds$ is finite for any $t \geq 0$. So, the process $(X_t, t \geq 0)$ does not explode in a finite time and there exists a global strong solution. \square

2.5. Singularity at $t = 0$. The last statement that we are going to prove in this paragraph is that a solution to the equation (1) with any initial condition at $t = 0$ (where the equation has a singularity) exists and is unique.

Proposition 2. *For any x_0 and almost every trajectory B_t of the Brownian motion a (continuous at $t = 0$) solution X_t to the equation (1) with the initial condition $X_0 = x_0$ exists on all the interval $[0, +\infty)$ and is unique.*

Proof. As Proposition 1 provides us global existence and uniqueness of solutions, starting from any arbitrary positive moment $r > 0$, it suffices to check the existence and uniqueness on some interval $[0, \delta)$. For the sake of simplicity of notation, suppose that $x_0 = 0$.

Let $\delta_1 > 0$ be such that for all $0 \leq t \leq \delta_1$, $|B_t| \leq \frac{1}{2}$ and $\delta_1 \sup_{|x| \leq 2} |\nabla W(x)| \leq \frac{1}{3}$. We work on the trajectories, which are staying inside $B_1(0)$, the unit ball centered in $x_0 = 0$. So, we consider $X : [0, \delta_1) \rightarrow B_1(0)$, $t \mapsto X_t$. Denote by μ_s^X the empirical measure of the process X . Then, the application $I : X \mapsto \tilde{X}$, such that

$$\tilde{X}_t = B_t + \int_0^t \nabla W * \mu_s^X(\tilde{X}_s) ds,$$

is well-defined on this space, and \tilde{X}_t also remains stuck in $B_1(0)$. Indeed, for any moment $t \leq \delta_1$, such that the solution \tilde{X} is defined on $[0, t]$ and stays in $B_1(0)$, we have

$$\begin{aligned} \left| \int_0^t \nabla W * \mu_s^X(\tilde{X}_s) ds \right| &= \left| \int_0^t \frac{1}{s} \int_0^s \nabla W(\tilde{X}_s - X_u) du ds \right| \\ &\leq \int_0^t \frac{1}{s} \int_0^s \sup_{|x| \leq 2} |\nabla W(x)| du ds \leq \delta_1 \sup_{|x| \leq 2} |\nabla W(x)| \leq \frac{1}{3}. \end{aligned}$$

Thus, if there existed a moment $t_0 \leq \delta_1$ such that $|\tilde{X}_{t_0}| \geq 7/8$ for the first time, then we would see that $|\tilde{X}_{t_0}| \leq 1/2 + 1/3$, which would contradict the bound $|\tilde{X}_{t_0}| \geq 7/8$. So, \tilde{X} stays in $B_1(0)$ for any $0 \leq t \leq \delta_1$.

Let us now show that for $\delta < \delta_1$ sufficiently small, the map I is a contraction on the space of continuous maps X from $[0, \delta]$ to $B_1(0)$ with $X_0 = 0$. Indeed, consider now two trajectories $X^{(1)}$ and $X^{(2)}$, realizing a coupling with the same Brownian motion, and their respective images (by I) $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$. Then, denoting by L the Lipschitz constant of ∇W on the 2-radius ball ; $L := \sup\{|\nabla^2 W(x)|; |x| \leq 2\}$, we have

$$\begin{aligned} |\tilde{X}_t^{(1)} - \tilde{X}_t^{(2)}| &= \left| \int_0^t \nabla W * \mu_s^{X^{(1)}}(\tilde{X}_s^{(1)}) ds - \int_0^t \nabla W * \mu_s^{X^{(2)}}(\tilde{X}_s^{(2)}) ds \right| \\ &= \left| \int_0^t \frac{1}{s} \int_0^s \nabla W(\tilde{X}_s^{(1)} - X_u^{(1)}) - \nabla W(\tilde{X}_s^{(2)} - X_u^{(2)}) du ds \right| \\ &\leq \int_0^t \frac{1}{s} \int_0^s |\nabla W(\tilde{X}_s^{(1)} - X_u^{(1)}) - \nabla W(\tilde{X}_s^{(2)} - X_u^{(2)})| du ds \\ &\leq \int_0^t \frac{1}{s} \int_0^s L(|\tilde{X}_s^{(1)} - \tilde{X}_s^{(2)}| + |X_u^{(1)} - X_u^{(2)}|) du ds \\ &\leq tL(\|\tilde{X}^{(1)} - \tilde{X}^{(2)}\|_{C([0, \delta])} + \|X^{(1)} - X^{(2)}\|_{C([0, \delta])}), \end{aligned}$$

where $\|X\|_{C([0, \delta])}$ is the norm of X on the space $C([0, \delta])$. As $t \leq \delta$, we conclude that $\|\tilde{X}^{(1)} - \tilde{X}^{(2)}\|_{C([0, \delta])} \leq \delta L(\|\tilde{X}^{(1)} - \tilde{X}^{(2)}\|_{C([0, \delta])} + \|X^{(1)} - X^{(2)}\|_{C([0, \delta])})$. As soon as $\delta L < 1$, we have

$$\|\tilde{X}^{(1)} - \tilde{X}^{(2)}\|_{C([0, \delta])} \leq \frac{\delta L}{1 - \delta L} \|X^{(1)} - X^{(2)}\|_{C([0, \delta])}.$$

We choose δ such that $\delta L < 1/3$ and then I is a contraction, as stated, with $Lip(I) \leq 1/2$. So, we have obtained existence and uniqueness of the solution on $[0, \delta]$. \square

3. FLOW $\dot{\mu} = \Pi(\mu) - \mu$

From now on, we precise the hypotheses on the symmetric polynomial P . We suppose that $P \geq 1$, with coefficients greater than 1, of degree k , such that for all $x, y \in \mathbb{R}^d$, we have $P(|x - y|) \leq P(|x|)P(|y|)$. Indeed, we choose $P(|x|) = A(1 + |x|^k)$, where A is a constant large enough.

3.1. Some useful measure spaces. As usual, we denote by $\mathcal{M}(\mathbb{R}^d)$ the space of signed (bounded) Borel measures on \mathbb{R}^d and by $\mathcal{P}(\mathbb{R}^d)$ its subspace of probability measures. We will need the following measure space:

$$(16) \quad \mathcal{M}(\mathbb{R}^d; P) := \left\{ \mu \in \mathcal{M}(\mathbb{R}^d); \int_{\mathbb{R}^d} P(|y|) |\mu|(dy) < \infty \right\},$$

where $|\mu|$ is the variation of μ (that is $|\mu| := \mu^+ + \mu^-$ with (μ^+, μ^-) the Hahn-Jordan decomposition of μ : $\mu = \mu^+ - \mu^-$). Belonging to this space will enable us to always get the integrability of P (and therefore of W , and its derivatives thanks to the domination condition (6)) with respect to the (random) measures to be considered. We endow this space with the following dual weighted supremum norm (or dual P -norm) defined for $\mu \in \mathcal{M}(\mathbb{R}^d; P)$ by

$$(17) \quad \|\mu\|_P := \sup_{\varphi \in \mathcal{C}(\mathbb{R}^d); |\varphi| \leq P} \left| \int_{\mathbb{R}^d} \varphi \, d\mu \right| = \int_{\mathbb{R}^d} P(|y|) |\mu|(dy).$$

We recall that $P(|x|) \geq 1$, so that $\|\mu\|_P \geq |\mu(\mathbb{R}^d)|$. This norm naturally arises in the approach to ergodic results for time-continuous Markov processes of Meyn & Tweedie [12]. It also makes $\mathcal{M}(\mathbb{R}^d; P)$ a Banach space.

Next, we consider $\mathcal{P}(\mathbb{R}^d; P) := \mathcal{M}(\mathbb{R}^d; P) \cap \mathcal{P}(\mathbb{R}^d)$. We remark that both $\mathcal{M}(\mathbb{R}^d; P)$ and $\mathcal{P}(\mathbb{R}^d; P)$ are non empty since they contain any probability measure with an exponential decreasing density.

3.2. Local existence. We start with an easy result that will be used many times:

Lemma 3. *For any $\mu \in \mathcal{M}(\mathbb{R}^d; P)$, we have for all $x \in \mathbb{R}^d$*

$$|W * \mu(x)| \leq \|\mu\|_P P(|x|).$$

Proof. Straightforward thanks to the domination condition (6). \square

Lemma 4. *On the space $\mathcal{M}_{+,P}$ of positive P -integrable measures, the center c is a continuous function.*

Proof. Let $\mu_1, \mu_2 \in \mathcal{M}_{+,P}$. Denoting by c_1 (resp. c_2) the center of μ_1 (resp. μ_2), we have

$$\nabla W * \mu_2(c_1) = \nabla W * \mu_1(c_1) + \nabla W * (\mu_2 - \mu_1)(c_1),$$

thus $|\nabla W * \mu_2(c_1)| \leq P(c_1) \|\mu_2 - \mu_1\|_P$. Joining the points c_1 and c_2 by a line, recalling that due to the uniform convexity of W , the second derivative of $W * \mu_t$ along this line is at least $C_W \mu_2(\mathbb{R}^d)$ and noticing that $\nabla W * \mu_2(c_2) = 0$, we obtain

$$(18) \quad |c_2 - c_1| \leq \frac{P(|c_1|)}{C_W \mu_2(\mathbb{R}^d)} \|\mu_2 - \mu_1\|_P. \quad \square$$

To avoid the difficulties created by the fact that probability measures cannot be thought as a “submanifold” of $\mathcal{M}(\mathbb{R}^d, P)$, we will pass to the integral interpretation of $\dot{\mu} = \Pi(\mu) - \mu$. For its study, for any $\kappa > 0$, we define

$$\mathcal{P}_\kappa(\mathbb{R}^d; P) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d; P); \|\mu\|_P = \int_{\mathbb{R}^d} P(|x|) \mu(dx) \leq \kappa \right\},$$

and consider on the space of probability measures $\mathcal{P}(\mathbb{R}^d; P)$ the integral equation

$$(19) \quad \Phi_t(\mu) = e^{-t} \mu + e^{-t} \int_0^t e^s \Pi(\Phi_s(\mu)) \, ds.$$

We will prove the existence of the semiflow $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; P) \rightarrow \mathcal{P}(\mathbb{R}^d; P)$ defined by (19). Since $\mathcal{P}(\mathbb{R}^d; P)$ is not a vector space, to prove the local existence of a solution, we will proceed directly by approximation. The following lemma will be used afterwards in order to find a good security cylinder.

Lemma 5. *For any $\kappa > 1$, the application Π restricted to $\mathcal{P}_\kappa(\mathbb{R}^d; P)$ is bounded and Lipschitz.*

Proof. First, we need to show that $\mu \mapsto Z(\mu)$ is bounded from below. For $\mu \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$, Lemma 3 asserts that $W * \mu(x) \leq \kappa P(|x|)$. So we get:

$$Z(\mu) = \int_{\mathbb{R}^d} e^{-W * \mu(x)} dx \geq \int_{\mathbb{R}^d} e^{-\kappa P(|x|)} dx$$

and thus, using that $W(x) \geq C_W |x|^2/2$, we have the following bound for $\Pi(\mu)$ (because $|x - y|^2 \geq |x|^2/4 - |y|^2$ and so $W * \mu(x) \geq \frac{C_W}{2} \left(\frac{|x|^2}{4} - \kappa \right)$):

$$(20) \quad \|\Pi(\mu)\|_P \leq \left(\int_{\mathbb{R}^d} e^{-\kappa P(|x|)} dx \right)^{-1} \int_{\mathbb{R}^d} P(|x|) e^{-\frac{C_W}{2}(|x|^2/4 - \kappa)} dx =: C_\kappa.$$

Note that Π is \mathcal{C}^1 on $\mathcal{P}(\mathbb{R}^d; P)$ endowed with the strong topology. As the set of probability measures has no interior points, we have to specify the meaning of \mathcal{C}^1 : there exists a continuous linear operator $D\Pi(\mu) : \mathcal{M}_0(\mathbb{R}^d; P) \rightarrow \mathcal{M}_0(\mathbb{R}^d; P)$, continuously depending on μ , such that $\|\Pi(\mu') - \Pi(\mu) - D\Pi(\mu)(\mu - \mu')\|_P = O(\|\mu - \mu'\|_P)$ provided that $\mu' \in \mathcal{P}(\mathbb{R}^d; P)$ and μ' converges toward μ . Indeed, it is easy to see that

$$(21) \quad \begin{aligned} D\Pi(\mu) \cdot \nu &:= -(W * \nu)\Pi(\mu) - \frac{DZ(\mu) \cdot \nu}{Z(\mu)^2} e^{-W * \mu} \\ &= -(W * \nu)\Pi(\mu) + \int_{\mathbb{R}^d} W * \nu(y) \frac{e^{-W * \mu(y)}}{Z(\mu)} dy \frac{e^{-W * \mu}}{Z(\mu)} \\ &= - \left(W * \nu - \int_{\mathbb{R}^d} W * \nu(y) \Pi(\mu)(dy) \right) \Pi(\mu). \end{aligned}$$

Now, note that the norms $\|D\Pi\|$ are uniformly bounded for $\mu \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$ (for any given κ). Indeed, fix $\nu \in \mathcal{M}_0(\mathbb{R}^d; P)$. Since $|W * \nu(x)| \leq \|\nu\|_P P(|x|)$, we find that

$$\|D\Pi(\mu) \cdot \nu\|_P \leq (1 + C_\kappa) \|\nu\|_P \int_{\mathbb{R}^d} P^2(|x|) \Pi(\mu)(dx).$$

But for $\mu \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$, the same computation used for the bound of $\Pi(\mu)$ enables to control the last integral. Hence, we get a bound (call it C'_κ) on the norm of the differential. Thus, Π is Lipschitz as stated. \square

Proposition 3. *For all $\mu \in \mathcal{P}(\mathbb{R}^d; P)$ the ODE has a local solution. This defines a \mathcal{C}^1 semiflow Φ for the strong topology.*

Proof. Let μ belong to $\mathcal{P}_\kappa(\mathbb{R}^d; P)$ where we choose $\kappa > 2\|\mu\|_P$. We introduce the classic Picard approximation scheme:

$$\begin{cases} \mu_t^{(0)} := \mu, \\ \mu_t^{(n)} := e^{-t} \mu + \int_0^t e^{s-t} \Pi(\mu_s^{(n-1)}) ds. \end{cases}$$

We set ε small enough such that $\|\mu\|_P + (1 - e^{-\varepsilon})C'_\kappa \leq \kappa$ and $(1 - e^{-\varepsilon})C'_\kappa < 1$ where both constants were defined in Lemma 5. Then, for all n , $\mu_t^{(n)}$ is defined and belongs to $\mathcal{P}_\kappa(\mathbb{R}^d; P)$, which makes $[0, \varepsilon) \times \mathcal{P}_\kappa(\mathbb{R}^d; P)$ a good security cylinder. We have, for $t < \varepsilon$,

$$\|\mu_t^{(n+1)} - \mu_t^{(n)}\|_P \leq (1 - e^{-\varepsilon})C'_\kappa \sup_{t < \varepsilon} \|\mu_t^{(n)} - \mu_t^{(n-1)}\|_P.$$

Now the series with general term $\sup_{t < \varepsilon} \|\mu_t^{(n+1)} - \mu_t^{(n)}\|_P$ converges and thus the sequence of functions $\mu^{(n)}$ is Cauchy for the topology of uniform convergence. Since $\mathcal{P}(\mathbb{R}^d; P)$ is complete, we have successfully built a solution on $[0, \varepsilon)$. It remains to show that the semiflow is smooth.

We have seen that the map Π is \mathcal{C}^1 for the strong topology. By induction, every Picard approximation $\mu \mapsto \mu_t^{(n)}$ is \mathcal{C}^1 and it is enough to take the limit uniformly in μ on $\mathcal{P}_\kappa(\mathbb{R}^d; P)$ to conclude. \square

3.3. Invariant sets, global existence and estimates. Throughout this section, μ_t and c_t respectively stand for the solution to the deterministic equation

$$(22) \quad \dot{\mu} = \Pi(\mu) - \mu$$

and its center, and *not* for the ones of the SDE considered previously.

To prove the global existence of the flow, we will proceed as follows. First, we prove that the probability measure $\Pi(\mu)$ is exponentially decreasing when one goes away from the center c_μ : for any $R > 0$, $\Pi(\mu)\{|x - c_\mu| \geq R\} \leq ae^{-bR}$ for some uniform constants a and b . This implies a uniform bound on the velocity of drift of the center: $|\dot{c}_t| \leq v_c$, where v_c is a uniform constant. Joined together, these two statements allow us to conclude that the set of exponentially decreasing (with respect to their centers) measures is positively invariant (see the definition below).

Definition 2. A subset A of $\mathcal{P}(\mathbb{R}^d; P)$ is *positively invariant*² for Φ provided $\Phi_t(A) \subset A$ for all $t \geq 0$.

3.3.1. *Deterministic center-drift estimates.* Prove now the exponential decrease for $\Pi(\mu)$.

Proposition 4. *There exists a positive constant C such that for all $\mu \in \mathcal{P}(\mathbb{R}; P)$, for all $R > 0$, we have $\Pi(\mu)(|x - c_\mu| \geq R) \leq Ce^{-C_W R}$.*

Proof. Note first, that imposing a condition $C \geq e^{2C_W}$, we can restrict ourselves only on $R \geq 2$: for $R < 2$, the estimate is trivial. The measure $\Pi(\mu)$ has the density $\frac{1}{Z(\mu)}e^{-W*\mu(x)}$. To avoid working with the normalization constant $Z(\mu)$, we will prove a stronger inequality, that is

$$\Pi(\mu)(|x - c_\mu| \geq R) \leq Ce^{-C_W R} \cdot \Pi(\mu)(|x - c_\mu| \leq 2),$$

which is equivalent to

$$\int_{|x - c_\mu| \geq R} e^{-W*\mu(x)} dx \leq Ce^{-C_W R} \int_{|x - c_\mu| \leq 2} e^{-W*\mu(x)} dx.$$

Pass to the polar coordinates, centered at the center c_μ : we want to prove that

$$\int_{\mathbb{S}^{d-1}} \int_R^\infty e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} d\lambda dv \leq Ce^{-C_W R} \int_{\mathbb{S}^{d-1}} \int_0^2 e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} d\lambda dv.$$

It suffices to prove such an inequality “directionwise”: for all $v \in \mathbb{S}^{d-1}$, for all $R \geq 2$

$$\int_R^\infty e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} d\lambda \leq Ce^{-C_W R} \int_0^2 e^{-W*\mu(c_\mu + \lambda v)} \lambda^{d-1} d\lambda.$$

But from the uniform convexity of W and the definition of the center, the function $f(\lambda) = W*\mu(c_\mu + \lambda v)$ satisfies $f'(0) = 0$ and $\forall r > 0$, $f''(r) \geq C_W$. Hence, f is monotone increasing on $[0, \infty)$, in particular,

$$(23) \quad \int_0^2 e^{-f(\lambda)} \lambda^{d-1} d\lambda \geq e^{-f(2)} \int_0^2 \lambda^{d-1} d\lambda =: C_1 e^{-f(2)}.$$

²Throughout this paper, the word “invariant” will mean only “positively invariant”. We will never launch trajectories of Φ_t in the past.

On the other hand, for all $\lambda \geq 2$, $f'(\lambda) \geq f'(2) \geq 2C_W$, and thus $f(\lambda) \geq 2C_W(\lambda - 2) + f(2)$. Hence,

$$(24) \quad \int_R^\infty e^{-f(\lambda)} \lambda^{d-1} d\lambda \leq e^{-f(2)} \int_R^\infty \lambda^{d-1} e^{-2C_W(\lambda-2)} d\lambda \leq C_2 R^{d-1} e^{-2C_W R} \cdot e^{-f(2)} \leq C_3 e^{-C_W R} \cdot e^{-f(2)}.$$

Comparing (23) and (24), we obtain the desired exponential decrease. \square

Consider now an arbitrary $\mu \in \mathcal{P}(\mathbb{R}^d; P)$. There exists a unique (maybe local) solution μ_t to (22) with $\mu_0 = \mu$. Then, an easy application of Implicit Function Theorem implies that the center c_t is differentiable, and that

$$\dot{c}_t = -(\nabla^2 W * \mu_t(c_t))^{-1} (\nabla W * \Pi(\mu_t)(c_t)).$$

This speed depends on the measure μ_t only. Thus, we can define the following function φ on $\mathcal{P}(\mathbb{R}^d; P)$:

$$(25) \quad \varphi(\mu) := -(\nabla^2 W * \mu(c_\mu))^{-1} (\nabla W * \Pi(\mu)(c_\mu)).$$

Proposition 5. *The function φ is bounded on $\mathcal{P}(\mathbb{R}^d; P)$: there exists v_c such that for all $\mu \in \mathcal{P}(\mathbb{R}^d; P)$, we have $|\varphi(\mu)| \leq v_c$.*

Proof. The norm of the matrix $(\nabla^2 W * \mu)^{-1}$ does not exceed $1/C_W$, so it suffices to prove the boundedness for $|\nabla W * \Pi(\mu)(c_\mu)|$. Now,

$$(26) \quad \begin{aligned} |\nabla W * \Pi(\mu)(c_\mu)| &\leq \int_{\mathbb{R}^d} |\nabla W(x - c_\mu)| \Pi(\mu)(dx) \leq \int_{\mathbb{R}^d} P(|x - c_\mu|) \Pi(\mu)(dx) \\ &= \int_0^\infty P(\lambda) d(\Pi(\mu)\{|x - c_\mu| \leq \lambda\}) \\ &\leq \int_0^\infty P(\lambda) d(1 - Ce^{-C_W \lambda}) = CC_W \int_0^\infty P(\lambda) e^{-C_W \lambda} d\lambda < \infty, \end{aligned}$$

where the last inequality (26) follows from Proposition 4. The obtained upper bound concludes the proof. \square

Proposition 6. *There exists κ_0 , such that for all $\kappa \geq \kappa_0$, the set $\{\mu; \mu^c \in \mathcal{P}_\kappa(\mathbb{R}^d; P)\}$ is invariant by the flow Φ .*

Proof. Consider the quantity $\|\mu_t^c\|_P$. We have $\frac{\partial}{\partial t} \|\mu_t^c\|_P = \frac{\partial}{\partial t} \int_{\mathbb{R}^d} P(|x - c_t|) d\mu_t(x)$. Remark that for all $\varepsilon > 0$, there exists $R > 0$ such that $P'(r) \leq \varepsilon P(r)$ for all $r \geq R$. As $\dot{\mu} = \Pi(\mu) - \mu$, so that

$$\begin{aligned} \frac{\partial}{\partial t} \|\mu_t^c\|_P &= \int_{\mathbb{R}^d} P'(|x - c_t|) \left(\frac{x - c_t}{|x - c_t|}, \dot{c}_t \right) \mu_t(dx) + \int_{\mathbb{R}^d} P(|x - c_t|) (\Pi(\mu_t) - \mu_t)(dx) \\ &\leq \varepsilon \int_{|x - c_t| \geq R} P(|x - c_t|) v_c \mu_t(dx) + \int_{|x - c_t| \leq R} P'(|x - c_t|) v_c \mu_t(dx) \\ &\quad + \int_{\mathbb{R}^d} P(|x - c_t|) \Pi(\mu_t)(dx) - \int_{\mathbb{R}^d} P(|x - c_t|) \mu_t(dx). \end{aligned}$$

By Proposition 4, the integral $\int_{\mathbb{R}^d} P(|x - c_t|) \Pi(\mu_t)(dx)$ is bounded by a positive constant C_P . So, we get the inequality

$$\begin{aligned} \frac{\partial}{\partial t} \|\mu_t^c\|_P &\leq \varepsilon \int_{|x-c_t| \geq R} P(|x - c_t|) v_c \mu_t(dx) + P'(R) v_c + C_P - \int_{\mathbb{R}^d} P(|x - c_t|) \mu_t(dx) \\ &\leq C_P - (1 - \varepsilon) \|\mu_t^c\|_P. \end{aligned}$$

In particular, $\frac{\partial}{\partial t} \|\mu_t^c\|_P$ is negative once $\|\mu_t^c\|_P \geq 2C_P$. Thus, applying Gronwall's lemma, for all $\kappa \geq 2C_P$, the set $\mathcal{P}_\kappa(\mathbb{R}^d; P)$ is positively invariant for the flow. \square

Now, we can conclude this paragraph with the following

Theorem 5 (of global existence). *For all $\mu \in \mathcal{P}(\mathbb{R}^d; P)$, the solution to the flow generated by $\dot{\mu} = \Pi(\mu) - \mu$ with the initial condition $\mu_0 = \mu$ exists for any $t \geq 0$.*

Proof. As $\|\mu\|_P < \infty$, the measure μ^c belongs to some $\mathcal{P}_\kappa(\mathbb{R}^d; P)$. Without any loss of generality, we can suppose $\kappa \geq \kappa_0$ (where κ_0 has been defined in Proposition 6). On the other hand, due to the boundedness and the Lipschitz condition on Π (see Lemma 5), the time for which the existence of solutions is guaranteed, is uniform over $\mathcal{P}_\kappa(\mathbb{R}^d; P)$ and hence, due to the translation-invariance of μ^c , for all $\{\mu; \mu^c \in \mathcal{P}_\kappa(\mathbb{R}^d; P)\}$. Finally, the positive invariance for Φ of $\{\mu; \mu^c \in \mathcal{P}_\kappa(\mathbb{R}^d; P)\}$, guaranteed by Proposition 6, allows to make such steps further and further, justifying the global existence of the solution. \square

3.3.2. A new metric: \mathcal{T}_P -metric. The strong norm that we have used previously is well-adapted to the proof of existence of solutions by the standard Cauchy-Lipschitz type arguments. However, its use has some drawbacks: the centering map $\mu \mapsto \mu^c$ is not continuous, radius one ball is not compact, etc. On the other hand, for a measure μ , the corresponding probability measure $\Pi(\mu)$ is defined using the convolution $W * \mu$. So, it would be rather natural to use a distance, looking like the one for the weak* topology, but allowing to control $W * \mu$ for our unbounded function W . This motivates the following

Definition 3. For $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d; P)$, we define the P -translation distance between them as

$$\mathcal{T}_P(\mu_1, \mu_2) := \inf \left\{ \int_0^1 \int_0^1 P(|f(s, \omega)|) |f'_s(s, \omega)| ds d\mathbb{P} \right\},$$

where the infimum is taken over the maps $f : [0, 1] \times \Omega \rightarrow \mathbb{R}$, such that $\{\text{law of } f(0, \cdot)\} = \mu_1$, and $\{\text{law of } f(1, \cdot)\} = \mu_2$.

Remark 4. In one dimension, we have the equivalent definition (the equivalence is left to the reader):

$$\mathcal{T}_P(\mu_1, \mu_2) := \int_{\mathbb{R}} P(|x|) |F_{\mu_1}(x) - F_{\mu_2}(x)| dx,$$

where $F_\mu(x) = \mu((-\infty, x])$ is the cumulative distribution function of μ .

It is clear from the definition that \mathcal{T}_P is a distance; and also taking into account that $|P'| \leq P$, one easily has

$$(27) \quad \|\mu_2\|_P \leq \|\mu_1\|_P + \mathcal{T}_P(\mu_1, \mu_2).$$

Thus, the set $\mathcal{P}(\mathbb{R}^d; P)$ is \mathcal{T}_P -complete. Indeed, a Cauchy sequence (μ_n) will have a weak limit μ and it is easy to check that $\|\mu\|_P = \lim_{n \rightarrow \infty} \|\mu_n\|_P < \infty$. So, $\mu \in \mathcal{P}(\mathbb{R}^d; P)$. Now, we are going to estimate the deviance of trajectories in terms of \mathcal{T}_P -metric. This will be useful in Section 5.

Lemma 6. 1) *The map c is Lipschitz in the sense of \mathcal{T}_P -metric:*

$$|c(\mu_1) - c(\mu_2)| \leq \frac{1}{C_W} \min(P(|c(\mu_1)|), P(|c(\mu_2)|)) \cdot \mathcal{T}_P(\mu_1, \mu_2);$$

2) *There exists $C > 0$ such that for all $r \in \mathbb{R}^d$, $\mathcal{T}_P(\mu, \mu(\cdot + r)) \leq C|r|^d P(|r|) \|\mu\|_P$;*

3)

$$\mathcal{T}_P(\mu(\cdot + r), \nu(\cdot + r)) \leq \sup_{x \geq 0} \frac{P(x + |r|)}{P(x)} \mathcal{T}_P(\mu, \nu) \leq \begin{cases} C|r| \mathcal{T}_P(\mu, \nu), & |r| \leq 1 \\ P(|r|) \mathcal{T}_P(\mu, \nu), & \forall |r|; \end{cases}$$

4) $\mu^c \mathcal{P}_\kappa(\mathbb{R}^d; P) \rightarrow \mathcal{P}_\kappa(\mathbb{R}^d; P)$ *is \mathcal{T}_P -Lipschitz.*

Proof. 1) See the proof of Lemma 4.

2) We have by definition of \mathcal{T}_P that

$$\mathcal{T}_P(\mu, \mu(\cdot + r)) \leq \int_{\mathbb{R}^d} P(|x|) \int_{|y-x| \leq |r|} \mu(dy) dx.$$

By Fubini's theorem, we find an upper bound for the latter: $\int_{\mathbb{R}^d} \int_{|y-x| \leq |r|} P(|x|) dx \mu(dy)$. As $P(|x|) \leq P(|x-y|)P(|y|)$, we deduce that

$$\mathcal{T}_P(\mu, \mu(\cdot + r)) \leq P(|r|) \int_{\mathbb{R}^d} \int_{|y-x| \leq |r|} dx P(|y|) \mu(dy) \leq C|r|^d P(|r|) \|\mu\|_P.$$

3) It results from a change of variable in the definition of \mathcal{T}_P . In dimension one, we see that

$$\mathcal{T}_P(\mu(\cdot + r), \nu(\cdot + r)) \leq \int \frac{P(|x-r|)}{P(|x|)} P(|x|) |\mu(-\infty, x) - \nu(-\infty, x)| dx \leq \sup_{x \geq 0} \frac{P(x+r)}{P(x)} \mathcal{T}_P(\mu, \nu).$$

In dimension $d \geq 1$, one uses also the definition of \mathcal{T}_P and we see that

$$\begin{aligned} \int_0^1 \int P(|sx + (1-s)y|) \mu(x+r) \nu(y+r) dx dy ds &\leq \sup_{x, y \in \mathbb{R}^d} \frac{P(|sx + (1-s)y - r|)}{P(|sx + (1-s)y|)} \mathcal{T}_P(\mu, \nu) \\ &\leq \sup_{x \geq 0} \frac{P(x+|r|)}{P(x)} \mathcal{T}_P(\mu, \nu). \end{aligned}$$

4) Suppose that $\mu, \nu \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$. Then, by the preceding points, we have

$$\begin{aligned} \mathcal{T}_P(\mu(\cdot + c_\mu), \nu(\cdot + c_\nu)) &\leq \mathcal{T}_P(\mu(\cdot + c_\mu), \nu(\cdot + c_\mu)) + \mathcal{T}_P(\nu(\cdot + c_\mu), \nu(\cdot + c_\nu)) \\ &\leq P(|c_\mu|) \mathcal{T}_P(\mu, \nu) + C|c_\mu - c_\nu|^d P(|c_\mu - c_\nu|) \|\nu\|_P \\ &\leq P(|c_\mu|) \mathcal{T}_P(\mu, \nu) \\ &\quad + C\kappa |c_\mu - c_\nu|^{d-1} P(|c_\mu - c_\nu|) \frac{1}{C_W} \min(P(|c_\mu|), P(|c_\nu|)) \mathcal{T}_P(\mu, \nu). \end{aligned}$$

Remark that, as $\mu, \nu \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$, there exists a positive constant $C(\kappa)$ such that

$$P(|c_\mu|) \left(1 + C\kappa |c_\mu - c_\nu|^{d-1} P(|c_\mu - c_\nu|) \frac{1}{C_W} \right) \leq C(\kappa)$$

and thus

$$\mathcal{T}_P(\mu(\cdot + c_\mu), \nu(\cdot + c_\nu)) \leq C(\kappa) \mathcal{T}_P(\mu, \nu). \quad \square$$

Proposition 7. *For all $\kappa \geq \kappa_0$, there exist $\beta, C > 0$ such that if $\mu_0, \nu_0 \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$, then*

$$\mathcal{T}_P((\mu_t)^c, (\nu_t)^c) \leq C e^{\beta t} \mathcal{T}_P(\mu_0, \nu_0).$$

Proof. First, by Lemma 6, there exists $C(\kappa)$ such that $\mathcal{T}_P((\mu_t)^c, (\nu_t)^c) \leq C(\kappa)\mathcal{T}_P(\mu_t, \nu_t)$. Let us now prove the upper bound for $\mathcal{T}_P(\mu_t, \nu_t)$.

Recall that $|\partial_t c(\mu_t)| \leq v_c$ and $|\partial_t c(\nu_t)| \leq v_c$ and hence for all $t \geq 0$:

$$|c(\mu_t)| \leq |c(\mu_0)| + tv_c \text{ and } |c(\nu_t)| \leq |c(\nu_0)| + tv_c.$$

Now let us estimate the norms $\|\mu_t\|_P$ and $\|\nu_t\|_P$. Actually, for any measure $\eta \in \mathcal{P}_\kappa(\mathbb{R}^d; P)$, one has $\|\Pi(\eta)\|_P \leq CP(c(\eta))$. Thus,

$$\|\mu_t\|_P \leq e^{-t}\|\mu_0\|_P + \int_0^t e^{-(t-s)}\|\Pi(\mu_s)\|_P ds \leq \max(\|\mu_0\|_P, CP(|c(\mu_0)| + tv_c)).$$

Consider now the distance $\mathcal{T}_P(\Pi(\mu_t), \Pi(\mu_s))$. Note first, that the normalization constants $Z(\mu_t), Z(\nu_t)$ can be estimated using the norms $\|\mu_t\|_P, \|\nu_t\|_P$ and that the difference between them can be estimated with the help of \mathcal{T}_P . Recall that

$$\begin{aligned} W * \mu(x) - W * \mu(c_\mu) &= \int_0^1 (\nabla W * \mu(t(x - c_\mu) + c_\mu), x - c_\mu) dt \\ &\leq P(|c_\mu| + 1)\|\mu\|_P|x - c_\mu| =: \beta|x - c_\mu|, \end{aligned}$$

and thus

$$\begin{aligned} Z(\mu) &= \int_{\mathbb{R}^d} e^{-W * \mu(x)} dx \geq e^{-W * \mu(c_\mu)} \int_{|x - c_\mu| \leq 1} e^{-(W * \mu(x) - W * \mu(c_\mu))} dx \\ &\geq e^{-W * \mu(c_\mu)} \int_{|x| \leq 1} e^{-\beta|x|} dx = e^{-W * \mu(c_\mu)} \frac{1 - e^{-\beta}}{\beta}. \end{aligned}$$

So, we have some estimates on $Z(\mu_t)$ and $Z(\nu_t)$ in terms of $\|\mu_t\|_P$ and $\|\nu_t\|_P$, which are in their turn already estimated. On the other hand, for $x \in \mathbb{R}^d$ we have up to a multiplicative constant

$$|W * \mu(x) - W * \nu(x)| \leq P(|x|)\mathcal{T}_P(\mu, \nu).$$

It implies that

$$(28) \quad \left| e^{-W * \mu(x)} - e^{-W * \nu(x)} \right| \leq \left| 1 - e^{-P(|x|)\mathcal{T}_P(\mu, \nu)} \right| e^{-\min(W * \mu(x), W * \nu(x))}.$$

Finally, $W * \mu(x) \geq \frac{C_W}{2}|x - c_\mu|^2$ (and the same estimate holds for ν). Thus, outside a large interval, the second term in (28) can be estimated from above. This provides us with (some) upper bound on $\mathcal{T}_P(\Pi(\mu), \Pi(\nu))$ in terms of $\mathcal{T}_P(\mu, \nu)$ and $\|\mu\|_P, \|\nu\|_P$. Solving the corresponding differential equation (22) and noting that $\partial_t \mathcal{T}_P(\mu_t, \nu_t) \leq \mathcal{T}_P(\Pi(\mu_t), \Pi(\nu_t)) - \mathcal{T}_P(\mu_t, \nu_t)$, we obtain a upper bound on $\mathcal{T}_P(\mu_t, \nu_t)$. Remark, however, that we need a smooth at 0 differential equation. So the radius for “cutting off” to estimate the difference $|\Pi(\mu)(x) - \Pi(\nu)(x)|$ is chosen uniformly in $t \in [0, 1]$. \square

4. FREE ENERGY FUNCTIONAL

Let us now pass to the physical interpretation of the flow $\dot{\mu} = \Pi(\mu) - \mu$. Namely, let us introduce the free energy functional.

Consider a drifted Brownian motion. Let an absolutely continuous probability measure $\mu = \mu(x)dx$ be given (by an abuse of notation, we denote the measure and its density by the same letter). We can imagine $\mu(x)$ as the density of a gas, particles of which implement the Brownian motion $\sqrt{2}dB_t$, as well as interact with the potential $W(x - y)$ in presence of an exterior potential V . Under the (maybe zero) confinement potential V and with the

self-interaction potential W , one defines the *free energy* of μ as the sum of its “entropy” and “potential energy”:

$$(29) \quad \mathcal{F}_{V,W}(\mu) := \int_{\mathbb{R}^d} \mu(x) \log \mu(x) \, dx + \int_{\mathbb{R}^d} V(x)\mu(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)W(x-y)\mu(y) \, dx \, dy.$$

It will play an important role as it will prove to be a Lyapunov function for the flow induced by the dynamical system (22).

In the cases of absence of confinement or interaction, the free energy reduces respectively to

$$(30) \quad \mathcal{F}(\mu) := \int_{\mathbb{R}^d} \mu(x) \log \mu(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)W(x-y)\mu(y) \, dx \, dy,$$

and to

$$(31) \quad \mathcal{F}_V(\mu) := \int_{\mathbb{R}^d} \mu(x) \log \mu(x) \, dx + \int_{\mathbb{R}^d} V(x)\mu(x) \, dx.$$

Let us compute the first derivative of \mathcal{F}_V , which we will use later. Actually, denote by $D\mathcal{F}_V$ its differential function. Let $\mu \in \mathcal{P}(\mathbb{R}^d; P)$ have a density with respect to the Lebesgue measure and $\nu \in \mathcal{M}_0(\mathbb{R}^d; P)$. We remind that the differential function is given by the following

$$(32) \quad D\mathcal{F}_V(\mu) \cdot \nu = \int_{\mathbb{R}^d} \nu(x) \log \mu(x) \, dx + \int_{\mathbb{R}^d} V(x)\nu(x) \, dx.$$

4.1. Fixed point and Lyapunov function. A general physical principle says that in the absence of any energy flow from outside the system, “the free energy decreases and converges to its minimum”. In our case, this principle is justified by the following statements:

Lemma 7. *For any potential V , the probability measure $Z^{-1}e^{-V}$ is the unique (and thus global) minimum of \mathcal{F}_V .*

Proof. We see that \mathcal{F}_V is a strictly convex function, as the function $p \log p$ is. It then admits a unique global minimum. Indeed, we get for all $\nu \in \mathcal{M}_0(\mathbb{R}^d; P)$, that

$$D\mathcal{F}_V(Z^{-1}e^{-V}) \cdot \nu = \int_{\mathbb{R}^d} \log(Z^{-1}e^{-V(x)})\nu(dx) + \int_{\mathbb{R}^d} V(x)\nu(dx) = - \int_{\mathbb{R}^d} \log(Z)\nu(dx) \equiv 0. \quad \square$$

Corollary 1. $\Pi(\mu)$ provides the unique (global) minimum of $\mathcal{F}_{W*\mu}$.

Indeed, we have the following

Proposition 8. *The function \mathcal{F} is a “Lyapunov function” on its domain of definition*

Proof. Note by $H(\mu) := \int_{\mathbb{R}^d} \mu(x) \log \mu(x) \, dx$ the “entropy” part of μ . Then $H(\mu)$ is convex as the function $\rho \log \rho$ is. Thus, if $\dot{\mu}_t = -\mu_t + \Pi(\mu_t)$,

$$(33) \quad \frac{d}{dt} H(\mu_t) \geq -H(\mu_t) + H(\Pi(\mu_t)).$$

On the other hand, for the interaction part, we have an exact equality:

$$(34) \quad \begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x-y)\mu_t(x)\mu_t(y) \, dx \, dy \right) &= - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x-y)\mu_t(x)\mu_t(y) \, dx \, dy \\ &+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x-y)\mu_t(x)\Pi(\mu_t)(y) \, dx \, dy. \end{aligned}$$

Adding (33) and (34) together, we obtain

$$(35) \quad \frac{d}{dt} \mathcal{F}(\mu_t) \geq -\mathcal{F}_{W*\mu_t}(\mu_t) + \mathcal{F}_{W*\mu_t}(\Pi(\mu_t)).$$

On the other hand, Lemma 7 states that $\Pi(\mu)$ is the (unique) minimum of the function $\Phi_\mu(\cdot) := \mathcal{F}_{W*\mu}(\cdot)$. So, we have $\frac{d}{dt} \mathcal{F}(\mu_t) \leq 0$ and the equality holds if and only if $\mu = \Pi(\mu)$, that is if and only if $\mu = \rho_\infty$. \square

Proposition 9. *The fixed point of Π exists, is unique up to a translation, and it is the unique (up to a translation) global minimum of \mathcal{F} .*

Proof. McCann [11] has proven that, assuming the strict convexity of W , \mathcal{F} admits a unique (global) minimum.

Note that, contrary to the case of Lemma 7, the function \mathcal{F} is *not* convex in the usual sense, due to the presence of the self-interaction energy part. An amazing fact, discovered by McCann, is that however \mathcal{F} is a *displacement convex* function. More precisely, given two probability measures μ_1, μ_2 , one can consider the interpolating path $(\rho_s)_{0 \leq s \leq 1}$ between them, minimizing the Wasserstein distance. Then, the map $s \mapsto \mathcal{F}(\rho_s)$ is convex.

Proposition 8 and equation (32) imply that $D\mathcal{F}(\mu) \cdot \nu = 0$ for all $\nu \in \mathcal{M}_0(\mathbb{R}^d; P)$ if and only if $\mu = \Pi(\mu)$, meaning that $\mu = \Pi(\mu)$ is a minimum for \mathcal{F} . So, we get

$$(36) \quad 0 \geq \mathcal{F}(\Pi(\mu)) - \mathcal{F}(\mu) \geq D\mathcal{F}(\mu) \cdot (\Pi(\mu) - \mu).$$

By displacement convexity, such a point is unique. \square

In the light of Corollary 1, we can interpret the flow $\dot{\mu} = \Pi(\mu) - \mu$ on $\mathcal{M}_{+,P}$ in the following way. A long tube is filled with W -interacting gas, separated in a plenty of very small cells. It is distributed with respect to the measure μ . Each unit of time, small parts (of proportion ε) of gas in these cells are separated, allowed to travel along the tube, and are proposed to equilibrate in the potential generated. This part of all the gas being small, its auto-interaction is neglectable, thus restricting us to the field $V := W * \mu$, generated by the major part of the particles, that stay fixed to their cells. The small part is then equilibrated to its weight ε times $\Pi(\mu)$. Then, it is separated again by the cells, thus the distribution after such step becomes

$$(1 - \varepsilon)\mu + \varepsilon\Pi(\mu) = \mu + \varepsilon(\Pi(\mu) - \mu).$$

On the other hand, this procedure does not require any work (in the physical sense!) to be done: the only actions are opening and closing the doors.

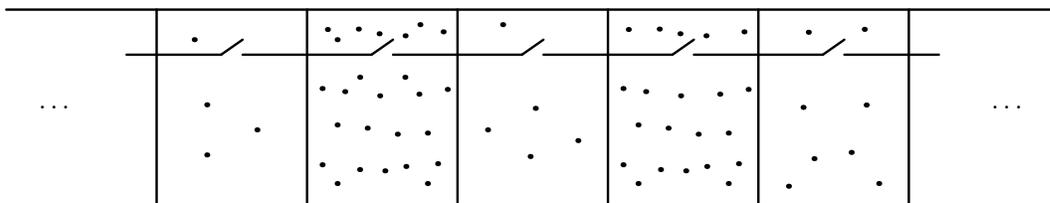


FIGURE 1. Gas: phase “separation”

So, due to the general principle, the free energy of the system $\mathcal{F}(\mu)$ should decrease. Indeed, \mathcal{F} is a Lyapunov function for the system $\dot{\mu} = \Pi(\mu) - \mu$ when it is defined. Though, it is defined only for measures that are absolutely continuous with respect to the Lebesgue measure (otherwise, $\mathcal{F}(\mu) = +\infty$).

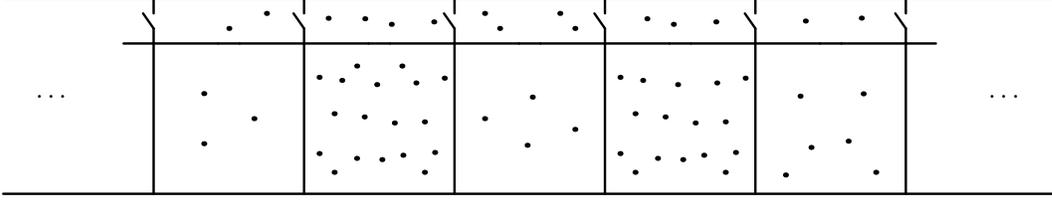


FIGURE 2. Gas: phase “re-distribution”

Remark 5. A tedious computation permits to show that the function $\mathcal{F} \circ \Pi$ is a general Lyapunov function for the system $\dot{\mu} = \Pi(\mu) - \mu$. Though, we must confess that we do not see any physical meaning behind the decrease of this function.

4.2. Convergence speed estimates.

4.2.1. *Absolutely continuous probability measures.* Assume that an absolutely continuous probability measure μ_0 belongs to a given set K_{α, C_2} . First, we need an estimate on the speed of convergence of the centered measure μ_t^c toward ρ_∞ . As $\mathcal{F}(\mu)$ obtains its minimal values exactly at the points of a form $\rho_\infty(x + c_\infty)dx$, it is natural to use the difference $\mathcal{F}(\mu|\rho_\infty) := \mathcal{F}(\mu) - \mathcal{F}(\rho_\infty)$ as a kind of a “distance” from μ^c to the set ρ_∞ . Indeed, it was shown in [6] that

$$(37) \quad W_2^2(\mu(\cdot + \bar{\mu}), \rho_\infty) \leq \frac{2}{C_W} \mathcal{F}(\mu, \rho_\infty),$$

where W_2^2 stands for the Wasserstein distance, and $\mu(\cdot + \bar{\mu})$ is the mean-centered translation of μ : we denote $\bar{\mu} := \int_{\mathbb{R}^d} x \mu(x) dx$.

We will now prove the following:

Proposition 10. *For any absolutely continuous $\mu_0 \in K_{\alpha, C_2}$ of finite energy, there exists a constant $C > 0$ such that $\mathcal{F}(\mu_t^c|\rho_\infty) \leq C e^{-a^{k+1}\sqrt{t}}$, where k is the degree of the polynomial P and C can be chosen uniformly over the set $\{\mathcal{F}(\mu_0) \leq A\}$ (for any fixed A).*

Proof. Recall that $\Pi(\mu)$ is the unique minimum of $\Phi_\mu = \mathcal{F}_{W^* \mu}$ and (35). So, to estimate the speed of decrease of $\mathcal{F}(\rho|\rho_\infty) := \mathcal{F}(\rho) - \mathcal{F}(\rho_\infty)$, one should estimate the difference $\Phi_{\mu_t}(\Pi(\mu_t)) - \Phi_{\mu_t}(\mu_t)$. Now, remark that

$$\begin{aligned} \Phi_\mu(\rho) &= \int \rho \log \rho + \int \int W(x - y) \mu(dx) \rho(dy) \\ &= \int \rho \log \rho + \frac{1}{2} \int \int W(x - y) \rho(dx) \rho(dy) \\ &\quad - \frac{1}{2} \int \int W(x - y) (\rho - \mu)(dx) (\rho - \mu)(dy) + \frac{1}{2} \int \int W(x - y) \mu(dx) \mu(dy) \\ &=: \mathcal{F}(\rho) - \frac{1}{2} \int \int W(x - y) (\rho - \mu)(dx) (\rho - \mu)(dy) + C(\mu). \end{aligned}$$

On the other hand, $\Pi(\mu)$ is the minimum of $\Phi_\mu(\rho)$. So, for any ρ , one has

$$\Phi_\mu(\Pi(\mu)) - \Phi_\mu(\mu) \leq \Phi_\mu(\rho) - \Phi_\mu(\mu) = \mathcal{F}(\rho) - \mathcal{F}(\mu) - \frac{1}{2} \int \int W(x - y) (\rho - \mu)(dx) (\rho - \mu)(dy).$$

Recall now, that \mathcal{F} is a displacement convex functional. Hence, if we denote by ν_s , $0 \leq s \leq 1$, the quadratic Wasserstein optimal W_2 -transport measure from $\nu_0 = \mu$ to $\nu_1 = \rho_\infty$, one obtains

$$\mathcal{F}(\nu_s | \rho_\infty) \leq (1-s)\mathcal{F}(\mu | \rho_\infty)$$

and thus

$$\Phi_\mu(\nu_s) - \Phi_\mu(\mu) \leq -s\mathcal{F}(\mu | \rho_\infty) - \frac{1}{2} \int \int W(x-y)(\nu_s - \mu)(dx)(\nu_s - \mu)(dy).$$

Let us now estimate the second right hand term of the inequality, namely $-\frac{1}{2} \int \int W(x-y)(\nu_s - \mu)(dx)(\nu_s - \mu)(dy)$. Indeed, let ξ_0, ξ_1 be two random variables providing the optimal transport, in the meaning $\xi_s := s\xi_1 + (1-s)\xi_0$, between the law of ξ_0 , that is $\nu_0 = \mu$, and the law of ξ_1 , that is $\nu_1 = \rho_\infty$. Let $(\eta_s, 0 \leq s \leq 1)$ be an independent copy of $(\xi_s, 0 \leq s \leq 1)$. Then,

$$\begin{aligned} & \int \int W(x-y)(\nu_s - \mu)(dx)(\nu_s - \mu)(dy) = \\ & = \mathbb{E} [W(\xi_0 - \eta_0) - W(\xi_s - \eta_0) - W(\xi_0 - \eta_s) + W(\xi_s - \eta_s)]. \end{aligned}$$

For any fixed L , we can divide this expectation into two parts: the one corresponding to $\max_{i,j \in \{0,1\}} (|\xi_i|, |\eta_j|) > L$ and the one with $|\xi_i|, |\eta_i| \leq L$ for $i = 0, 1$. We also remind that $\nu_i \in K_{\alpha, C_2}$ for $i = 0, 1$ and that P controls W as well as its first and second derivatives. So, there exists a positive constant \tilde{C} such that

$$\begin{aligned} & |\mathbb{E} [W(\xi_0 - \eta_0) - W(\xi_s - \eta_0) - W(\xi_0 - \eta_s) + W(\xi_s - \eta_s)]| \\ & \leq \left| \mathbb{E} [W(\xi_0 - \eta_0) - W(\xi_s - \eta_0) - W(\xi_0 - \eta_s) + W(\xi_s - \eta_s)] \mathbf{1}_{\left\{ \max_{i,j \in \{0,1\}} (|\xi_i|, |\eta_j|) \leq L \right\}} \right| \\ & + \int_L^\infty W(2l) dF_{\max(\xi_0, \xi_1, \eta_0, \eta_1)}(l) \\ & \leq \mathbb{E} \left[\max_{|x| \leq 4L} P(|x|) |\xi_0 - \xi_s| |\eta_0 - \eta_s| \mathbf{1}_{\left\{ \max_{i,j \in \{0,1\}} (|\xi_i|, |\eta_j|) \leq L \right\}} \right] + 4 \int_L^\infty P(2l) d(1 - C_2 e^{-\alpha l})^4 \\ & \leq s^2 P(4L) W_2^2(\nu_0, \nu_1) + \tilde{C} P(2L) e^{-\alpha L}. \end{aligned}$$

So, using the already mentioned comparison $W_2^2(\mu, \rho_\infty) \leq \frac{2}{C_W} \mathcal{F}(\mu | \rho_\infty)$ (see [6]), we have

$$\begin{aligned} \Phi_\mu(\nu_s) - \Phi_\mu(\mu) & \leq -s\mathcal{F}(\mu | \rho_\infty) + s^2 P(4L) W_2^2(\mu, \rho_\infty) + \tilde{C} P(2L) e^{-\alpha L} \\ & \leq -s\mathcal{F}(\mu | \rho_\infty) + \frac{2}{C_W} s^2 P(4L) \mathcal{F}(\mu | \rho_\infty) + \tilde{C} P(2L) e^{-\alpha L}. \end{aligned}$$

Choose now L and s in such a way that the last two terms become sufficiently smaller then the first one. Provided that \mathcal{F} is small enough, take $L = \frac{2}{\alpha} |\log \mathcal{F}(\mu | \rho_\infty)|$ and $s = \frac{C_W}{4P(4L)}$. Then there exists a positive constant C'' (depending on C_W, α) such that

$$(38) \quad -s + \frac{2}{C_W} s^2 P(4L) = -s/2 \leq -\frac{C''}{|\log \mathcal{F}(\mu | \rho_\infty)|^k}$$

and because $x \log x$ goes to zero as x does, we have

$$P(2L) e^{-\alpha L} = \mathcal{F}(\mu | \rho_\infty)^2 P\left(\frac{4}{\alpha} |\log \mathcal{F}(\mu | \rho_\infty)|\right) \leq \frac{C''}{2 |\log \mathcal{F}(\mu | \rho_\infty)|^k} \mathcal{F}(\mu | \rho_\infty).$$

Remark, that for any probability measure γ , we have $\partial_{\gamma-\mu}\mathcal{F}(\cdot|\rho_\infty)|_\mu \leq \Phi_\mu(\gamma) - \Phi_\mu(\mu)$. Taking these last inequalities together, we find

$$\frac{d}{dt}\mathcal{F}(\mu_t|\rho_\infty) \leq -\frac{C''}{2|\log \mathcal{F}(\mu_t|\rho_\infty)|^k}\mathcal{F}(\mu_t|\rho_\infty).$$

Solving the equation $\dot{z} = -\frac{C}{|\log z|^k}z$, we obtain $\log z = -((k+1)C(t-e))^{1/(k+1)}$. Thus, Gronwall's Lemma implies that there exists a constant a such that

$$\mathcal{F}(\mu_t^c|\rho_\infty) \leq e^{-a^{k+1}\sqrt[t]{t}}\mathcal{F}(\mu_0|\rho_\infty)$$

provided that $\mathcal{F}(\mu_0|\rho_\infty)$ is sufficiently small. \square

Remark 6. We have proved the result only for $\mathcal{F}(\mu_0|\rho_\infty)$ sufficiently small. Indeed, when this quantity is not small, then we also get the desired result because the interaction part $\int \int W(x-y)\rho(x)\rho(y)$ is bounded and the only part we need to control is $\int \rho \log \rho$. As we consider only exponentially-decreasing probability measures, the decrease of the free energy \mathcal{F} is immediately exponential.

4.2.2. *Non-absolutely continuous probability measures.* The preceding arguments have shown that the trajectories of $\dot{\mu}_t = \Pi(\mu_t) - \mu_t$ tend to an equilibrium $\rho_\infty(x)dx$, using that $\mathcal{F}(\mu_t^c|\rho_\infty)$ and thus $W_2^2(\mu_t^c, \rho_\infty)$ are decreasing as $e^{-a^{k+1}\sqrt[t]{t}}$. Unfortunately, the entropy of a general measure with a singular part is ill-defined, since it equals the infinity. However, we will need the estimates on a distance (in one of the senses, in fact \mathcal{T}_P) from μ_t^c to ρ_∞ that are valid for non-absolutely continuous μ . Indeed, initial measures we are going to consider are occupation measures (that are singular for $d \geq 2$ and that are difficult to control for $d = 1$). To do this, prove the following

Lemma 8. *Let μ_0^c be exponentially decreasing, then $\mathcal{T}_P(\mu_t^c, \rho_\infty) = O\left(e^{-a^{k+1}\sqrt[t]{t}}\right)$.*

Proof. Take $\tilde{\mu}_0 := \mu_0 * \mathbf{1}_{[-h/2, h/2]}$, where h is small and will be chosen later. Then, we get that $\mathcal{T}_P(\mu_0, \tilde{\mu}_0) \leq ah$ due to the exponential-decrease estimates. On the other hand, the density of $\tilde{\mu}_0$ never exceeds $1/h$, so its entropy is not greater than $|\log h|$. Given that, the potential energy is bounded due to the exponential-decrease and there exists a universal constant C such that we have

$$\mathcal{F}(\tilde{\mu}_0^c|\rho_\infty) \leq |\log h| + C.$$

Due to Remark 6, $\mathcal{F}(\tilde{\mu}_t^c|\rho_\infty)$ decreases exponentially while it is greater than one, and decreases as $e^{-at^{1/(k+1)}}$ when it is smaller. So, let $t_1 \equiv \log |\log h|$; for $t \leq t_1$, the value $\mathcal{F}(\tilde{\mu}_t^c|\rho_\infty)$ will reach 1, and afterwards, it will decrease as $e^{-a(t-t_1)^{1/(k+1)}}$, what implies the same estimate on $W_2^2(\tilde{\mu}_t^c, \rho_\infty)$, and thus³ on $\mathcal{T}_P(\tilde{\mu}_t^c, \rho_\infty)$ (in a way we have already used to compare the potential part and W_2 in Proposition 10). On the other hand, $\Pi(\mu)$ is \mathcal{T}_P -Lipschitz. So, applying Gronwall's lemma, the distance $\mathcal{T}_P(\tilde{\mu}_t^c, \mu_t^c)$ can grow at most exponentially. Finally, we see

$$\mathcal{T}_P(\mu_t^c, \rho_\infty) \leq \mathcal{T}_P(\mu_t^c, \tilde{\mu}_t^c) + \mathcal{T}_P(\tilde{\mu}_t^c, \rho_\infty) \leq e^{Lip(\Pi)t}h + e^{-a(t-t_1)^{1/(k+1)}}.$$

Remark that we choose a new h for each t . Taking $h = h(t) \sim e^{-2Lip(\Pi)t}$, we have the first term exponentially small in t while $t_1 \sim \log |\log h| \sim \log t$. So, once again, changing the constant, we have a decrease estimated as $e^{-at^{1/(k+1)}}$. \square

Remark 7. The preceding result is uniform in μ_0^c , for μ_0^c belonging to a compact set (for the \mathcal{T}_P -topology) of exponentially decreasing measures.

³with logarithms, which can be “consumed” by changing the constant a

4.3. Non-symmetric counter-example. When the interaction function is quadratic, we can express X in terms of a Brownian martingale added to a deterministic part. Consider a non-symmetric interaction potential $W(x) = \frac{1}{2}(x-1)^2$, so that the system becomes:

$$\begin{cases} dX_t = \left(-X_t + \frac{1}{1+t}Y_t\right) dt + d(B_t + t); & X_0 = 0 \\ dY_t = X_t dt; & Y_0 = 0 \end{cases}$$

Solving the corresponding homogeneous system and using a constant variation permits to get that the solution to the SDE (2) with the potential $W(x) := \frac{1}{2}(x-1)^2$ is given explicitly by

$$X_t = X_0 + \int_0^t [1 - (1+s)e^s(F(t) - F(s))] d(B_s + s)$$

and the empirical mean process reads

$$(39) \quad \bar{\mu}_t := \frac{1}{t} \int_0^t X_s ds = X_0 + \int_0^t \left[1 - (1+s)e^s \left(F(t) - F(s) + \frac{1}{1+t}e^{-t}\right)\right] d(B_s + s)$$

where $F(t) = \int_0^t e^{-s} \frac{ds}{1+s}$.

These expressions enable us to show that the variables $c_t := \bar{\mu}_t$ and X_t diverge almost surely. Indeed, to prove that, we only need to put the following asymptotic development of F in (39):

$$F(t) = F(\infty) - \frac{1}{1+t}e^{-t} + \frac{1}{(1+t)^2}e^{-t} + o\left(\frac{e^{-t}}{t^2}\right).$$

So the deterministic part of $\bar{\mu}_t$ diverges and we conclude.

Remark 8. Even in that case, we manage to prove that the process $(X_t - \bar{\mu}_t, t \geq 0)$ is ergodic, meaning that a.s. the centered-measure μ_t^c converges (weakly) to a Gaussian variable, but it is false that μ_t itself converges (because c_t does not).

5. PROOF OF THEOREM 3

5.1. Exponential decrease estimates. We shall now estimate the behaviour of the centered measures μ_t^c . First, we need one easy lemma.

Lemma 9. *Let Z be an Ornstein-Uhlenbeck process. Then, there exist $C_1, C_2 > 0$, such that for almost any trajectory Z_t , one has almost surely*

$$\exists T : \forall t \geq T, \forall r > 0 \quad \frac{1}{t} |\{s \leq t : |Z_s| > r\}| < C_1 e^{-C_2 r}.$$

Proof. The limit-quotient theorem implies that a.s. the empirical measure of the Ornstein-Uhlenbeck process Z converges weakly to the Gaussian probability measure γ_{OU} (that is $e^{-C_W|x|^2/2}$). Moreover, there exist $C, C' > 0$ such that, for all $r > 0$, $\int_{|x|>r} \gamma_{OU}(x) dx \leq C e^{-C' r}$. Consequently, there exist $C_1, C_2 > 0$ and T such that for all $t \geq T$, for all $r > 0$,

$$\frac{1}{t} |\{s \leq t : |Z_s| > r\}| < C_1 e^{-C_2 r}. \quad \square$$

The main result of this subsection is the following

Proposition 11. *There exist two constants $C_1, C_2 > 0$ such that a.s. at any sufficiently large moment of time, the following estimate of exponential decrease holds:*

$$\mu_t^c(\{y : |y| \geq r\}) \leq C_1 e^{-C_2 r}.$$

Proof. First, let us estimate the drift of the center. Namely, we know that

$$\dot{c}_t = \frac{1}{t} (\nabla^2 W * \mu_t(c_t))^{-1} \nabla W(X_t - c_t),$$

and thus, in the view of Lemma 1, (up to a multiplicative constant)

$$|\dot{c}_t| \leq \frac{1}{tC_W} P(|X_t - c_t|) \leq \frac{1}{tC_W} P(|Z_t|)$$

for the corresponding Ornstein-Uhlenbeck trajectory Z_t .

On the other hand, Z is a Harris recurrent process and $P(|Z|)$ is integrable with respect to the Gaussian measure, thus due to the limit-quotient (or Birkhoff) theorem, almost surely there exists a limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(|Z_s|) ds = \int_{\mathbb{R}^d} P(|z|) d\gamma_{OU}(z) =: I.$$

So, almost surely from some moment t_1 we have

$$\forall t > t_1 \quad \frac{1}{t} \int_0^t P(|Z_s|) ds \leq I + 1.$$

Therefore, after this moment we can estimate the displacement of the center between $t/2$ and t : $\forall t > t_1$

$$|c_{t/2} - c_t| \leq \int_{t/2}^t |\dot{c}_s| ds \leq \int_{t/2}^t \frac{1}{sC_W} P(|Z_s|) ds \leq \frac{1}{C_W t/2} \int_0^t P(|Z_s|) ds \leq 2(I + 1)/C_W =: C_3.$$

In fact, the same estimate holds for any t' between $t/2$ and t :

$$|c_{t'} - c_t| \leq C_3.$$

This immediately implies that for any $t > t_1$ and $n \in \mathbb{N}$ such that $2^{-n+1}t > t_1$, one has

$$|c_t - c_{t/2^n}| \leq C_3 n.$$

Now, we estimate the behaviour of μ_t . Indeed, the Ornstein-Uhlenbeck occupation measures exponentially decrease. Namely, by Lemma 9, there exist two positive constants C'_1, C'_2 , such that for almost any trajectory Z_t one has almost surely

$$\exists t_2 : \forall t \geq t_2 \forall r > 0 \quad \frac{1}{t} |\{s \leq t : |Z_s| > r\}| < C'_1 e^{-C'_2 r}.$$

Using this, for any $t \geq \max(t_1, t_2)$ we obtain an estimate on the measure $\mu_{[t/2, t]}$:

$$\begin{aligned} \mu_{[t/2, t]}(|y - c_t| > r) &= \frac{2}{t} |\{s \in [t/2, t] : |X_s - c_t| > r\}| \\ &\leq \frac{2}{t} |\{s \in [t/2, t] : |X_s - c_s| > r - |c_t - c_s|\}| \\ &\leq \frac{2}{t} |\{s \in [t/2, t] : |Z_s| > r - |c_t - c_s|\}| \\ &\leq \frac{2}{t} |\{s \in [0, t] : |Z_s| > r - C_3\}| \leq (2C'_1 e^{C_2 C_3}) e^{-C'_2 r} = C''_1 e^{-C'_2 r}, \end{aligned}$$

where $C''_1 := 2C'_1 e^{C_2 C_3}$. Note, that if the estimate

$$\mu_{[t/2, t]}(|y - c_t| > r) \leq C''_1 e^{-C'_2 r}$$

holds for some C'_2 , it holds also for any $C''_2 < C'_2$, so we can take C''_2 sufficiently small for $e^{C_3 C''_2} < 2$.

To finish the proof, for any $t \geq t_1$ decompose the measure μ_t as a sum

$$\mu_t = \frac{1}{2}\mu_{[t/2,t]} + \frac{1}{4}\mu_{[t/4,t/2]} + \cdots + \frac{1}{2^n}\mu_{[2^{-(n-1)}t, 2^{-n}t]} + \frac{1}{2^n}\mu_{2^{-n}t},$$

where n is the largest integer such that $2^{-n+1}t > \max(t_1, t_2) =: t_3$.

This implies, that

$$\begin{aligned} \mu_t(|y - c_t| > r) &= \sum_{j=1}^n \frac{1}{2^j} \mu_{[2^{-j+1}t, 2^{-j}t]}(|y - c_t| > r) + \frac{1}{2^n} \mu_{2^{-n}t}(|y - c_t| > r) \\ &\leq \sum_{j=1}^n \frac{1}{2^j} \mu_{[2^{-j+1}t, 2^{-j}t]}(|y - c_{2^{-j}t}| > r - |c_t - c_{2^{-j}t}|) + \frac{1}{2^n} \mu_{2^{-n}t}(|y - c_t| > r) \\ &\leq \sum_{j=1}^n \frac{1}{2^j} C_1'' e^{-C_2''(r-jC_3)} + \frac{1}{2^n} \mu_{2^{-n}t}(|y - c_t| > r) \\ &= C_1'' e^{-C_2''r} \sum_{j=1}^n \frac{1}{2^j} e^{C_2''C_3j} + \frac{1}{2^n} \mu_{2^{-n}t}(|y - c_t| > r). \end{aligned}$$

Due to the hypothesis on C''_2 , we have $\sum_{j=1}^{\infty} \left(\frac{e^{C_2''C_3}}{2}\right)^j < \infty$, and thus the first term can be estimated as $C_1''' e^{-C_2''r}$, where

$$C_1''' := C_1'' \sum_{j=1}^{\infty} \left(\frac{e^{C_2''C_3}}{2}\right)^j.$$

To estimate the second term, we note that $\text{supp } \mu_{2^{-n}t} \subset \text{supp } \mu_{t_3} \subset [-K, K]$ for some (random!) K , which does not depend on t . Thus, the second term is non-zero only if $r < K + |c_t|$, and using the bound $|c_t| < nC_3 + |c_{2^{-n}t}| \leq nC_3 + K$, we see that this happens only if $r < 2K + C_3n$.

Thus, either the second term is zero, or $r < 2K + C_3n$, in which case, rewriting the latter as $n < \frac{r-2K}{C_3}$ and using $e^{C_2''C_3} < 2$, we obtain

$$\frac{1}{2^n} \mu_{2^{-n}t}(|y - c_t| > r) \leq \frac{1}{2^n} < e^{-\frac{(r-2K)}{C_3} C_2''} = e^{2KC_2''} e^{-C_2''r}.$$

Moreover, due to the strict inequality $e^{C_2''C_3} < 2$ for any given K for n sufficiently big, one has

$$\frac{1}{2^n} \mu_{2^{-n}t}(|y - c_t| > r) \leq e^{-C_2''r}$$

(a multiplicative constant is minor with respect to the difference of exponent bases). Thus, for any t large enough, one has for any r ,

$$\mu_t(\{y; |y - c_t| > r\}) < C_1'' e^{-C_2''r}. \quad \square$$

5.2. Convergence for the random trajectories. Now, we are ready to establish the convergence (almost surely) of (random) occupation measures to a random measure $\rho_{\infty}(x)dx$. As for the deterministic flow, we shall first obtain the convergence of centered-measures⁴: $\mu_t^c \rightarrow \rho_{\infty}$, and then “integrate” it passing to the centers.

⁴From now on, μ_t denotes the true normalized occupation measure.

5.2.1. *Asymptotic pseudotrajectories.* The notion of asymptotic pseudotrajectories was first introduced by Benaim & Hirsch [2], and is particularly useful for analyzing the long-term behaviour of stochastic processes, considered as approximations to solutions of ordinary differential equation. We refer to this article for more details and just give the definition.

Definition 4. A continuous function $\xi : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}^d; P)$ is an *asymptotic pseudotrajectory* (or *asymptotic pseudo-orbit*) for the flow Φ (for the weak* topology of measures) if for all $T > 0$, we have

$$(40) \quad \lim_{t \rightarrow +\infty} \sup_{0 \leq s \leq T} \mathcal{T}_P(\xi_{t+s}, \Phi_s(\xi_t)) = 0.$$

The purpose here is to find an asymptotic pseudotrajectory for the flow Φ defined by (22). Indeed, we will show that the time-changed process μ_{e^t} (and not μ_t) is an asymptotic pseudotrajectory for Φ . The need for a time-change comes from the normalization of the occupation measure μ_t . We will prove in the following Subsections:

Theorem 6. *Almost surely, the function $t \mapsto \mu_{e^t}$ is an asymptotic pseudotrajectory for Φ .*

Strategy of the Proof. We shall consider an increasing sequence of moments $T_1 \leq T_2 \leq \dots$ (where T_n is increasing to the infinity), with $\Delta T_n := T_{n+1} - T_n \ll T_n$. More precisely, $\Delta T_n = \sqrt[3]{T_n}$. We will do the following:

- 1) Compare on $[T_n, T_{n+1}]$ the trajectories of the true drifted motion

$$(41) \quad dX_t^1 = \sqrt{2} dB_t - \nabla W * \mu_t(X_t^1) dt$$

with the ones where μ_t is replaced by μ_{T_n}

$$(42) \quad dX_t^2 = \sqrt{2} dB_t - \nabla W * \mu_{T_n}(X_t^2) dt;$$

- 2) Estimate the convergence speed of the occupation measure for the trajectories of (42) to $\Pi(\mu_{T_n})$;
- 3) Compare the obtained ‘‘Euler-method’’ flow

$$\tilde{\mu}_{[T_n, T_{n+1}]} = \tilde{\mu}_{T_n} + \frac{\Delta T_n}{T_{n+1}} (\tilde{\mu}_{[T_n, T_{n+1}]} - \tilde{\mu}_{T_n} + \text{error})$$

to the true $\dot{\mu} = \frac{1}{T_n} (\Pi(\mu) - \mu)$;

- 4) Conclude for the centered-measure μ_t^c ;
- 5) Conclude for the convergence of μ_t (in Section 6).

5.2.2. *Approximation motion: fix μ_t .* The main idea of this paragraph is to pass from (41) to (42). Namely, let X_t^1 and X_t^2 satisfy

$$\begin{cases} dX_t^1 = \sqrt{2} dB_t - \nabla W * \mu_t(X_t^1) dt \\ dX_t^2 = \sqrt{2} dB_t - \nabla W * \mu_{T_n}(X_t^2) dt, \end{cases}$$

where we emphasize that X^1 and X^2 are driven by the same Brownian motion.

Lemma 10. *For $T_n \leq t \leq T_{n+1}$, we have*

$$|X_t^1 - X_t^2| \leq e^{-C_W(t-T_n)} |X_{T_n}^1 - X_{T_n}^2| + \frac{\Delta T_n}{T_n C_W} P(2L_n),$$

where $L_n := \max_{T_n \leq t \leq T_{n+1}} |X_t^1 - c_{T_n}|$.

Remark 9. Given the estimates of Subsection 2.3, we see that $L_n \leq \log T_n$, for n large enough, for almost every trajectory.

Proof. We have that $X_t^1 - X_t^2$ is of class \mathcal{C}^1 , and we immediately compute $\frac{d}{dt}(X_t^1 - X_t^2) = -(\nabla W * \mu_t(X_t^1) - \nabla W * \mu_{T_n}(X_t^2))$. Adding and subtracting $\nabla W * \mu_{T_n}(X_t^1)$, we see

$$d(X_t^1 - X_t^2) = -[\nabla W * (\mu_t - \mu_{T_n})(X_t^1) - (\nabla W * \mu_{T_n}(X_t^2) - \nabla W * \mu_{T_n}(X_t^1))] dt.$$

The last term can be rewritten as

$$-(\nabla W * \mu_{T_n}(X_t^2) - \nabla W * \mu_{T_n}(X_t^1)) = \frac{1}{T_n} \int_0^{T_n} \int_0^1 \nabla^2 W|_{uX_t^2 + (1-u)X_t^1 - X_s^1} \cdot (X_t^1 - X_t^2) du ds.$$

Noting the first term as D_t , and putting a scalar product with $X_t^1 - X_t^2$, we see

$$\begin{aligned} \left(X_t^1 - X_t^2, \frac{d}{dt}(X_t^1 - X_t^2) \right) &= (D_t, X_t^1 - X_t^2) \\ &\quad - \frac{1}{T_n} \int_0^{T_n} \left(\int_0^1 \nabla^2 W|_{uX_t^2 + (1-u)X_t^1 - X_s^1} du \cdot (X_t^1 - X_t^2), X_t^1 - X_t^2 \right) ds \\ &\leq (D_t, X_t^1 - X_t^2) - C_W |X_t^1 - X_t^2|^2. \end{aligned}$$

Thus, $\frac{d}{dt}|X_t^1 - X_t^2|^2 \leq -2C_W |X_t^1 - X_t^2|^2 + 2|D_t||X_t^1 - X_t^2|$. Redividing by $2|X_t^1 - X_t^2|$, we obtain the desired

$$(43) \quad \frac{d}{dt}|X_t^1 - X_t^2| \leq |D_t| - C_W |X_t^1 - X_t^2|.$$

Solving (43) and taking into account that $|D_t| \leq P(2L_n) \frac{\Delta T_n}{T_n}$, as it is the difference between the forces generated at X_t^1 by μ_{T_n} and by $\mu_t = \mu_{T_n} + \frac{t-T_n}{t}(\mu_{[T_n, t]} - \mu_{T_n})$, we obtain the desired estimate for $T_n \leq t \leq T_{n+1}$. \square

Now, in order to use this new process, we have to estimate the difference between the old and the new occupation measures. Namely, denote by $\tilde{\mu}_{[T_n, T_{n+1}]}$ the occupation measure for X_t^2 on the interval of time $[T_n, T_{n+1}]$. In the next paragraph, choosing the initial condition $X_{T_n}^2$, we will (for some reasons) choose it within the ball $B_1(c_{T_n})$.

For any such initial condition, let us compare the empirical measures in terms of \mathcal{T}_P -distance, for the measures centered in c_{T_n} . Namely, prove the following for $\mathcal{T}_P^{c_{T_n}}(\nu, \mu) := \mathcal{T}_P(\nu(\cdot + c_{T_n}), \mu(\cdot + c_{T_n}))$

Proposition 12. *For any (family of) choices $X_{T_n}^2 \in B_1(c_{T_n})$, we have $\mathcal{T}_P^{c_{T_n}}(\mu_{[T_n, T_{n+1}]}, \Pi(\mu_{T_n})) = o(T_n^{-1/5})$.*

Remark 10. The words ‘‘family of’’ here correspond to the fact that $X_{T_1}^2, X_{T_2}^2, \dots$ can be chosen independently: we re-launch a new process on every new interval $[T_n, T_{n+1}]$.

Proof. Due to Lemma 10, we have

$$|X_t^1 - X_t^2| \leq e^{-C_W(t-T_n)} |X_{T_n}^1 - X_{T_n}^2| + \frac{\Delta T_n}{T_n C_W} P(2L_n) \leq e^{-C_W(t-T_n)} (L_n + 1) + \frac{\Delta T_n}{T_n C_W} P(2L_n),$$

where $L_n = \max_{T_n \leq t \leq T_{n+1}} |X_t^1 - c_{T_n}|$.

Recall that due to Remark 9, we have almost surely $L_n \leq \log T_n$ for all n sufficiently big. So, we have almost surely for every n sufficiently big,

$$|X_t^1 - X_t^2| \leq e^{-C_W(t-T_n)} (\log T_n + 1) + \frac{\Delta T_n}{T_n C_W} P(2 \log T_n),$$

Denote $\delta_n := \frac{1}{C_W}(\log T_n + \log(\log T_n + 1))$. Then, for all $t \in [T_n + \delta_n, T_{n+1}]$, one has

$$|X_t^1 - X_t^2| \leq \frac{1}{T_n} + \frac{\Delta T_n}{T_n C_W} P(2 \log T_n) \leq \frac{1}{T_n^{1/2}}$$

(the latter inequality holds provided n is sufficiently big).

On the other hand, for $t \in [T_n, T_n + \delta_n]$ one has $|X_t^1 - c_{T_n}| \leq L_n$ and $|X_t^2 - X_t^1| \leq L_n + \frac{\Delta T_n}{C_W T_n} P(2L_n) \leq 2L_n$. Thus,

$$|X_t^2 - c_{T_n}| \leq 3L_n \leq 3 \log T_n.$$

Together, these estimates imply for the c_{T_n} -centered transporting distance

$$\begin{aligned} \mathcal{T}_P^{c_{T_n}}(\mu_{[T_n, T_{n+1}]}, \tilde{\mu}_{[T_n, T_{n+1}]}) &\leq \\ &\leq \frac{1}{\Delta T_n} P(L_n) \left(\int_{T_n}^{T_n + \delta_n} |X_t^1 - X_t^2| dt + \int_{T_n + \delta_n}^{T_{n+1}} |X_t^1 - X_t^2| dt \right) \leq \\ &\leq \left(\frac{\delta_n}{\Delta T_n} + \frac{\text{const}}{T_n^{1/2}} \right) \cdot P(L_n) = O(T_n^{-1/3} (\log T_n)^{k+1}), \end{aligned}$$

as $\delta_n = O(\log T_n)$ and $L_n = O(\log T_n)$ almost surely. \square

5.2.3. Approximation: $\tilde{\mu}$ and $\Pi(\mu)$. In this paragraph, we will compare the occupation measure $\tilde{\mu}$ with $\Pi(\mu_{T_n})$. To do this, we will use Proposition 1.2 of the work Cattiaux & Guillin [7] (see also Wu [17]), stating that the trajectory mean of a function V is, with a probability close to 1 that can be exponentially-controlled, close to its stationary mean. Namely, this proposition says the following:

Proposition 13 (Cattiaux & Guillin [7]). *Given a process with a stationary measure m and an initial measure ν and a function V satisfying $|V| \leq 1$, for any $0 < \rho < 1$ and any $t > 0$, one has*

$$\mathbb{P}_\nu \left(\frac{1}{t} \int_0^t V(X_s) ds - \int V dm \geq \rho \right) \leq \left\| \frac{d\nu}{dm} \right\|_{L_2(m)} \cdot \exp \left(-\frac{t\rho^2}{8C_P \text{Var}_m(V)} \right).$$

Here the C_P denotes the Poincaré constant of the process.

So, we will use this proposition with the function V being the indicator function $V = \mathbf{1}_M$ of various sets M : it then allows to compare the occupation measure of the set M to its $\Pi(\mu_{T_n})$ -measure.

We know that $m = \Pi(\mu_{T_n})$ is the unique stationary measure for the drifted Brownian motion (42). Also, the Poincaré constant for this process is equal to $2C_W$ (see [1]).

Also, to proceed, we have to declare the initial measure $\nu = \nu_n$ for $X_{T_n}^2$, and we choose it to be the measure $\Pi(\mu_{T_n})$, restricted on the ball $B_1(c_{T_n})$ and then normalized accordingly. Then,

$$\left\| \frac{d\nu_n}{d\Pi(\mu_{T_n})} \right\|_{L_2(\Pi(\mu_{T_n}))} = \frac{1}{\Pi(\mu_{T_n})(B_1(c_{T_n}))} \leq c_E = \text{const},$$

the latter inequality is due to the exponential decrease of $\Pi(\mu_{T_n})$.

Having made these choices, we are going to prove the following

Proposition 14. *As $n \rightarrow \infty$, we have almost surely*

$$\mathcal{T}_P^{c_{T_n}}(\tilde{\mu}_{[T_n, T_{n+1}]}, \Pi(\mu)) = O((\Delta T_n)^{-\min(8C_W, \frac{1}{5d})}).$$

Proof. The estimates of the previous paragraph imply that the process X_t^2 on $[T_n, T_{n+1}]$ almost surely for all n sufficiently big stays inside the ball $B_{R_n}(c_{T_n})$, where $R_n := 3 \log T_n$.

Now, take this ball and cut it into some N_n parts M_1, \dots, M_{N_n} of diameter less than $\varepsilon_n := \frac{2dR_n}{\sqrt[d]{N_n}}$ (simply by cubic the grid with the step $2R_n/\sqrt[d]{N_n}$, that is, decomposing each of the coordinate segments of length $2R_n$ into $\sqrt[d]{N_n}$ parts). We will choose and fix the number N_n later.

For each of these parts, choose

$$\rho_j := \max \left(\frac{1}{N_n^2}, \frac{\Pi(\mu_{T_n})(M_j)}{N_n} \right).$$

Then, the probability that all the occupation measures $\tilde{\mu}_{[T_n, T_{n+1}]}(M_j)$ are ρ_j -close to their “theoretical” values $\Pi(\mu_{T_n})(M_j)$ is at least

$$1 - 2c_E \sum_{j=1}^{N_n} \exp \left(-\frac{\rho_j^2 \Delta T_n}{16C_W \text{Var}_{\Pi(\mu_{T_n})}(V_j)} \right).$$

As the variance $\text{Var}_{\Pi(\mu_{T_n})}(V_j)$ of the indicator function does not exceed $\Pi(\mu_{T_n})(V_j)$, we have a lower bound for the probability by

$$1 - 2c_E \sum_{j=1}^{N_n} \exp \left(-\frac{\rho_j \Delta T_n}{16C_W} \cdot \frac{\rho_j}{\Pi(\mu_{T_n})(V_j)} \right) \geq 1 - 2N_n c_E \exp \left(-\frac{\Delta T_n}{16C_W N_n^3} \right),$$

as $\frac{\rho_j}{\Pi(\mu_{T_n})(V_j)} \geq \frac{1}{N_n}$ and $\rho_j \geq \frac{1}{N_n^2}$.

So, taking $N_n = \sqrt[10]{T_n} \sim (\Delta T_n)^{3/10}$, we see that the series

$$\sum_n N_n \exp \left(-\frac{\Delta T_n}{16C_W N_n^3} \right) \sim \sum_n (\Delta T_n)^{3/10} \exp \left(-(\Delta T_n)^{1/10} \right)$$

converges, so almost surely for all n sufficiently big, all the closeness conditions on the occupation measures are satisfied: the measures $\tilde{\mu}_{[T_n, T_{n+1}]}(M_j)$ are a.s. ρ_j -close to $\Pi(\mu_{T_n})(M_j)$.

Now, let us estimate the c_{T_n} -centered distance $\mathcal{T}_P^{c_{T_n}}(\tilde{\mu}_{[T_n, T_{n+1}]}, \Pi(\mu_{T_n}))$, provided that these conditions are fulfilled.

Indeed, first let us transport inside each M_j the part $\min(\tilde{\mu}_{[T_n, T_{n+1}]}, \Pi(\mu_{T_n}))$: we pay at most $P(3 \log T_n) \varepsilon_n = O \left((\Delta T_n)^{-\frac{1}{5d}} \right)$. Next, bring the exterior part of $\Pi(\mu_{T_n})$ to the ball $B_{R_n}(c_{T_n})$: due to the exponential decrease estimates, we pay at most

$$\int_{R_n}^{\infty} P(r) d(1 - C e^{-C_W r}) \sim R_n^{k+1} e^{-C_W R_n} = O \left((\Delta T_n)^{-8C_W} \right)$$

as $R_n = 3 \log T_n$. Finally, let us re-distribute the parts left: we pay at most

$$\begin{aligned} \sum_{j=1}^{N_n} \rho_j R_n P(R_n) &= R_n P(R_n) \sum_{j=1}^{N_n} \max \left(\frac{1}{N_n^2}, \frac{\Pi(\mu_{T_n})(M_j)}{N_n} \right) \leq R_n P(R_n) \sum_{j=1}^{N_n} \left(\frac{1}{N_n^2} + \frac{\Pi(\mu_{T_n})(M_j)}{N_n} \right) \\ &\leq 2R_n P(R_n) \frac{1}{N_n} = O \left((\Delta T_n)^{-1/5} \right). \end{aligned}$$

Adding these three estimates, we obtain the desired $\mathcal{T}_P^{c_{T_n}}(\tilde{\mu}_{[T_n, T_{n+1}]}, \Pi(\mu_{T_n})) = O \left((\Delta T_n)^{-\beta} \right)$ with $\beta = \min(8C_W, (5d)^{-1})$. \square

5.2.4. *Convergence of μ_t^c .* Due to the Ornstein-Uhlenbeck estimates, it clearly suffices to check that $\mu_{T_n}^c \xrightarrow{n \rightarrow \infty} \rho_\infty$. From Subsections 5.2.2 and 5.2.3, almost surely, for all n large enough with $\beta := \min(8C_W, \frac{1}{5d})$,

$$\mathcal{T}_P(\mu_{[T_n, T_{n+1}]}, \Pi(\mu_{T_n})) \leq \mathcal{T}_P(\mu_{[T_n, T_{n+1}]}, \tilde{\mu}_{[T_n, T_{n+1}]}) + \mathcal{T}_P(\tilde{\mu}_{[T_n, T_{n+1}]}, \Pi(\mu_{T_n})) = O((\Delta T_n)^{-\beta}).$$

So, almost every trajectory of the process μ_{T_n} can be thought as Euler-method with steps ΔT_n for the differential equation $\dot{\mu}_t = \frac{1}{t}(\Pi(\tilde{\mu}_t) - \tilde{\mu}_t)$, perturbed with an error of $\Pi(\mu_t)$ determination at most $(\Delta T_n)^{-\beta}$. Indeed, we will choose $\Delta T_n = T_n^{1/3}$ in the following.

Lemma 11. *There exists $a > 0$ such that $\mathcal{T}_P(\tilde{\mu}_t^c, \rho_\infty) = O(e^{-a \sqrt[k+1]{\log t}})$ (where k is the degree of the polynomial P).*

Proof. Let us pass to the logarithmic coordinates. Let $\theta := \log t$, $m_\theta := \mu_{e^\theta}$, $\theta_n = \log T_n$, $\Delta\theta_n = \theta_{n+1} - \theta_n = \log\left(\frac{T_{n+1}}{T_n}\right) = \frac{\Delta T_n}{T_n} + O\left(\left(\frac{\Delta T_n}{T_n}\right)^2\right)$. Then,

$$\begin{aligned} m_{\theta_{n+1}} &= \frac{T_n}{T_{n+1}} m_{\theta_n} + \frac{\Delta T_n}{T_{n+1}} \mu_{[T_n, T_{n+1}]} \\ &= \frac{T_n}{T_{n+1}} m_{\theta_n} + \Delta\theta_n (\Pi(m_{\theta_n}) - m_{\theta_n}) + O_{\mathcal{T}_P} \left(\left(\frac{\Delta T_n}{T_n} \right)^2 + \frac{(\Delta T_n)^{-\beta}}{T_n} \right). \end{aligned}$$

Now, denote by $\Phi_a^b(\mu)$ the time- $(b-a)$ -map under $\dot{m} = \Pi(m) - m$ with the initial condition μ . Then, letting $Lip(\Pi)$ the Lipschitz constant of Π :

$$\begin{aligned} \mathcal{T}_P(\Phi_{\theta_n}^{\theta_{n+1}}(m_{\theta_n}), m_{\theta_n} + \Delta\theta_n \cdot (\Pi(m_{\theta_n}) - m_{\theta_n})) &= \mathcal{T}_P \left(\int_{\theta_n}^{\theta_{n+1}} e^{\theta - \theta_{n+1}} \Pi(m_\theta) d\theta, \Delta\theta_n \cdot \Pi(m_{\theta_n}) \right) \\ &\leq \Delta\theta_n \cdot Lip(\Pi) \cdot \max_{\theta_n \leq \theta \leq \theta_{n+1}} (\mathcal{T}_P(m_\theta, m_{\theta_n})). \end{aligned}$$

By the preceding decomposition of m_θ with respect to m_{θ_n} and due to the exponential decrease of the measures we are considering, there exists a positive constant C such that

$$\mathcal{T}_P \left(\Phi_{\theta_n}^{\theta_{n+1}}(m_{\theta_n}), m_{\theta_n} + \Delta\theta_n \cdot (\Pi(m_{\theta_n}) - m_{\theta_n}) \right) \leq \Delta\theta_n \cdot Lip(\Pi) \cdot \Delta\theta_n \cdot C.$$

As $\Delta\theta_n \sim \frac{\Delta T_n}{T_n}$, we get $(\Delta\theta_n)^2 \sim T_n^{-4/3}$. Finally, for all n large enough

$$\mathcal{T}_P \left(\Phi_{\theta_n}^{\theta_{n+1}}(m_{\theta_n}), m_{\theta_{n+1}} \right) \leq \Delta\theta_n \cdot T_n^{-\beta} + (\Delta\theta_n)^2 C \leq \alpha (\Delta\theta_n)^{1+3\beta/2}.$$

Once again, taking into account that Π is Lipschitz and thus that

$$\mathcal{T}_P(\Phi_a^b(m), \Pi_a^b(\tilde{m})) \leq e^{(b-a)Lip(\Pi)} \mathcal{T}_P(m, \tilde{m}),$$

we obtain the following estimate of deviation between $\Phi_{\theta_n}^{\theta_N}(m_{\theta_n})$ and m_{θ_N} (with $N \gg n$):

$$\begin{aligned} \mathcal{T}_P \left(\Phi_{\theta_n}^{\theta_N}(m_{\theta_n}), m_{\theta_N} \right) &\leq \sum_{i=n}^{N-1} \mathcal{T}_P \left(\Phi_{\theta_{i+1}}^{\theta_N}(\Phi_{\theta_i}^{\theta_{i+1}}(m_{\theta_i})), \Phi_{\theta_{i+1}}^{\theta_N}(m_{\theta_{i+1}}) \right) \\ &\leq \sum_{j=n}^{N-1} e^{(\theta_N - \theta_j)Lip(\Pi)} \alpha (\Delta\theta_j)^{1+3\beta/2} \\ &\leq e^{(\theta_N - \theta_n)Lip(\Pi)} \max_{n \leq j \leq N} (\Delta\theta_j)^{3\beta/2} \cdot \alpha (\theta_N - \theta_n). \end{aligned}$$

Fix $\delta > 0$. Taking $\theta_N = (1 + \delta)\theta_n$, we obtain an exponential of order $T_n^{\delta Lip(\Pi)}$ while $\max_{n \leq j \leq N} (\Delta\theta_j)^{3\beta/2} \sim \max_{n \leq j \leq N} T_j^{-\beta} = T_n^{-\beta}$. Thus $\mathcal{T}_P\left(\Phi_{\theta_n}^{\theta_N}(m_{\theta_n}), m_{\theta_N}\right)$ is of order $T_n^{-\beta+\beta'}$, where $\beta' = \delta Lip(\Pi)$ (choose for instance $\delta = \beta/(2Lip(\Pi))$). On the other hand, due to the convergence speed estimates,

$$(44) \quad \mathcal{T}_P\left(\left\{\Phi_{\theta_n}^{\theta_N}(m_{\theta_N})\right\}^c, \rho_\infty\right) \leq e^{-a(\theta_N - \theta_n)^{1/(k+1)}},$$

and so is of order $e^{-a(\delta \log T_n)^{1/(k+1)}}$. \square

We can now state and prove the following

Proposition 15. *There exists $a > 0$ such that $\mathcal{T}_P(\mu_t^c, \rho_\infty) = O\left(e^{-a k+1 \sqrt{\log t}}\right)$ (where k is the degree of the polynomial P).*

Proof. We keep the same notation as in the preceding Lemma. Finally, as $e^{-(\delta \log t)^{1/(k+1)}} \gg e^{-\beta \log t/2}$, we have the same estimate as (44) for $\mathcal{T}_P(m_{\theta_n}^c, \rho_\infty)$. Thus, $\mathcal{T}_P(m_{\theta_n}^c, \rho_\infty)$ goes to 0 at speed $e^{-a(\delta \theta_n)^{1/(k+1)}} = e^{-a(\delta \log T_n)^{1/(k+1)}}$. \square

Remark 11. We point out that we have a (rough) lower bound for a : $a^{k+1} \geq \min\left(4C_W, \frac{1}{10d}\right)$.

6. PROOF OF THEOREM 4

We will find a *compact* (for the \mathcal{T}_P -topology) set $K \subset \mathcal{P}(\mathbb{R}^d; P)$, of *centered* absolutely continuous measures, such that for a solution $(\mu_t, t \geq 0)$, starting from $\mu_0 \in K$, one has $\mu_t^c \in K$.

Proposition 16. *There exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, there exists $C_1 > 1$, such that for all $C_2 > C_1$, the set*

$$K_{\alpha, C_2} := \left\{ \mu / \forall R > 0, \mu(|x - c(\mu)| \geq R) \leq C_2 e^{-\alpha R} \right\}$$

is Φ_t -(positively) invariant.

Proof. We will construct such a constant C_1 explicitly. First, note that imposing on C_1 the condition $C_1 > e^{2\alpha}$, we guarantee that the inequality for any μ holds automatically for any $R \leq 2$. Also, it is clear that it suffices to consider a “small” interval of time, $0 \leq t \leq 1$, decomposing big intervals as a series of small ones if necessary. Now, let $(\mu_t, t \geq 0)$ be a local solution, starting from a measure μ_0 belonging to some K_{α, C_2} . Then,

$$(45) \quad \begin{aligned} \mu_t(|x - c_t| \geq R) &= e^{-t} \mu_0(|x - c_t| \geq R) + \int_0^t e^{-(t-s)} \Pi(\mu_s)(|x - c_t| \geq R) ds \\ &\leq e^{-t} \mu_0(|x - c_0| \geq R - |c_0 - c_t|) + \int_0^t e^{-(t-s)} \Pi(\mu_s)(|x - c_s| \geq R - |c_t - c_s|) ds. \end{aligned}$$

Recall that $|\dot{c}_t| \leq v_c$ and thus $|c_t - c_s| \leq v_c |t - s|$. Hence, the right hand side of (45) does not exceed

$$(46) \quad e^{-t} \mu_0(|x - c_0| \geq R - tv_c) + \int_0^t e^{-(t-s)} \cdot \Pi(\mu_s)(|x - c_s| \geq R - (t-s)v_c) ds.$$

Now, by hypothesis $\mu_0(|x-c_0| \geq R-tv_c) \leq C_2 e^{-\alpha(R-tv_c)}$ and by Proposition 4, $\Pi(\mu_s)(|x-c_s| \geq R-(t-s)v_c) \leq C e^{-C_W(R-(t-s)v_c)}$. Thus, (46) is less than

$$\begin{aligned}
& C_2 e^{-t} e^{-\alpha(R-tv_c)} + C \int_0^t e^{-(t-s)} e^{-C_W(R-(t-s)v_c)} ds \\
&= C_2 e^{-\alpha R} e^{-t(1-\alpha v_c)} + C e^{-C_W R} \int_0^t e^{-(t-s)} e^{C_W(t-s)v_c} ds \\
(47) \quad & \leq C_2 e^{-\alpha R} e^{-t(1-\alpha v_c)} + C e^{-C_W R} t e^{C_W v_c},
\end{aligned}$$

where we have used $t \leq 1$ and $e^{-(t-s)} \leq 1$ for the latter inequality. Let $\alpha_0 = \min\left(\frac{1}{2v_c}, C_W\right)$. Then, for any $0 < \alpha < \alpha_0$, $e^{-C_W R} \leq e^{-\alpha R}$ and hence the inequality (47) does not exceed $C_2 e^{-\alpha R} e^{-t(1-\alpha v_c)} + C e^{-\alpha R} t e^{C_W v_c}$. For some $\lambda > 0$, $e^{-t(1-\alpha v_c)} \leq 1 - \lambda t$ uniformly on $0 \leq t \leq 1$. Then,

$$\begin{aligned}
(48) \quad C_2 e^{-\alpha R} e^{-t(1-\alpha v_c)} + C e^{-\alpha R} t e^{C_W v_c} & \leq C_2 e^{-\alpha R} (1 - \lambda t) + C e^{-\alpha R} t e^{C_W v_c} \\
& \leq C_2 e^{-\alpha R} + t(C e^{C_W v_c} - C_2 \lambda) e^{-\alpha R}.
\end{aligned}$$

Choosing $C_2 \lambda > C e^{C_W v_c}$, we have that (48) is less than $C_2 e^{-\alpha R}$, concluding the proof of the invariance. \square

6.1. The deterministic flow: convergence of centers. Let μ_t be a trajectory of the deterministic flow $\dot{\mu} = \Pi(\mu) - \mu$. Assume that μ_0 belongs to a given set K_{α, C_2} . The aim of this paragraph is to show that there exists $c_\infty := \lim_{t \rightarrow \infty} c_t$ and thus, given the result of Subsection 4.2, that the deterministic μ_t converges to $\rho_\infty(x - c_\infty)dx$.

Proposition 17. *Assume that μ_0 belongs to a given set K_{α, C_2} . Let μ_t be a trajectory of the deterministic flow $\dot{\mu} = \Pi(\mu) - \mu$. Then the center of μ_t , denoted by c_t , admits a limit.*

Proof. We have obtained an estimate on the decrease of $\mathcal{F}(\mu_t^c | \rho_\infty)$ and thus on the W_2 -distance between μ_t^c and ρ_∞ . Now, note that they imply the estimates on $|\dot{c}_t|$. Indeed,

$$|\dot{c}_t| = \left| (\nabla^2 W * \mu_t(c_t))^{-1} \nabla W * \Pi(\mu_t)(c_t) \right| \leq \frac{|\nabla W * \Pi(\mu_t)(c_t)|}{C_W}.$$

As Π is Lipschitz, there exists a constant $C > 0$ such that $\mathcal{T}_P(\Pi(\mu_t^c), \rho_\infty) = \mathcal{T}_P(\Pi(\mu_t^c), \Pi(\rho_\infty)) \leq C \mathcal{T}_P(\mu_t^c, \rho_\infty)$. Given that, both $\Pi(\mu_t)$ and ρ_∞ are exponentially decreasing (see [6]). Decomposing $\nabla W * \Pi(\mu_t)(c_t) = \nabla W * (\Pi(\mu_t^c) - \rho_\infty)(0)$, by the same means, we see that for any $L > 1$

$$\begin{aligned}
|\nabla W * (\Pi(\mu_t^c) - \rho_\infty)(0)| &= |\mathbb{E}(\nabla W(\xi_0) - \nabla W(\xi_1))| \\
&\leq \tilde{C} P(2L) e^{-\alpha L} + \mathbb{E} \int_0^1 \|\nabla^2 W(\xi_s)\| ds |\xi_0 - \xi_1| \mathbf{1}_{|\xi_0|, |\xi_1| \leq L} \\
&\leq \tilde{C} P(2L) e^{-\alpha L} + P(L) \mathcal{T}_P(\mu_t^c, \rho_\infty).
\end{aligned}$$

As $\mathcal{T}_P(\mu_t^c, \rho_\infty) \leq e^{-at^{1/(k+1)}}$, choosing $L = t$, we find that

$$|\dot{c}_t| \leq \left(\tilde{C} P(2t) e^{-\alpha t} + P(t) e^{-at^{1/(k+1)}} \right) / C_W.$$

So, $\int_1^\infty |\dot{c}_t| dt$ converges and thus the limit of c_t exists. This finally implies the existence of $\lim_{t \rightarrow \infty} \mu_t(dx) = \rho_\infty(x - c_\infty)dx$ in the deterministic case. \square

6.2. Convergence of the random center.

Proposition 18. *Assume that μ_0 belongs to a given set K_{α, C_2} . Let μ_t be the empirical measure of the process (1). Then the center of the random μ_t , denoted by c_t , admits a limit.*

Proof. Recall that, due to the estimates of the preceding section, for some $\beta > 0$, one has (up to the change of variable $t \mapsto e^t$) for $T_n \leq t \leq T_{n+1}$, $|c_t - c_{T_n}| = O(T_n^{-\beta})$ a.s. Hence, to prove the convergence of the centers, it suffices to show the existence of the limit $\lim c_{T_n}$. For this, a sufficient condition would be that the series of general term $|c_{T_n} - c_{T_{n+1}}|$ converges. Now, as $\nabla W * \mu_{T_{n+1}}(c_{T_{n+1}}) = 0$ and $\nabla W * \mu_{T_{n+1}}(c_{T_n}) = \nabla W * \mu_{[T_n, T_{n+1}]}(c_{T_n})$, we get

$$|c_{T_{n+1}} - c_{T_n}| = \left| c \left(\frac{T_n}{T_{n+1}} \mu_{T_n} + \frac{\Delta T_n}{T_{n+1}} \mu_{[T_n, T_{n+1}]} \right) - c(\mu_{T_n}) \right| \leq \frac{1}{C_W} \cdot \frac{\Delta T_n}{T_{n+1}} |\nabla W * \mu_{[T_n, T_{n+1}]}(c_{T_n})|.$$

And we have the Lagrange-like estimate from Lemma 10. So, it is enough to show that for some $\beta' > 0$ almost surely $|W * \mu_{[T_n, T_{n+1}]}(c_{T_n})| = O(T_n^{-\beta'})$. Indeed, once we have this, then

$$\sum_n |c_{T_{n+1}} - c_{T_n}| \leq \sum_n T_n^{-\beta'} \frac{\Delta T_n}{T_{n+1}} < \infty$$

since T_n is of the order of $n^{1/3}$ and $\frac{\Delta T_n}{T_{n+1}}$ is equivalent to n^{-1} .

Due to the estimates from the preceding section, $\mu_{[T_n, T_{n+1}]}(c_{T_n} + \cdot)$ is close to $\Pi(\mu_{T_n}^c)$, and we have an upper bound on the \mathcal{T}_P -distance that is $O(T_n^{-\beta})$. In particular, this implies that $|\nabla W * \mu_{[T_n, T_{n+1}]}(c_{T_n})| = O(T_n^{-\beta}) + |\nabla W * \Pi(\mu_{T_n}^c)(c_{T_n})|$. But, once again, due to the preceding estimates of Subsection 5.2.4, $\mathcal{T}_P(\mu_{T_n}^c, \rho_\infty) = O(T_n^{-\gamma})$. On the other hand, as ρ_∞ is symmetric, we have $\nabla W * \Pi(\rho_\infty)(0) = 0$. This added to the fact that Π is Lipschitz implies

$$|\nabla W * \Pi(\mu_{T_n}^c)(c_{T_n})| \leq \mathcal{T}_P(\Pi(\mu_{T_n}^c), \Pi(\rho_\infty)) + |\nabla W * \Pi(\rho_\infty)(0)| \leq C \mathcal{T}_P(\mu_{T_n}^c, \rho_\infty) = O(T_n^{-\gamma}).$$

This concludes the proof of the estimate $|c_{T_n} - c_{T_{n+1}}| = O(T_n^{-\beta})$ and thus the proposition is proven. \square

REFERENCES

- [1] ANÉ C., BLACHÈRE S., CHAFAI D., FOUGÈRES P., GENTIL Y., MALRIEUX F., ROBERTO C. & SCHEFFER G. (2001), Sur les inégalités de Sobolev logarithmiques.
- [2] BENAÏM M. & HIRSCH M.W. (1996), Asymptotic pseudotrajectories and chain recurrent flows, *J. Dynam. Diff. Equ.* **8**, 141-176.
- [3] BENAÏM M., LEDOUX M. & RAIMOND O. (2002), Self-interacting diffusions, *Prob. Th. Rel. Fields* **122**, 1-41.
- [4] BENAÏM M. & RAIMOND O. (2005), Self-interacting diffusions III: symmetric interactions, *Ann. Prob.* **33**(5), 1716-1759.
- [5] BOLLEY F., GUILLIN A. & VILLANI C. (2007), Quantitative concentration inequalities for empirical measures on non compact spaces, *Prob. Th. Rel. Fields* **137**(3-4), 541-593.
- [6] CARRILLO J.A., MCCANN R.J. & VILLANI C. (2003), Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates, *Rev. math. Iberoam.* **19**(3), 971-1018.
- [7] CATTIAUX P. & GUILLIN A. (2008), Deviation bounds for additive functionals of Markov processes, *ESAIM PS* **12**, 12-29.
- [8] CATTIAUX P., GUILLIN A. & MALRIEU F. (2008), Probabilistic approach for granular media equations in the non uniformly convex case, *Prob. Th. Rel. Fields* **140**(1-2), 19-40.
- [9] DURRETT R.T. & ROGERS L.C.G. (1992), Asymptotic behaviour of Brownian polymers, *Prob. Th. Rel. Fields* **92**(3), 337-349.
- [10] KURTZMANN A. (2007), The ODE method for some self-interacting diffusions on \mathbb{R}^d , *Ann. Inst. Henri Poincaré, Prob. Stat.*, to appear.

- [11] McCANN R. (1997), A convexity principle for interacting gases, *Adv. Math.* **128**, 153-179.
- [12] MEYN S.P. & TWEEDIE R.L. (1993), *Markov Chains and Stochastic Stability*, Springer-Verlag.
- [13] PEMANTLE R. (2007), A survey of random processes with reinforcement, *Prob. Surveys* **4**, 1-79.
- [14] RAIMOND O. (1997), Self-attracting diffusions: case of constant interaction, *Prob. Theory Relat. Fields* **107**, 177-196.
- [15] ROGERS L.C.G. & WILLIAMS D. (2000), *Diffusions, Markov processes and Martingales*, 2nd edition, vol. 2 “Itô Calculus”, Cambridge Univ. Press.
- [16] VILLANI C. (2008), *Optimal Transport, Old and new*, Grundlehren der Math. Wissenschaften , Vol. 338, Springer.
- [17] WU L. (2000), A deviation inequality for non-reversible Markov process, *Ann. Inst. Henri Poincaré, Prob. Stat.* **36**(4), 435-445.
- [18] YOSIDA K. (1995), *Functional Analysis*, 6th edition, Springer.

VICTOR KLEPTSYN: IRMAR, Université Rennes 1, Campus de Beaulieu, F-35042 Rennes Cedex, France. Email: victor.kleptsyn@univ-rennes1.fr

ALINE KURTZMANN: Institut Elie Cartan, Université Nancy 1, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France. Email: aline.kurtzmann@iecn.u-nancy.fr