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Sara Brofferio, Bruno Schapira

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# POISSON BOUNDARY OF $GL_d(\mathbb{Q})$

SARA BROFFERIO AND BRUNO SCHAPIRA

ABSTRACT. We construct the Poisson boundary for a random walk supported by the general linear group on the rational numbers as the product of flag manifolds over the  $p$ -adic fields. To this purpose, we prove a law of large numbers using the Oseledets' multiplicative ergodic theorem.

## 1. INTRODUCTION

Denote by  $\mathcal{P}^*$  the set of prime numbers and let  $\mathcal{P} = \mathcal{P}^* \cup \{\infty\}$ . For  $p \in \mathcal{P}^*$ , denote by  $\mathbb{Q}_p$  the field of  $p$ -adic numbers, and set by convention  $\mathbb{Q}_\infty = \mathbb{R}$ .

If  $\mu$  is a probability measure on  $GL_d(\mathbb{Q}_p)$  with finite logarithmic moment, i.e.

$$\int (\ln^+ \|g\|_p + \ln^+ \|g^{-1}\|_p) d\mu(g) < +\infty,$$

the associated Lyapunov exponents are the real numbers  $\lambda_1(p) \geq \dots \geq \lambda_d(p)$  such that

$$\sum_{i=1}^k \lambda_i(p) = \lim_{n \rightarrow +\infty} \frac{1}{n} \int \ln \|\bigwedge^k g\|_p d\mu^{*n}(g),$$

where  $\mu^{*n}$  denotes the  $n$ -fold convolution of  $\mu$  and  $\wedge$  the exterior product. Denote by  $P_p$  the parabolic subgroup of  $GL_d(\mathbb{Q}_p)$  consisting of matrices  $(p_{i,j})$  with  $p_{i,j} = 0$ , when  $\lambda_i(p) < \lambda_j(p)$ , and let  $B_p := GL_d(\mathbb{Q}_p)/P_p$  be the associated flag manifold. The main result of this paper is the following:

**Theorem 1.1.** *Let  $\mu$  be a probability measure on  $GL_d(\mathbb{Q})$  such that*

$$\sum_{p \in \mathcal{P}} \int (\ln^+ \|g\|_p + \ln^+ \|g^{-1}\|_p) d\mu(g) < +\infty.$$

*Then there exists a unique probability measure  $\nu$  on the space*

$$\mathbb{B} := \prod_{p \in \mathcal{P}} B_p,$$

*such that  $(\mathbb{B}, \nu)$  is the Poisson boundary of  $(GL_d(\mathbb{Q}), \mu)$ .*

This theorem unifies and generalizes several results on Poisson boundary of rational matrices groups, known up to now. In particular, it has been proved separately by F. Ledrappier [15] and V. Kaimanovich [9] that the Poisson boundary of a random walk supported by  $SL_d(\mathbb{Z})$  is the real flag manifold  $B_\infty$ . This result is contained in Theorem 1.1 because, in this case, for all  $p \neq \infty$ , the associated Lyapunov exponents are all equal to zero, thus  $B_p$  is trivial. Furthermore since Theorem 1.1

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does not require any irreducibility condition, it also applies to the case of rational affine group and to rational triangular matrices previously threaded by the authors [3, 18].

We would like to remark that for general number fields (i.e. finite extensions of  $\mathbb{Q}$ ) a similar result can be proved adapting our methods (see in [18] hints to possible generalization).

Due to its generality, our result does not say much about  $\nu$  and its support. In particular it is not true that the restriction of  $\nu$  to each  $B_p$  has always full support. For instance if  $\mu$  is supported on the subgroup of upper triangular matrices, we know [3, 18] that  $\nu$  charges only one Bruhat cell of each  $B_p$ . But even this is not optimal since  $\mu$  could be supported on diagonal matrices and with all Lyapunov exponents distinct, but in this case the Poisson boundary would be trivial (one point). However irreducibility hypothesis can give information on the support of  $\mu$ . We have for instance the following triviality criterion:

**Corollary 1.1.** *Let  $p \in \mathcal{P}$ . If  $\lambda_1(p) = \lambda_d(p)$ , then the projection of  $\nu$  on  $B_p$  is trivial.*

*Conversely, if the projection of  $\nu$  on  $B_p$  is trivial and no proper subspace of  $\mathbb{Q}_p^d$  is fixed by the support of  $\mu$ , then  $\lambda_1(p) = \lambda_d(p)$ .*

There exists several results in the literature to decide whether the real Lyapunov exponents are all equal  $\lambda_1(\infty) = \lambda_d(\infty)$ . For instance, under irreducibility hypothesis this is equivalent to ask that the closed subgroup generated by  $\mu$  in  $GL_d(\mathbb{R})$  is amenable [8]. For other references and results on product of real random matrices, see also [2]. It seems very likely that similar results hold on  $p$ -adic setting.

A different question that is still open is to understand the behavior of the measure  $\nu$  on the product of the  $p$ -flag manifolds, and not only of its projection on each  $B_p$ . For instance: does  $\nu$  charge the whole product or is it supported by some "diagonal" sub-set? Is there some sort of correlation among the different  $p$ -adic components?

The main tool of the proof of Theorem 1.1 is to produce, using the multiplicative ergodic theorem of Oseledets, a Law of large numbers for random walks on  $GL_d(\mathbb{Q}_p)$  (not necessarily with rational coefficients, see Proposition 3.1). The Lyapunov exponents give the speeds of convergence and the boundary limit of the random walk on  $B_p$  the directions. This is done in Section 3, where we also use this result to prove that  $B_p$  and  $\mathbb{B}$  are  $\mu$ -boundaries.

In Section 4, we use entropy criterion due to Kaimanovich to establish the maximality of  $(\mathbb{B}, \nu)$  and prove the main theorem and its corollary.

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## 2. PRELIMINARIES

**2.1. General linear group over  $\mathbb{Q}_p$ .** If  $\mathbb{K}$  is a field, we denote by  $GL_d(\mathbb{K})$  the group of invertible matrices of size  $d$  with coefficients in  $\mathbb{K}$ . We denote by  $e$  the identity matrix.

For  $p \in \mathcal{P}$  and  $v = (v_1, \dots, v_d) \in \mathbb{Q}_p^d$ , we set

$$|v|_p = \max_i |v_i|_p, \text{ if } p \neq \infty \quad \text{and} \quad |v|_\infty = \sqrt{\sum_i |v_i|_\infty^2},$$

and if  $g \in GL_d(\mathbb{Q}_p)$  we set

$$\|g\|_p = \sup_{|v|_p=1} |gv|_p.$$

For any  $p \in \mathcal{P}$  and  $g, h \in GL_d(\mathbb{Q}_p)$  set

$$d_p(g, h) = \ln^+ \|g^{-1}h\|_p + \ln^+ \|h^{-1}g\|_p,$$

where  $\ln^+$  denotes the positive part of the function  $\ln$ . It is easily checked that  $d_p$  is symmetric and satisfies the triangular inequality. It is not a distance since the set of  $g \in GL_d(\mathbb{Q}_p)$  such that  $d_p(e, g) = 0$  is the compact subgroup of linear isometries of  $\mathbb{Q}_p^d$ . Furthermore  $d_p$  is left-invariant:

$$d_p(\gamma g, \gamma h) = d_p(g, h),$$

for all  $g, h, \gamma \in GL_d(\mathbb{Q}_p)$ .

For all  $g, h \in GL_d(\mathbb{Q})$ , let

$$d(g, h) = \sum_{p \in \mathcal{P}} d_p(g, h).$$

This defines a left-invariant pseudometric on  $GL_d(\mathbb{Q})$ .

**2.2. The flag manifold.** For each  $p \in \mathcal{P}$  fix the sequence of Lyapunov exponents  $\lambda_1(p) \geq \dots \geq \lambda_d(p)$ . The associated parabolic sub-group is

$$P_p = \{(p_{i,j}) \in GL_d(\mathbb{Q}_p) \mid p_{i,j} = 0 \text{ if } \lambda_i(p) < \lambda_j(p)\}$$

The flag manifold  $B_p := GL_d(\mathbb{Q}_p)/P_p$  is then a compact separable  $GL_d(\mathbb{Q}_p)$ -space.

We mention that there is a one to one map between  $B_p$  and the space of flags, viewed as the set of embedded sequences of sub-spaces of  $\mathbb{Q}_p^d$  of fixed dimensions. In fact

$$B_p = \{(V_1, \dots, V_r) \mid V_1 \leq \dots \leq V_r = \mathbb{Q}_p^d, \dim(V_i) = j_i \quad \forall i \leq r\},$$

where  $r$  is the number of distinct values taken by  $\lambda_1(p), \dots, \lambda_d(p)$ , and  $j_1, \dots, j_r$  are defined inductively by  $j_r = d$  and  $j_{i-1} = \max\{j < j_i \mid \lambda_j(p) > \lambda_{j_i}(p)\}$ , for  $2 \leq i \leq r$ . To see the correspondence between  $B_p$  and this space of flags, observe that  $GL_d(\mathbb{Q}_p)$  acts transitively on the flags and that the parabolic subgroup  $P_p$  is the stabilizer of the element  $(E_1, \dots, E_r)$ , where for all  $i$ ,  $E_i$  is the vector space generated by the first  $j_i$  vectors of the canonical basis.

Let

$$\mathbb{B} := \prod_{p \in \mathcal{P}} B_p,$$

equipped with the product topology. With the natural diagonal action,  $\mathbb{B}$  is a compact separable  $GL_d(\mathbb{Q})$ -space.

**2.3. Random walk and  $\mu$ -boundaries.** Let  $\mu$  be a probability measure on a locally compact separable group  $G$ . Let

$$(\Omega, \mathbb{P}) := (G, \mu)^{\otimes \mathbb{N}},$$

be the product of  $\mathbb{N}$  independent copies of  $(G, \mu)$  (here  $\mathbb{N}$  is the set of strictly positive integers). If  $w = (w_i, i \geq 1) \in \Omega$ , the *random walk* is the process defined by

$$x_n := w_1 \dots w_n \quad \forall n \geq 1 \quad \text{and} \quad x_0 := e.$$

Observe that under  $\mathbb{P}$ , for any fixed  $n$ , the law of  $x_n$  is  $\mu^{*n}$ , the  $n$ -th convolution power of  $\mu$ .

Assume that  $B$  is a compact separable space, endowed with a probability measure  $\nu$  and a continuous action of  $G$ . We say that  $\nu$  is  $\mu$ -stationary (also known as  $\mu$ -invariant or  $\mu$ -harmonic), if

$$\mu * \nu := \int_G (g\nu) d\mu(g) = \nu,$$

where for all  $g \in G$ ,  $g\nu$  is defined by

$$g\nu(f) = \int_B f(gz) d\nu(z),$$

for all continuous functions  $f$ . In this case, according to Furstenberg [6, 7], we say that  $(B, \nu)$  is a  $\mu$ -boundary if,  $\mathbb{P}$ -almost surely  $x_n\nu$  converges weakly to a dirac measure.

A  $\mu$ -boundary  $(B, \nu)$  is naturally associated to a measurable function  $\mathbf{b} = \mathbf{b}_B : \Omega \rightarrow B$  defined by

$$(1) \quad \lim_{n \rightarrow +\infty} x_n\nu = \delta_{\mathbf{b}(w)}.$$

Then  $\nu$  is the image of  $\mathbb{P}$  under  $\mathbf{b}$ .

Denote by  $\theta$  the shift transformation on  $\Omega$ : if  $w = (w_i, i \geq 1) \in \Omega$ , then

$$(\theta w)_i = w_{i+1} \quad i \geq 1.$$

The measure  $\mathbb{P}$  is  $\theta$ -invariant and it is easy to see that the function defined in (1) satisfies

$$w_1\mathbf{b}(\theta w) = \mathbf{b}(w).$$

This property characterizes functions that arise from  $\mu$ -boundaries, as follows from this known result (see for instance [10, 15]):

**Proposition 2.1.** *Let  $B$  be a compact separable  $G$ -space and let  $\mathbf{b} : \Omega \rightarrow B$  be a measurable map, such that  $\mathbb{P}$ -a.s. we have  $w_1\mathbf{b}(\theta w) = \mathbf{b}(w)$ . Let  $\nu$  be the law of  $\mathbf{b}$ . Then  $(B, \nu)$  is a  $\mu$ -boundary.*

*Proof.* We give here a proof for sake of completeness. By using the invariance of  $\mathbb{P}$  by  $\theta$  and the hypothesis on the map  $\mathbf{b}$  we get for every continuous functions  $f$  on  $B$

$$\begin{aligned} \nu(f) &= \int_{\Omega} f(\mathbf{b}(w)) d\mathbb{P}(w) \\ &= \int_{\Omega} f(w_1\mathbf{b}(\theta w)) d\mathbb{P}(w) \\ &= \int_G \left( \int_{\Omega} f(w_1\mathbf{b}(w')) d\mathbb{P}(w') \right) d\mu(w_1) = \mu * \nu(f), \end{aligned}$$

proving the invariance of  $\nu$  by  $\mu$ . The hypothesis on  $\mathbf{b}$  also shows that for any continuous function  $f$ , the sequence

$$M_n(w) := x_n \cdot \nu(f) \quad n \geq 1,$$

is a bounded martingale. Thus this sequence converges a.s and in  $L^1$  toward some limit, say  $\nu_{\infty}^w(f)$ . Since  $B$  is separable, this defines actually a random measure  $\nu_{\infty}^w$  on  $B$ , which is the weak limit of  $x_n \cdot \nu$ ,  $n \geq 1$ . Observe now that for all  $k \geq 1$ ,

$$\nu_{\infty}^w = x_k \nu_{\infty}^{\theta^k w} \quad \mathbb{P} - a.s.$$

Moreover  $\mathbb{E}[\nu_\infty^w] = \nu$ . Observe also that the dirac measure  $\delta_{\mathbf{b}(w)}$  has the same properties. As a consequence for any  $k \geq 1$ , any Borel subsets  $O_1, \dots, O_k \subset G$  and  $U \subset B$ ,

$$\begin{aligned} \mathbb{P}\nu_\infty^w[O_1 \times \dots \times O_k \times U] &= \int_{\Omega} \nu_\infty^w(U) 1(w_1 \in O_1, \dots, w_k \in O_k) d\mathbb{P}(w) \\ &= \int_{\Omega} w_1 \dots w_k \nu_\infty^{\theta^k w}(U) 1(w_1 \in O_1, \dots, w_k \in O_k) d\mathbb{P}(w) \\ &= \int_{\Omega} \nu((w_1 \dots w_k)^{-1}U) 1(w_1 \in O_1, \dots, w_k \in O_k) d\mathbb{P}(w). \end{aligned}$$

For the same reason

$$\mathbb{P}\delta_{\mathbf{b}(w)}[O_1 \times \dots \times O_k \times U] = \int_{\Omega} \nu((w_1 \dots w_k)^{-1}U) 1(w_1 \in O_1, \dots, w_k \in O_k) d\mathbb{P}(w).$$

Thus the two measures  $\mathbb{P}\nu_\infty^w$  and  $\mathbb{P}\delta_{\mathbf{b}(w)}$  defined on  $\Omega \times B$  coincide on  $\mathcal{F}_k := \sigma(w_1, \dots, w_k) \vee \mathcal{B}$ , for all  $k \geq 1$ , where  $\mathcal{B}$  denotes the Borel sets of  $B$ . Since the filtration  $(\mathcal{F}_k)_{k \geq 0}$  generates the  $\sigma$ -algebra of  $\Omega \times B$  on which are defined these measures, they are equal, proving that  $\nu_\infty^w$  is well a Dirac measure. This concludes the proof of the proposition.  $\square$

**2.4. Poisson boundary and asymptotic entropy.** The Poisson boundary  $(\mathbb{B}, \nu)$  is defined as the maximal  $\mu$ -boundary, i.e. it is the  $\mu$ -boundary such that any other  $\mu$ -boundary is one of its measurable  $G$ -equivariant quotients. A classical problem is to decide weather a space, that is known to be a  $\mu$ -boundary, is in fact the maximal one.

For countable groups, there exists powerful techniques based on the estimation of the entropy introduced by Kaimanovich and Vershik [12] and Derrienic [4] and developed by Kaimanovich (see [10] for details). Suppose that the measure  $\mu$  has finite entropy:

$$H(\mu) := - \sum_{g \in G} \mu(g) \ln \mu(g) < \infty.$$

If  $(B, \nu)$  is a  $\mu$ -boundary and  $z \in B$ , it is possible to define the law  $\mathbb{P}^z$  of  $w \in \Omega$  conditioned by  $\mathbf{b}(w) = z$ . Then for  $n \geq 0$ ,  $\mathbb{P}_n^z$  denotes the law of  $x_n$  under  $\mathbb{P}^z$ , i.e.

$$\mathbb{P}_n^z(g) = \mathbb{P}^z(x_n = g) = \mathbb{P}(x_n = g \mid \mathbf{b}(w) = z)$$

The conditional asymptotic entropy  $h^z$  is defined by

$$h^z := - \lim_{n \rightarrow +\infty} \frac{\log \mathbb{P}_n^z(x_n)}{n} \quad \mathbb{P}^z - a.s.$$

Then  $(B, \nu)$  is the Poisson boundary if, and only if,  $h^z$  is equal to zero for  $\nu$ -almost every  $z$ .

### 3. LAW OF LARGE NUMBERS AND $\mu$ -BOUNDARIES FOR $GL_d(\mathbb{Q}_p)$

In this section we can assume  $\mu$  to be a probability measure on  $GL_d(\mathbb{Q}_p)$ , not necessarily supported on matrices with rational coefficients. We are going to show that, under first moment hypothesis, the random walk on  $GL_d(\mathbb{Q}_p)$  satisfies a strong law of large numbers, in which the "speeds" of the drift are given by the Lyapunov exponents and the "directions" are given by an element of the associated flag manifold  $B_p$ . This approach was introduced by V. Kaimanovich in [11] for semisimple

Lie groups, as a group-geometrical version of the classical multiplicative ergodic theorem of Oseledets. (see also A. Karlsson and F. Ledrappier [13].)

Related to this result, we will see that  $B_p$ , endowed with the law of the "direction", is a  $\mu$ -boundary for the random walk.

**3.1. Oseledets' theorem and law of large numbers.** If  $p = \infty$ , let  $\Lambda_n = \Lambda_n(\infty)$  be the diagonal matrix of  $GL_d(\mathbb{R})$  with coefficients

$$(2) \quad (\Lambda_n)_{i,i} := e^{n\lambda_i(\infty)} \quad \forall i \leq d.$$

If  $p \in \mathcal{P}^*$ , let  $\Lambda_n = \Lambda_n(p)$  be the diagonal matrix with coefficients

$$(3) \quad (\Lambda_n)_{i,i} := p^{-\left[\frac{n\lambda_i(p)}{\ln p}\right]} \quad \forall i \leq d;$$

where  $[\cdot]$  is the integer part. In such a way,  $\Lambda_n$  has rational entries whose  $p$ -norms are close to the  $e^{n\lambda_i(p)}$ 's.

**Proposition 3.1.** *Suppose  $\mathbb{E}(d_p(1, w_1)) < \infty$ . Then there exists a measurable map*

$$\mathbf{b} = \mathbf{b}_p : \Omega \rightarrow B_p,$$

*such that  $\mathbb{P}$ -almost surely  $\mathbf{b}(w)$  is the unique element of  $B_p$  such that*

$$(4) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} d_p(x_n, b\Lambda_n) = 0,$$

*for any  $b$  in the class of  $\mathbf{b}(w)$ .*

To prove this proposition we use the following lemma that translates Oseledets' Theorem in our setting:

**Lemma 3.1.** *Suppose  $\mathbb{E}(d_p(1, w_1)) < \infty$ . Then there exists a measurable map*

$$\mathbf{b} = \mathbf{b}_p : \Omega \rightarrow B_p,$$

*such that  $\mathbb{P}$ -almost surely*

$$(5) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|x_n^{-1} b \Lambda_n\|_p = 0,$$

*for any  $b$  in the class of  $\mathbf{b}(w)$ .*

*Proof.* Let us first recall the multiplicative ergodic theorem, first proved by V.I. Oseledets [16] for real matrices and generalized by M.S. Raghunathan [17] to matrices on local fields. It says that  $\mathbb{P}$ -a.s. there exists a filtration of subspaces of  $\mathbb{Q}_p^d$ ,  $\{0\} = V^0(w) \subset V^1(w) \subset \dots \subset V^r(w) = \mathbb{Q}_p^d$ , such that

- (i) The map  $w \rightarrow V^i(w)$  is measurable for all  $i \leq r$ .
- (ii) For all  $1 \leq i \leq r$ , we have  $v \in V^i(w) \setminus V^{i-1}(w)$  if, and only if,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|w_n^{-1} \dots w_1^{-1} v\|_p = -\lambda_{j_i}(p),$$

where  $r$  and  $j_1, \dots, j_r$  are defined as in section 2.2.

Suppose that  $p \neq \infty$  (the real case is treated analogously). Denote by  $(e_1, \dots, e_d)$  the canonical basis of  $\mathbb{Q}_p^d$ . Consider a matrix  $b \in GL_d(\mathbb{Q}_p)$  such that for all  $i \leq r$ ,  $b$  sends the family  $(e_1, \dots, e_{j_i})$  into a basis of  $V^i(w)$ . Then  $b = [v_1 | \dots | v_d]$ , where for all  $i \leq r$ ,  $(v_1, \dots, v_{j_i})$  is a basis of  $V^i(w)$ . Observe that

$$x_n^{-1} b \Lambda_n = \left[ x_n^{-1} v_1 p^{-\left[\frac{n\lambda_1(p)}{\ln p}\right]} \mid \dots \mid x_n^{-1} v_d p^{-\left[\frac{n\lambda_d(p)}{\ln p}\right]} \right].$$

Then

$$\max_{k=1,\dots,d} \left( p^{\lfloor \frac{n\lambda_k(p)}{\ln p} \rfloor} |x_n^{-1} v_k|_p \right) \leq \|x_n^{-1} b \Lambda_n\|_p \leq d \max_{k=1,\dots,d} \left( p^{\lfloor \frac{n\lambda_k(p)}{\ln p} \rfloor} |x_n^{-1} v_k|_p \right).$$

Then, since  $w_n^{-1} \dots w_1^{-1} = x_n^{-1}$ , by (ii)

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|x_n^{-1} b \Lambda_n\|_p = 0.$$

To conclude the proof, just observe that two matrices  $b_1$  and  $b_2$  give two basis of the same filtration  $\{V_i(w)\}_i$  if, and only if,  $b_1^{-1} b_2$  is in the group  $P_p$ , thus such matrix  $b$  can be identified with an element of  $B_p$ .  $\square$

*Proof of Proposition 3.1.* Let  $\tilde{x}_n = (w_1^t)^{-1} \dots (w_n^t)^{-1} = (x_n^t)^{-1}$  be the random walk of law  $\tilde{\mu}$ , image of  $\mu$  under the map  $g \mapsto (g^t)^{-1}$ . Then the Lyapunov exponents associated to  $\tilde{\mu}$  are

$$\tilde{\lambda}_i = -\lambda_{d-i}.$$

Let  $\tilde{\Lambda}_n$  be the diagonal matrix constructed with the exponents  $\tilde{\lambda}_i$  as in (3). For  $\mathbb{P}$ -almost all  $w$  there exists a  $\tilde{b} \in GL_d(\mathbb{Q}_p)$  such that:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|x_n^t \tilde{b} \tilde{\Lambda}_n\|_p = 0.$$

Consider the matrix  $s = \begin{bmatrix} 0 & \dots & 1 \\ \vdots & \dots & \vdots \\ 1 & \dots & 0 \end{bmatrix}$  that transforms the basis  $(e_1, \dots, e_d)$  in

the basis  $(e_d, \dots, e_1)$ . Then

$$\begin{aligned} \|x_n^t \tilde{b} \tilde{\Lambda}_n\|_p &= \|x_n^t \tilde{b} s s^{-1} \tilde{\Lambda}_n s\|_p && \text{since } \|s\|_p = \|s^{-1}\|_p = 1 \\ &= \|x_n^t \tilde{b} s \Lambda_n^{-1}\|_p && \text{since } s^{-1} \tilde{\Lambda}_n s = \Lambda_n^{-1} \\ &= \|\Lambda_n^{-1} (\tilde{b} s)^t x_n\|_p && \text{since } \|g^t\|_p = \|g\|_p. \end{aligned}$$

Set  $\bar{b} = ((\tilde{b} s)^t)^{-1}$ , then

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|\Lambda_n^{-1} \bar{b}^{-1} x_n\|_p = 0.$$

We want to show now that if  $b$  is as in (5) then  $b$  and  $\bar{b}$  are in the same class in  $B_p$ . To do this observe that

$$(6) \quad u \in P_p \iff \lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|\Lambda_n^{-1} u \Lambda_n\|_p = 0 \iff \lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|\Lambda_n^{-1} u \Lambda_n\|_p \leq 0.$$

This can be proved by direct calculations using the fact that  $\max_{i,j} |g_{i,j}|_p \leq \|g\|_p \leq d^2 \max_{i,j} |g_{i,j}|_p$  for all  $p \in \mathcal{P}$ .

Then since

$$\ln \|\Lambda_n^{-1} \bar{b}^{-1} b \Lambda_n\|_p = \ln \|\Lambda_n^{-1} \bar{b}^{-1} x_n x_n^{-1} b \Lambda_n\|_p \leq \ln \|\Lambda_n^{-1} \bar{b}^{-1} x_n\|_p + \ln \|x_n^{-1} b \Lambda_n\|_p,$$

it follows immediately that  $\bar{b}^{-1} b \in P_p$ .

On the other hand

$$\|\Lambda_n^{-1} \bar{b}^{-1} x_n\|_p \|\Lambda_n^{-1} u \Lambda_n\|_p^{-1} \leq \|\Lambda_n^{-1} u^{-1} \bar{b}^{-1} x_n\|_p \leq \|\Lambda_n^{-1} u^{-1} \Lambda_n\|_p \|\Lambda_n^{-1} \bar{b}^{-1} x_n\|_p.$$

Then for every  $b_1 = \bar{b} u$  with  $u$  in the group  $P_p$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|\Lambda_n^{-1} b_1^{-1} x_n\|_p = 0.$$



Thus for all  $b \in \mathbf{b}(w)$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} d_p(x_n, b\Lambda_n) = \lim_{n \rightarrow +\infty} \frac{1}{n} (\ln^+ \|x_n^{-1} b \Lambda_n\|_p + \ln^+ \|\Lambda_n^{-1} b^{-1} x_n\|_p) = 0.$$

It just remains to see that the class  $\mathbf{b}(w)$  is the unique such that (4) holds. But if  $b_1$  and  $b_2$  are two matrices such that (4) holds, then

$$0 = \lim_{n \rightarrow +\infty} \frac{1}{n} d_p(b_1 \Lambda_n, b_2 \Lambda_n) = \lim_{n \rightarrow +\infty} \frac{1}{n} d_p(e, \Lambda_n^{-1} b_1^{-1} b_2 \Lambda_n),$$

and using once more (6) we conclude.  $\square$

**3.2. The spaces  $B_p$  and  $\mathbb{B}$  are  $\mu$ -boundaries.** It is easily checked, using left-invariance of  $d_p$ , that the function  $\mathbf{b}_p$  defined in Proposition 3.1 satisfies the hypothesis of Proposition 2.1. Then we immediately get

**Corollary 3.1.** *Let  $\mu$  be a probability measure on  $GL_d(\mathbb{Q}_p)$ . Suppose that  $\mathbb{E}(d_p(e, w_1)) < \infty$ , and let  $\nu_p$  be the law of  $\mathbf{b}_p$ . Then  $(B_p, \nu_p)$  is a  $\mu$ -boundary.*

*Let  $\mu$  be a probability measure on  $GL_d(\mathbb{Q})$ . Suppose that  $\mathbb{E}(d(e, w_1)) < \infty$ . Let  $\mathbf{b}$  be the map from  $\Omega$  to  $\mathbb{B} = \prod_{p \in \mathcal{P}} B_p$  defined by:*

$$\mathbf{b} : w \mapsto \mathbf{b}(w) = (\mathbf{b}_p(w))_{p \in \mathcal{P}}.$$

*Let  $\nu$  be the law of  $\mathbf{b}$ . Then  $(\mathbb{B}, \nu)$  is a  $\mu$ -boundary.*

#### 4. POISSON BOUNDARY OF $GL_d(\mathbb{Q})$

To prove that  $\mathbb{B}$  is the maximal  $\mu$ -boundary, we use the following lemma, which is a generalization of the ray criterion of V. Kaimanovich [10], already implicitly used in our previous works [3, 18].

**Lemma 4.1.** *Let  $\mu$  be a probability measure on a countable group  $G$  with finite entropy. Let  $(B, \nu)$  be a  $\mu$ -boundary and  $\mathbf{b}$  the associated boundary map. Suppose that for each  $n$  there exists a measurable map  $C_n$  from  $B$  to subsets of  $G$  such that:*

$$\lim_{n \rightarrow +\infty} \mathbb{P}(x_n \in C_n(\mathbf{b}(w))) = 1 \text{ and } \lim_{n \rightarrow +\infty} \frac{1}{n} \ln |C_n(z)| \leq \delta \quad \nu(dz)\text{-almost surely.}$$

*Then  $h^z \leq \delta$  for  $\nu$ -almost all  $z$ .*

*Proof.* Observe that

$$\mathbb{P}(x_n \in C_n(\mathbf{b}(w))) = \int_B \mathbb{P}_n^z [C_n(z)] \nu(dz) \rightarrow 1.$$

Thus, along a sub-sequence,  $\mathbb{P}_n^z [C_n(z)]$  converges to 1 for  $\nu$ -almost all  $z$ .

Recall that  $h^z$  is the  $\mathbb{P}^z$ -almost sure limit of  $-\ln \mathbb{P}_n^z(x_n)/n$ . Now for any  $\varepsilon > 0$  consider the set

$$A_n(z) = \{g \in G \mid -h^z - \varepsilon < \ln \mathbb{P}_n^z(g)/n < -h^z + \varepsilon\}.$$

Then  $\mathbb{P}_n^z(A_n(z) \cap C_n(z))$  converges to 1 on a sub-sequence, while, for large  $n$

$$\mathbb{P}_n^z(A_n(z) \cap C_n(z)) \leq e^{n(\varepsilon - h^z)} |C_n(z)| \leq e^{n(\varepsilon - h^z)} e^{n(\delta + \varepsilon)}.$$

Thus  $\delta - h^z + 2\varepsilon \geq 0$ . Since  $\varepsilon$  was arbitrarily chosen, we get  $h^z \leq \delta$ .  $\square$

In order to apply this lemma in our setting we need to show that the gauge on  $GL_d(\mathbb{Q})$  associated to the distance  $d$  grows at most exponentially:

**Lemma 4.2.** For  $g \in GL_d(\mathbb{Q})$  and  $R \geq 0$ , let

$$B(g, R) = \{h \in GL_d(\mathbb{Q}) \mid d(g, h) \leq R\}.$$

Then there exists a constant  $C > 0$  such that for all  $g$  and  $R$

$$|B(g, R)| \leq Ce^{CR}.$$

*Proof.* First observe that since  $d(e, g^{-1}h) = d(g, h)$ , we have  $g^{-1}B(g, R) = B(e, R)$ . Thus two balls with the same radius have the same cardinality, and we can restrict us without loss of generality to the case  $g = e$ .

Observe now that if  $h = (h_{i,j}) \in B(e, R)$  then for all couples of indices  $(i, j)$

$$\sum_{p \in \mathcal{P}} \ln^+ |h_{i,j}|_p \leq \sum_{p \in \mathcal{P}} \max_{i,j} \ln^+ |h_{i,j}|_p \leq d(e, h) \leq R.$$

It can be shown (see for instance [3]) that there exists  $C'$  such that for all  $R$

$$\left| \left\{ q \in \mathbb{Q} \mid \sum_{p \in \mathcal{P}} \ln^+ |q|_p < R \right\} \right| \leq C' e^{C'R}.$$

The desired result follows.  $\square$

*Proof of Theorem 1.1.* First observe that, since  $\mu$  has finite first moment with respect to an exponentially growing gauge, it has finite entropy.

For any  $p$ , consider the moment of the random walk with respect to  $d_p$ :

$$m_p = \int d_p(e, g) d\mu(g).$$

Observe that  $\sum_{p \in \mathcal{P}} m_p = \mathbb{E}(d(e, w_1)) < +\infty$ . Fix  $F$  a finite subset of  $\mathcal{P}$  and set  $m_{F^c} = \sum_{p \in F^c} m_p$ . By the law of large numbers,  $\mathbb{P}$ -almost surely

$$\frac{\sum_{p \in F^c} d_p(x_n, e)}{n} \leq \frac{\sum_{k=1}^n \sum_{p \in F^c} d_p(x_k, x_{k-1})}{n} = \frac{\sum_{k=1}^n \sum_{p \in F^c} d_p(w_k, e)}{n} \rightarrow m_{F^c}.$$

Fix  $\varepsilon > 0$  and  $\mathbf{b} = (b_p)_{p \in \mathcal{P}} \in \mathbb{B}$ , and set

$$C_n^{F, \varepsilon}(\mathbf{b}) = \left\{ g \in GL_d(\mathbb{Q}) \mid d_p(g, b_p \Lambda_n(p)) \leq n\varepsilon \forall p \in F, \sum_{p \in F^c} d_p(g, e) \leq n(m_{F^c} + \varepsilon) \right\}.$$

Then by Proposition 3.1

$$\mathbb{P}[x_n \in C_n^{F, \varepsilon}(\mathbf{b}(w))] \rightarrow 1.$$

To apply Lemma 4.1, we need to control the cardinality of  $C_n^{F, \varepsilon}$ . Suppose that  $C_n^{F, \varepsilon}(\mathbf{b})$  is nonempty and let  $g_0 \in C_n^{F, \varepsilon}(\mathbf{b})$ . Then for all  $g \in C_n^{F, \varepsilon}(\mathbf{b})$ ,

$$\begin{aligned} d(g_0, g) &= \sum_{p \in \mathcal{P}} d_p(g_0, g) \\ &\leq \sum_{p \in F} (d_p(g_0, b_p \Lambda_n(p)) + d_p(b_p \Lambda_n(p), g)) + \sum_{p \in \mathcal{P} - F} (d_p(g_0, e) + d_p(e, g)) \\ &\leq 2n(|F|\varepsilon + m_{F^c} + \varepsilon) \end{aligned}$$

Thus

$$\frac{1}{n} \ln |C_n^{F, \varepsilon}(\mathbf{b})| \leq \frac{1}{n} \ln |B(g_0, 2n(|F|\varepsilon + m_{F^c} + \varepsilon))| \leq 2nC(|F|\varepsilon + m_{F^c} + \varepsilon) + \frac{\ln C}{n}.$$

Thus for all finite  $F$  and all  $\varepsilon > 0$ ,

$$h^z \leq 2C(|F|\varepsilon + m_{F^c} + \varepsilon).$$

Letting  $\varepsilon$  go to zero and  $F$  grow to  $\mathcal{P}$  (in such a way  $m_{F^c}$  goes to zero), it follows that  $h^z = 0$  and thus that  $(\mathbb{B}, \nu)$  is the Poisson boundary.  $\square$

To conclude we prove our triviality criterion:

*Proof of Corollary 1.1.* It is immediate that if  $\lambda_1(p) = \lambda_d(p)$  then  $P_p = GL_d(\mathbb{Q}_p)$ , thus  $B_p$  is trivial.

Suppose now that the projection of  $\nu$  on  $B_p$  is trivial, but that  $\lambda_1(p) > \lambda_d(p)$ . In this case  $B_p$  is nontrivial and the projection of  $\nu$  on  $B_p$  is a Dirac measure whose mass is concentrated in a point  $b \in B_p$  that is fixed by the support of  $\mu$ . Then the support of  $\mu$  fixes all sub-spaces that compose the nontrivial flag associated to  $b$ . This contradicts the fact that no proper subspace of  $\mathbb{Q}_p^d$  is fixed by the support of  $\mu$ .  $\square$

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DÉPARTEMENT DE MATHÉMATIQUES, BÂT. 425, UNIVERSITÉ PARIS-SUD 11, F-91405 ORSAY, CEDEX, FRANCE.

*E-mail address:* `sara.brofferio@math.u-psud.fr`

*E-mail address:* `bruno.schapira@math.u-psud.fr`