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# Average site perimeter of directed animals on the two-dimensional lattices

Axel Bacher

February 17, 2011

## Abstract

We introduce new combinatorial (bijective) methods that enable us to compute the average value of three parameters of directed animals of a given area, including the site perimeter. Our results cover directed animals of any one-line source on the square lattice and its bounded variants, and we give counterparts for most of them in the triangular lattices. We thus prove conjectures by Conway and Le Borgne. The techniques used are based on Viennot's correspondence between directed animals and heaps of pieces (or elements of a partially commutative monoid).

## 1 Introduction

Let  $\Gamma$  be an oriented graph and  $S$  a nonempty finite set of vertices of  $\Gamma$ . A *directed animal* of source  $S$  on  $\Gamma$  is a finite set of vertices  $A$  that contains  $S$  and such that for every vertex  $v$  of  $A$ , there exists a vertex  $s$  of  $S$  and a path from  $s$  to  $v$  going only through vertices of  $A$ . The vertices of a directed animal  $A$  are called *sites*. The *area* of  $A$ , denoted by  $|A|$ , is the number of sites of  $A$ .

On Figure 1 are depicted single-source directed animals on the three two-dimensional regular lattices: the square lattice, the triangular lattice, and the honeycomb lattice.

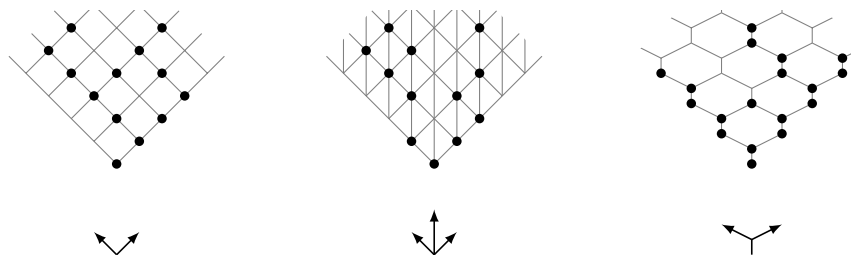


Figure 1: Single-source directed animals on the square, triangular and honeycomb lattices. All edges point upwards.

Single-source directed animals constitute a subclass of *animals* (an animal on a non-oriented graph  $\Gamma$  is simply a finite connected set of vertices of  $\Gamma$ ). While the enumeration of animals on any lattice is an open problem despite extensive research for decades, directed animals are fairly easier to enumerate. As we will not deal with general animals in this paper, we will abusively use the term *animal* instead of directed animal.

Single-source directed animals on the square and triangular lattices have been enumerated [6, 8, 2]. Specifically, let  $a(n)$  and  $\bar{a}(n)$  be the number of animals of source  $\{(0, 0)\}$  and area  $n$  on the square and triangular lattice, respectively. The generating functions of these numbers are:

$$\sum_{n \geq 1} a(n)t^n = \frac{1}{2} \left( \sqrt{\frac{1+t}{1-3t}} - 1 \right); \quad (1)$$

$$\sum_{n \geq 1} \bar{a}(n)t^n = \frac{1}{2} \left( \frac{1}{\sqrt{1-4t}} - 1 \right). \quad (2)$$

Even then, much remains unclear. The enumeration of directed animals on the honeycomb lattice is an open problem, and according to [9], the generating function is probably not D-finite; on the square and triangular lattices, comparatively very little is known when one tries to take into account parameters other than area.

Today, two enumeration methods account for almost every known result on directed animals. One of them is the *gas model* technique, originally used by Dhar [6]. This technique was further developed by Bousquet-Mélou [3]; see also [12, 1] for more recent work.

The method used in this paper is the second one, based on a correspondence, due to Viennot [13], between animals and other objects called *heaps of dominoes*. The basic idea is to replace each site of an animal by a  $2 \times 1$  domino, so that each domino either lies on the ground or sits on one or two other dominoes (Figure 2).

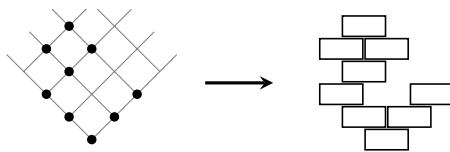


Figure 2: A directed animal on the square lattice can be turned into a heap by replacing each site by a  $2 \times 1$  domino.

As we will see later, this method works for the triangular lattice as well. However, no simple model of heaps of dominoes has been found to correspond to animals on the honeycomb lattice. This may explain the lack of knowledge on the subject.

The purpose of this paper is to study three other parameters of directed animals, introduced below, and illustrated in Figure 3:

- two sites of an animal on the square or triangular lattice are *adjacent* if they are of the form  $(i+1, j)$  and  $(i, j+1)$ . We denote by  $j(A)$  the number of pairs of adjacent sites of  $A$ .
- a *loop* consists of two adjacent sites  $(i+1, j)$  and  $(i, j+1)$ , along with a third site at  $(i+1, j+1)$ . We denote by  $\ell(A)$  the number of loops of  $A$ .
- a *neighbour* of an animal  $A$  of source  $S$  is a vertex  $v$  not in  $A$ , such that  $A \cup \{v\}$  is still a directed animal of source  $S$ . The number of neighbours of  $A$  is called the *site perimeter* of  $A$  and is denoted by  $p(A)$ .

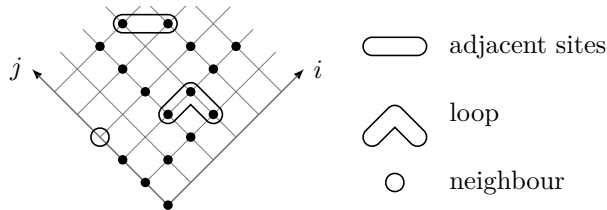


Figure 3: A directed animal on the square lattice with two adjacent sites, a loop, and a neighbour marked.

Taking, for instance, the site perimeter, we may consider the bivariate generating function counting single-source animals according to both area and perimeter on the square lattice:

$$A^p(t, u) = \sum_A t^{|A|} u^{p(A)}.$$

This generating function is not known, and is believed not to be D-finite [10]. Instead, we will consider the generating function giving the total number of neighbours in the animals of a fixed area:

$$\sum_A p(A) t^{|A|} = \frac{\partial A^p}{\partial u}(t, 1).$$

By dividing the total site perimeter of the animals of area  $n$  by the number of these animals, one gets the *average* site perimeter in animals of a fixed area. Alternatively, this generating function may be seen as counting single-source directed animals with a marked neighbour.

This function, and the ones that similarly give the average number of adjacent sites and loops, turns out to be easier to derive. Specifically, the value of the generating function counting the total number of loops of single-source animals on the square lattice was obtained by Bousquet-Mélou using gas model methods [3]:

$$\sum_A \ell(A) t^{|A|} = \frac{1}{2} \left( 1 - \frac{1 - 4t + t^2 + 4t^3}{\sqrt{1+t}(1-3t)^{3/2}} \right). \quad (3)$$

As for the total site perimeter on the square lattice, it was the object of a conjecture by Conway in 1996 [5]:

$$\sum_A p(A)t^{|A|} = \frac{1}{2t(1+t)} \left( -1 + t + t^2 + \frac{1 - 3t + 2t^2 + t^3 - 3t^4}{\sqrt{1+t}(1-3t)^{3/2}} \right). \quad (4)$$

Le Borgne [11] also conjectured the value of similar generating functions counting the site perimeter of animals on square and triangular lattices of bounded width.

In Section 4, we prove these conjectures and give a new proof of (3) using combinatorial methods; moreover, we show that the total number of adjacent sites is given by:

$$\sum_A j(A)t^{|A|} = \frac{1}{2t(1+t)} \left( 1 - \frac{1 - 4t + t^2 + 4t^3}{\sqrt{1+t}(1-3t)^{3/2}} \right). \quad (5)$$

Actually, our results are more general than that: the same methods can be used on different kinds of lattices, obtained by adding one or two vertical walls (the *half-lattice*, *cylindrical lattices* and *rectangular lattices*, defined in Section 3), and on animals with any fixed source.

Knowing, say, the total site perimeter of single-source animals of area  $n$  on the square lattice, we get their average perimeter by dividing by the number  $a(n)$  of these animals:

$$p(n) = \frac{1}{a(n)} \sum_{|A|=n} p(A).$$

This quantity may thus be computed using (1) and (4). The numbers of adjacent sites and loops are handled similarly. From these generating functions, singularity analysis [7] yields estimates on these quantities as  $n$  tends to infinity:

$$j(n) \sim \frac{n}{4}; \quad \ell(n) \sim \frac{n}{9}; \quad p(n) \sim \frac{3n}{4}.$$

The paper is organized as follows. In Section 2, we introduce in detail the notion of heaps of pieces and give several lemmas useful for animal enumeration. In Section 3, we enumerate directed animals of any source on several kinds of square and triangular lattices, according to area alone. In Section 4, we give a general method to derive the generating functions giving the average number of adjacent sites, number of loops, and site perimeter of directed animals on the square lattice, as well as counterparts of most of these results on the triangular lattice. We derive asymptotic results in Section 5. Finally, we illustrate our formulæ with a few examples in Section 6.

## 2 Heaps of pieces

The notion of heaps of pieces is due to Viennot, and this topic is covered in detail in [13]. We repeat the definitions for convenience, and make a few minor additions which we use later.

## 2.1 Basics

Intuitively, a heap is a finite set of pieces. It is built by dropping successively the pieces at certain positions, chosen from a given set. When the positions of two pieces overlap, the second piece falls on the first, like in Figure 2. A formal definition is given below.

**Definition 2.1.** Let  $Q$  be a set and  $\mathcal{C}$  a reflexive symmetric relation on  $Q$ . A *heap* of the model  $(Q, \mathcal{C})$  is a finite subset  $H$  of  $Q \times \mathbb{N}$  satisfying:

1. if  $(q, i)$  and  $(q', i)$  with  $q \neq q'$  are in  $H$ , then  $(q, q')$  is not in  $\mathcal{C}$ ;
2. if  $(q, i)$  is in  $H$  and  $i > 0$ , then there exists  $(q', i - 1)$  in  $H$  such that  $(q, q')$  is in  $\mathcal{C}$ .

The relation  $\mathcal{C}$  is called the *concurrency relation*, and two positions  $q$  and  $q'$  are *concurrent* if  $(q, q')$  is in  $\mathcal{C}$  (in the above intuitive definition, this means that they overlap). The elements of a heap are called *pieces*. If  $(q, i)$  is a piece of a heap,  $q$  is called its *position* and  $i$  its *height*.

The pieces of a heap are naturally equipped with a poset structure: define the relation  $\prec$  such that  $(q, i) \prec (q', i')$  whenever  $q$  and  $q'$  are concurrent and  $i < i'$ . Let  $\leq$  be the reflexive transitive closure of  $\prec$ . We say that a piece  $x$  is *below* a piece  $y$ , or  $y$  is *above*  $x$ , if  $x \leq y$ . The pieces  $x$  and  $y$  are *independent* if neither is above the other.

The partial order  $\leq$  may be viewed more intuitively: let  $H$  be a heap and  $x$  a piece of  $H$ . If one takes the piece  $x$  and push it upwards, it pushes along some pieces in the way. If one pushes it high enough, the moved pieces are exactly the pieces above  $x$ .

The pieces of a heap  $H$  that are minimal for the order  $\leq$  are called *minimal pieces*; they are exactly the pieces of height 0. The set of their positions is called the *base* of  $H$ , and denoted by  $b(H)$ . Likewise, the pieces that are maximal for  $\leq$  are called *maximal pieces*; we denote by  $m(H)$  the set of their positions.

We now define generating functions counting the heaps of a model; we denote by  $|H|$  the number of pieces of the heap  $H$ .

**Definition 2.2.** Let  $(Q, \mathcal{C})$  be a model of heaps and  $S$  a finite subset of  $Q$ . We denote by  $\mathcal{H}_S(t)$  and  $\mathcal{H}_{[S]}(t)$  the generating functions (provided they exist) of heaps respectively of base  $S$  and with base included in  $S$ :

$$\mathcal{H}_S(t) = \sum_{b(H)=S} t^{|H|}; \quad \mathcal{H}_{[S]}(t) = \sum_{b(H) \subseteq S} t^{|H|}.$$

These two generating functions are obviously linked by:

$$\mathcal{H}_{[S]}(t) = \sum_{T \subseteq S} \mathcal{H}_T(t);$$

Conversely, the inclusion-exclusion principle yields:

$$\mathcal{H}_S(t) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mathcal{H}_{[T]}(t).$$

Let us now assume that the set of positions  $Q$  of the model is finite. In this case, the generating functions above may be computed using a result due to Viennot [13], which we present below.

A heap is called *trivial* if all its pieces have height 0. This means that a trivial heap may be identified with the set of the positions of its pieces, which must be pairwise nonconcurrent. As  $Q$  is a finite set, there is only a finite number of trivial heaps. We denote by  $\mathcal{T}_{[S]}(t)$  the alternating generating function of trivial heaps included in  $S$ :

$$\mathcal{T}_{[S]}(t) = \sum_{\substack{T \subseteq S, \\ T \text{ trivial}}} (-t)^{|T|}.$$

Since  $Q$  is finite, this generating function is actually a polynomial, and is usually relatively easy to compute.

**Lemma 2.3** (Inversion Lemma). *Let  $(Q, \mathcal{C})$  be a finite heap model and  $S$  a subset of  $Q$ . The generating function of heaps of base included in  $S$  is:*

$$\mathcal{H}_{[S]}(t) = \frac{\mathcal{T}_{[Q \setminus S]}(t)}{\mathcal{T}_{[Q]}(t)}.$$

Thanks to this lemma, we see that in a finite model, the generating function  $\mathcal{H}_{[S]}(t)$  is a quotient of two polynomials, and is therefore rational.

## 2.2 Strict and inflated heaps

In this section, we define families of heaps which we use in the correspondence with directed animals. Let  $H$  be a heap, and let  $(q, i)$  and  $(q', i')$  be two pieces. We say that  $(q', i')$  *sits* on  $(q, i)$  if  $q$  and  $q'$  are concurrent and  $i' = i + 1$ . Thus, Condition 2 of Definition 2.1 states that any non-minimal piece must sit on another piece.

The objects obtained by relaxing this condition, keeping only Condition 1, are called *pre-heaps*.

**Definition 2.4.** A heap or pre-heap is *strict* if it has no piece sitting on another at the same position, i.e. no two pieces  $(q, i)$  and  $(q, i + 1)$ .

**Definition 2.5.** An *inflated heap* is a strict pre-heap  $H$ , such that for every piece  $(q, i)$  satisfying  $i > 0$ , at least one of the following pieces is in  $H$ :

- (a) a piece  $(q', i - 1)$ , such that  $q$  and  $q'$  are concurrent (and  $q \neq q'$ );
- (b) the piece  $(q, i - 2)$ .

Examples are found in Figure 7.

Let  $H$  be a heap. A *stack* of  $H$  is a maximal set of pieces all at the same position, with consecutive heights. Thus, a heap is strict if all its stacks have only one piece.

Any heap may be built in a unique manner from a strict heap by replacing each piece by a stack consisting of an arbitrary positive number of pieces; in



Figure 4: By replacing each piece of a strict heap by a stack of pieces, one gets a general heap; by inflating each stack, one gets an inflated heap.

turn, any inflated heap may be built from a heap by “inflating” each stack, pushing along all pieces above it (see Figure 4).

These remarks enable us to derive from Lemma 2.3 the generating functions of strict and inflated heaps. First, as inflating a stack does not change its base or number of pieces, the generating functions of inflated heaps are the same as those of general heaps.

**Notation 2.6.** Let  $(Q, \mathcal{C})$  be a model of heaps. We denote by  $H_S(t)$  and  $H_{[S]}(t)$  the generating functions of strict heaps with base  $S$  and base included in  $S$ , respectively<sup>1</sup>.

The construction of Figure 4 translates into a link between the generating functions of strict and general heaps:

$$\mathcal{H}_S(t) = H_S\left(\frac{t}{1-t}\right),$$

or equivalently:

$$H_S(t) = \mathcal{H}_S\left(\frac{t}{1+t}\right).$$

These links remain valid between generating functions of heaps with base included in  $S$ .

### 2.3 Factorized heaps

We now present a monoid structure, again due to Viennot, on the set of heaps of a given model. Let  $H$  be a heap and  $q$  a position. Let  $H \cdot q$  be the heap formed by dropping a piece at position  $q$  on top of  $H$ ; more formally, let  $H \cdot q$  be  $H \cup \{(q, i)\}$ , where  $i$  is the largest integer such that this is a heap.

In this way, a heap may be built one piece at a time. This may be done by adding the pieces in any order compatible with the partial order  $\leq$ . This idea is used to define the product of two heaps.

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<sup>1</sup>For the sake of clarity, all generating functions in this paper follow the same typographical pattern as Definitions 2.2 and 2.6: the generating functions of general (or inflated) heaps are denoted by calligraphic letters, while the ones of strict heaps are denoted by standard capital letters. Likewise, a subscript  $[S]$  in square brackets always indicates heaps with a base included in  $S$ , while a subscript  $S$  denotes heaps of base  $S$ .



**Definition 2.7.** Let  $H_1$  and  $H_2$  be two heaps. The product  $H_1 \cdot H_2$  is built by letting all pieces of  $H_2$  fall on  $H_1$ , in any order compatible with  $\leq$ .

A *factorized heap* is a heap  $H$ , with a distinguished factorization  $H = H_1 \cdot H_2$ . We denote such a heap  $(H = H_1 \cdot H_2)$  or  $(H_1 \cdot H_2)$ . A factorized heap  $(H = H_1 \cdot H_2)$  is *strict* if  $H$  is strict; it is *almost strict* if both  $H_1$  and  $H_2$  are strict.

The monoid structure induced by this product is isomorphic to the *partially commutative monoid* [4] on the alphabet  $Q$ , with concurrency relation  $\mathcal{C}$ . The product is illustrated in Figure 5.

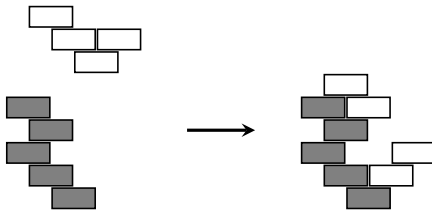


Figure 5: The product of two heaps is obtained by dropping the second heap on top of the first.

We now give a way to compute the base of a factorized heap. If  $H$  is a heap, let the *neighbourhood* of  $H$ , denoted  $v(H)$ , be the set of positions concurrent to at least one piece of  $H$ .

**Lemma 2.8.** Let  $(H = H_1 \cdot H_2)$  be a factorized heap. The base of the heap  $H$  is given by:

$$b(H) = b(H_1) \cup (b(H_2) \setminus v(H_1)).$$

*Proof.* As the heap  $H$  is built by dropping all pieces of  $H_1$ , then all pieces of  $H_2$ , no piece of  $H_1$  can be above a piece of  $H_2$ . Therefore, all minimal pieces of  $H_1$  are also minimal pieces in  $H$ .

A minimal piece of  $H_2$  is minimal in  $H$  if and only if it is not above a piece of  $H_1$ . This happens if and only if its position is not in the neighbourhood of  $H_1$ , hence the given formula.  $\square$

Given a heap  $H$  with base  $S$ , we may factorize  $H$  as  $H = S \cdot H_2$ . Lemma 2.8 asserts that the base of  $H_2$  is included in  $v(S)$ . This yields:

$$\mathcal{H}_S(t) = t^{|S|} \mathcal{H}_{[v(S)]}(t). \quad (6)$$

## 2.4 Heaps marked with a set of pieces

A number of our problems in animal enumeration can be seen as enumeration of heaps marked with a set of pieces; for example, computing the average number of adjacent sites in animals is linked to enumerating animals with two adjacent sites marked, which is in turn linked to enumerating heaps with some pieces marked. Here, we give a means to link such marked heaps to factorized heaps, which prove to be more manageable.

**Definition 2.9.** A *marked heap*  $(H, X)$  is a heap  $H$ , marked with a set  $X$  of *pairwise independent pieces*.

**Definition 2.10.** Let  $(H, X)$  be a marked heap. Let  $\mathcal{F}_\downarrow(H, X)$  be the factorized heap  $(H = H_1 \cdot H_2)$  where  $H_1$  consists of all pieces below at least a piece of  $X$ . Let  $\mathcal{F}_\uparrow(H, X)$  be the factorized heap  $(H = H_1 \cdot H_2)$  where  $H_2$  consists of all pieces above at least a piece of  $X$ .

**Definition 2.11.** An *almost strict marked heap* is a marked heap  $(H, X)$  such that no piece of  $H$  sits on an unmarked piece at the same position.

We call *marked stack* a stack of an almost strict marked heap containing a marked piece; such a stack may have one or two pieces, and the marked piece is always the lowest piece of the stack.

**Definition 2.12.** Let  $(H, X)$  be an almost strict marked heap. Let  $X^+$  be the set consisting of the highest piece of each marked stack. Define the following factorized heaps:

$$\begin{aligned} F_\downarrow(H, X) &= \mathcal{F}_\downarrow(H, X); \\ F_\uparrow(H, X) &= \mathcal{F}_\uparrow(H, X^+). \end{aligned}$$

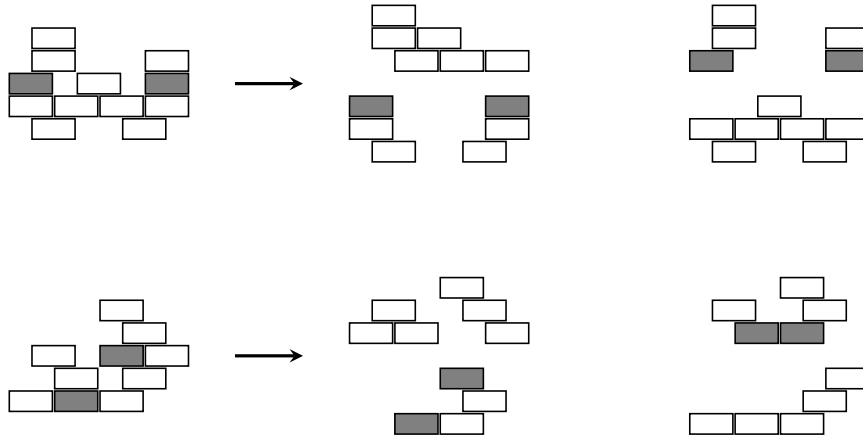


Figure 6: A marked heap and an almost strict marked heap (left), their image by  $\mathcal{F}_\downarrow$  and  $F_\downarrow$  (middle) and by  $\mathcal{F}_\uparrow$  and  $F_\uparrow$  (right).

**Lemma 2.13.** The mappings  $\mathcal{F}_\downarrow$  and  $\mathcal{F}_\uparrow$  (resp.  $F_\downarrow$  and  $F_\uparrow$ ) are bijections from the set of marked heaps to the set of factorized heaps (resp. almost strict marked heaps to almost strict factorized heaps).

Their inverse bijections are as follows. Let  $(H = H_1 \cdot H_2)$  be a factorized heap, let  $X$  be the set of maximal pieces of  $H_1$  and  $Y$  the set of minimal pieces of  $H_2$ . We have:

$$(H_1 \cdot H_2) = \mathcal{F}_\downarrow(H, X) = \mathcal{F}_\uparrow(H, Y).$$

If  $(H = H_1 \cdot H_2)$  is an almost strict factorized heap, let  $Y^-$  be the set consisting of the lowest piece of each stack of  $H$  containing a piece of  $Y$ . We have:

$$(H_1 \cdot H_2) = F_{\downarrow}(H, X) = F_{\uparrow}(H, Y^-).$$

*Proof.* Let us first do the case of marked heaps. This fact easily stems from the poset structure on the pieces of a heap: the set  $X$  of maximal pieces of  $H_1$  is the only set of pairwise independent pieces such that every piece of  $H_1$  is below a piece of  $X$ . The case of  $\mathcal{F}_{\uparrow}$  is identical.

We handle almost strict marked heaps similarly. Pulling downwards the lowest piece of each marked stack, or pushing upwards the highest piece, ensures that  $H_1$  and  $H_2$  are strict in the resulting factorized heap; conversely, as the pieces of  $X$  and  $Y^-$  are the lowest of their stacks,  $(H, X)$  and  $(H, Y^-)$  are almost strict marked heaps.  $\square$

As a first application, we give a means to compute the generating functions of heaps marked with one piece at a fixed position.

**Definition 2.14.** Let  $q$  be a position and  $S$  a set of positions. We denote by  $\mathcal{H}_{[S]}^{(q)}(t)$  the generating function of heaps with base included in  $S$ , marked with a piece at position  $q$ . We denote by  $\mathcal{V}_{[S]}^q(t)$  the set of heaps with base included in  $S$  avoiding  $q$ , i.e. such that no piece is concurrent to  $q$ . As usual, we use analogous notations for strict heaps.

**Lemma 2.15.** *The generating functions counting heaps of base included in  $S$  marked with a piece at position  $q$  is given by:*

$$\mathcal{H}_{[S]}^{(q)}(t) = \begin{cases} \mathcal{H}_{[S]}(t)\mathcal{H}_{\{q\}}(t) & \text{if } q \in S, \\ (\mathcal{H}_{[S]}(t) - \mathcal{V}_{[S]}^q(t))\mathcal{H}_{\{q\}}(t) & \text{otherwise;} \end{cases}$$

$$H_{[S]}^{(q)}(t) = \frac{1}{1+t} \begin{cases} H_{[S]}(t)H_{\{q\}}(t) & \text{if } q \in S, \\ (H_{[S]}(t) - V_{[S]}^q(t))H_{\{q\}}(t) & \text{otherwise.} \end{cases}$$

*Proof.* Let  $(H, x)$  be a heap with a marked piece at position  $q$ . We use the bijection  $\mathcal{F}_{\uparrow}$  to turn it into a factorized heap  $(H_1 \cdot H_2)$ . We know that  $H_2$  has base  $\{q\}$ , and that  $H_1 \cdot H_2$  has base included in  $S$ . According to Lemma 2.8, we have:

$$b(H_1 \cdot H_2) = b(H_1) \cup (\{q\} \setminus v(H_1)).$$

If  $q$  is in  $S$ , this simply means that  $b(H_1)$  is included in  $S$ ; if not, it means that  $q$  must be in  $v(H_1)$  as well, so that  $H_1$  must not avoid  $q$ . We thus get the result for general heaps.

Let  $H_{[S]}^{(q)*}(t)$  be the generating function of almost strict marked heaps, with exactly one marked piece at position  $q$ . The bijection  $\mathcal{F}_{\uparrow}$  turns these heaps into almost strict factorized heaps, on which we apply the same reasoning. Moreover, as only one piece may be duplicated, we have the identity:

$$H_{[S]}^{(q)*}(t) = (1+t)H_{[S]}^{(q)}(t).$$

This yields the second formula.  $\square$

### 3 Directed animals and heaps of dominoes

#### 3.1 Definitions

**Definition 3.1.** Let  $Q$  be either a subset of  $\mathbb{Z}$  or of the form  $\mathbb{Z}/m\mathbb{Z}$  with  $m$  an even number. The *square lattice* over  $Q$ , denoted by  $\Gamma_Q$ , is the oriented graph with vertices  $(q, i) \in Q \times \mathbb{N}$  such that  $q+i$  is even, and edges  $(q, i) \rightarrow (q+1, i+1)$  and  $(q, i) \rightarrow (q-1, i+1)$ . The *triangular lattice* over  $Q$ , denoted by  $\Delta_Q$ , is  $\Gamma_Q$  with additional edges  $(q, i) \rightarrow (q, i+2)$ .

**Definition 3.2.** Let  $Q$  be a subset or quotient of  $\mathbb{Z}$  in the same conditions as above. Let  $\mathcal{C}$  be the relation defined by  $(q, q') \in \mathcal{C}$  if and only if  $|q - q'| \leq 1$ . The model of heaps  $(Q, \mathcal{C})$  is called the model of *heaps of dominoes* with set of positions  $Q$ .

Up to a translation, there are four kinds of models and associated lattices:

- the *full model* is the model  $Q = \mathbb{Z}$ ;
- the *half model* is the model  $Q = \mathbb{N}$ ;
- the *cylindrical model* of width  $m$  is the model  $Q = \mathbb{Z}/m\mathbb{Z}$ , with  $m$  even;
- the *rectangular model* of width  $m$  is the model  $Q = \{0, \dots, m-1\}$ .

As seen in Figure 7, directed animals of source  $S$  on the square lattice  $\Gamma_Q$  are identical to strict heaps of dominoes of base  $S$  in the model  $(Q, \mathcal{C})$ , while directed animals on the triangular lattice  $\Delta_Q$  are inflated heaps.

We say that a heap  $H$  of dominoes is *aligned* if all its pieces  $(q, i)$  are such that  $q+i$  is even. In particular, all heaps and inflated heaps corresponding to directed animals are aligned.

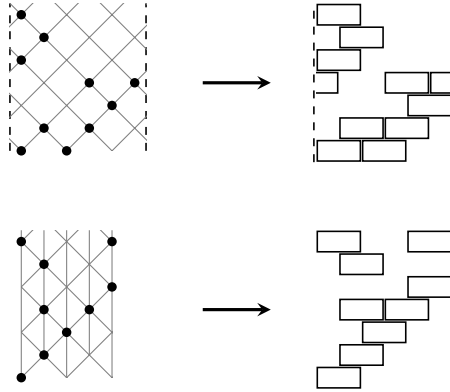


Figure 7: An animal on the square cylindrical lattice of width 6 and the triangular rectangular lattice of width 5, and their corresponding heaps of dominoes.

Let  $\Gamma_Q$  be a square lattice and  $\Delta_Q$  its associated triangular lattice; let  $S$  be a one-line source, i.e., a set of vertices of the form  $(q, 0)$ . We denote by  $A_S(t)$  and  $\mathcal{A}_S(t)$  the generating functions of animals of source  $S$  in the lattices  $\Gamma_Q$  and  $\Delta_Q$ , respectively. These two generating functions also count strict and inflated heaps of base  $S$  in the model of heaps of dominoes  $Q$ .

In this section, we give means to compute  $\mathcal{A}_S(t)$ ; the generating function  $A_S(t)$  is then given by performing the substitution  $t \mapsto \frac{t}{1+t}$ . Of course, we compute in the same way the generating functions  $\mathcal{A}_{[S]}(t)$  and  $A_{[S]}(t)$  of animals with a source included in  $S$ .

### 3.2 Bounded lattices

The bounded lattices correspond to finite models of heaps of dominoes, that is, the cylindrical models and the rectangular models. Let  $Q$  be a finite model and  $S$  be a set of positions. The identity (6) and Lemma 2.3 give the value of the generating function of heaps of base  $S$ :

$$\mathcal{A}_S(t) = t^{|S|} \frac{\mathcal{T}_{[Q \setminus v(S)]}(t)}{\mathcal{T}_{[Q]}(t)}.$$

All we need is therefore to compute the generating functions of trivial heaps. To do this, we define two sequences of polynomials.

**Definition 3.3.** Define the sequences  $(F_m(t))_{m \geq 0}$  and  $(\widehat{F}_m(t))_{m \geq 2}$  of polynomials by  $F_0(t) = 1$ ,  $F_1(t) = 1 - t$ , and for all  $m \geq 2$ :

$$\begin{aligned} F_m(t) &= F_{m-1}(t) - tF_{m-2}(t); \\ \widehat{F}_m(t) &= F_{m-1}(t) - tF_{m-3}(t). \end{aligned}$$

The polynomials  $F_m(t)$  are often called the *Fibonacci polynomials*. With these polynomials, we can compute the generating function  $\mathcal{T}_{[S]}(t)$ , in any finite model  $Q$  and for any set  $S$ , using the two lemmas below. We state them without proof, and refer to [13] for more detail. Examples are also given in Section 6.

**Lemma 3.4.** *Let  $m \geq 0$ . The generating function  $\mathcal{T}_{[Q]}(t)$  of trivial heaps in the rectangular model of width  $m$  is  $F_m(t)$ . If  $m$  is even, the generating function of trivial heaps in the cylindrical model of width  $m$  is  $\widehat{F}_m(t)$ .*

**Lemma 3.5.** *Let  $S$  be a finite set of positions; write  $S = S_1 \cup \dots \cup S_k$ , where the  $S_i$  are intervals of  $\mathbb{Z}$ , with  $k$  minimal. The generating function of trivial heaps included in  $S$  is:*

$$\mathcal{T}_{[S]}(t) = F_{|S_1|}(t) \cdots F_{|S_k|}(t).$$

We now give explicit formulæ for the special case  $S = \{0\}$ . In the following, we call *zero-source animals* the animals of source  $\{0, 0\}$  in any model. Let  $\mathcal{A}_m(t)$

and  $\mathcal{D}_m(t)$  be the generating functions of zero-source animals in the cylindrical and rectangular triangular lattices of width  $m$ , respectively. We have:

$$\mathcal{A}_m(t) = \frac{F_{m-1}(t)}{\widehat{F}_m(t)} - 1; \quad (7)$$

$$\mathcal{D}_m(t) = \frac{F_{m-1}(t)}{F_m(t)} - 1. \quad (8)$$

The generating functions  $A_m(t)$  and  $D_m(t)$  counting zero-source animals on the square lattices are derived by performing the substitution  $t \mapsto \frac{t}{1+t}$ .

### 3.3 Unbounded lattices

We now address the unbounded lattices, i.e. the full and half lattices. We start with zero-source animals, defined above, which are simplest. In the rectangular and half models, such animals are also called *half-animals*.

**Definition 3.6.** Let  $A(t)$  and  $D(t)$  be the generating functions of zero-source animals on the full square lattice and the half square lattice, respectively. Let  $\mathcal{A}(t)$  and  $\mathcal{D}(t)$  be their counterparts on the triangular lattices.

These four generating functions are given below (see [6, 8, 2]):

**Proposition 3.7.** *The generating functions of zero-source animals on the infinite models are:*

$$\mathcal{A}(t) = \frac{1}{2} \left( \frac{1}{\sqrt{1-4t}} - 1 \right); \quad (9)$$

$$\mathcal{D}(t) = \frac{1 - \sqrt{1-4t}}{2t} - 1; \quad (10)$$

$$A(t) = \frac{1}{2} \left( \sqrt{\frac{1+t}{1-3t}} - 1 \right); \quad (11)$$

$$D(t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t}. \quad (12)$$

Of course, as usual, the generating functions counting animals on the square lattices can be obtained by performing the substitution  $t \mapsto \frac{t}{1+t}$  in the ones counting animals on the triangular lattices.

**Definition 3.8.** In the full model, a *compact source* is a finite set of consecutive even positions. In the half model, a compact source is a finite set of consecutive even positions, starting at 0.

The next result gives the generating function of animals with a given compact source. The proof may be found in [2].

**Proposition 3.9.** *Let  $C$  be a compact source with  $k$  sites. The generating function of animals of source  $C$  on the full triangular lattice is:*

$$\mathcal{A}_C(t) = \mathcal{D}(t)^{k-1} \mathcal{A}(t).$$

*On the half lattice, this generating function is:*

$$\mathcal{A}_C(t) = \mathcal{D}(t)^k.$$

*Remark.* From this, it can be proved that the number of animals of area  $n$  with any compact source on the triangular lattice is  $4^{n-1}$ , and  $3^{n-1}$  on the square lattice. See [8, 2].

Finally, we are able to compute the generating function of animals with an arbitrary source  $S$ .

**Proposition 3.10.** *Let  $Q = \mathbb{Z}$  or  $\mathbb{N}$ . Let  $S \subseteq Q$  be a one-line source and  $C$  be the smallest compact source containing  $S$ . The generating function of animals of source  $S$  in  $\Delta_Q$  is:*

$$\mathcal{A}_S(t) = t^{|S|-|C|} \mathcal{T}_{[v(C) \setminus v(S)]}(t) \mathcal{A}_C(t).$$

*Proof.* Let us address the full lattice. Let  $Q_m$  be a cylindrical model large enough so that  $C \subseteq Q_m$ . The generating function of heaps of base  $S$  in the model  $Q_m$  is:

$$\mathcal{A}_{S,m}(t) = t^{|S|} \frac{\mathcal{T}_{[Q_m \setminus v(S)]}(t)}{\mathcal{T}_{[Q_m]}(t)}.$$

As  $C$  is the smallest compact source containing  $S$ ,  $v(C)$  is also the smallest interval containing  $v(S)$ , which ensures that no position of  $Q_m \setminus v(C)$  can be concurrent to  $v(C) \setminus v(S)$ . Therefore, Lemma 3.5 entails that:

$$\mathcal{T}_{[Q_m \setminus v(S)]}(t) = \mathcal{T}_{[v(C) \setminus v(S)]}(t) \mathcal{T}_{[Q_m \setminus v(C)]}(t),$$

and thus:

$$\mathcal{A}_{S,m}(t) = t^{|S|-|C|} \mathcal{T}_{[v(C) \setminus v(S)]}(t) \mathcal{A}_{C,m}(t).$$

We conclude by letting  $m$  tend to infinity.

In the case of the half-lattice, we repeat the same reasoning, this time taking for  $Q_m$  a rectangular model large enough so that  $C \subseteq Q_m$ .  $\square$

## 4 Average number of adjacent sites, loops and neighbours of directed animals

### 4.1 Notations and results

In this section,  $\Gamma$  is a square lattice (full, half, cylindrical or rectangular), and  $S$  is a one-line source of  $\Gamma$ . In Section 3, we have computed the generating

functions  $A_S(t)$  and  $A_{[S]}(t)$  counting directed animals on  $\Gamma$  of source  $S$  and with a source included in  $S$ .

We are now interested in three parameters of the directed animals: number of adjacent sites, number of loops, and site perimeter. The first two are defined in Section 1; we denote by  $j(A)$  and  $\ell(A)$  the number of pairs of adjacent sites and number of loops of the animal  $A$ , respectively.

**Definition 4.1.** Let  $A$  be an animal on  $\Gamma$  with a source included in  $S$ . An  $S$ -neighbour of  $A$  is a vertex  $v$  of  $\Gamma$  not in  $A$  such that  $A \cup \{v\}$  is still a directed animal with a source included in  $S$ .

Assume now that the graph  $\Gamma$  is embedded in a larger graph  $\Gamma'$ . An *internal*  $S$ -neighbour of  $A$  is an  $S$ -neighbour of  $A$  seen as an animal on  $\Gamma$ . An *external*  $S$ -neighbour of  $A$  is an  $S$ -neighbour of  $A$  seen as an animal on  $\Gamma'$ .

Finally, a vertex of  $\Gamma$  is at the *edge* of the lattice if it has outdegree 1.

For the purpose of this definition, we regard the half-lattice and the rectangular lattices as embedded in the full lattice. The full and cylindrical lattices are not naturally embedded in any larger graph, so in these lattices internal and external neighbours are identical. Moreover, if  $S$  is the source of  $A$ , the  $S$ -neighbours of  $A$  coincide with the usual neighbours of  $A$ , as defined in Section 1.

Assuming no ambiguity on the set  $S$ , we denote by  $p_i(A)$  the number of internal  $S$ -neighbours of  $A$  (or internal site perimeter) and by  $p_e(A)$  its number of external  $S$ -neighbours (or external site perimeter). We also denote by  $e(A)$  the number of sites of  $A$  at the edge of the lattice  $\Gamma$ .

The generating functions defined below are linked to the average value of each parameter in animals of a given area.

**Definition 4.2.** Define the following generating functions, counting animals with a source included in  $S$ :

- Let  $J_{[S]}(t)$  be the generating function of animals marked with two adjacent sites.
- Let  $L_{[S]}(t)$  be the generating function of animals marked with a loop.
- Let  $P_{[S]}^e(t)$  be the generating function of animals marked with an external  $S$ -neighbour.
- Let  $P_{[S]}^i(t)$  be the generating function of animals marked with an internal  $S$ -neighbour.

Also define  $J_S(t)$ ,  $L_S(t)$ ,  $P_S^e(t)$  and  $P_S^i(t)$  the analogous generating functions of animals of source  $S$ .

To compute these generating functions, we again use the correspondence between animals and heaps of dominoes. We denote by  $(Q, \mathcal{C})$  the model of heaps of dominoes associated with the lattice  $\Gamma$ ; we also denote by  $S$  the aligned set of positions associated to the source  $S$ .



We now define some generating functions counting heaps. As usual, a generating function with a subscript  $[S]$  counts heaps with a base included in  $S$ , and one with a subscript  $S$  counts heaps with base  $S$ , so this precision will often be omitted from the definition.

In some cases, we consider heaps having a minimal piece outside  $S$ ; we call such a piece an *illegal minimal piece*. Note that an illegal minimal piece does not have to be aligned with  $S$ .

**Definition 4.3.** Define the following generating functions:

- Let  $M_{[S]}(t)$  be the generating function of strict heaps marked with a piece at a position  $q$ , such that  $q + 2$  is in  $Q$ .
- If  $q$  is in  $S$ , let  $W_{[S]}^q(t)$  be the generating function of strict heaps with a minimal piece at position  $q$  and an illegal minimal piece at  $q + 2$ . Let  $W_{[S]}(t)$  be the sum of  $W_{[S]}^q(t)$  over all  $q \in S$  such that  $q + 2 \notin S$ .
- Let  $E_{[S]}(t)$  be the generating function of strict heaps marked with a piece at a position  $q$ , such that either  $q - 1$  or  $q + 1$  is not in  $Q$ .

Also define the analogues  $M_S(t)$ ,  $W_S(t)$  and  $E_S(t)$ ; thus,  $W_S(t)$  is the generating functions of strict heaps of base  $S \cup \{q + 2\}$ , such that  $q$  is in  $S$  but  $q + 2$  is not.

We now show that these generating functions can be computed using previous results. First, we have:

$$W_{[S]}(t) = \sum_{\substack{q \in S \\ q+2 \notin S}} \sum_{\substack{T \subseteq S \\ q \in T}} A_{T \cup \{q+2\}}(t), \quad (13)$$

where  $A_{T \cup \{q+2\}}(t)$  counts strict heaps with base  $T \cup \{q + 2\}$ , and is computed using the results of Section 3. Next, the generating function of strict heaps with base included in  $S$  marked with a single piece is  $tA'_{[S]}(t)$ . Thus, we have:

$$M_{[S]}(t) = tA'_{[S]}(t) - \sum_{q+2 \notin Q} A_{[S]}^{(q)}(t); \quad (14)$$

$$E_{[S]}(t) = \sum_{q-1 \notin Q \text{ or } q+1 \notin Q} A_{[S]}^{(q)}(t), \quad (15)$$

where  $A_{[S]}^{(q)}(t)$  counts strict heaps with base included in  $S$  marked with a piece at position  $q$  and is computed using Lemma 2.15. In all three equations, the sum goes over a finite number of positions  $q$ , which ensures that all three generating functions can be computed. In this regard, the full and cylindrical models are the simplest, as  $M_{[S]}(t)$  is equal to  $tA'_{[S]}(t)$  and  $E_{[S]}(t)$  is zero.

**Theorem 4.4.** *In square lattices, the generating functions counting the total number of adjacent pieces, loops and site perimeters of the animals with source*

included in  $S$  are given by:

$$J_{[S]}(t) = \frac{tM_{[S]}(t) - W_{[S]}(t)}{1+t}; \quad (16)$$

$$L_{[S]}(t) = t(1+t)J_{[S]}(t); \quad (17)$$

$$P_{[S]}^e(t) = |S|A_{[S]}(t) + tA'_{[S]}(t) - J_{[S]}(t); \quad (18)$$

$$P_{[S]}^i(t) = P_{[S]}^e(t) - E_{[S]}(t). \quad (19)$$

Moreover, the corresponding generating functions for animals of source  $S$  are:

$$J_S(t) = \frac{tM_S(t) + j(S)A_S(t) - W_S(t)}{1+t}; \quad (20)$$

$$L_S(t) = t(1+t)J_S(t); \quad (21)$$

$$P_S^e(t) = |S|A_S(t) + tA'_S(t) - J_S(t); \quad (22)$$

$$P_S^i(t) = P_S^e(t) - E_S(t), \quad (23)$$

where  $j(S)$  denotes the number of pairs of adjacent sites in the source  $S$ .

Applications of this theorem are given in Section 6.

## 4.2 Site perimeters

We first prove the four identities (18), (19), (22) and (23) dealing with the internal and external site perimeter.

First, we remark that a vertex  $(q, i)$  of  $\Gamma$  has outdegree 1 if and only if either  $q-1$  or  $q+1$  is not in  $Q$ . Thus, the generating function  $E_{[S]}(t)$  satisfies:

$$E_{[S]}(t) = \sum_A e(A)t^{|A|}, \quad (24)$$

where the sum goes over all animals of source included in  $S$ . The same goes for the generating function  $E_S(t)$ .

**Lemma 4.5.** *The number of external and internal  $S$ -neighbours of every directed animal  $A$  with source included in  $S$  satisfy:*

$$\begin{aligned} p_e(A) &= |S| + |A| - j(A); \\ p_i(A) &= p_e(A) - e(A). \end{aligned}$$

By summing the identities of this lemma over all animals of source included in  $S$ , and using (24), we prove the identities (18) and (19). By summing them over all animals of source  $S$ , we get (22) and (23).

*Proof.* When dealing with the external site perimeter, the lattice  $\Gamma$  is embedded in a lattice  $\Gamma'$  which is either the full lattice or a cylindrical lattice. Let  $Z$  be the number of pairs of vertices  $(v, w)$  such that  $v$  is a site of  $A$  and  $w$  is a *child*

of  $v$  (i.e.,  $v \rightarrow w$  is an edge of  $\Gamma'$ ), whether in  $A$  or not. As every vertex has outdegree 2, we have

$$Z = 2|A|.$$

Now, as  $A$  is a directed animal, a child of a site of  $A$  is either a site of  $A$  or an external  $S$ -neighbour of  $A$ . The only sites and  $S$ -neighbours of  $A$  not counted are the ones in  $S$ ; moreover, two sites have a child in common if and only if they are adjacent. Hence:

$$Z = |A| + p_e(A) + j(A) - |S|,$$

which yields the announced formula for  $p_e(A)$ .

If  $\Gamma$  is either the half-lattice or a rectangular lattice, then each site on the edge of the lattice has one external neighbour not in  $\Gamma$ . Thus, we have:

$$p_e(A) = p_i(A) + e(A). \quad \square$$

### 4.3 Average number of adjacent sites and loops

To prove the remaining identities of Theorem 4.4, dealing with the number of adjacent sites and loops, we use several bijections between various sets of heaps marked with certain pieces. Rather than strict marked heaps, it is convenient here to use almost strict marked heaps (see Definition 2.11).

Although the proof seems complicated due to the high number of sets and associated generating functions that we must consider, each bijection is actually very simple, consisting in adding/removing a single piece.

Each of these sets of heaps is illustrated in Figures 8 and 9 with the relevant bijections.

**Definition 4.6.** Define the following sets of almost strict marked heaps, assumed to have a base included in  $S$ :

- Let  $\mathbb{J}_{[S]}^*$  be the set of almost strict heaps marked with two adjacent pieces.
- Let  $\mathbb{L}_{[S]}^*$  be the set of almost strict heaps marked with the top piece of a loop.
- Let  $\mathbb{M}_{[S]}^*$  be the set of almost strict heaps marked with a piece at a position  $q$ , such that  $q + 2$  is in  $Q$ .
- Let  $\mathbb{W}_{[S]}^*$  be the set of almost strict heaps marked with a minimal piece at a position  $q$ , and having an illegal minimal piece at position  $q + 2$ .
- Let  $\mathbb{I}_{[S]}^{2*}$  (resp.  $\mathbb{I}_{[S]}^{3*}$ ) be the set of almost strict heaps marked with two independent pieces at positions  $q$  and  $q + 2$  (resp.  $q + 3$ ).
- Let  $\mathbb{X}_{[S]}^{2*}$  (resp.  $\mathbb{X}_{[S]}^{3*}$ ) be the set of almost strict heaps marked with a piece  $x$  at a position  $q$ , and having an illegal minimal piece at position  $q + 2$  (resp.  $q + 3$ ) independent from  $x$ .

Let  $J_{[S]}^*(t)$ ,  $L_{[S]}^*(t)$ ,  $M_{[S]}^*(t)$ ,  $W_{[S]}^*(t)$ ,  $I_{[S]}^{2*}(t)$ ,  $I_{[S]}^{3*}(t)$ ,  $X_{[S]}^{2*}(t)$ ,  $X_{[S]}^{3*}(t)$  be the generating functions of these sets.

The asterisks  $*$  are used above to denote sets of almost strict marked heaps; we link these heaps to strict marked heaps later. Note that we have the inclusions  $\mathbb{J}_{[S]}^* \subseteq \mathbb{I}_{[S]}^{2*}$  and  $\mathbb{W}_{[S]}^* \subseteq \mathbb{X}_{[S]}^{2*}$ .

**Lemma 4.7.** *The following identity holds:*

$$L_{[S]}^*(t) = tJ_{[S]}^*(t).$$

*Proof.* To prove this result, we use a first bijection, which removes exactly one piece:

$$\Phi_0: \mathbb{L}_{[S]}^* \rightarrow \mathbb{J}_{[S]}^*.$$

Let  $(H, \{x\})$  be a heap of  $\mathbb{L}_{[S]}^*$ . Pull the marked piece  $x$  downwards to form the factorized heap  $(H_1 \cdot H_2) = F_{\downarrow}(H, \{x\})$ , such that  $x$  is the only maximal piece of  $H_1$ . Let  $H_1 = H'_1 \cdot x$  and  $H' = H'_1 \cdot H_2$ . As  $x$  is a loop, the heap  $H'_1$  has two maximal pieces  $y$  and  $z$ , which are adjacent (see Figure 8, left). Moreover, as  $H_1$  and  $H'_1$  have same base and neighbourhood, Lemma 2.8 guarantees that  $H$  and  $H'$  have the same base. We may thus set:

$$\Phi_0(H, \{x\}) = (H', \{y, z\}).$$

This operation is easily reversible, by putting back the piece  $x$ .  $\square$

**Lemma 4.8.** *The following identity holds:*

$$I_{[S]}^{2*}(t) - J_{[S]}^*(t) = tI_{[S]}^{3*}(t).$$

*Proof.* We again prove this result with a bijection removing one piece:

$$\Phi_1: \mathbb{I}_{[S]}^{2*} \setminus \mathbb{J}_{[S]}^* \rightarrow \mathbb{I}_{[S]}^{3*},$$

Let  $(H, \{x, y\})$  be a heap of  $\mathbb{I}_{[S]}^{2*} \setminus \mathbb{J}_{[S]}^*$ . We set  $(H_1 \cdot H_2) = F_{\downarrow}(H, \{x, y\})$ , pulling the pieces  $x$  and  $y$  downwards.

As the pieces  $x$  and  $y$  are not adjacent, one of them (say,  $x$ ) is higher than the other. Let  $H_1 = H'_1 \cdot x$ . As  $H_1$  is strict,  $H'_1$  must have a second maximal piece  $z$ , such that the positions of  $y$  and  $z$  are at distance 3 (Figure 8, middle). Let  $H' = H'_1 \cdot H_2$ . As  $H_1$  and  $H'_1$  have same base and neighbourhood, we may set:

$$\Phi_1(H, \{x, y\}) = (H', \{y, z\}).$$

This operation is reversible: let  $(H', \{y, z\})$  be a heap of  $\mathbb{I}_{[S]}^{3*}$ , and let  $(H_1 \cdot H_2) = F_{\downarrow}(H', \{y, z\})$ . As the set of positions  $S$  is aligned, the heap  $H'_1$  is also aligned. Therefore,  $y$  and  $z$  cannot be at the same height; say,  $z$  is higher. We set  $H_1 = H'_1 \cdot x$  so that  $x$  sits on  $z$  and the positions of  $x$  and  $y$  are at distance 2, and  $H = H_1 \cdot H_2$ ; thus, we have  $\Phi_1(H, \{x, y\}) = (H', \{y, z\})$ .  $\square$

**Lemma 4.9.** *The following identity holds:*

$$X_{[S]}^{2*}(t) - W_{[S]}^*(t) = tX_{[S]}^{3*}(t).$$

The generating function  $W_T(t)$  may be found in Definition 4.3.

*Proof.* We use another bijection removing one piece:

$$\Phi_2: \mathbb{X}_{[S]}^{2*} \setminus \mathbb{W}_{[S]}^* \rightarrow \mathbb{X}_{[S]}^{3*},$$

Let  $(H, \{x\})$  be a heap of  $\mathbb{X}_{[S]}^{2*} \setminus \mathbb{W}_{[S]}^*$ , and let  $y$  be the illegal minimal piece of  $H$ . We pull the pieces  $x$  and  $y$  downwards, forming the factorized heap  $(H_1 \cdot H_2)$ . As  $x$  is not a minimal piece of  $H_1$ , it must sit on another piece  $z$ , at position  $q - 1$  (Figure 8, right). Let  $H_1 = H'_1 \cdot x$  and  $H' = H'_1 \cdot H_2$ . Again,  $H_1$  and  $H'_1$  have the same base and neighbourhood, so that  $y$  is still the only illegal minimal piece of  $H'$ . We set:

$$\Phi_2(H, \{x\}) = (H', \{z\}).$$

This operation is easily reversible by putting back the piece  $x$ .  $\square$

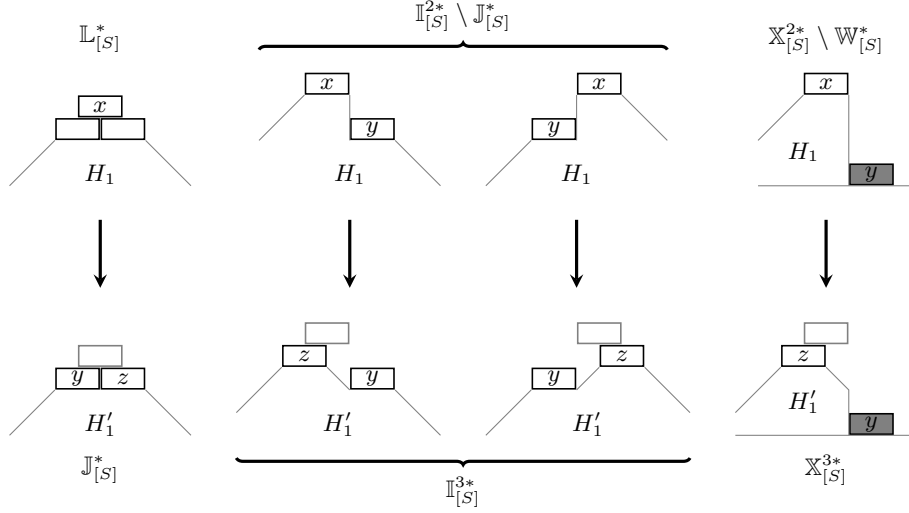


Figure 8: On the left, the bijection  $\Phi_0$ : removing the piece  $x$  uncovers two adjacent pieces  $y$  and  $z$ . In the middle, the bijection  $\Phi_1$ : removing the piece  $x$  uncovers a piece  $z$ , with position at distance 3 from  $y$  and higher than  $y$ . On the right, the bijection  $\Phi_2$ , with the illegal minimal piece  $y$  colored gray. Removing the piece  $x$  uncovers a piece  $z$ , with position at distance 3 from  $y$ .

**Lemma 4.10.** *The following identity holds:*

$$I_{[S]}^{2*}(t) + X_{[S]}^{2*}(t) = t \left( M_{[S]}^*(t) + I_{[S]}^{3*}(t) + X_{[S]}^{3*}(t) \right).$$

*Proof.* We use a fourth and final bijection removing one piece:

$$\Phi_3: \mathbb{I}_{[S]}^{2*} \cup \mathbb{X}_{[S]}^{2*} \rightarrow \mathbb{M}_{[S]}^* \cup \mathbb{I}_{[S]}^{3*} \cup \mathbb{X}_{[S]}^{3*}.$$

In this proof, if  $x$  is a piece, we write  $x^+$  to denote the highest piece of the stack of  $x$ , and  $x^-$  to denote the lowest piece in the stack of  $x$  (see Section 2.4).

Let  $H$  be a heap of  $\mathbb{I}_{[S]}^{2*}$  or  $\mathbb{X}_{[S]}^{2*}$ . In the first case, let  $x$  be the left-hand marked piece, and  $y$  the right-hand one. In the second, let  $x$  be the marked piece and  $y$  the illegal maximal piece. In both cases, we set  $(H_1, H_2) = F_{\uparrow}(H, \{x, y\})$ , pushing upwards the pieces  $x^+$  and  $y^+$ .

Now, let  $H_2 = y^+ \cdot H'_2$  and  $H' = H_1 \cdot H'_2$ . We distinguish three cases, illustrated in Figure 9:

- (a) The piece  $x^+$  is the only minimal piece of  $H'_2$ : the heap  $(H', \{x\})$  is in  $\mathbb{M}_{[S]}^*$ .
- (b) The heap  $H'_2$  has a minimal piece  $z$  at position  $q+3$ , and  $z$  is not an illegal minimal piece of  $H'$ : the heap  $(H', \{x, z^-\})$  is in  $\mathbb{I}_{[S]}^{3*}$ .
- (c) The heap  $H'_2$  has a minimal piece  $z$  at position  $q+3$ , and  $z$  is an illegal minimal piece of  $H'$ : the heap  $(H', \{x\})$  is in  $\mathbb{X}_{[S]}^{3*}$ .

Once again, this operation is easily reversible by putting back the piece  $y^+$  and checking whether it is an illegal minimal piece.  $\square$

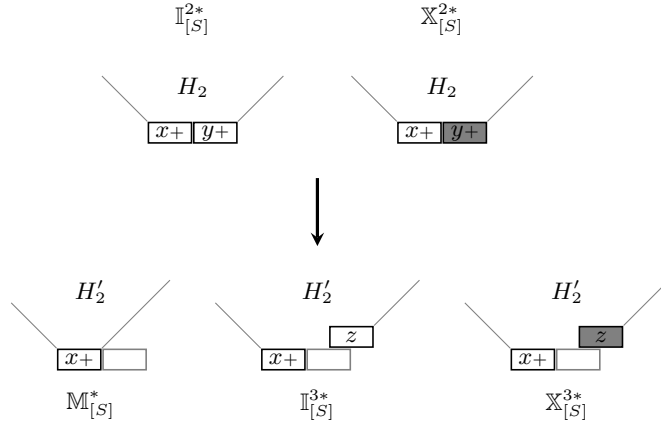


Figure 9: The bijection  $\Phi_3$ ; illegal minimal pieces are colored gray. After removing the piece  $y^+$ , either a new minimal piece  $z$  of  $H_2$  is uncovered at position 3 or not. If it is, the heap  $H'$  may be in  $\mathbb{I}_{[S]}^{3*}$  or  $\mathbb{X}_{[S]}^{3*}$ , depending on whether  $z$  is an illegal minimal piece. If not, the heap  $H'$  is in  $\mathbb{M}_{[S]}^*$ .

Finally, to prove the identity (20), we need a last lemma, given below.

**Lemma 4.11.** *The following identity holds:*

$$W_{[S]}(t) = \sum_{T \subseteq S} (W_T(t) - j(T)A_T(t)).$$

*Proof.* Consider the generating function:

$$\sum_{T \subseteq S} W_T(t).$$

We write  $W_T(t)$  as the sum of all  $W_T^q(t)$ , for  $q \in T$  and  $q+2 \notin T$ . We then split the sum according to whether  $q+2$  is in  $S$  or not:

$$\sum_{T \subseteq S} W_T(t) = \sum_{T \subseteq S} \left( \sum_{\substack{q \in T \\ q+2 \notin S}} W_T^q(t) \right) + \sum_{T \subseteq S} \left( \sum_{\substack{q \in T \\ q+2 \notin T \\ q+2 \in S}} W_T^q(t) \right).$$

In the first term of the right-hand side of this equation, we recognize the generating function  $W_{[S]}(t)$ . We rewrite the second term using the fact that  $W_T^q(t) = A_{T \cup \{q+2\}}(t)$  and by posing  $T' = T \cup \{q+2\}$ :

$$\begin{aligned} \sum_{T \subseteq S} W_T(t) &= W_{[S]}(t) + \sum_{T' \subseteq S} \left( \sum_{\substack{q \in T' \\ q+2 \in T'}} A_{T'}(t) \right); \\ &= W_{[S]}(t) + \sum_{T' \subseteq S} j(T')A_{T'}(t). \end{aligned}$$

The lemma follows. □

With the above lemmas, we are now able to prove the theorem.

*Proof of Theorem 4.4.* The identities (18), (19), (22) and (23) dealing with the site perimeters are proved in Section 4.2.

To prove the remaining identities, we first link the generating function counting almost strict marked heaps with the ones counting strict marked heaps. As each marked piece accounts for a  $1+t$  factor, we have:

$$\begin{aligned} J_{[S]}^*(t) &= (1+t)^2 J_{[S]}(t); \\ L_{[S]}^*(t) &= (1+t) L_{[S]}(t); \\ M_{[S]}^*(t) &= (1+t) M_{[S]}(t); \\ W_{[S]}^*(t) &= (1+t) W_{[S]}(t). \end{aligned}$$

Moreover, putting together the identities of Lemmas 4.7, 4.8, 4.9 and 4.10, we find:

$$\begin{aligned} L_{[S]}^*(t) &= t J_{[S]}^*(t); \\ J_{[S]}^*(t) &= t M_{[S]}^*(t) - W_{[S]}^*(t). \end{aligned}$$

Thus, we derive the first two identities of the theorem, (16) and (17).

To prove the identities dealing with animals of source  $S$ , we remark that the generating function  $J_{[S]}(t)$  verifies:

$$J_{[S]}(t) = \sum_{T \subseteq S} J_T(t).$$

The generating functions  $L_{[S]}(t)$  and  $M_{[S]}(t)$  also behave in this manner. Using the inclusion-exclusion principle, this means that the equation (21) giving  $L_S(t)$  is a consequence of (17).

To address the generating function  $W_{[S]}(t)$ , we use Lemma 4.11, rewriting (16) as:

$$\sum_{T \subseteq S} \left( J_T(t) - \frac{tM_T(t) + j(T)A_T(t) - W_T(t)}{1+t} \right) = 0.$$

The identity (20) is then derived using the inclusion-exclusion principle.  $\square$

#### 4.4 Triangular lattices

Let  $\Delta$  be the triangular lattice corresponding to  $\Gamma$ . A number of our results on the animals of  $\Gamma$  have counterparts on the animals of  $\Delta$ . The results and proofs are very similar, and we go into slightly less detail.

Given an animal  $A$  on  $\Delta$ , we define its number  $j(A)$  of adjacent sites and its number  $\ell(A)$  of loops. In this paper, a loop is still defined by two adjacent sites capped by another site. Note that this definition is different from the one used by Bousquet-Mélou [3], who found similar results.

With our methods, we have been unable to address the site perimeter, which is not surprising as the generating function of animals marked with a neighbour is believed to be non-algebraic on the unbounded lattices [5]. The best we could do is to compute bounds on the perimeter, using manipulations similar to Lemma 4.5, although we do not give further details in this paper.

As on the square lattice, we begin by defining generating functions, which are analogues to the ones of Definitions 4.2, 4.3 and 4.6.

**Definition 4.12.** Let  $S \subseteq Q$  be an aligned set of positions. Define the following generating functions, counting animals with a source included in  $S$  (or inflated heaps with base included in  $S$ ):

- $\mathcal{J}_{[S]}(t)$  and  $\mathcal{L}_{[S]}(t)$ , the generating functions of animals with a source included in  $S$ , marked respectively with two adjacent sites and a loop;
- $\mathcal{M}_{[S]}(t)$  the generating function of inflated heaps marked with a piece at a position  $q$  such that  $q+2 \in S$ ;
- $\mathcal{W}_{[S]}(t)$  the generating function of inflated heaps with a minimal piece at position  $q$  and an illegal minimal piece at position  $q+2$ ;
- $\mathcal{I}_{[S]}^2(t)$  (resp.  $\mathcal{I}_{[S]}^3(t)$ ) the generating function of inflated heaps marked with two independent pieces at positions  $q$  and  $q+2$  (resp.  $q$  and  $q+3$ ).



Also let  $\mathcal{J}_S(t)$ ,  $\mathcal{L}_S(t)$ ,  $\mathcal{M}_S(t)$  and  $\mathcal{W}_S(t)$  be the analogous generating functions counting animals of source  $S$  and heaps of base  $S$ .

**Theorem 4.13.** *The generating functions counting the total number of adjacent sites and loops of animals of a given area on the triangular lattice are:*

$$\mathcal{J}_{[S]}(t) = \frac{t\mathcal{M}_{[S]}(t) - \mathcal{W}_{[S]}(t)}{1+t}; \quad (25)$$

$$\mathcal{L}_{[S]}(t) = t\mathcal{J}_{[S]}(t); \quad (26)$$

$$\mathcal{J}_S(t) = \frac{t\mathcal{M}_S(t) + j(S)\mathcal{A}_S(t) - \mathcal{W}_S(t)}{1+t}; \quad (27)$$

$$\mathcal{L}_S(t) = t\mathcal{J}_S(t). \quad (28)$$

To prove this theorem, we use the results of Section 4.3, along with an additional bijection.

**Lemma 4.14.** *The following identity holds:*

$$\mathcal{I}_{[S]}^2(t) - \mathcal{J}_{[S]}(t) = \frac{t}{1-t} \left( 2\mathcal{J}_{[S]}(t) + \mathcal{I}_{[S]}^3(t) \right).$$

*Proof.* We use a bijection  $\Psi$ , analogue to  $\Phi_1$  (see Lemma 4.8) and illustrated in Figure 10. Let  $(H, \{x, y\})$  be a marked inflated heap, such that  $x$  and  $y$  are independent, with positions at distance 2, and not at the same height (say,  $x$  is higher). We use the bijection  $\mathcal{F}_\downarrow$  to form a factorized heap  $(H_1 \cdot H_2)$ . We then remove from  $H_1$  all pieces of the stack of  $x$  that are higher than  $y$ , thus forming the heap  $H'_1$ . There are two possibilities:

- (a)  $H'_1$  has two maximal pieces,  $y$  and  $z$ , which are adjacent;
- (b)  $H'_1$  has two maximal pieces,  $y$  and  $z$ , with positions at distance 3.

The inverse bijection is done by putting back the stack of  $x$ , which can have an arbitrary number of pieces. In case (b), as the inflated heap  $H'_1$  is aligned, the pieces  $y$  and  $z$  cannot be at the same height. Therefore,  $z$  must be the higher maximal piece. In case (a), however,  $z$  can be either the left maximal piece or the right, leading to the factor 2 on the term  $\mathcal{J}_{[S]}(t)$ .  $\square$

*Proof of Theorem 4.13.* First, we derive the identities (26) and (28) dealing with loops, using a method identical to the proof of Lemma 4.7. When dealing with general heaps, there is no  $1+t$  factor due to the duplication of marked pieces.

We now prove the identity (25). Let  $(H, \{x\})$  be a heap counted by  $\mathcal{M}_{[S]}(t)$ . We use the bijection  $\mathcal{F}_\downarrow$  to pull downwards the piece  $x$ , creating a factorized heap. We remark that such factorized heaps may be built by replacing each piece of an almost strict factorized heap by an arbitrary stack, leading to the link:

$$\mathcal{M}_{[S]}(t) = M_{[S]}^* \left( \frac{t}{1-t} \right).$$

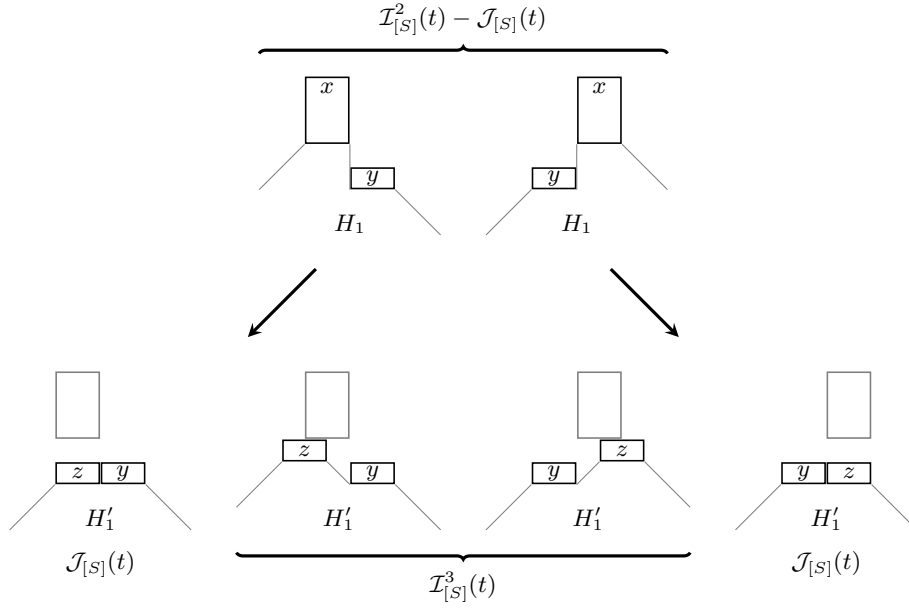


Figure 10: The bijection  $\Psi$ : we remove all pieces of the stack of  $x$  which are higher than  $y$ . This uncovers a piece  $z$ , either adjacent with  $y$  or with a position at distance 3 and higher than  $y$ .

The generating functions  $\mathcal{I}_{[S]}^2(t)$  and  $\mathcal{I}_{[S]}^3(t)$  are also given in this manner. As for  $\mathcal{W}_{[S]}(t)$ , it satisfies:

$$\mathcal{W}_{[S]}(t) = W_{[S]}\left(\frac{t}{1-t}\right) = (1-t)W_{[S]}^*\left(\frac{t}{1-t}\right).$$

Taking the identities of Lemmas 4.9 and 4.10 together and performing the substitution  $t \mapsto \frac{t}{1-t}$ , we thus find:

$$\mathcal{I}_{[S]}^2(t) + \mathcal{W}_{[S]}(t) = \frac{t}{1-t} \left( \mathcal{M}_{[S]}(t) + \mathcal{I}_{[S]}^3(t) \right).$$

Using now Lemma 4.14, this boils down to (25). Performing the same substitution on the identity of Lemma 4.11 yields:

$$\mathcal{W}_{[S]}(t) = \sum_{T \subseteq S} (\mathcal{W}_T(t) - j(T)\mathcal{A}_T(t)).$$

The last identity (27) is thus derived using an inclusion-exclusion argument.  $\square$

## 5 Asymptotic results

### 5.1 Animals according to area

Here, we derive asymptotic estimates from the results of Section 3. First, consider the polynomials  $F_m(t)$  and  $\widehat{F}_m(t)$ , defined in Definition 3.3. Let  $\rho_m$  and  $\sigma_m$  be their respective smallest root.

**Lemma 5.1.** *For all  $m \geq 0$ , the polynomials  $F_m(t)$  and  $\widehat{F}_m(t)$  have only real, simple roots. Their smallest roots  $\rho_m$  and  $\sigma_m$  verify:*

$$\frac{1}{\rho_m} = 4 \cos^2 \frac{\pi}{m+2};$$

$$\frac{1}{\sigma_m} = 4 \cos^2 \frac{\pi}{2m}.$$

*Proof.* We check by induction on  $m$  that the degrees of  $F_m(t)$  and  $\widehat{F}_m(t)$  are  $\lceil \frac{m}{2} \rceil$  and  $\lfloor \frac{m}{2} \rfloor$ , respectively. We also check by induction the following identities:

$$F_m\left(\frac{1}{4 \cos^2 \theta}\right) = \frac{\sin[(m+2)\theta]}{(2 \cos \theta)^{m+1} \sin \theta};$$

$$\widehat{F}_m\left(\frac{1}{4 \cos^2 \theta}\right) = \frac{2 \cos(m\theta)}{(2 \cos \theta)^m}.$$

By choosing appropriate values of  $\theta$  in the interval  $[0, \pi/2)$ , these identities account for all the roots of the polynomials. Thus, we prove that the roots are real and simple, and we derive the value of the smallest root.  $\square$

Now, let  $\Gamma$  be a square lattice and  $\Delta$  be its associated triangular lattice; let  $S$  be a one-line source. We denote by  $a(n)$  and  $\bar{a}(n)$  the number of animals of area  $n$  of source  $S$  on the lattices  $\Gamma$  and  $\Delta$  respectively. The result below is simply obtained by performing singularity analysis [7] on the formulæ of Section 3.

**Proposition 5.2.** *The general form of the asymptotic behaviour of  $a(n)$  and  $\bar{a}(n)$  is:*

$$a(n) \sim \lambda \mu^n n^\nu; \quad \bar{a}(n) \sim \bar{\lambda} \bar{\mu}^n n^\nu,$$

where the constants  $\bar{\mu}$  and  $\nu$  are:

- in the full lattice,  $\bar{\mu} = 4$  and  $\nu = -1/2$ ;
- in the half lattice,  $\bar{\mu} = 4$  and  $\nu = -3/2$ ;
- in the cylindrical lattice of width  $m$ ,  $\bar{\mu} = 1/\sigma_m$  and  $\nu = 0$ ;
- in the rectangular lattice of width  $m$ ,  $\bar{\mu} = 1/\rho_m$  and  $\nu = 0$ .

Moreover, in each case,  $\mu$  is equal to  $\bar{\mu} - 1$  and  $\lambda$  and  $\bar{\lambda}$  depend on the source  $S$ .

Notably, changing the source  $S$  only changes the behaviour of  $a(n)$  and  $\bar{a}(n)$  by a multiplicative constant.

## 5.2 Average number of adjacent sites and loops and average perimeter

Let now  $j(n)$  be the average number of adjacent sites in the animals of source  $S$  and area  $n$  in the lattice  $\Gamma$ . Let  $\ell(n)$ ,  $p_e(n)$ ,  $p_i(n)$  be their average number of loops, external perimeter, and internal perimeter; let  $\bar{j}(n)$  and  $\bar{\ell}(n)$  be the analogous quantities in the lattice  $\Delta$ .

**Corollary 5.3.** *Assume that  $\Gamma$  is either the full lattice, the half lattice, or a cylindrical bounded lattice. As  $n$  tends to infinity, we have the following estimates:*

$$\begin{aligned} j(n) &\sim \frac{n}{\mu+1}; & \bar{j}(n) &\sim \frac{n}{\bar{\mu}+1}; \\ \ell(n) &\sim \frac{n}{\mu^2}; & \bar{\ell}(n) &\sim \frac{n}{\bar{\mu}(\bar{\mu}+1)}; \\ p_i(n) &\sim p_e(n) \sim \frac{\mu}{\mu+1}n. \end{aligned}$$

In the unbounded lattices, the growth constants are  $\bar{\mu} = 4$  and  $\mu = 3$ . Thus, these estimates become:

$$j(n) \sim \frac{n}{4}; \quad \ell(n) \sim \frac{n}{9}; \quad p(n) \sim \frac{3n}{4}; \quad \bar{j}(n) \sim \frac{n}{5}; \quad \bar{\ell}(n) \sim \frac{n}{20}.$$

*Proof.* Let us begin with the number of adjacent pieces in the square lattice. This number is given by the identity (20):

$$J_S(t) = \frac{tM_S(t) + j(S)A_S(t) - W_S(t)}{1+t},$$

where the generating functions are defined in Definitions 4.2 and 4.3. We examine the coefficients of these generating functions.

- The  $n$ th coefficient of  $J_S(t)$  is  $j(n)a(n)$ .
- As, in the full, half, and cylindrical models, the position  $q+2$  is always in  $Q$  as soon as  $q$  is, the generating function  $M_S(t)$  simply counts animals marked with any site; its  $n$ th coefficient is  $na(n)$ .
- As a corollary to Proposition 5.2, the  $n$ th coefficient of both  $A_S(t)$  and  $W_S(t)$  is  $O(a(n))$ .

From this, it follows that the dominant term in the right-hand side is that of  $M_S(t)$ . We perform singularity analysis, letting  $t$  tend to the singularity  $1/\mu$ . We obtain, as  $n$  tends to infinity:

$$j(n)a(n) \sim \frac{1/\mu}{1+1/\mu}na(n).$$

The result follows; the other estimates are obtained with a similar analysis on the equations of Theorems 4.4 and 4.13.  $\square$

## 6 Examples

### 6.1 Single-source animals on the full lattice

We start with the simplest case, that of single-source animals on the full lattices.

**Corollary 6.1.** *The generating functions counting the total number of adjacent sites, number of loops and site perimeter of the single-source directed animals on the full square lattice are respectively given by:*

$$\begin{aligned} J(t) &= \frac{1}{2t(1+t)} \left( 1 - \frac{1-4t+t^2+4t^3}{\sqrt{1+t}(1-3t)^{3/2}} \right); \\ L(t) &= \frac{1}{2} \left( 1 - \frac{1-4t+t^2+4t^3}{\sqrt{1+t}(1-3t)^{3/2}} \right); \\ P(t) &= \frac{1}{2t(1+t)} \left( -1+t+t^2 + \frac{1-3t+2t^2+t^3-3t^4}{\sqrt{1+t}(1-3t)^{3/2}} \right). \end{aligned}$$

The value of  $P(t)$  was conjectured by Conway [5], and the value of  $L(t)$  was proved by Bousquet-Mélou using a gas model method [3].

*Proof.* We use Theorem 4.4 to derive the generating functions. First, we use (20) to compute  $J(t) \equiv J_{\{0\}}(t)$ , which gives:

$$J_{\{0\}}(t) = \frac{tM_{\{0\}}(t) + j(\{0\})A_{\{0\}}(t) - W_{\{0\}}(t)}{1+t}.$$

The generating function  $M_{\{0\}}(t)$  is simply equal to  $tA'(t)$ ,  $j(\{0\})$  is zero, and  $W_{\{0\}}(t)$  is equal to  $A_{\{0,2\}}(t)$ , in turn equal to  $D(t)A(t)$  using Proposition 3.9. This yields the announced formula; the other two generating functions follow from equations (21) and (22).  $\square$

Similarly, Theorem 4.13 instantiates on single-source animals on the triangular full lattice. We omit the proof, which is identical to the square lattice case.

**Corollary 6.2.** *The generating functions counting the total number of adjacent sites and number of loops of single-source animals on the full triangular lattice are:*

$$\begin{aligned} \mathcal{J}(t) &= \frac{1}{2t(1+t)} \left( 1-t - \frac{1-7t+12t^2-2t^3}{(1-4t)^{3/2}} \right); \\ \mathcal{L}(t) &= \frac{1}{2(1+t)} \left( 1-t - \frac{1-7t+12t^2-2t^3}{(1-4t)^{3/2}} \right); \end{aligned}$$

This time, the value of  $\mathcal{L}(t)$  is different from the one found by Bousquet-Mélou [3], who used a different definition of loops.

## 6.2 Compact-source animals on the full lattice

As an illustration of how to deal with non-single source animals, we consider animals with any compact source (see Definition 3.8). Recall that the number of such animals of area  $n$  is  $3^{n-1}$  on the square full lattice, so that the generating function is:

$$A_c(t) = \frac{t}{1-3t}.$$

We only give the result for the number of adjacent sites of animals on the square full lattice, but other configurations can be handled similarly.

**Corollary 6.3.** *The generating function counting the total number of adjacent sites of the compact-source directed animals on the full square lattice is:*

$$J_c(t) = \frac{1}{2} \left( \frac{1-2t}{\sqrt{1+t}(1-3t)^{3/2}} - \frac{1-3t-2t^2}{(1+t)(1-3t)^2} \right).$$

*Proof.* Let  $C$  be a compact source with  $k$  sites. The generating function  $A_C(t)$  is, according to Proposition 3.9:

$$A_C(t) = D(t)^{k-1} A(t).$$

Moreover,  $M_C(t)$  is simply equal to  $tA'_C(t)$ ,  $j(C)$  is  $k-1$ , and  $W_C(t)$  counts animals with a compact source with  $k+1$  sites, and is thus equal to  $D(t)A_C(t)$ . Therefore:

$$J_C(t) = \frac{t^2 A'_C(t) + (k-1)A_C(t) - D(t)A_C(t)}{1+t}.$$

We sum this identity for all  $k \geq 0$ :

$$J_c(t) = \frac{1}{1+t} \left( t^2 A'_c(t) + \frac{A(t)D(t)}{(1-D(t))^2} - D(t)A_c(t) \right).$$

This completes the proof.  $\square$

## 6.3 Half-animals on the square rectangular lattices

Finally, we present our results on the external and internal site perimeter of half-animals (that is, animals of source  $\{0\}$ ) on the square rectangular lattices. The former was the object of a conjecture by Le Borgne [11]; from our formula, one can prove this conjecture.

We denote by  $D_m(t)$  the generating function of half-animals in square rectangular lattice of width  $m$ .

**Corollary 6.4.** *The generating functions giving the total external and internal site perimeter of half-animals on the square rectangular lattice of width  $m$  are:*

$$\begin{aligned} P_m^e(t) &= D_m(t) + \frac{t}{1+t} D'_m(t) + \frac{1}{1+t} D_m(t)^2; \\ P_m^i(t) &= \frac{t}{1+t} D_m(t) + \frac{t}{1+t} D'_m(t) - \frac{1}{1+t} D_m(t) (D_m(t) - D_{m-2}(t)), \end{aligned}$$

where the generating function  $D_m(t)$  is derived from (8).

By letting  $m$  tend to infinity, we obtain the generating functions of animals on the half lattice:

$$P^e(t) = D(t) + \frac{t}{1+t}D'(t) + \frac{1}{1+t}D(t)^2;$$

$$P^i(t) = \frac{t}{1+t}D(t) + \frac{t}{1+t}D'(t).$$

*Proof.* Let  $Q$  be the rectangular model  $\{0, \dots, m-1\}$  of width  $m$ . By symmetry, instead of considering animals with a source at position 0, we consider them to have a source at position  $m-1$ . This does not change the site perimeter of the animals.

The generating functions  $P_m^e(t) \equiv P_{\{m-1\}}^e(t)$  and  $P_m^i(t) \equiv P_{\{m-1\}}^i(t)$  are given by (22) and (23), which in turn require us to compute the generating function  $J_{\{m-1\}}(t)$ . The number  $j(\{m-1\})$  is again zero; moreover, as the position  $m+1$  is not in  $Q$ ,  $W_{\{m-1\}}(t)$  is also zero. Thus, all we need to compute are the generating functions  $M_{\{m-1\}}(t)$  and  $E_{\{m-1\}}(t)$ .

Let  $D_m^{(q)}(t)$  be the generating function of half-animals marked with a site at position  $q$ . Using Definition 4.2, we find:

$$M_{\{m-1\}}(t) = tD_m'(t) - D_m^{(m-1)}(t) - D_m^{(m-2)}(t);$$

$$E_{\{m-1\}}(t) = D_m^{(m-1)}(t) + D_m^{(0)}(t).$$

Finally, we derive the generating functions  $D_m^{(q)}(t)$  using Lemma 2.15; in the notations of this lemma,  $D_m^{(q)}(t)$  is equal to  $H_{[\{m-1\}]}^{(q)}(t)$  as a marked heap cannot be empty. We must compute the following generating functions:

- $H_{\{m-1\}}(t)$  and  $H_{\{0\}}(t)$  are both equal to  $D_m(t)$  by symmetry;
- the only heap of base included in  $\{m-1\}$  avoiding  $m-2$  is the empty heap, so that  $V_{[\{m-1\}]}^{m-2}(t) = 1$ ;
- as a strict pyramid of base  $m-1$  is either a single piece or a piece topped by a pyramid of base  $m-2$ , we have  $H_{\{m-2\}}(t) = \frac{D_m(t)}{t} - 1$ ;
- as a pyramid of base  $m-1$  avoiding 0 lives in the model  $\{2, \dots, m-1\}$ , which has  $m-2$  positions, we have  $V_{[\{m-1\}]}^0(t) = 1 + D_{m-2}(t)$ .

From this, we find:

$$D_m^{(m-1)}(t) = \frac{1}{1+t} \left( 1 + D_m(t) \right) D_m(t);$$

$$D_m^{(m-2)}(t) = \frac{1}{1+t} D_m(t) \left( \frac{D_m(t)}{t} - 1 \right);$$

$$D_m^{(0)}(t) = \frac{1}{1+t} \left( D_m(t) - D_{m-2}(t) \right) D_m(t).$$

Injecting these values into (20), (22) and (23), we get the announced results.  $\square$

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