Cell centered Galerkin methods

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Abstract

In this work we propose a new approach to obtain/analyze cell centered methods yielding at the same time convergence rates and convergence to minimal regularity solutions. The approach merges ideas from Finite Volume and Discontinuous Galerkin methods and it provides a unified perspective.

Résumé

Dans cette note, on propose une nouvelle approche pour obtenir/analyser des méthodes centrées aux mailles. L'analyse permet d'estimer à la fois le taux de convergence des méthodes et de prouver leur convergence vers des solutions à régularité minimale. Cette approche combine les avancées récentes dans les méthodes de Volumes Finis et de Galerkine discontinues.

1 Introduction

Dans cette note on propose une nouvelle approche pour obtenir et analyser des méthodes centrées aux mailles pour un problème de diffusion hétérogène et anisotrope. La méthode se compose de plusieurs étapes : (i) on définit un interpolateur de trace consistant pour des fonctions suffisamment régulières ; (ii) à partir des valeurs de trace, on reconstruit un gradient par maille qui est aussi consistant ; (iii) le gradient est utilisé pour définir un sous-espace de l'espace des fonctions affines par morceaux sur le maillage, qui est en suite utilisé dans une formulation inspirée des méthodes de Galerkine discontinues. Cette classe de méthodes bénéficie aussi bien des techniques d'analyse par estimation d'erreurs propres à la communauté Éléments Finis que de celles par compacité propres à la communauté Volumes Finis.

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2 The continuous setting

Let $\Omega \in \mathbb{R}^d$ be a bounded polygonal domain and let $\nu \in [L^\infty(\Omega)]^{d,d}$ a symmetric uniformly positive definite tensor field piecewise constant on a partition of Ω, say P_Ω, into bounded polygonal subdomains. For $f \in L^2(\Omega)$, consider the problem

$$-\nabla\cdot(\nu\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Letting $a(u,v) \stackrel{\text{def}}{=} \int_\Omega \nabla u\cdot\nabla v$, the weak formulation reads

$$\text{Find } u \in H^1_0(\Omega) \text{ s.t. } a(u,v) = \int_\Omega fu \text{ for all } v \in H^1_0(\Omega). \tag{2}$$

3 Discretization

Let $\mathcal{H} \subset \mathbb{R}^+_*$ be a countable set having 0 as an accumulation point and let $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ define an admissible mesh family in the sense of [4] parametrized by the meshsize h and compatible with P_Ω. Let $h \in \mathcal{H}$ be given. For each $K \in \mathcal{T}_h$, let $x_T \in T \setminus \partial T$ be such that T is star-shaped with respect to x_T (the *cell center*) and, for all face $F \in \mathcal{F}_h$, set $\widehat{x}_F \stackrel{\text{def}}{=} \int_F x/|F|$. The set of faces belonging to an element $T \in \mathcal{T}_h$ will be denoted by $\mathcal{F}_h^T$, the set of boundary faces by $\mathcal{F}_h^b$. For each internal face we select an arbitrary direction for the normal $\mathrm{n}_F \equiv \mathrm{n}_{T_1,F}$ and denote by T_1 the element out of which n_F points. For all $T \in \mathcal{T}_h$, $F \in \mathcal{F}_h^T$, set $d_{T,F} \stackrel{\text{def}}{=} \mathrm{dist}(x_T, F)$ and let $d_F \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T}_h^F} d_{T,F}$, $\mathcal{T}_h^F$ being the set of mesh element(s) sharing F. In what follows we shall assume that, for all $F \in \mathcal{F}_h$, d_F is uniformly bounded by $\mathrm{diam}(F)$ both from above and from below.

Let $\mathbb{V}^{\mathcal{T}_h} \stackrel{\text{def}}{=} \mathbb{R}^{\mathrm{card}(\mathcal{T}_h)}$, $\mathbb{V}^{\mathcal{F}_h} \stackrel{\text{def}}{=} \mathbb{R}^{\mathrm{card}(\mathcal{F}_h)}$. We call a *trace interpolator* $\mathbf{I}$ a linear bounded bijective operator from $\mathbb{V}^{\mathcal{T}_h}$ to $\mathbb{X} \subset \mathbb{V}^{\mathcal{F}_h}$ such that, for all $\mathbb{V}^{\mathcal{T}_h} \ni \mathbf{V} = \{v_T\}_{T\in\mathcal{T}_h}$, $\mathbb{X} \ni \mathbf{I}(\mathbf{V}) = \{I_F(\mathbf{V})\}_{F\in\mathcal{F}_h}$ with $I_F(\mathbf{V}) = 0$ if $F \in \mathcal{F}_h^b$. The bijectivity requirement on the interpolation operator $\mathbf{I}$ can usually be reformulated in terms of the inversibility of (local) matrices (cf., e.g., [1, §3.1]). Also, we denote by $\Pi_{\mathbb{V}^{\mathcal{T}_h}} : W \to \mathbb{V}^{\mathcal{T}_h}$ the operator that maps every element $w \in W$ onto $\Pi_{\mathbb{V}^{\mathcal{T}_h}}(w) \stackrel{\text{def}}{=} \{w(x_T)\}_{T\in\mathcal{T}_h}$. Let

$$W \stackrel{\text{def}}{=} \{w \in H^1_0(\Omega)\,;\, \nu\nabla w \in H(\mathrm{div};\Omega) \text{ and } w|_T \in H^2(\overline{T}) \text{ for all } T \in \mathcal{T}_h\}.$$

The trace interpolators considered hereafter are assumed to fulfill the following consistency requirement: For all $w \in W$, there exists $C_w \in \mathbb{R}^+_*$ depending on the mesh regularity parameters, on some bounded norm of w but not on the

meshsize h such that

$$\forall T \in \mathcal{T}_h,\, \forall F \in \mathcal{F}_h^T, \quad |w(\widehat{x}_F) - (\mathbf{I} \circ \Pi_{\mathbb{V}^{\mathcal{T}_h}})(w)| \le C_w h d_{T,F}. \tag{3}$$

Remark 1. The above property holds, e.g., for the trace values provided by the L-interpolator analyzed in [1]. Other interpolators matching property (3) (possibly with additional assumptions on ν and/or on $\mathcal{T}_h$) include the two-point interpolator (for ν-orthogonal meshes), the barycentric interpolator of [7] (provided ν is homogeneous), the O-interpolator of Aavatsmark and coworkers. Hybrid/mixed Finite Volume (FV) methods resembling those discussed in [6] and references therein can also be obtained using the procedure of §3. For such methods, the interpolation procedure amounts to eliminating cell unknowns in favour of face unknowns (such elimination need not be implemented).

The trace interpolator is the key ingredient to construct a constistent gradient approximation. For all $\mathbf{V} \in \mathbb{V}^{\mathcal{T}_h}$, we let

$$\mathbf{G}_T(\mathbf{V}) \stackrel{\text{def}}{=} \frac{1}{|T|} \sum_{F \in \mathcal{F}_h^T} |F| \, (I_F(\mathbf{V}) - v_T) \, \mathrm{n}_{T,F}.$$

For $k \ge 0$, let P_h^k denote the piecewise polynomial space of local degree k, i.e. $P_h^k \stackrel{\text{def}}{=} \{p_h \in L^2(\Omega)\,;\, \forall T \in \mathcal{T}_h,\, p_h|_T \in \mathbb{P}_d^k\}$. Consider the following approximation space:

$$V_h \stackrel{\text{def}}{=} \left\{ v_h \in P_h^1\,;\, \exists \mathbf{V} \in \mathbb{V}^{\mathcal{T}_h},\, \forall T \in \mathcal{T}_h,\, v_h(x_T) = v_T \text{ and } \nabla v_h|_T = \mathbf{G}_T(\mathbf{V}) \right\}.$$

For every $v_h \in V_h$, there holds $v_h|_T(x) = v_T + \mathbf{G}_T(\mathbf{V}) \cdot (x - x_T)$ for all $T \in \mathcal{T}_h$ and $\mathbf{V}$ the element of $\mathbb{V}^{\mathcal{T}_h}$ associated to v_h. The space V_h is thus a subspace of P_h^1 with one single affine test function per element whose value at x_T is prescribed and whose slope depends on the values at neighbouring cell centers, on the mesh and, possibly, on the problem data via the interpolator $\mathbf{I}$.

Remark 2. Owing to the bijectivity of $\mathbf{I}$, each set of cell center values $\mathbf{V} \in \mathbb{V}^{\mathcal{T}_h}$ defines a unique element of V_h.

The space V_h is equipped with the following norm:

$$\|v_h\|_h^2 \stackrel{\text{def}}{=} \sum_{T \in \mathcal{T}_h} \|\nabla v_h\|_{[L^2(T)]^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{d_F} \|[\![v_h]\!]\|_{L^2(F)}^2,$$

where the jump operator $[\![\cdot]\!]$ is defined as follows:

$$[\![v_h]\!] \stackrel{\text{def}}{=} \begin{cases} v_h|_{T_1} - v_h|_{T_2} & \text{for all } F \in \mathcal{F}_h^{T_1} \cap \mathcal{F}_h^{T_2}, \\ v_h|_T & \text{for all } F \in \mathcal{F}_h^T \cap \mathcal{F}_h^b. \end{cases}$$

The above norm naturally extends to the space $V(h) \stackrel{\text{def}}{=} W + V_h$. Thanks to (3), the following remarkable approximation property holds:

Lemma 3 (Approximation estimate)**.** *For all $w \in W$, there exists $C_w \in \mathbb{R}_*^+$ depending on the mesh regularity parameters, on some bounded norm of w but not on the meshsize h such that*

$$\inf_{w_h \in V_h} \|w - w_h\|_h \le C_w h. \tag{4}$$

Proof. Let $\mathbf{W} \stackrel{\text{def}}{=} \Pi_{\mathbb{V}^{\mathcal{T}_h}}(w)$ and be w_h the unique element of V_h associated to $\mathbf{W}$. Then, proceeding as in [2, Lemma 3.10] and using (3), one can prove that $\|\nabla w - \nabla_h w_h\|_{[L^2(\Omega)]^d} \le C_w h$. For the second term, observe that, for all $F \in \mathcal{F}_h$ and for almost every $x \in F$, $[\![w_h]\!] = [\![w_h - w]\!]$ and use a classical trace inequality together with the local regularity of w to expand $w|_T$, $T \in \mathcal{T}_h$, around x_T to conclude. □

Consider the following bilinear form inspired from the SIPG discontinuous Galerkin (dG) method of Arnold (see [3] for a review):

$$\begin{aligned} a_h(u_h, v_h) \stackrel{\text{def}}{=} &\int_\Omega \nu \nabla_h u_h \cdot \nabla_h v_h - \sum_{F \in \mathcal{F}_h} \int_F [\{\nu \nabla_h u_h\} \cdot \mathrm{n}_F [\![v_h]\!] + [\![u_h]\!] \{\nu \nabla_h v_h\} \cdot \mathrm{n}_F] \\ &+ \sum_{F \in \mathcal{F}_h} \frac{\eta}{d_F} \int_F [\![u_h]\!][\![v_h]\!], \end{aligned}$$

where ∇_h denotes the broken gradient, $\eta \in \mathbb{R}_*^+$ and

$$\{v_h\} \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2}(v_h|_{T_1} + v_h|_{T_2}) & \text{for all } F \in \mathcal{F}_h^{T_1} \cap \mathcal{F}_h^{T_2}, \\ v_h|_T & \text{for all } F \in \mathcal{F}_h^T \cap \mathcal{F}_h^b. \end{cases}$$

Remark 4. The bilinear form a_h yields a globally conservative method, since the characteristic function of Ω is in V_h. Its implementation does not require cubatures. Indeed, the consistency and symmetry terms can be rewritten as $\sum_{F \in \mathcal{F}_h} |F| \{\nu \nabla_h u_h\} \cdot \mathrm{n}_F [\![v_h(\widehat{x}_F)]\!]$ and $\sum_{F \in \mathcal{F}_h} |F| \{\nu \nabla_h v_h\} \cdot \mathrm{n}_F [\![u_h(\widehat{x}_F)]\!]$ respectively and the penalty term can be safely approximated using the middle point rule by $\sum_{F \in \mathcal{F}_h} \frac{\eta |F|}{d_F} [\![u_h(\widehat{x}_F)]\!][\![v_h(\widehat{x}_F)]\!]$.

We consider the following approximation of (2):

$$\text{Find } u_h \in V_h \text{ s.t. } a_h(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h. \tag{5}$$

Problem (5) has *only one* unknown per cell, as the elimination of face unknowns has been carried out in the interpolation process. Such an operation can be performed in practice when, for a given $\mathbf{V} \in \mathbb{V}^{\mathcal{T}_h}$, the dependence of each $I_F(\mathbf{V})$ on $\mathbf{V}$ is limited to a few cell values (which is the case, e.g., for the L-interpolator). On the other hand, when each $I_F(\mathbf{V})$ potentially depends on all the values $\{v_T\}_{T \in \mathcal{T}_h}$ (as it is the case for the interpolator obtained by expressing face unknowns in terms of cell unknowns in the hybrid methods of [6]), such an elimination need not be implemented in practice.

4 Main results and comments

Define, for all $w \in V(h)$, $\|w\|_{h,*}^2 \stackrel{\text{def}}{=} \|w\|_h^2 + \sum_{T\in\mathcal{T}_h} \mathrm{diam}(T)^2 |w|_{H^2(T)}^2$ and observe that, owing to (4), $\inf_{w_h\in V_h} \|w - w_h\|_{h,*} \le C_w h$.

Theorem 5 (Convergence to piecewise regular solutions)**.** *Let u solve (2) and assume, moreover, that $u \in W$. Denote by $\{u_h\}_{h\in\mathcal{H}}$ the sequence of solutions of (5) on the admissible mesh family $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, there exists $C_{u,\nu} \in \mathbb{R}_*^+$ depending on the mesh regularity parameters, on some bounded norm of u and on ν but not on the meshsize h such that*

$$\|u - u_h\|_h \le C_{\nu,u} h. \tag{6}$$

Proof. Upon choosing η large enough (see [3]), a_h is unconditionally uniformly coercive and, for, all $(w, v_h) \in W \times V_h$, $a_h(w, v_h) \le C\|w\|_{h,*}\|v_h\|_h$ with $C \in \mathbb{R}_*^+$ independent of the meshsize h. Finally, a_h is consistent in the finite element sense, i.e., $a_h(u, v_h) = -\int_\Omega \triangle u v_h$. The result follows. □

Remark 6. The dependence on ν in estimate (6) can be removed proceeding as in [5]. The estimate (6) is optimal for the $\|\cdot\|_h$ norm. Under usual elliptic regularity assumptions, the Aubin-Nitsche trick yields $\|u - u_h\|_{L^2(\Omega)} \le C_{\nu,u} h^2$.

For all $v_h \in V_h$ and for all $F \in \mathcal{F}_h$, let r_F solve the following problem:

$$\text{Find } r_F([\![v_h]\!]) \in [P_h^0]^d \text{ s.t. } \int_\Omega r_F([\![v_h]\!])\cdot\tau_h = \int_F [\![v_h]\!]\{\tau_h\}\cdot \mathrm{n}_F \text{ for all } \tau_h \in [P_h^0]^d,$$

and set $R([\![v_h]\!]) \stackrel{\text{def}}{=} \sum_{F\in\mathcal{F}_h} r_F([\![v_h]\!])$. Then, proceeding as in [4, Theorem 2.2] one can prove that, for all sequences $\{v_h\}_{h\in\mathcal{H}}$ bounded in the $\|\cdot\|_h$ norm, (i) there is $v \in H_0^1(\Omega)$ such that $v_h \to v$ in $L^2(\Omega)$ and (ii) $\nabla_h v_h - R([\![v_h]\!])$ weakly converges to ∇v in $[L^2(\Omega)]^d$. Following the guidelines of [4, Theorem 3.1] and using a suitable test space (see, e.g., [1, Lemma 6] for the L-interpolator), one can prove that

Theorem 7 (Convergence to solutions with minimal regularity)**.** *Let u solve (2). Denote by $\{u_h\}_{h\in\mathcal{H}}$ the sequence of solutions of (5) on the admissible mesh family $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $h \to 0$, (i) $u_h \to u$ in $L^2(\Omega)$, (ii) $\nabla_h u_h \to \nabla u$ and $\nabla_h u_h - R([\![u_h]\!]) \to \nabla u$ in $[L^2(\Omega)]^d$ and (iii) $\sum_{F\in\mathcal{F}_h} d_F^{-1} \|[\![u_h]\!]\|_{L^2(F)}^2 \to 0$.*

Some comments are of order. The derivation of the method outlined in §3 remains valid taking inspiration from any alternative dG method for problem (1); see, e.g., [3]. Moreover, provided suitable interpolation operators satisfying estimates of the form (3) are found, the technique exemplified in §3 virtually extends extended to any linear problem for which a dG method has been devised (a brief literature review is provided in the introduction of [4]). At the

Table 1
Numerical example on a triangular mesh family.

h	$\|u-u_h\|_{L^2(\Omega)}$	order	$\|u-u_h\|_h$	order
2.5000e-01	1.1864e-02	–	3.7139e-01	–
1.2500e-01	2.9512e-03	2.00	1.7427e-01	1.09
6.2500e-02	7.4631e-04	1.98	8.3669e-02	1.06
3.1250e-02	1.8824e-04	1.99	4.0943e-02	1.03
1.5625e-02	4.7304e-05	1.99	2.0241e-02	1.02

same time, the analysis techniques developped by Eymard, Gallouët, Herbin in the context of FV methods and later extended to dG methods in [4] also apply, so that valuable tools for the analysis of nonlinear problems are also available.

References

[1] L. Agélas, D. A. Di Pietro, and J. Droniou, *The G method for heterogeneous anisotropic diffusion on general meshes*, M2AN Math. Model. Numer. Anal. (2009), To appear. Preprint available at http://hal.archives-ouvertes.fr/hal-00342739/fr.

[2] R. Agélas, D. A. Di Pietro, R. Eymard, and R. Masson, *An abstract analysis framework for nonconforming approximations of anisotropic heterogeneous diffusion*, Preprint available at http://hal.archives-ouvertes.fr/hal-00318390/fr, September 2008, Submitted.

[3] D.N. Arnold, F. Brezzi, B. Cockburn, and D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2002), no. 5, 1749–1779.

[4] D. A. Di Pietro and A. Ern, *Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations*, Math. Comp. (2009), To appear. Preprint available at http://hal.archives-ouvertes.fr/hal-00278925/fr/.

[5] D. A. Di Pietro, A. Ern, and J.-L. Guermond, *Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection*, SIAM J. Numer. Anal. **46** (2008), no. 2, 805–831.

[6] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin, *A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods*, Preprint available at http://hal.archives-ouvertes.fr/hal-00346077/fr/, November 2008, Submitted.

[7] R. Eymard, T. Gallouët, and R. Herbin, *A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis*, C. R. Math. Acad. Sci. **344** (2007), no. 6, 3–10.