On certain time-space fractional evolution equations

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Abstract

In this article, we present first a short proof of the Cazenave, Dickstein and Weissler result [3] on the blowing-up solutions to a temporally nonlocal nonlinear parabolic equation. We then apply our technique of proof to tackle nonlinear systems of temporally nonlocal coupled fractional parabolic equations. Furthermore, we consider hyperbolic nonlocal equation and system.

Keywords: Parabolic equation, hyperbolic equation, blow-up, critical exponent, fractional integrals and derivatives.

1 Introduction

The aim of this paper is two fold:
First, we investigate the nonlinear parabolic equation with a nonlocal in time nonlinearity

$$\text{(NLDE)} \quad \begin{cases} \partial_t u + (-\Delta)^{\beta/2}|u|^m = \int_0^t (t-s)^{-\gamma}|u(s)|^p \, ds, , \quad x \in \mathbb{R}^N, \ t > 0, \ u > 0, \\ \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $0 < \beta \leq 2$, $0 < \gamma < 1$, $1 \leq m < p$ and the nonlocal operator $(-\Delta)^{\beta/2}$ defined by

$$(-\Delta)^{\beta/2} v(x,t) := \mathcal{F}^{-1}\left(|\xi|^\beta \mathcal{F}(v)(\xi,t)\right)(x,t)$$

where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ its inverse.
Then, we extend our analysis to the system

$$\text{(NLDS)} \quad \begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma}|v|^p\,ds, \;\; x \in \mathbb{R}^N, \;\; t > 0, \\ \\ v_t - \Delta v = \displaystyle\int_0^t (t-s)^{-\delta}|u|^q\,ds, \end{cases} \tag{1.2}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \;\; x \in \mathbb{R}^N.$$

Here $0 < \gamma < 1$, $0 < \delta < 1$, $1 < pq$ are real numbers.
In system (NLDS), we take $-\Delta u$ and $-\Delta v$ rather than $(-\Delta)^{\beta/2}|u|^m$ and $(-\Delta)^{\alpha/2}|u|^n$ just for clarity. It will be clear that analysis can handle such diffusion terms.
In the second part, we extend our analysis to the nonlocal nonlinear hyperbolic equation

$$\text{(NLWE)} \quad \begin{cases} u_{tt} + (-\Delta)^{\alpha/2}|u| = \displaystyle\int_0^t (t-s)^{-\gamma}|u|^p\,ds, \;\; x \in \mathbb{R}^N, \; t > 0, \\ u(x,0) = u_0(x)\;, \; u_t(x,0) = u_1(x), \qquad\qquad x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $0 < \alpha \leq 2$, $0 < \gamma < 1$ and $p > 1$.
We then naturally, present results concerning the system

$$\text{(NLWS)} \quad \begin{cases} u_{tt} + (-\Delta)^{\alpha/2}|u| = \displaystyle\int_0^t (t-s)^{-\mu}|v|^p\,ds, \quad x \in \mathbb{R}^N, \; t > 0, \\ \\ v_{tt} + (-\Delta)^{\alpha/2}|v| = \displaystyle\int_0^t (t-s)^{-\nu}|u|^q\,ds, \quad x \in \mathbb{R}^N, \; t > 0, \end{cases}$$

subject to the initial data

$$\begin{array}{rcl} u(x,0) & = & u_0(x) \quad , \quad u_t(x,0) = u_1(x), \\ v(x,0) & = & v_0(x) \quad , \quad v_t(x,0) = v_1(x). \end{array}$$

In all the equations above, Γ is the gamma function of Euler.

Our article is motivated by the recent and very interesting article of Cazenave, Dickstein and Weissler which deals with the global existence and blow-up for the parabolic equation with nonlocal in time nonlinearity

$$\partial_t u - \Delta u = \int_0^t (t-s)^{-\gamma} u(s)^p\,ds, \quad x \in \mathbb{R}^N, t > 0, \tag{1.4}$$

where $0 < \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^N)$.
They proved that, if we set

$$p_\gamma = 1 + \frac{4 - 2\gamma}{(N - 2 + 2\gamma)_+}$$

and

$$p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0, +\infty],$$

(i) If $p \leq p_*$,
and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.

(ii) If $p > p_*$,
and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ $\left(\text{where } q_{sc} = \frac{N(p-1)}{4-2\gamma}\right)$ with $||u_0||_{L^{q_{sc}}}$ sufficiently small, then u exists globally.

Their study reveals that for equation (1.4) the critical exponent in Fujita's sense p_* is not the one predicted by scaling.
They also consider Eq. (1.4) in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and showed that for $u_0 \in C_0(\Omega)$

(i) If $p\gamma \leq 1$,
and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.

(ii) If $p\gamma > 1$,
and $||u_0||_{L^\infty}$ sufficiently small, then u exists globally.

Needless to say that the equation considered by Cazenave, Dickstein and Weissler is a genuine extension to the one considered by Fujita in his pioneering work [5].
In this article, concerning blowing-up solutions, we present a different proof that the one presented by Cazenave, Dickstein and Weissler, and for the more general equation (NLDE) when considered with a nonlinearity of the form u^p, reads

$$u_t + (-\Delta)^{\beta/2} u^m = u^p.$$

This equation has been considered by Nagasawa and Sirao [] for $m = 1$, and then by Kirane and Qafsaoui [] in the general case.
Observe that the equation without the reaction term u^p :

$$u_t + (-\Delta)^{\beta/2} u^m = 0 \tag{1.5}$$

has been considered by Bologna, Tsallis and Grigolini in [] in one space dimension but for $-\infty < \beta \leq 2$ and $-1 < m$. The operator $(-\Delta)^{\beta/2}$ describes anomalous

diffusion; this a superdiffusion equation in the termenology of Zaslavsky [].
At this stage, let us observe that Eq. (1.4) can be written formally in the form:

$$D^{\alpha}_{0|t}u_t + D^{\alpha}_{0|t}(-\Delta)^{\beta/2}u = u^p, \tag{1.6}$$

where $D^{\alpha}_{0|t}$ is the left-sided Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined by (see (2.30) p.37 [19])

$$D^{\alpha}_{0|t}f(t) := \frac{d}{dt}\int_0^t \frac{f(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}}\,ds \tag{1.7}$$

for every absolutely continuous function $f(t)$; we have set in (1.6) $\alpha = 1-\gamma \in (0,1)$. The Eq. (1.6) without the nonlinearity u^p, i.e.,

$$D^{\alpha}_{0|t}u_t + D^{\alpha}_{0|t}(-\Delta)^{\beta/2}u = 0$$

is not an usual one.
In [], Sokolov and Klafter discussed modern extensions of the usual heat equation $u_t - \Delta u = 0$ to include heavy tailed particle jumps (in the case of random walk model of particle motion)and random waiting times, leading to fractional diffusion equations, the simplest of which is

$$D^{\alpha}_{0|t}u_{=} - D(-\Delta)^{\beta/2}u.$$

($u(x,t)$ is the probability distribution).
The fractional Laplacian reflects heavy tailed waiting times, where the probability of waiting longer than t falls like $t^{-\alpha}$. Heavy tails in space lead to superdiffusion, where a plume of particles spreads faster that the classical $\sqrt{t}$ rate associated with Brownian motion. Heavy tails in time lead to subdiffusion, since long waiting times retard particle motion.
Much more can be said; see the valuable contributions of Zaslavsky [], Baleseu [], Sokolov, Klafter and Blumen[].
It transpires then clearly that either Eq. (1.6) (and hence Eq. (1.4)) doesn't fit in its diffusive part in the existing models derived in Physical-Chemistry or a pure diffusive term in space is missing like $-\Delta u$ or alike one.
At this stage, we mention some literature exists on equations of the form

$$u_t = \Delta u^m + f(u(x,t), \int_0^t g(x,t,s,u(x,s))\,ds)$$

$t > 0, x \in \Omega \subseteq \mathbb{R}^N$, which serve as models for nonlinear diffusion phenomena with memory effects. In population dynamics, for example, one encounters

$$g(x,t,s,u(x,s)) = K(t-s)h(u(x,s))$$

but with regular kernels $K(t-s)$, and so usually

$$f(u, \int_0^t g(x,t,s,u(x,s))\,ds) = (a-bu)u - \int_0^t K(t-s)u(x,s)\,ds.$$

Other works concern the nonlocal in time equation

$$u_t - \Delta u = u^q \int_0^t u^p(x,s)\,ds$$

which for $p = q = 1$ plays a role in the theory of nuclear reactor kinetics as reported by Kastenberg in [].
More close to our equation, we can mention the article of Miao and Yang [] which is concerned with the problem

$$\begin{cases} D_{0|t}^{1+\alpha} u = \Delta u + f(u), \ \ x \in \mathbb{R}^N, \ \ t > 0, \\ \\ u(x,0) = u_0(x), \ \ x \in \mathbb{R}^N. \end{cases}$$

The article of Kirane, Laskri and Tatar [] treats the more general equation

$$\begin{cases} D_{0|t}^{\alpha} u + (-\Delta)^{\beta/2} u = h(x,t)u^p, \ \ x \in \mathbb{R}^N, \ \ t > 0, \\ \\ u(x,0) = u_0(x), \ \ x \in \mathbb{R}^N. \end{cases}$$

Let us notice that in [], systems of equations of the latter equation are also considered.
Concerning our results, we will show that

- For $\int_{\mathbb{R}^N} u_0(x)\,dx > 0$, if

$$1 < p \le p_F := 1 + \frac{(2-\gamma)(N(m-1)+\beta)}{(N-\beta+\beta\gamma)_+} \qquad \text{or} \qquad p < \frac{m}{\gamma},$$

  then solutions to Eq. (NLDE) blow-up in finite time.

- For $\int_{\mathbb{R}^N} u_0(x)\,dx > 0$, $\int_{\mathbb{R}^N} v_0(x)\,dx > 0$, if

$$\frac{N}{2} \le \max\left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1} \ ; \ \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\},$$

  or

$$p < \frac{1}{\gamma} \qquad \text{and} \qquad q < \frac{1}{\delta},$$

  then solutions to the system (NLDS) blow-up in finite time.

- For $\int_{\mathbb{R}^N} u_0(x)\,dx > 0$, if

$$p \;\leq\; 1 \;+\; \frac{(3-\gamma)\alpha}{(2N-2\alpha+\gamma\alpha)^+},$$

or

$$p < \frac{1}{\gamma},$$

then (NLWE) doesn't admit global solutions other than the trivial one.

- For $\int_{\mathbb{R}^N} u_0(x)\,dx > 0$, $\int_{\mathbb{R}^N} v_0(x)\,dx > 0$, $\int_{\mathbb{R}^N} u_t(x,0)\,dx \geq 0$, $\int_{\mathbb{R}^N} v_t(x,0)\,dx \geq 0$, if

$$N \;\leq\; \max\{\Upsilon_1\,;\,\Upsilon_2\}\,,$$

or

$$p < \frac{1}{\mu} \qquad \text{and} \qquad q < \frac{1}{\nu},$$

where

$$\Upsilon_1 := \frac{(p(3-\nu)+pq(2-\mu)+1)\,\alpha}{2(pq-1)} \;\;;\;\; \Upsilon_2 := \frac{(q(3-\mu)+pq(2-\nu)+1)\,\alpha}{2(pq-1)},$$

then no global solution to system (NLWS) do exist.

Our method of proof is the test function method of Pohozaev, Mitidieri, Tesei and Véron [].

2 Preliminaries

In this section, we present some definitions and results concerning fractional derivatives that will be used in the sequel.
If Γ stands for the usual gamma function of Euler, for $f \in L^1(\mathbb{R}_+)$, we define, for $0 < \alpha < 1$,

$$D^{\alpha}_{0|t} f(t) := D J^{1-\alpha}_{0|t} f(t),$$

where $D := \frac{d}{dt}$ and

$$J^{\alpha}_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds.$$

If

$$J^{\alpha}_{t|T} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)\,ds$$

then, for every $f, g \in C([0,T])$, such that $D^{\alpha}_{0|t} f(t), D^{\alpha}_{t|T} g(t)$ exist and are continuous for all $t \in [0,T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [19])

$$\int_0^T \left(D^{\alpha}_{0|t} f\right)(s) g(s)\, ds \;=\; \int_0^T f(s) \left(D^{\alpha}_{t|T} g\right)(s)\, ds.$$

Note also that

$$-D.D^{\alpha}_{t|T} f = D^{1+\alpha}_{t|T} f.$$

3 The Case of a Parabolic Equation

In this section, we consider the nonlocal porous-medium fractional problem

$$\begin{cases} u_t = (-\Delta)^{\beta/2} |u|^m + \displaystyle\int_0^t \frac{|u(s)|^p}{(t-s)^{\gamma}}\, ds, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $u_0 \in C_0(\mathbb{R}^N)$, $0 < \gamma < 1$, $0 < \beta \leq 2$, and $1 \leq m < p$.
Our analysis is based on the observation that the differential Equation (3.1) can be written in the form:

$$u_t = (-\Delta)^{\beta/2} |u|^m + \Gamma(\alpha) J^{\alpha}_{0|t} \left(|u|^p\right), \tag{3.2}$$

where $\alpha := 1 - \gamma \in (0,1)$. We have $D^{\alpha}_{0|t} J^{\alpha}_{0|t} = Id$.

Applying $D^{\alpha}_{0|t}$ to (3.2), we obtain

$$D^{\alpha}_{0|t} u_t + (-\Delta)^{\beta/2} D^{\alpha}_{0|t} |u|^m = \Gamma(\alpha) |u|^p. \tag{3.3}$$

Now, we want to derive a nonexistence result for Eq.(3.3). We start with

Definition 1 *We say that u is a weak solution of (2.1), if*

$$u \in C([0, T_{max}), C_0(\mathbb{R}^N))$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \left(\int_0^t (t-s)^{-\gamma} |u|^p\, ds \right) \varphi\, dx\, dt \;=\; & -\int_0^T \int_{\mathbb{R}^N} u\, D\varphi\, dx\, dt \;-\; \int_{\mathbb{R}^N} u_0(x) \varphi(0)\, dx \\ & + \int_0^T \int_{\mathbb{R}^N} |u|^m\, (-\Delta)^{\beta/2} \varphi\, dx\, dt, \end{aligned} \tag{3.4}$$

for any nonnegative test function $\varphi \in C^{\infty}([0,T], C^{\infty}_c(\mathbb{R}^N))$, with $\varphi(T,x) = 0$, for all $0 < T < T_{max}$, where T_{max} is the maximal time of local existence proved in [3].

Definition 2 *We say that u is a weak solution of* (2.5), *if*

$$u \in C([0,T_{max}), C_0(\mathbb{R}^N))$$

and

$$\begin{aligned}\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N}|u|^p\,\varphi\,dx\,dt &= \int_0^T\int_{\mathbb{R}^N}u\,D_{t|T}^{1+\alpha}\varphi\,dx\,dt-\int_{\mathbb{R}^N}u_0(x)\left(D_{t|T}^{\alpha}\varphi\right)(0)\,dx\\ &+\int_0^T\int_{\mathbb{R}^N}|u|^m\,(-\Delta)^{\beta/2}D_{t|T}^{\alpha}\varphi\,dx\,dt, \qquad (3.5)\end{aligned}$$

for any nonnegative test function $\varphi \in C^\infty([0,T], C_c^\infty(\mathbb{R}^N))$, with $D_{t|T}^{\alpha}\varphi(T,x)=0$, for all $0<T<T_{max}$.

Theorem 1 *Let*

$$u \in C([0,T_{max}), C_0(\mathbb{R}^N)).$$

Then the following assertions are equivalent:

1. *u is a weak solution of* (3.1),
2. *u is a weak solution of* (3.3).

Proof. Necessity. Let u be a weak solution of (2.1) and $\varphi \in C^\infty([0,T], C_c^\infty\mathbb{R}^N)$ be such that $D_{t|T}^{\alpha}\varphi(T,x)=0$, for all $0<T<T_{max}$. If we take $D_{t|T}^{\alpha}\varphi$ as a test function in (3.4) and use (**??**), we obtain

$$\begin{aligned}\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N}J_{0|t}^{\alpha}|u|^p\,D_{t|T}^{\alpha}\varphi\,dx\,dt &= \int_0^T\int_{\mathbb{R}^N}u\,D_{t|T}^{1+\alpha}\varphi\,dx\,dt\;-\;\int_{\mathbb{R}^N}u_0(x)D_{t|T}^{\alpha}\varphi(0)\,dx\\ &+\int_0^T\int_{\mathbb{R}^N}|u|^m\,(-\Delta)^{\beta/2}D_{t|T}^{\alpha}\varphi\,dx\,dt.\end{aligned}$$

Using the formula of integration by parts for fractional derivatives (**??**) and the fact that $D_{0|t}^{\alpha}J_{0|t}^{\alpha}=Id$, we deduce that u is a weak solution of (3.3).
Sufficiency. As $D_{t|T}^{\alpha}J_{t|T}^{\alpha}\varphi = \varphi$ and $D_{t|T}^{1+\alpha}J_{t|T}^{\alpha}\varphi = -DD_{t|T}^{\alpha}J_{t|T}^{\alpha}\varphi = -D\varphi$, we can take $J_{t|T}^{\alpha}\varphi$ as a test function in (3.5).
Using the formula for fractional integration by parts (see (2.20) p.34 in [19])

$$\int_0^T f(t)\left(J_{0|t}^{\alpha}g\right)(t)\,dt = \int_0^T\left(J_{t|T}^{\alpha}f\right)(t)g(t)\,dt, \qquad (3.6)$$

for every $f\in L^p(0,T)$, $g\in L^q(0,T)$, such that $1/p+1/q\le 1+\alpha$, $p\ge 1$, $q\ge 1$, with $p\neq 1$, $q\neq 1$ in the case $1/p+1/q=1+\alpha$, for all $T>0$, we obtain (3.4) and this for every $\varphi\in C^\infty([0,T],C_c^\infty\mathbb{R}^N)$ with $\varphi(T,x)=0$, for all $0<T<T_{max}$; this completes the proof. □

Theorem 2 *Let* $\int_{\mathbb{R}^N} u_0(x)\,dx > 0$. *If*

$$p \leq 1 + \frac{(2-\gamma)(N(m-1)+\beta)}{(N-\beta+\beta\gamma)^+} := p_F \tag{3.7}$$

or

$$p < \frac{m}{\gamma},$$

then any solution to (3.1), *with initial condition* u_0, *blows-up in a finite time. Note that in the critical case we take only the case* $p\theta > \frac{N\theta m}{N-\beta}$ *and* $N > \beta$.

Proof. The proof is by contradiction. Suppose that u is a nonnegative global weak solution to (3.1).
Later on, we will use the following results.
If $\psi(t) = \left(1 - \frac{t}{T}\right)^{\sigma}$, $\sigma >> 1$, then

$$D^{\alpha}_{t|T}\psi(t) = \frac{\sigma\Gamma(\sigma-\alpha)}{\Gamma(\sigma-2\alpha+1)} T^{-\sigma}(T-t)^{\sigma-\alpha}, \quad \text{for all } \ T > 0, \alpha > 0. \tag{3.8}$$

Hence

$$\left(D^{\alpha}_{t|T}\psi\right)(T) = 0 \quad ; \quad \left(D^{\alpha}_{t|T}\psi\right)(0) = C\,T^{-\alpha},$$

for $C = \frac{\sigma\Gamma(\sigma-\alpha)}{\Gamma(\sigma-2\alpha+1)}$.
Observe that

$$\begin{aligned}\int_0^{T^*}\int_{\mathbb{R}^N} \varphi\,(-\Delta)^{\beta/2} D^{\alpha}_{0|t}|u|^m(x,t)\,dx\,dt &= \int_0^{T^*}\int_{\mathbb{R}^N} (-\Delta)^{\beta/2}\varphi\; D^{\alpha}_{0|t}|u|^m(x,t)\,dx\,dt \\ &= \int_0^{T^*}\int_{\mathbb{R}^N} |u|^m(x,t)\;(-\Delta)^{\beta/2} D^{\alpha}_{0|t}\varphi\,dx\,dt\end{aligned}$$

for every test function φ.
Now, we take $\varphi(x,t) = (\varphi_1(x))^{\ell}\,\varphi_2(t)$ as a test function with $\varphi_1(x) := \Phi\left(\frac{|x|}{T^{\theta}}\right)$, $\varphi_2(t) := \Psi\left(\frac{t}{BT}\right)$, where $\ell \geq \frac{p}{p-m}$, $\theta := \frac{p+\alpha-m(1+\alpha)}{\beta(p-1)}$ and Φ be a smooth non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $B < T < T^*$, $|\Phi'(r)| \leq C_1/r$, for all $r > 0$, and

$$\Psi(r) = \begin{cases} (1-r)^{\eta} & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 1, \end{cases}$$

with $\eta \geq \max\left\{\frac{\alpha p+1}{p-1}; \frac{\alpha p+m-p}{p-m}\right\}$.

Here $B > 0$ is a fixed constant which plays some role in the critical case $p = p_F$, while in the case $p < p_F$ we take $B = 1$. Then, using the fact that $D^{\mu}_{t|T^*}\varphi_2(t) = D^{\mu}_{t|T}\varphi_2(t)$, for every $\mu > 0$, $t \in [0, T^*]$ and Ju's inequality (see Appendix)

$$(-\Delta)^{\beta/2}\left(\varphi_1^{\ell}\right) \leq \ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}\left(\varphi_1\right),$$

we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt \; + \; C \, T^{-\alpha} \int_{\Omega_1} \left(\varphi_1(x)\right)^{\ell} u_0(x) \, dx \\
&\leq \; C \int_0^T \int_{\Omega_1} |u| \left(\varphi_1(x)\right)^{\ell} \left| D^{1+\alpha}_{t|T}\varphi_2(t) \right| \, dx \, dt \\
&+ C \int_0^T \int_{\Omega_1} |u|^m \; \left(\varphi_1(x)\right)^{\ell-1} \left| (-\Delta_x)^{\beta/2}\varphi_1(x) \; D^{\alpha}_{t|T}\varphi_2(t) \right| \, dx \, dt \\
= \; & C \int_0^T \int_{\Omega_1} |u| \varphi^{1/p} \varphi^{-1/p} \left(\varphi_1(x)\right)^{\ell} \left| D^{1+\alpha}_{t|T}\varphi_2(t) \right| \, dx \, dt \\
&+ C \int_0^T \int_{\Omega_1} |u|^m \varphi^{m/p} \varphi^{-m/p} \; \left(\varphi_1(x)\right)^{\ell-1} \left| (-\Delta_x)^{\beta/2}\varphi_1(x) \; D^{\alpha}_{t|T}\varphi_2(t) \right| \, dx \, dt, \quad (3.9)
\end{aligned}$$

where

$$\Omega_1 := \left\{ x \in \mathbb{R}^N \; ; \; |x| \leq 2T^{\theta} \right\}.$$

Therefore, using the Young inequality

$$ab \leq \; \frac{1}{2q} \, a^{\,q} \; + \; \frac{2^{q'-1}}{q'} \, b^{\,q'} \quad \text{with} \quad a \geq 0, b \geq 0, q > 1 \quad \text{where} \quad qq' = q + q', \qquad (3.10)$$

and taking account of the positivity of the norm in L^1 of the initial data, we obtain the estimate

$$\begin{aligned}
\int_0^T \int_{\Omega_1} |u|^p \, \varphi \; &\leq \; C \int_0^T \int_{\Omega_1} \left(\varphi_1(x)\right)^{\ell} \left(\varphi_2(t)\right)^{-p'/p} \left| D^{1+\alpha}_{t|T}\varphi_2(t) \right|^{p'} \\
&+ \; C \int_0^T \int_{\Omega_1} \left(\varphi_1(x)\right)^{\ell - \frac{p}{p-m}} \left(\varphi_2(t)\right)^{-\frac{m}{p-m}} \left| (-\Delta_x)^{\beta/2}\varphi_1(x) D^{\alpha}_{t|T}\varphi_2(t) \right|^{\frac{p}{p-m}}.
\end{aligned}$$

At this stage, we introduce the new variables: $\tau = T^{-1}t$, $\xi = T^{-\theta}x$ and the formula (3.8) in the right hand-side of the precedent inequality to obtain:

$$\int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt \leq \; C \, T^{-\delta}, \qquad (3.11)$$

where $\delta := (1+\alpha)p' - 1 - N\theta$, $C = C(|\Omega_2| \, , \, |\Omega_3|)$, where $|\Omega_i|$ stands for the measure of Ω_i, for $i = 2, 3$, with

$$\Omega_2 := \left\{ \xi \in \mathbb{R}^N \; ; \; |\xi| \leq 2 \right\} \quad , \quad \Omega_3 := \left\{ \tau \geq 0 \; ; \; \tau \leq 1 \right\}.$$

Now, we choose

$$\delta \geq 0 \quad \Longleftrightarrow \quad p \leq 1 + \frac{(2-\gamma)(N(m-1)+\beta)}{(N-\beta+\beta\gamma)^+} =: p_F. \tag{3.12}$$

As we want to let T goes to infinity in (3.11), we have to distinguish two cases:
• The sub-critical case $p < p_F$: As $\int_{\mathbb{R}^N} u_0(x)\,dx > 0$, We have from (3.11)

$$\int_0^T \int_{\Omega_1} |u|^p \, \varphi(x,t)\,dx\,dt \leq \; C\,T^{-\delta}, \tag{3.13}$$

therefore

$$\lim_{T\to+\infty} \int_0^T \int_{|x|\leq 2T^\theta} |u|^p \varphi(x,t)\,dx\,dt = 0.$$

Using the dominated convergence theorem, we infer

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p \,dx\,dt = 0 \qquad \Longrightarrow \qquad u = 0 \;\; \text{(a.e.)}.$$

Contradiction.

• The critical case $p = p_F$: From (3.13), we obtain

$$\int_0^{\infty} \int_{\mathbb{R}^N} |u|^p \,dx\,dt \leq C < \infty. \tag{3.14}$$

On the other hand, we recall here that $B \neq 1$ sufficiently large. Then from inequality (3.9) and using the Young inequality (3.10), we have

$$\begin{aligned} \int_0^{BT} \int_{\Omega_1} |u|^p \, \varphi \,dx\,dt \;\; &\leq \;\; C \int_0^{BT} \int_{\Omega_1} \varphi_1^{\ell} \varphi_2^{-p'/p} |D^{1+\alpha}_{t|BT} \varphi_2|^{p'} \,dx\,dt \\ &+ \;\; C \int_0^{BT} \int_{\Omega_1} \varphi_1^{\ell - \frac{p}{p-m}} \varphi_2^{-\frac{m}{p-m}} |(-\Delta_x)^{\beta/2} \varphi_1 \; D^{\alpha}_{t|BT} \varphi_2|^{\frac{p}{p-m}} \,dx\,dt. \end{aligned}$$

Now, by taking the new variables $\xi = T^{-\theta}x$, $\tau = (BT)^{-1}t$ and using the critical exponent p_F, into account of (3.14), we obtain

$$\int_0^{BT} \int_{\Omega_1} |u|^p \, \varphi \leq \; CB^{1-(1+\alpha)p'} \; + \; CB^{1-\alpha\frac{p}{p-m}}.$$

Using that $p\theta > \frac{N\theta m}{N-\beta}$, $N > \beta$ and taking the limit when $B \longrightarrow \infty$, into account the dominated convergence theorem, leads to:

$$\int_0^{\infty} \int_{\mathbb{R}^N} |u|^p \,dx\,dt = 0. \tag{3.15}$$

This completes the proof in the critical case.

• For the case $p < \frac{m}{\gamma}$, we repeat the same technique as in the subcritical case $p < p_F$ by choosing the following test function $\varphi(x,t) = (\varphi_1(x))^{\ell}\,\varphi_2(t)$ where $\varphi_1(x) = \Phi\left(\frac{|x|}{T}\right)$ and $\varphi_2(t) = \Psi\left(\frac{t}{B}\right)$ with $T < B < T^*$,with the same functions Φ and Ψ defined above. We obtain then

$$\begin{aligned}
&\int_0^B \int_{\Sigma_1} |u|^p\,\varphi\,dx\,dt \;+\; C\,B^{-\alpha}\int_{\Sigma_1}(\varphi_1(x))^{\ell}\,u_0(x)\,dx\\
&\leq\; C\int_0^B\int_{\Sigma_1}|u|\,(\varphi_1(x))^{\ell}\left|D_{t|B}^{1+\alpha}\varphi_2(t)\right|\,dx\,dt\\
&+ C\int_0^B\int_{\Sigma_1}|u|\,(\varphi_1(x))^{\ell-1}\left|(-\Delta_x)^{\beta/2}\varphi_1(x)\;D_{t|B}^{\alpha}\varphi_2(t)\right|\,dx\,dt,
\end{aligned}$$

where

$$\Sigma_1 := \left\{x\in\mathbb{R}^N\;;\;|x|\leq 2T\right\}.$$

Now, by Young's inequality, (3.10) and the positivity of $\int u_0$, we get

$$\begin{aligned}
\int_0^B\int_{\Sigma_1}|u|^p\,\varphi \;&\leq\; C\int_0^B\int_{\Sigma_1}(\varphi_1(x))^{\ell}\,(\varphi_2(t))^{-p'/p}\left|D_{t|B}^{1+\alpha}\varphi_2(t)\right|^{p'}\\
&+\; C\int_0^B\int_{\Sigma_1}(\varphi_1(x))^{\ell-\frac{p}{p-m}}\,(\varphi_2(t))^{-\frac{m}{p-m}}\left|(-\Delta_x)^{\beta/2}\varphi_1 D_{t|B}^{\alpha}\varphi_2\right|^{\frac{p}{p-m}}.
\end{aligned}$$

Then, the new variables $\xi = T^{-1}x$, $\tau = B^{-1}t$ and (3.8) lead to the estimate

$$\int_0^B\int_{\Sigma_1}|u|^p(x,t)\,\varphi(x,t)\,dx\,dt \leq\; C\,B^{1-(1+\alpha)p'}T^N \;+\; C\,B^{1-\alpha\frac{p}{p-m}}T^{N-\beta\frac{p}{p-m}}.$$

As $p<\frac{m}{\gamma} \iff 1-\alpha\frac{p}{p-m}<0$, then by taking the limit as $B\longrightarrow\infty$, we get

$$\int_0^{\infty}\int_{\Sigma_1}|u|^p(x,t)\,(\varphi_1(x))^{\ell}\,dx\,dt = 0.$$

Finally, by taking $T\longrightarrow\infty$, we infer the contradiction. □

Remarks.

- We can extend our analysis to the equation

$$u_t = -(-\Delta)^{\beta/2}|u|^m + \int_0^t \frac{\psi(s)|u|(s)^p}{(t-s)^{\gamma}}\,ds, \qquad x\in\mathbb{R}^N, \tag{3.16}$$

  with $\psi\in L^1_{Loc}(\mathbb{R}_+)$, $\psi(t)\geq 0$ for all $t\geq 0$, $\psi\not\equiv 0$, $\psi(\tau BT)\geq C>0$, for any $T>0$, $\tau\in[0,1]$, $1\leq m<p$, and $0<\gamma<1$.

- If we take $m=1$ and $\beta=2$ we obtain the same result that [3].

4 The Case of 2×2 Parabolic Systems

In this section, we consider the system

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma} |v|^p \, ds, \\ \\ v_t - \Delta v = \displaystyle\int_0^t (t-s)^{-\delta} |u|^q \, ds, \end{cases} \tag{4.1}$$

where $pq > 1, 0 < \gamma, \delta < 1$ are real numbers, subject to the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^N.$$

We can write the system (4.1) in the form

$$\begin{cases} D^{\alpha}_{0|t} u_t - \Delta D^{\alpha}_{0|t} u = \Gamma(\alpha) |v|^p, \\ \\ D^{\beta}_{0|t} v_t - \Delta D^{\beta}_{0|t} v = \Gamma(\beta) |u|^q. \end{cases} \tag{4.2}$$

where $\alpha := 1 - \gamma \in (0,1)$ and $\beta := 1 - \delta \in (0,1)$.

We start with the

Definition 3 *We say that the pair (u,v) is a weak solution of (4.2), if*

$$u \in L^q_{Loc}(\mathbb{R}_+ \times \mathbb{R}^N), \; v \in L^p_{Loc}(\mathbb{R}_+ \times \mathbb{R}^N)$$

and

$$\begin{aligned} \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^N} |v|^p \, \varphi \, dx \, dt &= \int_0^T \int_{\mathbb{R}^N} u \, D^{1+\alpha}_{t|T} \varphi \, dx \, dt - \int_{\mathbb{R}^N} u_0(x) \left(D^{\alpha}_{t|T} \varphi \right)(0) \, dx \\ &- \int_0^T \int_{\mathbb{R}^N} u \, \Delta D^{\alpha}_{t|T} \varphi \, dx \, dt, \end{aligned}$$

$$\begin{aligned} \Gamma(\beta) \int_0^T \int_{\mathbb{R}^N} |u|^q \, \varphi \, dx \, dt &= \int_0^T \int_{\mathbb{R}^N} v \, D^{1+\beta}_{t|T} \varphi \, dx \, dt - \int_{\mathbb{R}^N} v_0(x) \left(D^{\beta}_{t|T} \varphi \right)(0) \, dx \\ &- \int_0^T \int_{\mathbb{R}^N} v \, \Delta D^{\beta}_{t|T} \varphi \, dx \, dt, \end{aligned}$$

for any nonnegative test function $\varphi \in C^{\infty}([0,T], C^{\infty}_c(\mathbb{R}^N))$, with $D^{\alpha}_{t|T} \varphi(T,x) = D^{\beta}_{t|T} \varphi(T,x) = 0$, for all $T > 0$.

Our result concerning the system (4.1) is the

Theorem 3 *(Local existence)*
If $u_0, v_0 \in C([0,T], C_0(\mathbb{R}^N))$, there exist a time $0 < T_{max} \leq +\infty$ and solution (u,v) of the system (4.1) in $C([0,T_{max}), C_0(\mathbb{R}^N) \bigoplus C_0(\mathbb{R}^N))$.

Proof. usually the proof support to the fixe point theorem. We begin by remembering of the mild solution of the system (4.1) :

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\gamma} |v|^p(\sigma)\, d\sigma\, ds, \\ \\ v(t) = e^{t\Delta}v_0 + \displaystyle\int_0^t e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\delta} |u|^q(\sigma)\, d\sigma\, ds. \end{cases} \tag{4.3}$$

Let

$$E_T = \left\{ (u,v) \in L^\infty((0,T), C_0(\mathbb{R}^N) \oplus C_0(\mathbb{R}^N)); \;\; |||(u,v)||| \leq 2\left(||u_0||_\infty + ||v_0||_\infty\right) \right\}, \tag{4.4}$$

for arbitrary $T > 0$, where $|||.|||$ is the norm of the compact vectorial space E_T define by:

$$|||(u,v)||| := ||u||_{L^\infty((0,T)\times\mathbb{R}^N)} + ||v||_{L^\infty((0,T)\times\mathbb{R}^N)}.$$

After that we define a new close space in E_T, as we obtain a nonnegativity solution, by $P_T := \{(u,v) \in E_T \; ; \; u \geq 0, v \geq 0\}$. Now we take the new function $\Psi(u,v) := (\Psi_1(u,v), \Psi_2(u,v))$, where

$$\Psi_1(u,v) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\gamma} v^p(\sigma)\, d\sigma\, ds,$$

and

$$\Psi_2(u,v) := e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\delta} u^q(\sigma)\, d\sigma\, ds.$$

Here we mean by $\oplus$ the following sense: $(u,v) \in L^\infty((0,T), C_0(\mathbb{R}^N) \oplus C_0(\mathbb{R}^N)) \Longleftrightarrow u, v \in L^\infty((0,T), C_0(\mathbb{R}^N))$.
Either $u, v \in L^\infty((0,T), C_0(\mathbb{R}^N))$, then

$$\begin{aligned} |||\Psi(u,v)||| &:= ||\Psi_1(u,v)||_{L^\infty((0,T)\times\mathbb{R}^N)} + ||\Psi_2(u,v)||_{L^\infty((0,T)\times\mathbb{R}^N)} \\ &:= ||e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\gamma} v^p(\sigma)\, d\sigma\, ds||_{L^\infty((0,T)\times\mathbb{R}^N)} \\ &+ ||e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\delta} u^q(\sigma)\, d\sigma\, ds||_{L^\infty((0,T)\times\mathbb{R}^N)} \\ &\leq ||e^{t\Delta}u_0||_{L^\infty((0,T)\times\mathbb{R}^N)} + || \int_0^t ||e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\gamma} v^p(\sigma)\, d\sigma||_{L^\infty(\mathbb{R}^N)}\, ds||_{L^\infty((0,T))} \end{aligned}$$

$$
\begin{aligned}
&+ \quad ||e^{t\Delta}v_0||_{L^\infty((0,T)\times\mathbb{R}^N)} + || \int_0^t ||e^{(t-s)\Delta} \int_0^s (s-\sigma)^{-\delta} u^q(\sigma)\, d\sigma||_{L^\infty(\mathbb{R}^N)}\, ds||_{L^\infty((0,T))} \\
&\leq \quad ||u_0||_{L^\infty(\mathbb{R}^N)} + || \int_0^t \int_0^s (s-\sigma)^{-\gamma} ||v^p(\sigma)||_{L^\infty(\mathbb{R}^N)}\, d\sigma\, ds||_{L^\infty((0,T))} \\
&+ \quad ||v_0||_{L^\infty(\mathbb{R}^N)} + || \int_0^t \int_0^s (s-\sigma)^{-\delta} ||u^q(\sigma)||_{L^\infty(\mathbb{R}^N)}\, d\sigma\, ds||_{L^\infty((0,T))},
\end{aligned}
$$

by using the contraction property for the semi-group of the Laplacian operator

$$
||e^{t\Delta}u||_{L^\infty(\mathbb{R}^N)} \leq ||u||_{L^\infty(\mathbb{R}^N)} \quad \text{for all } t \geq 0.
$$

Hence by change of the Lebesgue theorem, we have

$$
\begin{aligned}
|||\Psi(u,v)||| &\leq ||u_0||_{L^\infty(\mathbb{R}^N)} + || \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} ||v^p(\sigma)||_{L^\infty(\mathbb{R}^N)}\, ds\, d\sigma||_{L^\infty((0,T))} \\
&+ ||v_0||_{L^\infty(\mathbb{R}^N)} + || \int_0^t \int_\sigma^t (s-\sigma)^{-\delta} ||u^q(\sigma)||_{L^\infty(\mathbb{R}^N)}\, ds\, d\sigma||_{L^\infty((0,T))} \\
&= ||u_0||_{L^\infty(\mathbb{R}^N)} + \frac{1}{1-\gamma} || \int_0^t (t-\sigma)^{1-\gamma} ||v^p(\sigma)||_{L^\infty(\mathbb{R}^N)}\, d\sigma||_{L^\infty((0,T))} \\
&+ ||v_0||_{L^\infty(\mathbb{R}^N)} + \frac{1}{1-\delta} || \int_0^t (t-\sigma)^{1-\delta} ||u^q(\sigma)||_{L^\infty(\mathbb{R}^N)}\, d\sigma||_{L^\infty((0,T))} \\
&\leq ||u_0||_{L^\infty(\mathbb{R}^N)} + \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)} ||v||^p_{L^\infty((0,T)\times\mathbb{R}^N)} \\
&+ ||v_0||_{L^\infty(\mathbb{R}^N)} + \frac{T^{2-\delta}}{(1-\delta)(2-\delta)} ||u||^q_{L^\infty((0,T)\times\mathbb{R}^N)} \\
&\leq (||u_0||_{L^\infty(\mathbb{R}^N)} + ||v_0||_{L^\infty(\mathbb{R}^N)}) \\
&+ \Lambda(||u_0||_{L^\infty(\mathbb{R}^N)} + ||v_0||_{L^\infty(\mathbb{R}^N)}),
\end{aligned}
$$

where

$$
\Lambda := 2\max\{2^{p-1}||u_0||^{p-1}_{L^\infty}; 2^{q-1}||v_0||^{q-1}_{L^\infty}\} \left(\frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)} + \frac{T^{2-\delta}}{(1-\delta)(2-\delta)} \right).
$$

Then, if we choose the time T small enough such that $\Lambda \leq 1$, we infer

$$
|||\Psi(u,v)||| \leq 2\left(||u_0||_{L^\infty} + ||v_0||_{L^\infty}\right)
$$

and by using the positivity of the semi-group $e^{t\Delta}$, we obtain $\Psi(u,v) \in P_T$.
Next, we take $u, \widetilde{u}, v, \widetilde{v} \in L^\infty((0,T), C_0(\mathbb{R}^N))$ with $u(t=0) = \widetilde{u}(t=0)$ and $v(t=0) = \widetilde{v}(t=0)$, we obtain

$$
|||\Psi(u,v) - \Psi(\widetilde{u},\widetilde{v})||| \quad \leq \quad || \int_0^t \int_0^s (s-\sigma)^{-\gamma} ||v^p(\sigma) - \widetilde{v}^p(\sigma)||_{L^\infty(\mathbb{R}^N)}\, d\sigma\, ds||_{L^\infty((0,T))}
$$

$$
\begin{aligned}
&+ \quad || \int_0^t \int_0^s (s-\sigma)^{-\delta} ||u^q(\sigma) - \widetilde{u}^q(\sigma)||_{L^\infty(\mathbb{R}^N)} \, d\sigma \, ds||_{L^\infty((0,T))} \\
&= \quad || \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} ||v^p(\sigma) - \widetilde{v}^p(\sigma)||_{L^\infty(\mathbb{R}^N)} \, ds \, d\sigma||_{L^\infty((0,T))} \\
&+ \quad || \int_0^t \int_\sigma^t (s-\sigma)^{-\delta} ||u^q(\sigma) - \widetilde{u}^q(\sigma)||_{L^\infty(\mathbb{R}^N)} \, ds \, d\sigma||_{L^\infty((0,T))} \\
&= \quad \frac{1}{1-\gamma} || \int_0^t (t-\sigma)^{1-\gamma} ||v^p(\sigma) - \widetilde{v}^p(\sigma)||_{L^\infty(\mathbb{R}^N)} \, d\sigma||_{L^\infty((0,T))} \\
&+ \quad \frac{1}{1-\delta} || \int_0^t (t-\sigma)^{1-\delta} ||u^q(\sigma) - \widetilde{u}^q(\sigma)||_{L^\infty(\mathbb{R}^N)} \, d\sigma||_{L^\infty((0,T))} \\
&\leq \quad \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)} ||v - \widetilde{v}||^p_{L^\infty((0,T)\times\mathbb{R}^N)} \\
&+ \quad \frac{T^{2-\delta}}{(1-\delta)(2-\delta)} ||u - \widetilde{u}||^q_{L^\infty((0,T)\times\mathbb{R}^N)} \\
&\leq \quad \frac{\Lambda}{2} |||(u-\widetilde{u}), (v-\widetilde{v})||| \\
&\leq \quad \frac{1}{2} |||(u-\widetilde{u}), (v-\widetilde{v})|||.
\end{aligned}
$$

Then by the Point fix theorem, the system (5.1) admits a unique solution $(u, v) \in L^\infty((0,T), C_0(\mathbb{R}^N) \oplus C_0(\mathbb{R}^N))$. Note that its easily to proof that the solution $(u, v) \in C((0,T), C_0(\mathbb{R}^N) \oplus C_0(\mathbb{R}^N))$. Moreover, we choose the maximal time $T_{\max}$ as

$$T_{\max} := \sup\{T > 0; \ (u, v) \text{ is a solutionn to the system (5.1) in } C((0,T), C_0(\mathbb{R}^N) \oplus C_0(\mathbb{R}^N))\}.$$

Theorem 4 *Let $u_0, v_0 \not\equiv 0$. If*

$$\frac{N}{2} \leq \max\left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1} \ ; \ \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\} \tag{4.5}$$

or

$$p < \frac{1}{\gamma} \qquad \text{and} \qquad q < \frac{1}{\delta}$$

and $u_0, v_0 \geq 0$, then any nonnegative solution (u, v) (i.e. $u \geq 0$ and $v \geq 0$) of the system (4.1) blows-up in a finite time.
For the critical case in (4.5), we need that $N > 2$ and $pq > \frac{N}{N-2} > 1$.

Proof. Let (u, v) be a nonnegative solution of the system (4.1) with $u \in L^q\left([0, T^*), L^q_{Loc}(\mathbb{R}^N)\right)$ and $v \in L^p\left([0, T^*), L^p_{Loc}(\mathbb{R}^N)\right)$ for fixed $T^* > 0$.
Now, if we take as a test function $\varphi = (\varphi_1(x))^{2} \varphi_2(t)$, with $\varphi_1(x) := \Phi\left(\frac{|x|}{T^{1/2}}\right)$ and $\varphi_2(t) := \Psi\left(\frac{t}{BT}\right)$, where

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2. \end{cases}$$

non-increasing smooth function, $0 \le \Phi \le 1$, $B < T < T^*$, $|\Phi^{'}(r)| \le C_2/r$, for all $r > 0$, and

$$\Psi(r) = \begin{cases} (1-r)^{\zeta} & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 1, \end{cases}$$

where $\zeta \ge \frac{(1+\alpha)(1+\beta)pq}{(p-1)(q-1)}$ is large enough such that $\Psi \in C^{\infty}$, we obtain

$$\begin{aligned} \int_0^T \int_{\Omega_1} v^p \, \varphi \, dx \, dt &= C \int_0^T \int_{\Omega_1} u \, (\varphi_1)^2 \, D_{t|T}^{1+\alpha} \varphi_2 - C \int_0^T \int_{\Omega_1} u \, \Delta(\varphi_1^2) D_{t|T}^{\alpha}(\varphi_2) \\ &- C \, T^{-\alpha} \int_{\mathbb{R}^N} (\varphi_1(x))^2 \, u_0 \, dx. \end{aligned}$$

Note that as in the last section, we assume that $B = 1$ in this case and $B \neq 1$ in the case of equality in (4.5).
Now, as $-\Delta\varphi_1^2 \le -2\varphi_1 \Delta\varphi_1$, we may write

$$\begin{aligned} &\int_0^T \int_{\Omega_1} v^p \, \varphi \, + \, C \, T^{-\alpha} \int_{\Omega_1} (\varphi_1(x))^2 \, u_0(x) \\ &\le \, C \int_0^T \int_{\Omega_1} u \, (\varphi_1(x))^2 \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| + C \int_0^T \int_{\Omega_2} u \, \varphi_1(x) \, |\Delta\varphi_1(x)| \left| D_{t|T}^{\alpha} \varphi_2(t) \right| \end{aligned} \tag{4.6}$$

which via the Hölder inequality leads to

$$\begin{aligned} &\int_0^T \int_{\Omega_1} v^p \, \varphi \, + \, C \, T^{-\alpha} \int_{\Omega_1} (\varphi_1(x))^2 \, u_0(x) \, dx \\ &\le \, C \left(\int_0^T \int_{\Omega_1} u^q \, \varphi \right)^{1/q} \mathcal{A}, \end{aligned}$$

where we have set

$$\mathcal{A} := \left(\int_0^T \int_{\Omega_1} \varphi_1^2 \varphi_2^{-q^{'}/q} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{q^{'}} \right)^{1/q^{'}} + \left(\int_0^T \int_{\Omega_2} \varphi_1^{\frac{q-2}{q-1}} \varphi_2^{-q^{'}/q} \left| \Delta_x \varphi_1 \right|^{q^{'}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^{q^{'}} \right)^{1/q^{'}}.$$

Via the scaled variables (ξ, τ) as above and taking account of (3.8), we obtain

$$\int_0^T \int_{\Omega_1} \varphi_1^2 \varphi_2^{-q^{'}/q} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{q^{'}} \, \le \, C \, T^{-(1+\alpha)q^{'}+1+\frac{N}{2}},$$

and

$$\int_0^T \int_{\Omega_2} \varphi_1^{\frac{q-2}{q-1}} \varphi_2^{-q'/q} \left|\Delta_x \varphi_1\right|^{q'} \left|D_{t|T}^{\alpha} \varphi_2\right|^{q'} \leq C\, T^{-(1+\alpha)q'+1+\frac{N}{2}}.$$

Whereupon

$$\mathcal{A} \leq C\, T^{-1-\alpha+\left(1+\frac{N}{2}\right)\frac{1}{q'}}.$$

Similarly, we obtain

$$\int_0^T \int_{\Omega_1} u^q\, \varphi \,+\, C\, T^{-\beta} \int_{\Omega_1} \left(\varphi_1(x)\right)^2 v_0 \leq \left(\int_0^T \int_{\Omega_1} v^p\, \varphi\right)^{1/p} \mathcal{B},$$

where

$$\begin{aligned} \mathcal{B} \;&:=\; \left(\int_0^T \int_{\Omega_1} \varphi_1^2 \varphi_2^{-p'/p} \left|D_{t|T}^{1+\beta} \varphi_2\right|^{p'}\right)^{1/p'} + \left(\int_0^T \int_{\Omega_2} \varphi_1^{\frac{p-2}{p-1}} \varphi_2^{-p'/p} \left|\Delta_x \varphi_1\right|^{p'} \left|D_{t|T}^{\beta} \varphi_2\right|^{p'}\right)^{1/p'} \\ &\leq\; C\, T^{-1-\beta+\left(1+\frac{N}{2}\right)\frac{1}{p'}}. \end{aligned}$$

For later use, using the non-negativity of initial condition, we may write

$$\begin{cases} \displaystyle\int_0^T \int_{\Omega_1} v^p\, \varphi\, dx\, dt \;\leq\; \left(\int_0^T \int_{\Omega_1} u^q\, \varphi\, dx\, dt\right)^{1/q} \mathcal{A}, \\ \displaystyle\int_0^T \int_{\Omega_1} u^q\, \varphi\, dx\, dt \leq \left(\int_0^T \int_{\Omega_1} v^p\, \varphi\, dx\, dt\right)^{1/q} \mathcal{B}. \end{cases} \tag{4.7}$$

The former estimates can be arranged into:

$$\begin{cases} \displaystyle\int_0^T \int_{\Omega_1} v^p\, \varphi\, dx\, dt \leq \left(\int_0^T \int_{\Omega_1} v^p\, \varphi\, dx\, dt\right)^{1/pq} \mathcal{B}^{1/q} \mathcal{A}, \\ \displaystyle\int_0^T \int_{\Omega_1} u^q\, \varphi\, dx\, dt \leq \left(\int_0^T \int_{\Omega_1} u^q\, \varphi\, dx\, dt\right)^{1/pq} \mathcal{A}^{1/p} \mathcal{B}. \end{cases} \tag{4.8}$$

Finally

$$\begin{cases} \left(\displaystyle\int_0^T \int_{\Omega_1} v^p\, \varphi\, dx\, dt\right)^{1-\frac{1}{pq}} \leq C\, T^{\theta_1}, \\ \left(\displaystyle\int_0^T \int_{\Omega_1} u^q\, \varphi\, dx\, dt\right)^{1-\frac{1}{pq}} \leq C\, T^{\theta_2}, \end{cases} \tag{4.9}$$

with

$$\theta_1 := \left(-q^{'} - \alpha q^{'} + 1 + \frac{N}{2}\right)\frac{1}{q^{'}} + \left(-p^{'} - \beta p^{'} + 1 + \frac{N}{2}\right)\frac{1}{qp^{'}}, \tag{4.10}$$

and

$$\theta_2 := \left(-q^{'} - \alpha q^{'} + 1 + \frac{N}{2}\right)\frac{1}{pq^{'}} + \left(-p^{'} - \beta p^{'} + 1 + \frac{N}{2}\right)\frac{1}{p^{'}}. \tag{4.11}$$

Note that the inequality (4.5) is equivalent to $\theta_1 \leq 0$ or $\theta_2 \leq 0$.

• If $\theta_1 < 0$ or $\theta_2 < 0$, then by taking the limit as T goes to infinity in (4.9), we conclude that $u \equiv 0$ or $v \equiv 0$ and (4.7) implies that $u \equiv v \equiv 0$. Contradiction.

• In the critical case, if $\theta_1 = 0$ or $\theta_2 = 0$, we have from (4.9)

$$\int_0^\infty \int_{\mathbb{R}^N} v^p(x,t)\,dx\,dt < +\infty \implies \lim_{T\longrightarrow\infty} \int_0^{BT} \int_{\Omega_2} v^p(x,t)\,\varphi(x,t)\,dx\,dt = 0, \tag{4.12}$$

or

$$\int_0^\infty \int_{\mathbb{R}^N} u^q(x,t)\,dx\,dt < +\infty \implies \lim_{T\longrightarrow\infty} \int_0^{BT} \int_{\Omega_2} u^q(x,t)\,\varphi(x,t)\,dx\,dt = 0. \tag{4.13}$$

Here $B \neq 1$. Then again by taking the Hölder inequality in the right hand-side of (4.6) and the non-negativity of initial data, we have

$$\begin{aligned}
\int_0^{BT} \int_{\Omega_1} v^p\,\varphi \;&\leq\; C\left(\int_0^{BT} \int_{\Omega_1} u^q\,\varphi\right)^{1/q} \left(\int_0^{BT} \int_{\Omega_1} \varphi_1^2 \varphi_2^{-q^{'}/q} \left|D^{1+\alpha}_{t|BT}\varphi_2\right|^{q^{'}}\right)^{1/q^{'}} \\
&+\; C\left(\int_0^{BT} \int_{\Omega_2} u^q\,\varphi\right)^{1/q} \left(\int_0^{BT} \int_{\Omega_2} \varphi_1^{\frac{q-2}{q-1}} \varphi_2^{-q^{'}/q} \left|\Delta_x \varphi_1\right|^{q^{'}} \left|D^{\alpha}_{t|BT}\varphi_2\right|^{q^{'}}\right)^{1/q^{'}}.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\int_0^{BT} \int_{\Omega_1} u^q\,\varphi \;&\leq\; C\left(\int_0^{BT} \int_{\Omega_1} v^p\,\varphi\right)^{1/p} \left(\int_0^{BT} \int_{\Omega_1} \varphi_1^2 \varphi_2^{-p^{'}/p} \left|D^{1+\beta}_{t|BT}\varphi_2\right|^{p^{'}}\right)^{1/p^{'}} \\
&+\; C\left(\int_0^{BT} \int_{\Omega_2} v^p\,\varphi\right)^{1/p} \left(\int_0^{BT} \int_{\Omega_2} \varphi_1^{\frac{p-2}{p-1}} \varphi_2^{-p^{'}/p} \left|\Delta_x \varphi_1\right|^{p^{'}} \left|D^{\beta}_{t|BT}\varphi_2\right|^{p^{'}}\right)^{1/p^{'}}.
\end{aligned}$$

Then the new variables $\xi = T^{-1/2}x$ and $\tau = (BT)^{-1}t$, allow us to write

$$\begin{aligned}
\int_0^{BT} \int_{\Omega_1} v^p\,\varphi \;&\leq\; CT^{\left(1+N/2-(1+\alpha)q^{'}\right)\frac{1}{q^{'}}} B^{\left(1-(1+\alpha)q^{'}\right)\frac{1}{q^{'}}} \left(\int_0^{BT} \int_{\Omega_1} u^q\,\varphi\right)^{1/q} \\
&+\; CT^{\left(1+N/2-(1+\alpha)q^{'}\right)\frac{1}{q^{'}}} B^{\left(1-\alpha q^{'}\right)\frac{1}{q^{'}}} \left(\int_0^{BT} \int_{\Omega_2} u^q\,\varphi\right)^{1/q}
\end{aligned} \tag{4.14}$$

and

$$\int_0^{BT}\int_{\Omega_1} u^q\,\varphi \;\leq\; CT^{\left(1+N/2-(1+\beta)p'\right)\frac{1}{p'}}B^{\left(1-(1+\beta)p'\right)\frac{1}{p'}}\left(\int_0^{BT}\int_{\Omega_1} v^q\,\varphi\right)^{1/p} + CT^{\left(1+N/2-(1+\beta)p'\right)\frac{1}{p'}}B^{\left(1-\beta p'\right)\frac{1}{p'}}\left(\int_0^{BT}\int_{\Omega_2} v^p\,\varphi\right)^{1/p}. \quad (4.15)$$

Now, by combination to (4.14) and (4.15) and using the critical exponent ($\theta_1 = 0$ or $\theta_2 = 0$), we infer

$$\begin{aligned}\int_0^{BT}\int_{\Omega_1} v^p\,\varphi \;\leq\;& CB^{\left(1-(1+\beta)p'\right)\frac{1}{qp'}+\left(1-(1+\alpha)q'\right)\frac{1}{q'}}\left(\int_0^{BT}\int_{\Omega_1} v^p\,\varphi\right)^{1/pq}\\ &+ CB^{\left(1-\beta p'\right)\frac{1}{qp'}+\left(1-(1+\alpha)q'\right)\frac{1}{q'}}\left(\int_0^{BT}\int_{\Omega_2} v^p\,\varphi\right)^{1/pq}\\ &+ CB^{\left(1-(1+\beta)p'\right)\frac{1}{qp'}+\left(1-\alpha q'\right)\frac{1}{q'}}\left(\int_0^{BT}\int_{\Omega_1} v^p\,\varphi\right)^{1/pq}\\ &+ CB^{\left(1-\beta p'\right)\frac{1}{qp'}+\left(1-\alpha q'\right)\frac{1}{q'}}\left(\int_0^{BT}\int_{\Omega_2} v^p\,\varphi\right)^{1/pq} \quad (4.16)\end{aligned}$$

or

$$\begin{aligned}\int_0^{BT}\int_{\Omega_1} u^q\,\varphi \;\leq\;& CB^{\left(1-(1+\alpha)q'\right)\frac{1}{pq'}+\left(1-(1+\beta)p'\right)\frac{1}{p'}}\left(\int_0^{BT}\int_{\Omega_1} u^q\,\varphi\right)^{1/pq}\\ &+ CB^{\left(1-\alpha q'\right)\frac{1}{pq'}+\left(1-(1+\beta)p'\right)\frac{1}{p'}}\left(\int_0^{BT}\int_{\Omega_2} u^q\,\varphi\right)^{1/pq}\\ &+ CB^{\left(1-(1+\alpha)q'\right)\frac{1}{pq'}+\left(1-\beta p'\right)\frac{1}{p'}}\left(\int_0^{BT}\int_{\Omega_1} u^q\,\varphi\right)^{1/pq}\\ &+ CB^{\left(1-\alpha q'\right)\frac{1}{pq'}+\left(1-\beta p'\right)\frac{1}{p'}}\left(\int_0^{BT}\int_{\Omega_2} u^q\,\varphi\right)^{1/pq}. \quad (4.17)\end{aligned}$$

Moreover, taking the limit as T goes to infinity in (4.16) and (4.17) and using ((4.12) and (4.13)) and the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}\left(\int_0^{\infty}\int_{\mathbb{R}^N} v^p\,\varphi(x,t)\,dx\,dt\right)^{1-1/pq} \leq\;& CB^{\left(1-(1+\beta)p'\right)\frac{1}{qp'}+\left(1-(1+\alpha)q'\right)\frac{1}{q'}}\\ &+ CB^{\left(1-(1+\beta)p'\right)\frac{1}{qp'}+\left(1-\alpha q'\right)\frac{1}{q'}} \quad (4.18)\end{aligned}$$

or

$$\left(\int_0^{\infty}\int_{\mathbb{R}^N} u^q\,\varphi(x,t)\,dx\,dt\right)^{1-1/pq} \leq\; CB^{\left(1-(1+\alpha)q'\right)\frac{1}{pq'}+\left(1-(1+\beta)p'\right)\frac{1}{p'}}$$

$$+ \quad CB^{\left(1-(1+\alpha)q^{'}\right)\frac{1}{pq^{'}}+\left(1-\beta p^{'}\right)\frac{1}{p^{'}}}. \qquad (4.19)$$

Again with using the critical exponent ($\theta_1 = 0$ or $\theta_2 = 0$) and the fact that $pq > \frac{N}{N-2}$, $N > 2$ we infer after letting B going to infinity, that

$$\left(\int_0^{\infty}\int_{\mathbb{R}^N} v^p\ \varphi(x,t)\,dx\,dt\right)^{1-1/pq} = 0$$

or

$$\left(\int_0^{\infty}\int_{\mathbb{R}^N} u^q\ \varphi(x,t)\,dx\,dt\right)^{1-1/pq} = 0.$$

Then $u = 0$ or $v = 0$ a.e. and by (4.14) and (4.15) we have $u = v = 0$ a.e.;

• If $p < \frac{1}{\gamma}$ and $q < \frac{1}{\delta}$, by the similar process doing in the subcritical case and using the following new test function $\varphi(x,t) = (\varphi_3(x))^2\,\varphi_4(t)$ with $\varphi_3(x) := \Phi\left(\frac{|x|}{T}\right)$, $\varphi_4(t) := \Psi\left(\frac{t}{B}\right)$ where Φ and Ψ are the same and $T < B < T^*$, we obtain

$$\begin{cases} \left(\int_0^{T}\int_{\Sigma_1} v^p\ \varphi(x,t)\,dx\,dt\right)^{1-\frac{1}{pq}} \leq\ C\mathcal{J}^{1/q}\mathcal{I}, \\ \\ \left(\int_0^{T}\int_{\Sigma_1} u^q\ \varphi(x,t)\,dx\,dt\right)^{1-\frac{1}{pq}} \leq\ \mathcal{I}^{1/p}\mathcal{J}, \end{cases} \qquad (4.20)$$

where

$$\mathcal{I} := \left(\int_0^{B}\int_{\Sigma_1} \varphi_3^2\varphi_4^{-q^{'}/q}\left|D_{t|B}^{1+\alpha}\varphi_4\right|^{q^{'}}\right)^{1/q^{'}} + \left(\int_0^{B}\int_{\Sigma_2} \varphi_3^{\frac{q-2}{q-1}}\varphi_4^{-q^{'}/q}\left|\Delta_x\varphi_3\right|^{q^{'}}\left|D_{t|B}^{\alpha}\varphi_4\right|^{q^{'}}\right)^{1/q^{'}}$$

and

$$\mathcal{J} := \left(\int_0^{B}\int_{\Sigma_1} \varphi_3^2\varphi_4^{-p^{'}/p}\left|D_{t|B}^{1+\beta}\varphi_4\right|^{p^{'}}\right)^{1/p^{'}} + \left(\int_0^{B}\int_{\Sigma_2} \varphi_3^{\frac{p-2}{p-1}}\varphi_4^{-p^{'}/p}\left|\Delta_x\varphi_3\right|^{p^{'}}\left|D_{t|B}^{\beta}\varphi_4\right|^{p^{'}}\right)^{1/p^{'}}.$$

Now, if we take the new variables $\xi = T^{-1}x$ and $\tau = B^{-1}t$, we get

$$\mathcal{I} \leq B^{1-(1+\alpha)q^{'}}T^N\ +\ B^{1-\alpha q^{'}}T^{N-2q^{'}}$$

and

$$\mathcal{J} \leq B^{1-(1+\beta)p^{'}}T^N\ +\ B^{1-\beta p^{'}}T^{N-2p^{'}}.$$

Finally, using the conditions $p < \frac{1}{\gamma}$ and $q < \frac{1}{\delta}$, and passing to the limit as B going to infinity, we infer $\mathcal{I}$ and $\mathcal{J}$ going to 0 and by consequence $u = v = 0$ a.e.;

this ends the proof.

The same strategy allows us to obtain nonexistence results for nonlocal hyperbolic equations and systems.

5 A Fractional Hyperbolic Equation

In this section, we consider the hyperbolic problem

$$\begin{cases} u_{tt} + (-\Delta)^{\alpha/2} u = \displaystyle\int_0^t (t-s)^{-\gamma} u^p \, ds, & x \in \mathbb{R}^N, \ t > 0, \\ u_t(x,0) = f_1(x), & x \in \mathbb{R}^N, \\ u(x,0) = f_2(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.1}$$

with $\alpha \in (0,2]$, $\gamma \in (0,1)$ and $p > 1$ is a real number. In(5.1), $u \geq 0$.

Like in the last sections, we write equation (5.1) in the form

$$D^{\beta}_{0|t} \, u_{tt} + (-\Delta)^{\alpha/2} D^{\beta}_{0|t} \, u \ = \ \Gamma(\beta) u^p, \tag{5.2}$$

where $\beta := 1 - \gamma \in (0,1)$.

The definition for a weak solution for (5.1) we adopt is

Definition 4 *We say that u is a very weak solution of* (5.1), *if*

$$u \in L^p_{Loc}\left(\mathbb{R}_+ \times \mathbb{R}^N\right),$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \left(\int_0^s (t-s)^{-\gamma} u^p \, ds \right) \varphi \ &= \ \int_0^T \int_{\mathbb{R}^N} u \, D^2 \varphi - \int_{\mathbb{R}^N} f_1(x) \varphi(0,x) \\ &+ \ \int_{\mathbb{R}^N} f_2(x) D\varphi(0,x) + \int_0^T \int_{\mathbb{R}^N} u \, (-\Delta)^{\alpha/2} \varphi, \end{aligned}$$

for any nonnegative test function $\varphi \in C^{\infty}([0,T], C^{\infty}_c(\mathbb{R}^N))$, with $\varphi(T,x) = D\varphi(T,x) = 0$, for all $x \in \mathbb{R}^N$, $T > 0$.

Definition 5 *We say that u is a weak solution of* (5.2), *if*

$$u \in L^p_{Loc}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$$

and

$$\Gamma(\beta)\int_0^T\int_{\mathbb{R}^N} u^p\,\varphi \;=\; \int_0^T\int_{\mathbb{R}^N} u\,D_{t|T}^{2+\beta}\varphi - \int_{\mathbb{R}^N} f_1(x)\left(D_{t|T}^{\beta}\varphi\right)(0)$$
$$-\;\int_{\mathbb{R}^N} f_2(x)\left(D_{t|T}^{1+\beta}\varphi\right)(0) + \int_0^T\int_{\mathbb{R}^N} u\,(-\Delta)^{\alpha/2}D_{t|T}^{\beta}\varphi, \quad (5.3)$$

for any nonnegative test function $\varphi \in C^\infty([0,T], C_c^\infty(\mathbb{R}^N))$, *with* $D_{t|T}^{1+\beta}\varphi(T,x) = 0$ *and* $D_{t|T}^{\beta}\varphi(T,x) = 0$, *for all* $x \in \mathbb{R}^N$, $T > 0$.

Note that the weak solutions in (5.1) and (5.2) are equivalent.

Now, we are ready to state our theorem for problem (5.1).

Theorem 5 *Let* $f_1(x), f_2(x) \geq 0$ *for all* $x \in \mathbb{R}^N$. *If*

$$p \;\leq\; 1 \;+\; \frac{(3-\gamma)\alpha}{(2N-2\alpha+\gamma\alpha)^+}, \quad (5.4)$$

or

$$p < \frac{1}{\gamma}$$

then problem (5.1) *admits no global nonnegative weak solutions other than the trivial one.*
Note that in the critical case we are interest in the dimensions $N > \alpha$ *and* $p > \frac{N}{N-\alpha}$.

Proof. The proof proceeds by contradiction. Let u be a nontrivial nonnegative global weak solution of the equation (5.1). Then u verifies the weak formulation (5.3) in $L^p\left([0,T^*), L^p_{Loc}\mathbb{R}^N\right)$, for $T^* > 0$.
Setting as a test function in (5.3) $\varphi(x,t) = (\varphi_1(x))^{\ell}\,\varphi_2(t)$, such that $\varphi_1(x) := \Phi\left(\frac{|x|}{T^{2/\alpha}}\right)$ and $\varphi_2(t) := \Psi\left(\frac{t}{BT}\right)$, where Φ is the same function as in the proof of theorem 1, $\ell := \frac{2p-1}{p-1}$ and

$$\Psi(r) = \begin{cases} (1-r)^k & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 1, \end{cases}$$

with $k \geq \frac{(2+\beta)p}{p-1}$ sufficiently large.
Note that, as above, we take $B = 1$ in the subcritical case $p < 1 \;+\; \frac{(3-\gamma)\alpha}{(2N-2\alpha+\gamma\alpha)^+}$, while $B \neq 1$ with $B < T$ in the critical case.

Using Ju's inequality (see Appendix)

$$(-\Delta)^{\delta/2}\psi^q \;\le\; q\psi^{q-1}\,(-\Delta)^{\delta/2}\psi, \quad \text{for any } \delta\in[0,2],\; q\ge 1, \psi\ge 0,\; \psi\in C_c^\infty(\mathbb{R}^N),$$

we have

$$\begin{aligned}&\int_0^T\int_{\Omega_6} u^p\,\varphi \;+\; C\,T^{-\beta}\int_{\Omega_6}(\varphi_1(x))^\ell\, f_1(x) \;+\; C\,T^{-\beta-1}\int_{\Omega_6}(\varphi_1(x))^\ell\, f_2(x)\\ &\le\; C\int_0^T\int_{\Omega_6} u\,\varphi_1^\ell\left|D_{t|T}^{2+\beta}\varphi_2\right|\,dx\,dt \;+\; C\int_0^T\int_{\Omega_6} u\,\varphi_1^{\ell-1}\left|(-\Delta)^{\alpha/2}\varphi_1\right|\left|D_{t|T}^{\beta}\varphi_2\right|\,dx\,dt\end{aligned}$$

where

$$\Omega_6 := \left\{x\in\mathbb{R}^N\;;\;\; |x|\le 2T^{\frac{2}{\alpha}}\right\}.$$

Now, using Young's inequality (3.10) and the non-negativity of f_1 and f_2, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega_6} u^p\,\varphi \;&\le\; C\int_0^T\int_{\Omega_6}\varphi_1^\ell\varphi_2^{-p'/p}\left|D_{t|T}^{2+\beta}\varphi_2\right|^{p'}\\ &+\; C\int_0^T\int_{\Omega_6}\varphi_1\varphi_2^{-p'/p}\left|(-\Delta)^{\alpha/2}\varphi_1\right|^{p'}\left|D_{t|T}^{\beta}\varphi_2\right|^{p'}. \qquad (5.5)\end{aligned}$$

Setting $\tau = T^{-1}t$, $\xi = T^{-2/\alpha}x$, we deduce that

$$\int_0^T\int_{\Omega_6} u^p\,\varphi\,dx\,dt \;\le\; C\,T^{\theta}, \qquad (5.6)$$

where $\theta := -2p' - \beta p' + 1 + \frac{2N}{\alpha}$.
Note that

$$\theta \;\le\; 0 \qquad\Longleftrightarrow\qquad p \;\le\; 1 \;+\; \frac{(3-\gamma)\alpha}{(2N-2\alpha+\gamma\alpha)^+} \;=:\; p^*.$$

- For $p<p^*$, we have

$$\lim_{T\longrightarrow\infty}\int_0^T\int_{\Omega_6} u^p\,\varphi(x,t)\,dx\,dt \;=\; 0 \quad\Longrightarrow \int_0^\infty\int_{\mathbb{R}^N} u^p\,dt\,dx,$$

and consequently $u=0$ (a.e.). Contradiction.

• For $p = p^*$, we need in this case the constrain ($N > \alpha$, $p > \frac{N}{N-\alpha}$) and we recall that $B \neq 1$ with $B < T$. From (5.6), we have

$$\int_0^\infty \int_{\mathbb{R}^N} u^p \, dx \, dt \; < \infty. \tag{5.7}$$

From the similar estimate in the subcritical case, we obtain

$$\begin{aligned} \int_0^{BT} \int_{\Omega_6} u^p \, \varphi \;\; &\leq \;\; C \int_0^{BT} \int_{\Omega_6} \varphi_1^{\ell} \varphi_2^{-p^{'}/p} \left| D^{2+\beta}_{t|BT} \varphi_2 \right|^{p^{'}} \\ &+ \;\; C \int_0^{BT} \int_{\Omega_6} \varphi_1 \varphi_2^{-p^{'}/p} \left| (-\Delta)^{\alpha/2} \varphi_1 \right|^{p^{'}} \left| D^{\beta}_{t|BT} \varphi_2 \right|^{p^{'}}. \end{aligned} \tag{5.8}$$

Then, by setting the new variables $\zeta = T^{-2/\alpha} x$, $\lambda = (BT)^{-1} t$ and using the critical exponent, we conclude

$$\int_0^{BT} \int_{\Omega_6} u^p(x,t) \, \varphi(x,t) \, dx \, dt \leq \; C B^{1-(2+\beta)p^{'}} \; + \; C B^{1-\beta p^{'}}. \tag{5.9}$$

Finally, using (5.9) and the fact that $p > \frac{N}{N-\alpha}$, $N > \alpha$ implies, using critical case, that $1 - \beta p^{'} < 0$, we obtain via the dominated convergence theorem, after passing to the limit as $B \longrightarrow \infty$,

$$\int_0^\infty \int_{\mathbb{R}^N} u^p \, dx \, dt \; = \; 0.$$

This leads to $u = 0$ a.e.; contradiction.

• For $p < \frac{1}{\gamma}$, we take as test function $\varphi(x,t) = \varphi_1(x)^l \varphi_2(t)$ with $\varphi_1(x) := \Phi\left(\frac{|x|}{T}\right)$ and $\varphi_2(t) := \Psi\left(\frac{t}{B}\right)$ where Φ and Ψ are the same function introduced in the subcritical case, then using Young's inequality, we obtain

$$\begin{aligned} \int_0^{B} \int_{\Sigma_3} u^p \, \varphi \;\; &\leq \;\; C \int_0^{B} \int_{\Sigma_3} \varphi_1^{\ell} \varphi_2^{-p^{'}/p} \left| D^{2+\beta}_{t|B} \varphi_2 \right|^{p^{'}} \\ &+ \;\; C \int_0^{B} \int_{\Sigma_3} \varphi_1 \varphi_2^{-p^{'}/p} \left| (-\Delta)^{\alpha/2} \varphi_1 \right|^{p^{'}} \left| D^{\beta}_{t|B} \varphi_2 \right|^{p^{'}}, \end{aligned}$$

where

$$\Sigma_3 := \left\{ x \in \mathbb{R}^N \; ; \; |x| \leq 2T \right\}.$$

As usually, taking a new variables $\eta := B^{-1} t$ and $\mu = T^{-1} x$, we get

$$\int_0^{B} \int_{\Sigma_3} u^p \, \varphi \leq C B^{1-(2+\beta)p^{'}} T^N \; + \; C B^{1-\beta p^{'}} T^{N-\alpha p^{'}}.$$

Then, as $p < \frac{1}{\gamma} \Longleftrightarrow 1 - \beta p^{'} < 0$ and by taking the limit when B then after T goes to infinity, we infer that $u = 0$ a.e. and completes the proof. □

Remarks. We may use the same analysis to study the nonlocal hyperbolic equation with fractional power of the Laplacian

$$u_{tt} + (-\Delta)^{\lambda/2} u^m \;=\; \int_0^t (t-s)^{-\mu} u^\nu \, ds, \qquad u \geq 0, \tag{5.10}$$

with $0 < \lambda \leq 2$, $1 \leq m < \nu$ and $0 < \mu < 1$.

For equation (5.8), the «critical» exponent is:

$$\nu^* \;:=\; 1 + \frac{(3-\mu)(N(m-1)+\lambda)}{(2N - 2\lambda + \lambda\mu)^+}. \tag{5.11}$$

6 A Fractional Hyperbolic System

In this section, we extend our analysis to the nonlocal fractional hyperbolic system

$$\begin{cases} u_{tt} + (-\Delta)^{\alpha/2} u = \displaystyle\int_0^t (t-s)^{-\mu} v^p \, ds, \\ \\ v_{tt} + (-\Delta)^{\alpha/2} v = \displaystyle\int_0^t (t-s)^{-\nu} u^q \, ds, \end{cases} \tag{6.1}$$

with initial conditions

$$\begin{cases} u_t(x,0) = g_1(x), \quad u(x,0) = g_2(x), \\ v_t(x,0) = h_1(x), \quad v(x,0) = h_2(x). \end{cases} \tag{6.2}$$

The system (6.1) is equivalent to

$$\begin{cases} D^\beta_{0|t}\, u_{tt} + (-\Delta)^{\alpha/2}\, D^\beta_{0|t} u = \; \Gamma(\beta) v^p, \\ \\ D^\gamma_{0|t}\, v_{tt} + (-\Delta)^{\alpha/2}\, D^\gamma_{0|t} v = \; \Gamma(\gamma) u^q, \end{cases} \tag{6.3}$$

where $\beta := 1 - \mu \in (0,1)$, $\gamma := 1 - \nu \in (0,1)$ and $pq > 1$.
Noting that (u, v) in (6.1) and (6.3) supposed as a nonnegative solution.

Definition 6 *We say that (u,v) is a weak solution of system* (6.3) *if*

$$u \;\in\; L^q_{Loc}(\mathbb{R}_+ \times \mathbb{R}^N) \quad , \quad v \;\in\; L^p_{Loc}(\mathbb{R}_+ \times \mathbb{R}^N)$$

and

$$\begin{aligned}\Gamma(\beta)\int_0^T\int_{\mathbb{R}^N} v^p\,\varphi &= \int_0^T\int_{\mathbb{R}^N} u\,D_{t|T}^{2+\beta}\varphi - \int_{\mathbb{R}^N} g_1(x)\left(D_{t|T}^{\beta}\varphi\right)(0)\\ &- \int_{\mathbb{R}^N} g_2(x)\left(D_{t|T}^{1+\beta}\varphi\right)(0) + \int_0^T\int_{\mathbb{R}^N} u\,(-\Delta)^{\alpha/2}D_{t|T}^{\beta}\varphi,\end{aligned}$$

$$\begin{aligned}\Gamma(\gamma)\int_0^T\int_{\mathbb{R}^N} u^q\,\varphi &= \int_0^T\int_{\mathbb{R}^N} v\,D_{t|T}^{2+\gamma}\varphi - \int_{\mathbb{R}^N} h_1(x)\left(D_{t|T}^{\gamma}\varphi\right)(0)\\ &- \int_{\mathbb{R}^N} h_2(x)\left(D_{t|T}^{1+\gamma}\varphi\right)(0) + \int_0^T\int_{\mathbb{R}^N} v\,(-\Delta)^{\alpha/2}D_{t|T}^{\gamma}\varphi,\end{aligned}$$

for any nonnegative test function $\varphi \in C^{\infty}([0,T], C_c^{\infty}(\mathbb{R}^N))$, *with* $D_{t|T}^{\beta}\varphi(T,x) = D_{t|T}^{1+\beta}\varphi(T,x) = 0$ *and* $D_{t|T}^{\gamma}\varphi(T,x) = D_{t|T}^{1+\gamma}\varphi(T,x) = 0$, *for all* $T > 0$.

Theorem 6 *Let* $g_1(x), g_2(x), h_1(x), h_2(x) \geq 0$, *for each* $x \in \mathbb{R}^N$. *If*

$$N \;\leq\; \max\{\Upsilon_1\,;\;\Upsilon_2\}\,, \tag{6.4}$$

or

$$p < \frac{1}{\mu} \qquad \textit{and} \qquad q < \frac{1}{\nu},$$

where

$$\Upsilon_1 := \frac{(p(3-\nu)+pq(2-\mu)+1)\,\alpha}{2(pq-1)}\;\;;\;\;\Upsilon_2 := \frac{(q(3-\mu)+pq(2-\nu)+1)\,\alpha}{2(pq-1)},$$

then the system (6.1) *has no global no trivial nonnegative weak solutions.*
For the critical case, we take must impose the constrain $N > \alpha$ *and* $\frac{pq-1}{p(1+p)(1+q)} > \frac{\alpha}{N}$.

Proof. Suppose, by contradiction, that u, v is a non-negative nontrivial global weak solution of the system (6.1) with $u \in L^q\left([0,T^*), L^q_{Loc}\mathbb{R}^N\right)$ and $v \in L^p\left([0,T^*), L^p_{Loc}\mathbb{R}^N\right)$ for $T^* > 0$.
Setting as a test function $\varphi(x,t) = (\varphi_1(x))^m\,\varphi_2(t)$ with $\varphi_1(x) := \Phi\left(\frac{|x|}{T^{2/\alpha}}\right)$ and $\varphi_2(t) := \Psi\left(\frac{t}{BT}\right)$, where Φ is as before, $m = \frac{(2p-1)(2q-1)}{(p-1)(q-1)}$ and

$$\Psi(r) = \begin{cases}(1-r)^n & \text{if } 0 \leq r \leq 1,\\ 0 & \text{if } r \geq 1,\end{cases}$$

with $n \geq \frac{(2+\beta)(2+\gamma)pq}{(p-1)(q-1)}$ large enough and $B = 1$ in the sub-critical case.

Then by Ju's and Hölder's inequality, and taking into account the nonnegativity of the initial datum (6.2), we deduce that

$$\begin{cases} \displaystyle\int_0^T \int_{\Omega_6} v^p \, \varphi \leq \left(\int_0^T \int_{\Omega_6} u^q \, \varphi \right)^{1/q} \mathcal{F}, \\ \displaystyle\int_0^T \int_{\Omega_6} u^q \, \varphi \leq \left(\int_0^T \int_{\Omega_6} v^p \, \varphi \right)^{1/p} \mathcal{G}, \end{cases} \tag{6.5}$$

where

$$\begin{aligned} \mathcal{F} \; := \; & \left(\int_0^T \int_{\Omega_6} \varphi_1^{(m-q')} \varphi_2^{-q'/q} \left| (-\Delta)^{\alpha/2} \varphi_1 \right|^{q'} \left| D_{t|T}^{\beta} \varphi_2 \right|^{q'} \right)^{1/q'} \\ + \; & \left(\int_0^T \int_{\Omega_6} \varphi_1^{m} \varphi_2^{-q'/q} \left| D_{t|T}^{2+\beta} \varphi_2 \right|^{q'} \right)^{1/q'}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{G} \; := \; & \left(\int_0^T \int_{\Omega_6} \varphi_1^{(m-p')} \varphi_2^{-p'/p} \left| (-\Delta)^{\alpha/2} \varphi_1 \right|^{p'} \left| D_{t|T}^{\gamma} \varphi_2 \right|^{p'} \right)^{1/p'} \\ + \; & \left(\int_0^T \int_{\Omega_6} \varphi_1^{m} \varphi_2^{-p'/p} \left| D_{t|T}^{2+\gamma} \varphi_2 \right|^{p'} \right)^{1/p'}. \end{aligned}$$

Using the scaled variables $\tau = T^{-1}t$, $\xi = T^{-\frac{2}{\alpha}}x$, we obtain

$$\begin{cases} \mathcal{F} \; \leq \; C \, T^{\sigma}, \\ \mathcal{G} \; \leq \; C \, T^{\delta}, \end{cases} \tag{6.6}$$

where C is a positive constant that may change from line to line and

$$\sigma := \left(-(2+\beta)q' + 1 + \frac{2N}{\alpha} \right) \frac{1}{q'} \quad ; \quad \delta := \left(-(2+\gamma)p' + 1 + \frac{2N}{\alpha} \right) \frac{1}{p'}.$$

Therefore, thanks to (6.5) and (6.6), we have

$$\begin{cases} \left(\displaystyle\int_0^T \int_{\Omega_6} v^p \, \varphi \right)^{1-\frac{1}{pq}} \leq \; C \, T^{\zeta}, \\ \left(\displaystyle\int_0^T \int_{\Omega_6} u^q \, \varphi \right)^{1-\frac{1}{pq}} \leq \; C \, T^{\eta}. \end{cases} \tag{6.7}$$

Where

$$\zeta := \left(-(2+\gamma)p^{'} + 1 + \frac{2N}{\alpha}\right)\frac{1}{p^{'}q} \ + \ \left(-(2+\beta)q^{'} + 1 + \frac{2N}{\alpha}\right)\frac{1}{q^{'}},$$

$$\eta := \left(-(2+\beta)q^{'} + 1 + \frac{2N}{\alpha}\right)\frac{1}{q^{'}p} \ + \ \left(-(2+\gamma)p^{'} + 1 + \frac{2N}{\alpha}\right)\frac{1}{p^{'}}.$$

Clearly

$$\zeta \ \leq 0 \ \text{ or } \ \eta \ \leq 0 \quad \Longleftrightarrow \quad N \ \leq \ \Upsilon_1 \ \text{ or } \ N \ \leq \ \Upsilon_2.$$

• For $N \ < \ \Upsilon_1$ or $N \ < \ \Upsilon_2$. From (6.7) and letting T goes to ∞, we deduce that $u \equiv 0$ or $v \equiv 0$; whereupon (6.5) implies that $u \equiv v \equiv 0$. Contradiction.

• For the case $N \ = \ \Upsilon_1$ or $N \ = \ \Upsilon_2$, we take $N > \alpha$ and $\frac{pq-1}{p(1+p)(1+q)} > \frac{\alpha}{N}$. In other hand, from (6.7), we have

$$\int_0^\infty \int_{\mathbb{R}^N} v^p \ < \ \infty,$$

or

$$\int_0^\infty \int_{\mathbb{R}^N} u^q \ < \ \infty.$$

Finally from (6.5), by elementary computations like in the critical case of section (5) and by the technique used in the same case of section (4), we get $u \equiv v \equiv 0$.

• For the case $p < \frac{1}{\mu}$ and $q < \frac{1}{\nu}$, we repeat with the same test function used in the fractional hyperbolic equation case but with the similar process in the parabolic system case. This end the proof. □

APPENDIX

We will prove Ju's inequality (see proposition 3.3 in [10]), in dimension N where $\delta \in [0, 2]$ and $q \geq 1$, for all nonnegative Schwartz function ψ (in general case)

$$(-\Delta)^{\delta/2}\psi^q \leq q\psi^{q-1}(-\Delta)^{\delta/2}\psi. \tag{6.8}$$

Indeed, the cases $\delta = 0$ and $\delta = 2$ are obvious.

- If $\delta \in (0, 1)$, we have (see Th. 1 in [4])

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{\psi(x+z) - \psi(x)}{|z|^{N+\delta}}\, dz, \quad \forall\, x \in \mathbb{R}^N, \tag{6.9}$$

where $c_N(\delta) = \frac{\delta\Gamma(\frac{N+\delta}{2})}{2\pi^{\frac{N}{2}+\delta}\Gamma(1-\frac{\delta}{2})}$. Then

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N}\frac{(\psi(x))^{q-1}\psi(x+z)-(\psi(x))^q}{|z|^{N+\delta}}\,dz.$$

The case $q=1$ is clear. If $q>1$, then by Young's inequality we have

$$(\psi(x))^{q-1}\psi(x+z) \le \frac{q-1}{q}(\psi(x))^q + \frac{1}{q}(\psi(x+z))^q.$$

Therefore, we infer

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) \ge \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N}\frac{(\psi(x+z))^q-(\psi(x))^q}{|z|^{N+\delta}}\,dz = \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.$$

- If $\delta\in(1,2)$, we have (see Th. 1 in [4])

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N}\frac{\psi(x+z)-\psi(x)-\nabla\psi(x).z}{|z|^{N+\delta}}\,dz, \quad \forall\, x\in\mathbb{R}^N. \tag{6.10}$$

Then by the similar process, we have

$$\begin{aligned}(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) &\ge -c_N(\delta)\int_{\mathbb{R}^N}\frac{\frac{1}{q}\left((\psi(x+z))^q-(\psi(x))^q\right)-(\nabla\psi(x).z)(\psi(x))^{q-1}}{|z|^{N+\delta}}\,dz\\ &= \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N}\frac{(\psi(x+z))^q-(\psi(x))^q-\nabla(\psi(x))^q.z}{|z|^{N+\delta}}\,dz\\ &= \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.\end{aligned}$$

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