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# On the time schedule of Brownian Flights

Athanasios BATAKIS and Michel ZINSMEISTER

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**Abstract:** *We are interested on the statistics of the duration of Brownian diffusions started at distance  $\epsilon$  from a given boundary and stopped when they hit back the interface.*

## 1 Introduction

The motivation of the following work has its origin in experimental physics. Some long molecules are solvable in a liquid (for instance imogolite in water or DNA in lithium), the molecules forming the liquid show an intermittent dynamics, alternating diffusion in the bulb and adsorption on the long molecules.

NMR (nuclear magnetic resonance) apparatus, via relaxation methods, allows to measure the statistics of the “flights” of liquid molecules in the bulk, ie. statistics of the diffusion phase of the dynamics. More precisely, it allows to study the rare events, ie. statistics of long flights.

In [GKL<sup>+</sup>06] a connection is established between the statistics of the long flight lengths and the geometry of the long molecules (more precisely their Minkowski dimension). This connection has been made rigorous in [BLZ09]. These two papers concern almost exclusively lengths. But the NMR relaxation allows only to measure time. Some heuristic link between time and length was used in [GKL<sup>+</sup>06]. The aim of this paper is to make this heuristics rigorous.

## 2 Setting and Preliminary Facts

Let  $\Omega$  be a domain in  $\mathbb{R}^d$  with compact boundary. For  $r > 0$  consider the collection  $\mathcal{S}_r$  of Whitney cubes intersecting  $\Gamma_r$ , the  $r$ -level surface for the distance to the boundary  $\Gamma_r = \{x \in \Omega ; \text{dist}(x, \partial\Omega) = r\}$  (to recall definition of Whitney cubes see Proposition 2.1 below).

**Definition 2.1** *Let  $\Omega$  be a domain in  $\mathbb{R}^d$  and fix  $\epsilon > 0$ . We will call Brownian flight the random process  $F_t, t \geq 0$  consisting of picking at random with uniform law one of the  $\mathcal{S}_\epsilon$*

dyadic Whitney cubes and then starting Brownian motion  $B_t$  at the center of the cube. We note  $\tau_\Omega = \inf\{t ; F_t \notin \Omega\}$  the lifetime of this process.

We can choose a different definition of the Brownian flights; namely we can choose the starting point uniformly on  $\Gamma_r$ . The resulting process has exactly the same properties.

Under the  $\Delta$ -regularity condition (1) reminded below we get

**Theorem 2.2** *There exists  $c > 0$  such that*

$$\frac{1}{c} \left( \frac{\epsilon}{\sqrt{t}} \right)^{d_M+2-d} \leq \mathbb{P}(\tau_\Omega > t)$$

and

$$\mathbb{P}(\tau_\Omega > t) \leq c \left( \frac{\epsilon}{\sqrt{t}} \right)^{d_M+2-d} \left| \log \left( \frac{\epsilon}{\sqrt{t}} \right) \right|^{\frac{d}{2}},$$

for all  $0 < \epsilon < t$ .

where  $d$  is the dimension of  $\mathbb{R}^d$  and  $d_M$  the Minkowski dimension of  $\partial\Omega$ .

Let us note  $\beta_s = \text{total time Brownian flight spent in the Minkowski sausage } \{x \in \Omega ; \text{dist}(x, \partial\Omega) \leq s\}$  and  $\delta\beta_s = \beta_s - \beta_{s/2}$ .

Let us recall the definition of a (dyadic) Whitney decomposition (cf. [Gra08], p. 463). For a cube  $Q$  we note  $\ell(Q)$  the side-length of  $Q$  and, for  $\lambda > 0$ ,  $\lambda Q$  the cube of the same center and of  $\lambda$  times the side-length of  $Q$ .

**Proposition 2.1** *Given any non-empty open proper subset  $\Omega$  of  $\mathbb{R}^d$ , there exists a family of closed dyadic cubes  $\{Q_j\}_j$  such that*

- $\bigcup_j Q_j = \Omega$  and the cubes  $Q_j$ 's have disjoint interiors
- $\sqrt{d}\ell(Q_j) \leq \text{dist}(Q_j, \partial\Omega) \leq 4\sqrt{d}\ell(Q_j)$
- if  $Q_j$  and  $Q_k$  touch then  $\ell(Q_j) \leq 4\ell(Q_k)$
- for a given Whitney cube  $Q_j$  there are at most  $12^d$  Whitney cubes  $Q_k$ 's that touch  $Q_j$ .

We note  $\mathcal{Q}_k = \mathcal{Q}_{2^k}$ , the collection of Whitney cubes of size  $2^k$ . For  $k \leq \log_2(\sqrt{t})$  we write  $\delta\tilde{\beta}_k$  for the total lifetime of Brownian flight  $F_t$  inside the union of all cubes of  $\mathcal{Q}_k$ . Without loss of generality we can suppose  $\epsilon, r$  both smaller than 1.

Remark that by definition of dyadic Whitney cubes there exist  $k^*$  depending only on the dimension of the space (in the remaining we take  $k^* = \lceil \log_2(8\sqrt{d}) \rceil + 1$ ) such that for all  $k \in \mathbb{Z}$ ,

$$\bigcup_{Q \in \mathcal{Q}_k} Q \subset \bigcup_{k \leq l \leq k+k^*} \bigcup_{Q \in \mathcal{S}_{2^l}} Q \quad \text{and} \quad \bigcup_{Q \in \mathcal{S}_{2^k}} Q \subset \bigcup_{k-k^* \leq l \leq k} \bigcup_{Q \in \mathcal{Q}_l} Q$$

which implies that total lifetime of Brownian flight  $F_t$  inside  $\bigcup_{Q \in \mathcal{S}_{2k}} Q$  is bounded above by

$\sum_{k-k^* \leq l \leq k} \delta \tilde{\beta}_l$  and we can get a lower control by the same reasoning. This is implicitly used in the proof of lemma 2.6.

If  $D$  is any open set, for a measurable  $F \subset \partial D$  and  $x \in D$  we note  $\omega(x, F, D)$  the harmonic measure of  $F$  at  $x$  inside  $D$ , i.e. probability that Brownian motion started at  $x$  exits  $D$  through  $F$ .

We suppose that the domain  $\Omega$  satisfies the  $\Delta$ -regularity condition (see also [JW88], [Anc86], [HKM93]): let  $x$  be any point of  $\Omega$  whose distance to the boundary  $r$  is less than 1. Then

$$\omega(x, \partial\Omega, \mathbb{B}(x, 2r) \cap \Omega) \geq L, \quad (1)$$

where  $L$  does not depend on  $x$ . In [BLZ09] we have proven the following.

**Theorem 2.3** *Choose  $Q$  at random with uniform law in  $\mathcal{Q}_\varepsilon$ . The probability for a Brownian motion started at any point  $x$  of  $Q$  to exit  $\Omega$  at distance greater than  $r$  from the starting point is comparable to  $\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon} \left(\frac{r}{\varepsilon}\right)^{n-2}$ .*

In order to exhibit the ideas of the proof let us first state the following independent lemma which is nevertheless of its own interest.

**Lemma 2.4** *Under the  $\Delta$ -regularity hypothesis*

1. *The probability that BM touches more than  $N$  Whitney cubes of a given size decreases as  $Cp^N$ , with  $0 < p < 1$ ,  $C$  a positive constant.*
2. *The probability that BM started at distance  $\varepsilon < r$  from the boundary exits  $\Omega$  at distance greater than  $R$  from the starting point without leaving the Minkowski sausage  $\{x \in \Omega ; \text{dist}(x, \partial\Omega) \leq r\}$  is bounded above by  $cp^{R/r}$  where  $c > 0$  and  $0 < p < 1$  are constants (depending only on the constant that appear in the  $\Delta$ -regularity hypothesis and on  $d$ )*

The proof of the lemma relies on an annuli reasoning.

**Proof** We prove the first statement: the proof of the second is essentially the same. Let  $(B_t)_{t>0}$  be Brownian motion started at any point  $x \in \Omega$  and choose  $k \in \mathbb{Z}$ . Choose any  $Q \in \mathcal{Q}_k$  and let  $\lambda Q$  be the cube of the same center but  $\lambda$  times the side-length  $\ell(Q)$  of  $Q$ . By the definition of Whitney cubes, there is a  $\lambda \leq 8\sqrt{d}$  depending only on  $d$  such that

$$\frac{\lambda}{2}\ell(Q) \leq \text{dist}(Q, \partial\Omega) \leq \frac{\lambda}{2}\ell(Q).$$

Suppose that there exists  $t_0 > 0$  such that  $B_{t_0} \in Q$ . By the  $\Delta$ -regularity condition (1), the probability that there exists  $t_1 > t_0$  with  $B_{[t_0, t_1]} \subset \Omega$  and  $B_{t_1} \notin \lambda Q$  is bounded above by  $p < 1$  depending only on  $L, \lambda$ :

$$\mathbb{P}\left(\exists t_1 > t_0 ; B_{[t_0, t_1]} \subset \Omega \text{ and } B_{t_1} \notin \lambda Q \mid \exists t_0 > 0 ; B_{t_0} \in Q\right) < p \quad (2)$$

On the other hand, the number of Whitney cubes of  $\mathcal{Q}_k$  lying inside  $\lambda Q$  is bounded by a constant  $c_1 = c_1(d)$ . The probability that there exists a Whitney cube  $Q_1 \in \mathcal{Q}_k$  outside  $\lambda Q$  that is visited by Brownian motion is hence bounded above by  $p < 1$ .

We study probability that there exist Whitney cubes  $Q_1, \dots, Q_m \in \mathcal{Q}_k$  such that  $Q_1 \cap \lambda Q = Q_2 \cap \lambda Q_1 = \dots = Q_m \cap \lambda Q_{m-1} = \emptyset$  all visited by Brownian motion. It is sufficient to prove that this probability decays exponentially.

By the strong Markov property the probability that there exists  $t_m > t_{m-1} > \dots > t_0$  such that  $B_{t_0} \in Q, B_{t_1} \in Q_1, \dots, B_{t_m} \in Q_m$  is given by

$$\begin{aligned} & \mathbb{P}(\exists t_m > t_{m-1} > \dots > t_0 ; \text{ and } Q, \dots, Q_m \text{ as above such that } B_{t_m} \in Q_m, \dots, B_{t_0} \in Q) \\ &= \mathbb{P}(\exists t_m > t_{m-1} ; B_{t_m} \in Q_m \mid \exists t_{m-1} > \dots > t_0 B_{t_{m-1}} \in Q_{m-1}, \dots, B_{t_0} \in Q) \cdot \\ & \mathbb{P}(\exists t_{m-1} > \dots > t_0 B_{t_{m-1}} \in Q_{m-1}, \dots, B_{t_0} \in Q) \\ &= \mathbb{P}(\exists t_m > t_{m-1} ; B_{t_m} \in Q_m \text{ with } Q_m \cap \lambda Q_{m-1} = \emptyset \mid \exists t_{m-1} ; B_{t_{m-1}} \in Q_{m-1}) \cdot \\ & \mathbb{P}(\exists t_{m-1} > \dots > t_0 B_{t_{m-1}} \in Q_{m-1}, \dots, B_{t_0} \in Q) \end{aligned}$$

Now, by (2),

$$\mathbb{P}(\exists t_m > t_{m-1} ; B_{t_m} \in Q_m \text{ with } Q_m \cap \lambda Q_{m-1} = \emptyset \mid \exists t_{m-1} ; B_{t_{m-1}} \in Q_{m-1}) < p.$$

By induction we get that

$$\mathbb{P}(\exists t_m > t_{m-1} > \dots > t_0 ; \text{ and } Q, \dots, Q_m \text{ as above such that } B_{t_m} \in Q_m, \dots, B_{t_0} \in Q) < p^m$$

and hence the lemma.  $\bullet$

For a given dyadic Whitney cube  $Q$  we note  $\tilde{Q}$  the union of  $Q$  with all Whitney cubes  $Q'$  verifying

$$Q' \cap \lambda Q \neq \emptyset \text{ and } Q \cap \lambda Q' \neq \emptyset.$$

We can easily check that there are less than  $(100\sqrt{d})^d$  such cubes  $Q'$  of size at most  $\ell(Q)/12$  (the constants are not optimal). We say that the  $k$ -level layers are visited more than  $n$  times if there exist  $t_0 < s_1 < t_1 < \dots < s_n < t_n$  satisfying

$$B_{t_j} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \text{ and } B_{s_j} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q},$$

for all  $j = 1, \dots, n$ . For any  $k \in \mathbb{Z}$  note

$$\nu_k = \sup\{n \in \mathbb{N} ; \text{ the } k\text{-level layers are visited more than } n \text{ times}\}$$

**Lemma 2.5** *There exists  $0 < p < 1$  and a positive constant  $C$  such that, given  $k \in \mathbb{Z}$ , for all  $n \in \mathbb{N}$*

$$\mathbb{P}(\nu_k > n) \leq p^n \mathbb{P}(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^k}; B_{t_0} \in Q).$$

**Proof** The arguments are similar as in lemma 2.4. We only need to prove that  $\mathbb{P}(\nu_k > 1) < p$  and apply strong Markov property. We have

$$\begin{aligned} \mathbb{P}(\nu_k > 1) &\leq \mathbb{P}\left(\exists 0 < t_0 < s_1 < t_1, Q \in \mathcal{S}_{2^k}; B_{t_0} \in Q, B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}, B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q\right) \\ &= \mathbb{P}\left(\exists t_1 > s_1 > t_0; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}, B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \mid \exists t_0 > 0; B_{t_0} \in Q \in \mathcal{S}_{2^k}\right) \\ &\times \mathbb{P}(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^k}; B_{t_0} \in Q) \end{aligned}$$

To abbreviate formulas we note  $\mathbb{P}_c(\cdot) = \mathbb{P}(\cdot \mid \exists t_0 > 0; B_{t_0} \in Q \in \mathcal{S}_{2^k})$ . With this notation,

$$\begin{aligned} \mathbb{P}_c\left(\exists t_1 > s_1 > t_0; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}, B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q\right) &= \\ \mathbb{P}_c\left(\exists t_1 > s_1; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \mid A\right) \mathbb{P}_c(A) &+ \mathbb{P}_c\left(\exists t_1 > s_1; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \mid B\right) \mathbb{P}_c(B) \end{aligned}$$

where

$$\begin{aligned} A &= \left\{ \exists s_1 > t_0; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \lambda Q \right\} \text{ and} \\ B &= \left\{ \exists s_1 > t_0; B_{s_1} \in Q', \lambda Q' \cap \bigcup_{Q \in \mathcal{S}_{2^k}} Q = \emptyset, B_{s_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} \lambda Q \right\} \end{aligned}$$

form a partition of the event  $\left\{ \exists s_1 > t_0; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q} \right\}$ .

By (2),  $\mathbb{P}_c(A) \leq p$ . Similarly, by the strong Markov property of Brownian motion and by (2),

$$\mathbb{P}_c\left(\exists t_1 > s_1; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \mid B\right) = \mathbb{P}\left(\exists t_1 > s_1; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \mid B_{s_1} \in Q'\right) \leq p.$$

We deduce that

$$\mathbb{P}_c\left(\exists t_1 > s_1 > t_0; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}, B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q\right) \leq \mathbb{P}_c(A) + p(1 - \mathbb{P}_c(A)) \leq 2p - p^2 < 1$$

and the lemma is proven. •

We also derive the following

**Lemma 2.6** *There exists a constant  $C$  and  $0 < p < 1$  depending only on dimension and on  $L$  such that for all  $t > 0$ ,  $k \in \mathbb{Z}$  and  $N \in \mathbb{N}$  we have*

$$\begin{aligned} \mathbb{P}(\delta\beta_{2^k} > t) &\leq Cp^N \mathbb{P}(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^{k-k^*-2}}; B_{t_0} \in Q) + \\ &\mathbb{P}\left(\exists Q \in \mathcal{Q}_k \cup \dots \cup \mathcal{Q}_{k-k^*-2}; \exists 0 < s_1 < s_2 \text{ with } B_{[s_1, s_2]} \subset \tilde{Q} \text{ and } s_2 - s_1 > t/N\right) \end{aligned}$$

**Proof** Given  $t > 0$ ,  $k \in \mathbb{Z}$ , by lemma 2.5 we have for all  $N$

$$\begin{aligned} \mathbb{P}(\delta\beta_{2^k} > t) &\leq \mathbb{P}(\delta\beta_{2^k} > t, \nu_k + \dots + \nu_{k-k^*-2} > N) + \mathbb{P}(\delta\beta_{2^k} > t, \nu_k + \dots + \nu_{k-k^*-2} \leq N) \\ &< (k^* + 2) \left(p^{\frac{1}{k^*+2}}\right)^N \mathbb{P}(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^{k-k^*-2}}; B_{t_0} \in Q) \\ &+ \mathbb{P}(\delta\beta_{2^k} > t, \nu_k + \dots + \nu_{k-k^*-2} \leq N). \end{aligned}$$

Let us estimate this last term. It is clear that, for all  $k \in \mathbb{Z}$ ,

$$\{x \in \Omega; 2^{k-1} \leq \text{dist}(x, \partial\Omega) \leq 2^k\} \subset \bigcup_{j=k-k^*-2}^k \bigcup_{Q \in \mathcal{Q}_j} Q.$$

Therefore, using definition of  $\nu_k$  we get

$$\begin{aligned} &\mathbb{P}(\delta\beta_{2^k} > t, \nu_k + \dots + \nu_{k-k^*-2} \leq N) \\ &\leq \mathbb{P}(\exists l \leq N, Q_1, \dots, Q_l \in \bigcup_{j=k-k^*-2}^k \mathcal{Q}_j; Q_s \cap \tilde{Q}_{s-1} = \emptyset, \forall s = 2, \dots, l \text{ and} \\ &\exists t_1 < s_1 \leq t_2 < s_2 < \dots \leq t_l < s_l; B_{[t_i, s_i]} \subset \tilde{Q}_s \forall s = 1, \dots, l \text{ and } \sum_{i=1}^l s_i - t_i > t). \end{aligned}$$

Since  $l \leq N$  we get that  $\mathbb{P}(\delta\beta_{2^k} > t, \nu_k + \dots + \nu_{k-k^*-2} \leq N)$  is bounded above by

$$\mathbb{P}\left(\exists Q \in \mathcal{Q}_k \cup \dots \cup \mathcal{Q}_{k-k^*-2}; \exists 0 < t_{i_0} < s_{i_0} \text{ with } B_{[t_{i_0}, s_{i_0}]} \subset \tilde{Q} \text{ and } s_{i_0} - t_{i_0} > t/N\right),$$

which completes the proof. •

### 3 Upper bound estimates

We first establish the following lemma.

**Lemma 3.1** *Let  $Q_k$  be a dyadic cube of size  $2^k$  and  $O$  a point of  $Q_k$ . Consider  $B_t$  a Brownian motion started at  $O$  and  $\tau_{\tilde{Q}_k}$  the exit time of  $B_t$  from  $\tilde{Q}_k$ . There exists a constant  $c > 0$  such that for any Borel set  $F \subset \partial\Omega$*

$$\mathbb{P}(\tau_{\tilde{Q}_k} > t) \leq c \left(\frac{2^k}{\sqrt{t}}\right)^d$$

**Proof** Recall that given a ball of radius  $r$  and Brownian motion started at the center of the ball, the probability that the exit time of the process from the ball exceeds  $t$  is given by the formula  $\left(\int_0^{\frac{r}{\sqrt{t}}} e^{-\frac{u^2}{2}} du\right)^d$ . This estimate, combined with the fact that the diameter of  $\tilde{Q}_k$  is bounded above by a constant times  $2^k$ , implies the statement. •

Let us prove the upper estimates of lifetime of Brownian flights in Theorem 2.2.

We clearly have

$$\mathbb{P}(\tau_\Omega > t) \leq \mathbb{P}\left(\sum_{k=-\infty}^{\log_2(\sqrt{t})} \delta\beta_{2^k} = \tau_\Omega > t\right) + \mathbb{P}\left(\tau_\Omega > \sum_{k=-\infty}^{\log_2(\sqrt{t})} \delta\beta_{r_k}\right)$$

The second term is equal to the probability that Brownian flight gets to distance  $> \sqrt{t}$  from the surface before coming back to it. This probability is hence equivalent to  $\frac{\#\mathcal{S}_{\sqrt{t}}}{\#\mathcal{S}_\varepsilon} \left(\frac{\sqrt{t}}{\varepsilon}\right)^{n-2}$ .

Now the probability  $\mathbb{P}\left(\sum_{k=-\infty}^{\log_2(\sqrt{t})} \delta\beta_{2^k} = \tau_\Omega > t\right)$  (namely that the Brownian flight did not get at distance greater than  $\sqrt{t}$  but lived more than  $t$  time) is bounded by

$$\sum_{j=k_0}^{k_1} \mathbb{P}(\delta\beta_j > 2^{j-k_1}t) + \mathbb{P}(\beta_{k_0} > t/2),$$

$k_1$  and  $k_0$  being the integer parts of  $\log_2(\sqrt{t}) + 2$  and  $\log_2(\sqrt{\varepsilon})$  respectively.

Let us first deal with the sum. By lemmas 2.6, 3.1 and the strong Markov property

$$\begin{aligned} \mathbb{P}(\delta\beta_j > 2^{j-k_1-1}t) &\leq cp^N \mathbb{P}(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^{j-k_1-k^*-3}}; B_{t_0} \in Q) + \\ &\mathbb{P}\left(\exists Q \in \mathcal{Q}_{j-k_1-1} \cup \dots \cup \mathcal{Q}_{j-k_1-k^*-3} \exists s_1, s_2 \in [0, \tau_\Omega); F_{[s_1, s_2]} \subset \tilde{Q}, s_2 - s_1 > \frac{2^{j-k_1-1}t}{N}\right) \\ &\leq cp^N + \mathbb{P}(\exists Q \in \mathcal{Q}_{j-k_1-1} \cup \dots \cup \mathcal{Q}_{j-k_1-k^*-3}, \exists s_1 > 0; F_{s_1} \in Q) \times \\ &\mathbb{P}\left(\exists s_2 > s_1; F_{[s_1, s_2]} \subset \tilde{Q}, s_2 - s_1 > \frac{2^{j-k_1-1}t}{N} \middle| F_{s_1} \in Q \in \mathcal{Q}_{j-k_1-1} \cup \dots \cup \mathcal{Q}_{j-k_1-k^*-3}\right) \\ &\leq cp^N + c \left(\frac{2^j}{\sqrt{\frac{2^{j-k_1-1}t}{N}}}\right)^d \left(\frac{\varepsilon}{2^j}\right)^{d_{M+2-d}}, \end{aligned}$$

where  $c$  is a constant depending on the dimension  $d$ . We obtain

$$\begin{aligned} \sum_{j=k_0}^{k_1} \mathbb{P}(\delta\beta_j > 2^{j-k_1-1}t) &\leq \sum_{j=k_0}^{k_1} c \left(\frac{2^j}{\sqrt{\frac{2^{j-k_1-1}t}{N}}}\right)^d \left(\frac{\varepsilon}{2^j}\right)^{d_{M+2-d}} + cp^N(k_1 - k_0) \\ &\leq cN^{\frac{d}{2}} \left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_{M+2-d}} + cp^N \log\left(\frac{\varepsilon}{\sqrt{t}}\right) \end{aligned}$$

The second term can be bounded in a similar way: consider dyadic cubes  $C_j$  of size  $2^{k_0+1}$  of disjoint interiors touching the boundary of  $\Omega$  and covering the Minkowski sausage  $\{x \in \Omega ; \text{dist}(x, \partial\Omega) < 2^{k_0}\}$ . Once more, using the same arguments as in lemma 2.4, one can show that the probability that Brownian motion visits more than  $N$  such cubes decreases exponentially. Therefore

$$\mathbb{P}(\beta_{k_0} > t/2) \leq \mathbb{P}\left(\exists C_j \exists s_2 > s_1 + t/2N \text{ s.t. } F_{[s_1, s_2]} \subset \tilde{C}_j\right) + cp^N \leq c \left(\frac{\epsilon}{\sqrt{t/2N}}\right)^d + cp^N$$

To finish the proof we only need to optimize on  $N$  ( $N \approx \log\left(\frac{\epsilon}{\sqrt{t}}\right) / \log p$  will do).

## 4 Lower Bound estimates

Following the same reasoning for  $k_1 = \lceil \log_2 \sqrt{t} \rceil + 1$  we get

$$\mathbb{P}(\tau_\Omega > t) \geq \mathbb{P}(\exists s_1 > 0 \text{ s.t. } F_{s_1} \in \bigcup_{Q \in \mathcal{Q}_{k_1}} Q \text{ and } \exists s_2 > s_1 + t \text{ s.t. } F_{[s_1, s_2]} \in \bigcup_{Q \in \mathcal{Q}_{k_1}} 2Q)$$

Using strong Markov property the this probability can be written as the product of  $\mathbb{P}(\exists s_1 > 0 \text{ s.t. } F_{s_1} \in \mathcal{Q}_{k_1})$  with  $\mathbb{P}(\exists s_2 > s_1 + t \text{ s.t. } F_{[s_1, s_2]} \in \bigcup_{Q \in \mathcal{Q}_{k_1}} 2Q)$ . The second one is greater than the probability that Brownian motion exits a cube of size  $2^{k_1}$  at time greater than  $t$  and the first one is simply the probability that Brownian flight got to  $\mathcal{Q}_{k_1}$  which is equivalent to  $\left(\frac{\epsilon}{\sqrt{t}}\right)^{d_M+2-d}$  and the proof is complete.

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