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TRANSPORTATION DISTANCES ON THE CIRCLE

JULIEN RABIN, JULIE DELON, AND YANN GOUSSEAU

ABSTRACT. In this contribution, we study Monge-Kantorovich distances between discrete set of points on the unit circle S^1 , when the ground distance between two points x and y on the circle is defined as $c(x, y) = \min(|x - y|, 1 - |x - y|)$. We first prove that computing a Monge-Kantorovich distance between two given sets of pairwise different points boils down to cut the circle at a well chosen point and to compute the same distance on the real line.

This result is then used to prove a formula on the Earth Mover's Distance [3], which is a particular Monge-Kantorovich distance. This formula asserts that the Earth Mover's Distance between two discrete circular normalized histograms $f = (f[i])_{i=0, \dots, N-1}$ and $g = (g[i])_{i=0, \dots, N-1}$ on N bins can be computed by

$$(1) \quad \text{CEMD}(f, g) = \min_{k \in \{0, \dots, N-1\}} \|F_k - G_k\|_1,$$

where F_k and G_k are the cumulative histograms of f and g starting at the k^{th} quantization bin. This formula is used in recent papers [1, 2] on the matching of local features between images, where the Earth Mover's Distance is used to compare circular histograms of gradient orientations.

1. INTRODUCTION

Consider two discrete set of points $\{x_1, \dots, x_P\}$ and $\{y_1, \dots, y_P\}$ on the unit circle S^1 , and the corresponding discrete distributions

$$f = \frac{1}{P} \sum_{k=1}^P \delta_{x_k}, \text{ and } g = \frac{1}{P} \sum_{k=1}^P \delta_{y_k},$$

where the notations x_k, y_k are used equally for points on the unit circle or for their coordinates in $[0, 1[$. For any $\lambda > 0$, the λ -Monge-Kantorovich distance between f and g is defined as

$$(2) \quad \text{MK}_\lambda(f, g) := \min_{(\alpha_{i,j}) \in \mathcal{M}} \left(\sum_{i=1}^N \sum_{j=1}^N \alpha_{i,j} c(x_i, y_j)^\lambda \right)^{\frac{1}{\lambda}}, \text{ where}$$

$$(3) \quad \mathcal{M} = \{(\alpha_{i,j}); \alpha_{i,j} \geq 0, \sum_j \alpha_{i,j} = \frac{1}{P}, \sum_i \alpha_{i,j} = \frac{1}{P}\}$$

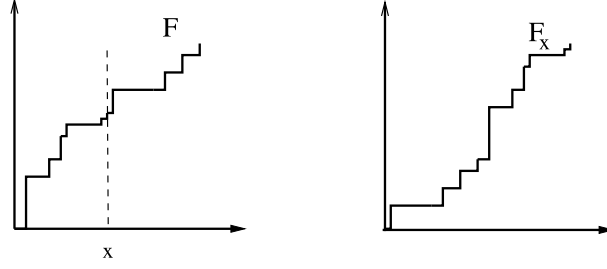
and where $c(\cdot, \cdot)$ is the periodic distance on $[0, 1[$

$$(4) \quad c(x, y) = \min(|x - y|, 1 - |x - y|).$$

The Earth Mover's Distance [3] is the distance obtained when $\lambda = 1$.

Now, it is well known (see for example the introduction of [4]) that the λ -Monge-Kantorovich distance between f and g equals

$$(5) \quad \text{MK}_\lambda(f, g) := \min_{\sigma \in \Sigma_P} W_\sigma^\lambda(f, g)$$

FIGURE 1. F on the left and F_x on the right.

where Σ_P is the set of permutations of $\{1, \dots, P\}$ and where

$$(6) \quad W_\sigma^\lambda(f, g) := \frac{1}{P} \left(\sum_k c(x_k, y_{\sigma(k)})^\lambda \right)^{\frac{1}{\lambda}}$$

is the cost of transporting f to g with the permutation σ . In other words, finding the optimal transportation between f and g boils down to find the optimal permutation σ between the points $\{x_k\}$ and $\{y_k\}$.

1.1. Paths. If x and y are two different points of S^1 , we consider $\gamma(x, y)$ **the geodesic path linking x and y on S^1** (the path is supposed open : it does not contain x and y). This path is always unique except in the case where x and y are in opposite positions on the circle. In this case, we choose $\gamma(x, y)$ as the path going from x to y in the trigonometric direction. A path $\gamma(x, y)$ is said to be **positive** if it goes from x to y in the trigonometric direction. If the path goes from x to y in the opposite direction, it is said to be **negative**.

1.2. Cumulative distribution functions. We define the cumulative distribution function of $f = \frac{1}{P} \sum_{k=1}^P \delta_{x_k}$ on $[0, 1[$ as

$$(7) \quad \forall y \in [0, 1[, \quad F(y) = \frac{1}{P} \sum_{k=1}^P \mathbb{1}_{\{x_k \in [0, y]\}}.$$

Observe that on $[0, 1[$ seen as a the unit circle S^1 , no strict order can be defined between points. We can define as many cumulative distribution functions as there are starting points on the circle. If x is a point in $[0, 1[$, the x -cumulative distribution function F_x of f can be defined by choosing x as the reference point on the circle S^1 and by cumulating the mass in the trigonometric order from this new reference point. More precisely, F_x can be computed as

$$(8) \quad \forall y \in [0, 1[, \quad F_x(y) = \begin{cases} \frac{1}{P} \sum_{k=1}^P \mathbb{1}_{\{x_k \in [x, y+x]\}} & \text{if } y < 1 - x, \\ \frac{1}{P} \sum_{k=1}^P \mathbb{1}_{\{x_k \in [x, 1] \cup [0, y-1+x]\}} & \text{if } y \geq 1 - x. \end{cases}$$

Observe that F_x is an increasing function and that for each (x, y) in $[0, 1]^2$, $F_x(y) = F(y+x-1) + 1 - F(x)$ if $y \geq 1-x$, and $F_x(y) = F(y+x) - F(x)$ if $y < 1-x$. An example of a cumulative distribution F and its corresponding x -cumulative distribution F_x is shown on Figure 1.

2. COMPUTING THE MONGE-KANTOROVICH DISTANCE ON THE CIRCLE

In the following, we show that the Monge-Kantorovich distance on the circle can be computed very easily for $\lambda = 1$ (this case corresponds to the EMD distance). In short, we show that if σ is an optimal permutation for (5), there is always a point on the circle which is not contained in any optimal path of σ . This result is proven first for $\lambda > 1$ and for any optimal permutation σ , then for $\lambda = 1$ and a well chosen optimal permutation. The main consequence of this result is that the computation of the Monge-Kantorovich distance on the circle boils down to the computation of the same distance on an interval of \mathbb{R} .

Proposition 1. *Assume that $\lambda > 1$. Let x_1, \dots, x_P and y_1, \dots, y_P be P points in $[0, 1[$. Assume that all these points are pairwise different. Then for each permutation σ of Σ_P which minimizes (5) with the cost (4), there exists $k \in \{1, \dots, P\}$ such that for all $l \neq k$, $x_k \notin \gamma(x_l, y_{\sigma(l)})$.*

The proof of this proposition needs the following lemma, which describes some properties of the geodesic paths $\gamma(x_l, y_{\sigma(l)})$ obtained when σ is a minimizer of (5) with $\lambda > 1$.

Lemma 1. *Assume that $\lambda > 1$. Let σ be a minimizer of (5) and let $\gamma_l = \gamma(x_l, y_{\sigma(l)})$ and $\gamma_k = \gamma(x_k, y_{\sigma(k)})$ (with $l \neq k$) be two geodesic paths for the assignment defined by σ . Assume also that $x_l \neq x_k$ and $y_{\sigma(l)} \neq y_{\sigma(k)}$. Then, one of the following holds:*

- $\gamma_l \cap \gamma_k = \emptyset$;
- $\gamma_l \cap \gamma_k \neq \emptyset$ and in this case γ_l and γ_k have the same direction (both positive or both negative) and neither of them is contained in the other.

Proof. Assume that $\gamma_l \cap \gamma_k \neq \emptyset$. If $\gamma_l \cap \gamma_k$ is equal to $\gamma(x_l, x_k)$ or $\gamma(y_{\sigma(l)}, y_{\sigma(k)})$, then

$$c(x_l, y_{\sigma(l)})^\lambda + c(x_k, y_{\sigma(k)})^\lambda > c(x_l, y_{\sigma(k)})^\lambda + c(x_k, y_{\sigma(l)})^\lambda,$$

which contradicts the optimality of σ . Moreover, if for example the path γ_l is included in γ_k , then the strict convexity of the function $x \mapsto |x|^\lambda$ (for $\lambda > 1$) implies

$$c(x_l, y_{\sigma(l)})^\lambda + c(x_k, y_{\sigma(k)})^\lambda > c(x_l, y_{\sigma(k)})^\lambda + c(x_k, y_{\sigma(l)})^\lambda,$$

which also contradicts the optimality of σ . Thus, $\gamma_l \cap \gamma_k$ is equal to $\gamma(x_l, y_{\sigma(k)})$ or to $\gamma(x_k, y_{\sigma(l)})$ and it follows that γ_k and γ_l are either both positive or both negative. \square

Proof of Proposition 1 Let σ be a minimizer of (5). In the following, we will denote by γ_l the geodesic path $\gamma(x_l, y_{\sigma(l)})$. We can assume without loss of generality that the points x_1, \dots, x_P are in trigonometric order on the circle.

Assume that for each $l \in \{1, \dots, P\}$, there exists $q(l) \neq l$ such that x_l belongs to the open path $\gamma_{q(l)}$. Then, for each l , we have $\gamma_{q(l)} \cap \gamma_l \neq \emptyset$, which means that the geodesic paths $\gamma_{q(l)}$ and γ_l are either both positive or both negative (from lemma 1). Assume for instance that they are both positive and let us show that in this case $x_l \in \gamma_{l-1}$ (with $l-1 = P$ if $l = 0$). If $q(l) = l-1$, there is nothing to prove. If $q(l) \neq l-1$, it means in particular that $x_{q(l)}, x_{l-1}, x_l$ are in trigonometric order on the circle. Since $\gamma_{q(l)}$ is a positive path starting from $x_{q(l)}$ and containing x_l , it follows that $\gamma_{q(l)}$ contains x_{l-1} (recall that the points are assumed to be pairwise different, in particular $x_{l-1} \neq x_{q(l)}$). Thus $\gamma_{l-1} \cap \gamma_{q(l)} \neq \emptyset$, which implies that γ_{l-1} is positive. Now, x_l must be in γ_{l-1} , otherwise we would have $\gamma_{l-1} \subset \gamma_{q(l)}$, which is impossible from lemma 1. Thus, if the paths $\gamma_{q(l)}$ and γ_l are both positive, $x_l \in \gamma_{l-1}$.

In the same way, if $\gamma_{q(l)}$ and γ_l are both negative, then $x_l \in \gamma_{l+1}$. In any case, for each $l \in \{1, \dots, P\}$, $x_l \in \gamma_{l-1} \cup \gamma_{l+1}$ (with the obvious convention $\gamma_{P+1} = \gamma_1, \gamma_0 = \gamma_P$).

Now, suppose that for a given $k \in \{1, \dots, P\}$, x_k is in γ_{k-1} . Then, γ_{k-1} and γ_k have the same direction. From lemma 1, it follows that x_{k-1} cannot be contained in γ_k . Since we know that $x_{k-1} \in \gamma_{k-2} \cup \gamma_k$, x_{k-1} must be in γ_{k-2} . Recursively, for each $l \in \{1, \dots, P\}$, $x_l \in \gamma_{l-1}$. It follows that for each $l \in \{1, \dots, P\}$, $c(x_l, y_{\sigma(l-1)}) < c(x_{l-1}, y_{\sigma(l-1)})$, and then

$$(9) \quad \sum_{l=1}^P c(x_l, y_{\sigma(l)})^\lambda > \sum_{l=1}^P c(x_{l+1}, y_{\sigma(l)})^\lambda,$$

which contradicts the fact that σ is a minimizer of (5). We come to the same conclusion if for a given $k \in \{1, \dots, P\}$, x_k is in γ_{k+1} \square

The same result can be proven for $\lambda = 1$ with the difference that it is only satisfied for a good choice of the permutation σ which minimizes (5), and not for all of these permutations. This result can be seen as a limit version of proposition 1 when $\lambda \rightarrow 1$.

Corollary 1. *Assume that $\lambda = 1$. Let x_1, \dots, x_P and y_1, \dots, y_P be P points in $[0, 1]$. Assume that all these points are pairwise different. Then there exists a permutation σ of Σ_P which minimizes (5) and a point $x_k \in \{x_1, \dots, x_P\}$ such that for all $l \neq k$, $x_k \notin \gamma(x_l, y_{\sigma(l)})$.*

Proof. We know that for any $\lambda > 1$, if σ_λ minimizes the cost $\sigma \mapsto W_\sigma^\lambda(f, g)$, there exists $k \in \{1, \dots, P\}$ such that for all $l \neq k$, $x_k \notin \gamma_l = \gamma(x_l, y_{\sigma_\lambda(l)})$.

Now, observe that if σ and the points $x_1, \dots, x_P, y_1, \dots, y_P$ are fixed, $W_\sigma^\lambda(f, g)$ is a continuous function of λ . Thus, for each $\varepsilon > 0$, there exists $\beta > 1$, such that for all $\lambda \in [1, \beta]$, $|W_\sigma^\lambda(f, g) - W_\sigma^1(f, g)| \leq \varepsilon$. Since Σ_P is a finite set, we can chose β close enough to 1 such that this property holds for every σ in Σ_P . We can also chose β such that $|\min_\sigma W_\sigma^1(f, g) - \min_\sigma W_\sigma^\lambda(f, g)| \leq \varepsilon$ (the minimum of a finite set of continuous functions is a continuous function). It follows that for $\lambda \in [1, \beta]$,

$$\begin{aligned} |\min_\sigma W_\sigma^1(f, g) - W_{\sigma_\lambda}^1(f, g)| &\leq |\min_\sigma W_\sigma^1(f, g) - \min_\sigma W_\sigma^\lambda(f, g)| \\ &\quad + |W_{\sigma_\lambda}^\lambda(f, g) - W_{\sigma_\lambda}^1(f, g)| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus, when λ is close enough to 1, a minimizer σ_λ of $W_\sigma^\lambda(f, g)$ is also a minimizer of $W_\sigma^1(f, g)$. Which proves that there exists at least one minimizer σ of $\sigma \mapsto W_\sigma^1(f, g)$ such that $x_k \notin \gamma(x_l, y_{\sigma_\lambda(l)})$ for some $k \in \{1, \dots, P\}$ and all $l \neq k$. \square

We can now draw a first conclusion on the form of the Monge Kantorovich distance when all points are pairwise different.

Corollary 2. *Assume that x_1, \dots, x_P and y_1, \dots, y_P are pairwise different. Then,*

$$(10) \quad \forall \lambda \geq 1, \quad \text{MK}_\lambda(f, g) = \left(\inf_{x \in S^1} \int |F_x^{-1} - G_x^{-1}|^\lambda \right)^{1/\lambda},$$

where F_x^{-1} and G_x^{-1} are the pseudo-inverses of the increasing functions F_x and G_x , defined as $F_x^{-1}(y) = \inf\{t; F_x(t) > y\}$ and $G_x^{-1}(y) = \inf\{t; G_x(t) > y\}$.

Proof. Proposition 1 and the above corollary show that for any $\lambda \geq 1$, if x_1, \dots, x_P and y_1, \dots, y_P are pairwise different, we can choose some optimal permutation σ for which there is some point x_k which is not contained in any path (recall that paths are defined

as open: they do not contain their boundaries). Since all points are supposed pairwise different, the only path meeting all the neighborhoods of x_k is γ_k . It follows that there exists some open set on one side of x_k and not containing x_k which does not cross any path of the optimal permutation σ . The middle x of this open set is not contained in any path of σ . We can thus cut the circle S^1 at x and reduce the transportation problem on the circle to the transportation problem on the real line. The optimal permutation σ is thus given by the sorting of the points (formula (72) in [4]), taking x as the reference point on the circle. This means that when points are pairwise different, we have

$$(11) \quad \forall \lambda \geq 1, \quad \text{MK}_\lambda(f, g) = \left(\inf_{x \in S^1} \int |F_x^{-1} - G_x^{-1}|^\lambda \right)^{1/\lambda},$$

where F_x^{-1} and G_x^{-1} are the pseudo-inverses of the increasing functions F_x and G_x . \square

Observe that in the case where $\lambda = 1$, this result can be rewritten

$$(12) \quad \text{MK}_1(f, g) = \inf_{x \in S^1} \|F_x - G_x\|_1.$$

In order to generalize this result to the case where the points x_1, \dots, x_P and y_1, \dots, y_P can coincide, we first show that $\text{MK}_1(f, g)$ can also be written $\min_{l \in \{-P, \dots, P\}} \|F - G - \frac{l}{P}\|_1$, which will be easier to handle. Obviously, the quantity $\|F_x - G_x\|_1$ can be written as

$$\begin{aligned} \|F_x - G_x\|_1 &= \int_0^{1-x} |F(t+x) - F(x) - G(t+x) + G(x)| dt \\ &\quad + \int_{1-x}^1 |F(t+x-1) - F(x) - G(t+x-1) + G(x)| dt \\ &= \int_x^1 |F(t) - F(x) - G(t) + G(x)| dt + \int_0^x |F(t) - F(x) - G(t) + G(x)| dt \\ &= \|F - F(x) - G + G(x)\|_1. \end{aligned}$$

Now, observe that $F(x) - G(x)$ takes its values in $\{\frac{l}{P}; l = -P, \dots, P\}$. Moreover, since $F - G$ takes only a finite number of values, it is easy to see that $\|F - G - \frac{l}{P}\|_1$ reaches its minimum at one (non necessarily unique) of these values. Thus,

$$\text{MK}_1(f, g) = \min_{l \in \{-P, \dots, P\}} \|F - G - \frac{l}{P}\|_1.$$

In the following corollary, we show that this formula can be generalized to the case where some of the points x_1, \dots, x_P and y_1, \dots, y_P coincide.

Corollary 3. *Let $f = \frac{1}{P} \sum_{k=1}^P \delta_{x_k}$ and $g = \frac{1}{P} \sum_{k=1}^P \delta_{y_k}$ be two discrete distributions on S^1 (some of the points x_1, \dots, x_P and y_1, \dots, y_P can coincide). Then, the 1-Monge Kantorovich distance between f and g can be computed as*

$$\text{MK}_1(f, g) = \min_{l \in \{-P, \dots, P\}} \|F - G - \frac{l}{P}\|_1.$$

Proof. Let $x_1, \dots, x_P, y_1, \dots, y_P$ be $2P$ points on S^1 , not necessarily different. For every $\varepsilon > 0$, we can build $x_1^\varepsilon, \dots, x_P^\varepsilon, y_1^\varepsilon, \dots, y_P^\varepsilon$, all pairwise different, such that $\forall k \in \{1, \dots, P\}$, $c(x_k^\varepsilon, x_k) \leq \varepsilon$ and $c(y_k^\varepsilon, y_k) \leq \varepsilon$. Let $f^\varepsilon = \frac{1}{P} \sum_{k=1}^P \delta_{x_k^\varepsilon}$ and $g^\varepsilon = \frac{1}{P} \sum_{k=1}^P \delta_{y_k^\varepsilon}$

be the corresponding distributions. Then,

(13)

$$\forall \sigma \in \Sigma_P, \quad |W_\sigma^1(f^\varepsilon, g^\varepsilon) - W_\sigma^1(f, g)| \leq \sum_{k=1}^P |c(x_k^\varepsilon, y_{\sigma(k)}^\varepsilon) - c(x_k, y_{\sigma(k)})| \leq 2P\varepsilon = C\varepsilon.$$

Since MK_1 is a minimum over a finite set, it follows that

$$\text{MK}_1(f^\varepsilon, g^\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \text{MK}_1(f, g).$$

Now, if we write F^ε and G^ε the cumulative distribution functions of the distributions f^ε and g^ε , then

$$\forall l \in \{-P, \dots, P\}, \quad \|F^\varepsilon - G^\varepsilon - \frac{l}{P}\|_1 \xrightarrow{\varepsilon \rightarrow 0} \|F - G - \frac{l}{P}\|_1.$$

Since l takes a finite number of values,

$$\min_{l \in \{-P, \dots, P\}} \|F^\varepsilon - G^\varepsilon - \frac{l}{P}\|_1 \xrightarrow{\varepsilon \rightarrow 0} \min_{l \in \{-P, \dots, P\}} \|F - G - \frac{l}{P}\|_1.$$

From these two convergences, we can conclude that

$$(14) \quad \text{MK}_1(f, g) = \min_{l \in \{-P, \dots, P\}} \|F - G - \frac{l}{P}\|_1.$$

□

Observe that an alternative proof of this last result when $\lambda = 1$ was proposed by Werman et al. in [5], with some inaccuracies in the case where points may coincide.

Now, using the same sequence of equalities as before, and noting that $\|F - G - \frac{l}{P}\|_1$ must reach its minimum at one of the values $\|F - G - F(x) + G(x)\|_1$, we can conclude that formula (12) is satisfied in the case where points may coincide.

3. CONSEQUENCES IN THE CASE OF DISCRETE HISTOGRAMS

Consider two discrete histograms $f = (f[i])_{i=0 \dots N-1}$ and $g = (g[i])_{i=0 \dots N-1}$, sampled on N equal bins. We suppose that f and g are normalized, in the sense that $\sum_{i=0}^{N-1} f[i] = \sum_{i=0}^{N-1} g[i] = 1$. Now, assume that these histograms are **circular**, which means that the bins 0 and $N - 1$ are neighbors. In this case, these two histograms can be considered as probability distributions on the unit circle S^1 , or equivalently as periodic distributions of period 1 on \mathbb{R} . These distributions can then be written as

$$f = \sum_{i=0}^{N-1} f[i] \delta_{i/N}, \quad \text{and} \quad g = \sum_{k=0}^{N-1} g[k] \delta_{k/N},$$

Now, if all the weights $f[i]$ and $g[k]$ are rational, which is the case for numerical histograms, it is always possible to replicate these points as many times as necessary in order to write

$$f = \frac{1}{P} \sum_{k=1}^P \delta_{x_k}, \quad \text{and} \quad g = \frac{1}{P} \sum_{k=1}^P \delta_{y_k},$$

where $\{x_1, \dots, x_P\}$ and $\{y_1, \dots, y_P\}$ are two sets of points on the unit circle S^1 . As a result, several of these points may coincide. For instance, if $N = 2$ and $f = (3/4, 1/4)$, we can write $f = \frac{1}{4}(\delta_0 + \delta_0 + \delta_0 + \delta_{\frac{1}{2}})$.

The circular Earth Mover's Distance between the discrete histograms f and g can thus be computed as

$$(15) \quad \text{CEMD}(f, g) = \text{MK}_1(f, g) = \min_{l \in \{-P, \dots, P\}} \|F - G - \frac{l}{P}\|_1 = \min_{l \in \{-P, \dots, P\}} \sum_{k=-P}^P \left| \frac{k-l}{P} \right| \omega_k,$$

where $\omega_k = \text{measure}\{z \in [0, 1[; F(z) - G(z) = \frac{k}{P}\}$. In other words, the value $\frac{l}{P}$ where this quantity is minimum can be computed as a weighted median of the values $F(z) - G(z)$. In the case of a discrete histograms, the points x_k and y_k are located on a regular grid of N bins. Consequently,

$$(16) \quad \text{MK}_1(f, g) = \frac{1}{N} \min_{j \in \{0, \dots, N-1\}} \sum_{i=0}^{N-1} |F[i] - G[i] - F[j] + G[j]| = \frac{1}{N} \sum_{i=0}^{N-1} |F[i] - G[i] - \mu|,$$

where μ is the median of all the values $F[i] - G[i]$.

In the same way, for discrete histograms, formula (12) becomes

$$(17) \quad \text{CEMD}(f, g) = \min_{k \in \{0, \dots, N-1\}} \|F_k - G_k\|_1,$$

where F_k and G_k are the cumulative histograms of f and g starting at the k^{th} quantization bin. For each k in $\{0, \dots, N-1\}$

$$F_k[i] = \begin{cases} \sum_{j=k}^{k+i} f[j] & \text{if } 0 \leq i \leq N-k-1 \\ \sum_{j=k}^{N-1} f[j] + \sum_{j=0}^{k+i-N} f[j] & \text{if } N-k \leq i \leq N-1 \end{cases}.$$

The definition is similar for G_k by replacing f by g . Note that $F_k[i] = F[k+i] - F[k-1]$ if $i \leq N-k-1$ and $F_k[i] = F[k+i-N+1] + 1 - F[k-1]$ if $i > N-k-1$.

In other words, the distance $\text{CEMD}(f, g)$ is also the minimum in k of the L^1 distance between F_k and G_k , the cumulative histograms of f and g starting at the k^{th} quantization bin.

Observe that this formula remains valid for any translated version of F_k and G_k . In particular, if we define

$$\tilde{F}_k[i] = \begin{cases} \sum_{j=k}^i f[j] & \text{if } k \leq i \leq N-1 \\ \sum_{j=k}^{N-1} f[j] + \sum_{j=0}^i f[j] & \text{if } 0 \leq i \leq k-1 \end{cases},$$

the distance $\text{CEMD}(f, g)$ can also be computed as

$$(18) \quad \text{CEMD}(f, g) = \min_{k \in \{0, \dots, N-1\}} \|\tilde{F}_k - \tilde{G}_k\|_1.$$

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