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On certain time - and space - fractional evolution equations

Ahmad Z. FINO^{†*} and Mokhtar KIRANE[†]

Abstract

In this article, we present first a new technique to prove, in a general case, the recent result of Cazenave, Dickstein and Weissler [5] on the blowing-up solutions to a temporally nonlocal nonlinear parabolic equation.

Then, we study the blow-up rate and the global existence in time of the solutions. Furthermore, we show necessary conditions for global existence.

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1 Introduction

In this paper, we investigate the nonlinear parabolic equation with a nonlocal in time nonlinearity

$$\begin{cases} u_t + (-\Delta)^{\beta/2}u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1}u(s) ds & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $u_0 \in C_0(\mathbb{R}^N)$, $N \geq 1$, $0 < \beta \leq 2$, $0 < \gamma < 1$, $p > 1$ and the nonlocal operator $(-\Delta)^{\beta/2}$ defined by

$$(-\Delta)^{\beta/2}v(x) := \mathcal{F}^{-1} \left(|\xi|^\beta \mathcal{F}(v)(\xi) \right) (x)$$

for every $v \in D((-\Delta)^{\beta/2}) = H^\beta(\mathbb{R}^N)$, where $H^\beta(\mathbb{R}^N)$ is the fractional Sobolev space of order β , \mathcal{F} stands for the Fourier transform and \mathcal{F}^{-1} for its inverse, Γ is the Euler gamma

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function and $C_0(\mathbb{R}^N)$ denotes the space of all continuous functions decay to zero at infinity. When Eq. (1.1) is considered with a nonlinearity of the form u^p , it reads

$$u_t + (-\Delta)^{\beta/2}u = u^p.$$

This equation has been considered by Nagasawa and Sirao [22], Kobayashi [16], Fino and Karch [8], Birkner, Lopez-Mimbela and Wakolbinger [2], and by Kirane and Qafsaoui [15], in the general case.

The fractional laplacian $(-\Delta)^{\beta/2}$ is related to Lévy flights in physics. Many observations and experiments related to Lévy flights (super-diffusion), e.g., collective slip diffusion on solid surfaces, quantum optics or Richardson turbulent diffusion, have performed in recent years. The symmetric β -stable processes ($\beta \in (0, 2)$) are the basic characteristics for a class of jumping Lévy's processes. Compared with the continuous Brownian motion ($\beta = 2$), symmetric β -stable processes have infinite jumps in an arbitrary time interval. The large jumps of these processes make their variances and expectations infinite according to $\beta \in (0, 2)$ and $\beta \in (0, 1]$, respectively (see [13]). Let us also mention that when $\beta = 3/2$, the symmetric β -stable processes appear in the study of stellar dynamics (see [6]).

Our article is motivated by the recent and very interesting paper by Cazenave, Dickstein and Weissler [5] which deals with the global existence and blow-up for the parabolic equation with nonlocal in time nonlinearity

$$u_t - \Delta u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s) ds \quad x \in \mathbb{R}^N, t > 0, \quad (1.2)$$

where $0 \leq \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^N)$, which is a particular case of (1.1) with $\beta = 2$. They proved that, if we set,

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma)_+}$$

and

$$p_* = \max \left\{ \frac{1}{\gamma}, p_\gamma \right\} \in (0, +\infty],$$

where $(\cdot)_+$ is the positive part, we have the following results:

- (i) If $\gamma \neq 0$, $p \leq p_*$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.
- (ii) If $\gamma \neq 0$, $p > p_*$, and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) with $\|u_0\|_{L^{q_{sc}}}$ sufficiently small, then u exists globally.

If $\gamma = 0$ then all nontrivial positive solutions blow-up as proved by Souplet in [25]. Their study reveals the surprising fact that for equation (1.2) the critical exponent in Fujita's sense p_* is not the one predicted by scaling.

This can be explained by the fact that their equation can be formally converted into

$$D_{0t}^\alpha u_t - D_{0t}^\alpha \Delta u = |u|^{p-1} u, \quad (1.3)$$

where $D_{0|t}^\alpha$ is the left-sided Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ defined in (2.6) below (we have set in (1.3), $\alpha = 1 - \gamma \in (0, 1)$).

Eq. (1.3) clearly interpolates the heat equation and the wave equation and as is well known scaling are efficient for detecting the Fujita exponent only for equations of parabolic type. Needless to say that the equation considered by Cazenave, Dickstein and Weissler [5] is a genuine extension of the one considered by Fujita in his pioneering work [9].

In this article, concerning blowing-up solutions, we present a different proof from the one presented in [5], and for the more general equation (1.1). Our proof is more versatile and can be applied to more nonlinear equations.

Our analysis is based on the observation that the nonlinear differential Equation (1.1) can be written in the form:

$$u_t + (-\Delta)^{\beta/2} u = J_{0|t}^\alpha (|u|^{p-1} u), \quad (1.4)$$

where $\alpha := 1 - \gamma \in (0, 1)$ and $J_{0|t}^\alpha$ is the Riemann-Liouville fractional integral defined in (2.8).

We will show that:

(1) For $u_0 \geq 0$, $u_0 \not\equiv 0$, and $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, if

$$p \leq 1 + \frac{\beta(2 - \gamma)}{(N - \beta + \beta\gamma)_+} \quad \text{or} \quad p < \frac{1}{\gamma},$$

then all solutions of problem (1.1) blow-up in finite time.

(2) For $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, where $p_{sc} := N(p - 1)/\beta(2 - \gamma)$, if

$$p > \max \left\{ 1 + \frac{\beta(2 - \gamma)}{(N - \beta + \beta\gamma)_+}; \frac{1}{\gamma} \right\},$$

and $\|u_0\|_{L^{p_{sc}}}$ sufficiently small, then u exists globally.

The method used to prove the blow-up theorem is the test function method considered by Mitidieri and Pohozaev [19, 20] and Kirane et al. [10, 14].

Furthermore, in the case $\beta = 2$, we derive the blow-up rate estimates for the parabolic equation (1.1). We shall prove that, if u is the blow-up solution of (1.1) at the finite time $T^* > 0$, then there are constants $c, C > 0$ such that $c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(\cdot, t) \leq C(T^* - t)^{-\alpha_1}$ for $1 < p \leq 1 + 2(2 - \gamma)/(N - 2 + 2\gamma)_+$ or $1 < p < 1/\gamma$ and all $t \in (0, T^*)$, where $\alpha_1 := (2 - \gamma)/(p - 1)$. We use a scaling argument to reduce the problems of blow-up rate to Fujita-type theorems (it is similar to ‘‘blow-up analysis’’ in elliptic problems to reduce the problems of a priori bounds to Liouville-type theorems). As far as we know, this method was first applied to parabolic problems by Hu [11], and sequently, was used in various parabolic equations and systems (see [3, 7]). We notice that in the limiting case when $\gamma \rightarrow 0$, we obtain the rate constant $2/(p - 1)$ found by P. Souplet [26].

The organization of this paper is as follows. In section 2, we present some definitions and properties. In Section 3, we derive the local existence of solutions for the parabolic equation (1.1). Section 4 contains the blow-up result of solutions for (1.1). Section 5 is dedicated to the blow-up rate of solutions. Global existence is studied in Section 6. Finally, a necessary condition for local and global existence is given in Section 7.

2 Preliminaries

In this section, we present some definitions and results concerning fractional laplacians, fractional integrals and fractional derivatives that will be used hereafter.

First, if we take the usual linear fractional equation

$$u_t + (-\Delta)^{\beta/2} = 0, \quad \beta \in (0, 2], \quad x \in \mathbb{R}^N, \quad t > 0, \quad (2.1)$$

then, the fundamental solution S_β of Eq. (2.1) can be represented via the Fourier transform

$$S_\beta(t) := S_\beta(x, t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi - t|\xi|^\beta} d\xi. \quad (2.2)$$

It is well-known that this function satisfies

$$S_\beta(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad S_\beta(x, t) \geq 0, \quad \int_{\mathbb{R}^N} S_\beta(x, t) dx = 1, \quad (2.3)$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using the Young inequality for the convolution and the following self-similar form $S_\beta(x, t) = t^{-N/\beta} S_\beta(xt^{-1/\beta}, 1)$, we have

$$\|S_\beta(t) * v\|_q \leq Ct^{-\frac{N}{\beta}(\frac{1}{r} - \frac{1}{q})} \|v\|_r, \quad (2.4)$$

for all $v \in L^r(\mathbb{R}^N)$ and all $1 \leq r \leq q \leq \infty$, $t > 0$.

Moreover, as $(-\Delta)^{\beta/2}$ is a self-adjoint operator with $D((-\Delta)^{\beta/2}) = H^\beta(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^{\beta/2}v(x) dx = \int_{\mathbb{R}^N} v(x)(-\Delta)^{\beta/2}u(x) dx, \quad (2.5)$$

for all $u, v \in H^\beta(\mathbb{R}^N)$.

Next, if $AC[0, T]$ is the space of all functions which are absolutely continuous on $[0, T]$ with $0 < T < \infty$, then, for $f \in AC[0, T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D_{0|t}^\alpha f(t)$ and $D_{t|T}^\alpha f(t)$ of order $\alpha \in (0, 1)$ are defined by (see [13])

$$D_{0|t}^\alpha f(t) := DJ_{0|t}^{1-\alpha} f(t), \quad (2.6)$$

$$D_{t|T}^\alpha f(t) := -\frac{1}{\Gamma(1-\alpha)} D \int_t^T (s-t)^{-\alpha} f(s) ds, \quad (2.7)$$

for all $t \in [0, T]$, where $D := d/(dt)$ is the usual derivative, and

$$J_{0|t}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (2.8)$$

is the Riemann-Liouville fractional integral defined in [13].

Now, for every $f, g \in C([0, T])$, such that $D_{0|t}^\alpha f(t), D_{t|T}^\alpha g(t)$ exist and are continuous, for all $t \in [0, T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [24])

$$\int_0^T \left(D_{0|t}^\alpha f \right) (t) g(t) dt = \int_0^T f(t) \left(D_{t|T}^\alpha g \right) (t) dt. \quad (2.9)$$

Note also that, for all $f \in AC^2[0, T]$, we have (see (2.2.30) in [13])

$$-D.D_{t|T}^\alpha f = D_{t|T}^{1+\alpha} f, \quad (2.10)$$

where

$$AC^2[0, T] := \{ f : [0, T] \rightarrow \mathbb{R} \text{ and } Df \in AC[0, T]. \}$$

Moreover, for all $1 \leq q \leq \infty$, the following equality (see [13, Lemma 2.4 p.74])

$$D_{0|t}^\alpha J_{0|t}^\alpha = Id_{L^q(0, T)} \quad (2.11)$$

holds almost everywhere on $[0, T]$. Later on, we will use the following results.

• If $w_1(t) = (1 - t/T)_+^\sigma$, $t \geq 0$, $T > 0$, $\sigma \gg 1$, then

$$D_{t|T}^\alpha w_1(t) = \frac{(1 - \alpha + \sigma)\Gamma(\sigma + 1)}{\Gamma(2 - \alpha + \sigma)} T^{-\alpha} \left(1 - \frac{t}{T} \right)_+^{\sigma - \alpha}, \quad (2.12)$$

$$D_{t|T}^{\alpha+1} w_1(t) = \frac{(1 - \alpha + \sigma)(\sigma - \alpha)\Gamma(\sigma + 1)}{\Gamma(2 - \alpha + \sigma)} T^{-(\alpha+1)} \left(1 - \frac{t}{T} \right)_+^{\sigma - \alpha - 1}, \quad (2.13)$$

for all $\alpha \in (0, 1)$; so

$$\left(D_{t|T}^\alpha w_1 \right) (T) = 0 \quad ; \quad \left(D_{t|T}^\alpha w_1 \right) (0) = C T^{-\alpha}, \quad (2.14)$$

where $C = (1 - \alpha + \sigma)\Gamma(\sigma + 1)/\Gamma(2 - \alpha + \sigma)$; indeed, using the Euler change of variable $y = (s - t)/(T - t)$, we get

$$\begin{aligned} D_{t|T}^\alpha w_1(t) &:= -\frac{1}{\Gamma(1 - \alpha)} D \left[\int_t^T (s - t)^{-\alpha} \left(1 - \frac{s}{T} \right)^\sigma ds \right] \\ &= -\frac{T^{-\sigma}}{\Gamma(1 - \alpha)} D \left[(T - t)^{1 - \alpha + \sigma} \int_0^1 (y)^{-\alpha} (1 - y)^\sigma ds \right] \\ &= \frac{(1 - \alpha + \sigma)B(1 - \alpha; \sigma + 1)}{\Gamma(1 - \alpha)} T^{-\sigma} (T - t)^{\sigma - \alpha}, \end{aligned}$$

where $B(., .)$ stands for the beta function. Then, (2.12) follows using the relation

$$B(1 - \alpha; \sigma + 1) = \frac{\Gamma(1 - \alpha)\Gamma(\sigma + 1)}{\Gamma(2 - \alpha + \sigma)}.$$

Moreover, using (2.10) and (2.12) to obtain (2.13).

• If $w_2(t) = (1 - t^2/T^2)_+^\ell$, $T > 0$, $\ell \gg 1$, then, using the change of variable $y = (s - t)/(T - t)$, we have

$$D_{t|T}^\alpha w_2(t) = \frac{T^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} C_1(\ell, k, \alpha) \left(1 - \frac{t}{T}\right)^{\ell+k-\alpha}, \quad (2.15)$$

$$D_{t|T}^{1+\alpha} w_2(t) = \frac{T^{-\alpha-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} C_2(\ell, k, \alpha) \left(1 - \frac{t}{T}\right)^{\ell+k-\alpha-1}, \quad (2.16)$$

for all $-T \leq t \leq T$, $\alpha \in (0, 1)$, where

$$\begin{cases} C_1(\ell, k, \alpha) := c_\ell^k (1 - \alpha + \ell + k) 2^{\ell-k} (-1)^k \frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)}, \\ C_2(\ell, k, \alpha) := (\ell + k - \alpha) C_1(\ell, k, \alpha), \\ c_\ell^k := \frac{\ell!}{(\ell-k)!k!}; \end{cases}$$

so

$$\left(D_{t|T}^\alpha w_2\right)(T) = 0 \quad ; \quad \left(D_{t|T}^\alpha w_2\right)(-T) = C_3(\ell, k, \alpha) T^{-\alpha}, \quad (2.17)$$

where

$$C_3(\ell, k, \alpha) := \frac{2^{2\ell-\alpha}(-1)^\ell}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} c_\ell^k (1 - \alpha + \ell + k) \frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)}.$$

3 Local existence

This section is dedicated to prove the local existence and uniqueness of mild solutions to the problem (1.1). So, we start by giving the definition of the mild solution to (1.1).

Definition 1 (*Mild solution*)

Let $u_0 \in L^\infty(\mathbb{R}^N)$, $0 < \beta \leq 2$, $p > 1$ and $T > 0$. We say that $u \in L^\infty([0, T], L^\infty(\mathbb{R}^N))$ is a mild solution of the problem (1.1) if u satisfies the following integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s) J_{0|s}^\alpha (|u|^{p-1}u) ds, \quad t \in [0, T], \quad (3.1)$$

where $T(t) := e^{-t(-\Delta)^{\beta/2}}$. As $(-\Delta)^{\beta/2}$ is a positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$, $T(t)$ is a strongly continuous semigroup on $L^2(\mathbb{R}^N)$ generated by the fractional power $-(-\Delta)^{\beta/2}$ (see Yosida [27]). It holds $T(t)v = S_\beta(t) * v$ (see [21, Prop.3.3]), for all $v \in L^2(\mathbb{R}^N)$, $t > 0$, where S_β is given by (2.2) and $u * v$ is the convolution of u and v . \square

Theorem 1 (*Local existence*)

Given $u_0 \in C_0(\mathbb{R}^N)$ and $p > 1$, there exist a maximal time $T_{\max} > 0$ and a unique mild solution $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$ to the problem (1.1). In addition, if $u_0 \geq 0$, $u_0 \not\equiv 0$, then $u(t) > 0$ for all $0 < t < T_{\max}$.

Moreover, if $u_0 \in L^r(\mathbb{R}^N)$, for $1 \leq r < \infty$, then $u \in C([0, T_{\max}), L^r(\mathbb{R}^N))$.

Proof For arbitrary $T > 0$, we define a Banach space E_T such that

$$E_T := \{u \in L^\infty((0, T), C_0(\mathbb{R}^N)); \|u\|_1 \leq 2\|u_0\|_{L^\infty}\},$$

where $\|\cdot\|_1 := \|\cdot\|_{L^\infty((0, T), L^\infty(\mathbb{R}^N))}$. Next, for every $u \in E_T$, we define

$$\Psi(u) := T(t)u_0 + \int_0^t T(t-s)J_{0|s}^\alpha (|u|^{p-1}u) ds.$$

As usual, we prove the local existence by the Banach fixed point theorem.

• **$\Psi : \mathbf{E}_T \rightarrow \mathbf{E}_T$** : Let $u \in E_T$, using (2.4), we obtain with $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^N)}$

$$\begin{aligned} \|\Psi(u)\|_1 &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\sigma)^{-\gamma} \|u(\sigma)\|_\infty^p d\sigma ds \right\|_{L^\infty(0, T)} \\ &= \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} \|u(\sigma)\|_\infty^p ds d\sigma \right\|_{L^\infty(0, T)} \\ &\leq \|u_0\|_\infty + \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \|u\|_1^p \\ &\leq \|u_0\|_\infty + \frac{T^{2-\gamma} 2^p \|u_0\|_{L^\infty}^{p-1}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \|u_0\|_\infty. \end{aligned}$$

Now, if we choose T small enough such that

$$\frac{T^{2-\gamma} 2^p \|u_0\|_\infty^{p-1}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \leq 1, \quad (3.2)$$

we conclude that $\|\Psi(u)\|_1 \leq 2\|u_0\|_\infty$, and then $\Psi(u) \in E_T$.

• **Ψ is a contraction map**: For $u, v \in E_T$, taking account of (2.4), we have

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_1 &\leq \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\sigma)^{-\gamma} \||u|^{p-1}u(\sigma) - |v|^{p-1}v(\sigma)\|_\infty d\sigma ds \right\|_{L^\infty(0, T)} \\ &= \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} \||u|^{p-1}u(\sigma) - |v|^{p-1}v(\sigma)\|_\infty ds d\sigma \right\|_{L^\infty(0, T)} \\ &\leq \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \||u|^{p-1}u - |v|^{p-1}v\|_1 \\ &\leq \frac{C(p)2^{p-1}\|u_0\|_\infty^{p-1}T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \|u - v\|_1 \\ &\leq \frac{1}{2} \|u - v\|_1, \end{aligned}$$

thanks to the following inequality

$$\||u|^{p-1}u - |v|^{p-1}v\| \leq C(p)|u - v|(|u|^{p-1} + |v|^{p-1}),$$

for the choice of T such that

$$\frac{T^{2-\gamma} 2^p \|u_0\|_\infty^{p-1} \max(C(p), 1)}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \leq 1. \quad (3.3)$$

Then, by the Banach fixed point theorem, there exists a unique mild solution $u \in \Pi_T$, where $\Pi_T := L^\infty((0, T), C_0(\mathbb{R}^N))$, to the problem (1.1).

Next, using the uniqueness of solutions, we conclude the existence of a solution on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup \{T > 0 : u \text{ is a mild solution to (1.1) in } \Pi_T\}.$$

Note that, using the continuity of the semigroup $T(t)$, we can easily conclude that $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$.

• **Positivity of solutions:** If $u_0 \geq 0$, then we can construct a nonnegative solution on some interval $[0, T]$ by applying the fixed-point argument in the set $E_T^+ = \{u \in E_T; u \geq 0\}$. In particular, it follows from (3.1) that $u(t) \geq T(t)u_0 > 0$ on $(0, T]$. It is not difficult to deduce that u stays positive on $(0, T_{\max})$.

• **Regularity of solutions:** If $u_0 \in L^r(\mathbb{R}^N)$, for $1 \leq r < \infty$, then by repeating the fixed point argument in the space

$$E_{T,r} := \{u \in L^\infty((0, T), C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)); \|u\|_1 \leq 2\|u_0\|_{L^\infty}, \|u\|_{\infty,r} \leq 2\|u_0\|_{L^r}\},$$

instead of E_T , where $\|\cdot\|_{\infty,r} := \|\cdot\|_{L^\infty((0,T), L^r(\mathbb{R}^N))}$, and by estimating $\|u^p\|_{L^r(\mathbb{R}^N)}$ by $\|u\|_{L^\infty(\mathbb{R}^N)}^{p-1} \|u\|_{L^r(\mathbb{R}^N)}$ in the contraction mapping argument, using (2.4), we obtain a unique solution in $E_{T,r}$; we conclude then that

$$u \in C([0, T_{\max}), C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)).$$

□

In the case of $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ (the case which will be used below), we say that u is a global solution if $T_{\max} = \infty$ and when $T_{\max} < \infty$ we say that u blows up in a finite time or that u is non-global and in this case we have

$$\lim_{t \rightarrow T_{\max}} \left(\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} + \|u(\cdot, t)\|_{L^2(\mathbb{R}^N)} \right) = \infty.$$

4 Blow-up of solutions

Now, we want to derive a blow-up result for Eq. (1.1). To do this we show that under some conditions, a mild solution is a weak solution. Hereafter

$$\int_{Q_T} = \int_0^T \int_{\mathbb{R}^N} dx dt \quad \text{for all } T > 0; \quad \int_{\mathbb{R}^N} = \int_{\mathbb{R}^N} dx.$$

Definition 2 (*Weak solution*)

Let $T > 0$, $0 < \beta \leq 2$ and $u \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$. We say that u is a weak solution of the problem (1.1) if

$$\begin{aligned} \int_{\mathbb{R}^N} u_0(x)\varphi(x, 0) + \int_{Q_T} J_{0|t}^\alpha(|u|^{p-1}u)(x, t)\varphi(x, t) &= \int_{Q_T} u(x, t)(-\Delta)^{\beta/2}\varphi(x, t) \\ &- \int_{Q_T} u(x, t)\varphi_t(x, t), \end{aligned} \quad (4.1)$$

for all $\varphi \in C^1([0, T], H^\beta(\mathbb{R}^N))$ such that $\varphi(x, T) = 0$, $x \in \mathbb{R}^N$, where $\alpha := 1 - \gamma \in (0, 1)$.

Lemma 1

Let $u \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ be a mild solution of (1.1), then u is a weak solution of (1.1), for all $0 < \beta \leq 2$ and all $T > 0$.

Proof Let $T > 0$, $0 < \beta \leq 2$, $u \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ be a solution of (3.1) and $\varphi \in C^1([0, T], H^\beta(\mathbb{R}^N))$ such that $\varphi(x, T) = 0$ for all $x \in \mathbb{R}^N$. Then after multiplying (3.1) by φ and integrate over \mathbb{R}^N , we have

$$\int_{\mathbb{R}^N} u(x, t)\varphi(x, t) = \int_{\mathbb{R}^N} T(t)u_0(x)\varphi(x, t) + \int_{\mathbb{R}^N} \left(\int_0^t T(t-s)J_{0|s}^\alpha(|u|^{p-1}u)(x, s) ds \right) \varphi(x, t).$$

So after differentiation in time we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} u(x, t)\varphi(x, t) &= \int_{\mathbb{R}^N} \frac{d}{dt} (T(t)u_0(x)\varphi(x, t)) \\ &+ \int_{\mathbb{R}^N} \frac{d}{dt} \int_0^t T(t-s)J_{0|s}^\alpha(|u|^{p-1}u)(x, s) ds \varphi(x, t). \end{aligned} \quad (4.2)$$

Now, using (2.5) and the property of the semigroup $T(t)$ ([4, Chapter 3]) with the negative self-adjoint operator $A = -(-\Delta)^{\beta/2}$, we have:

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt} (T(t)u_0(x)\varphi(x, t)) &= \int_{\mathbb{R}^N} A(T(t)u_0(x))\varphi(x, t) + \int_{\mathbb{R}^N} T(t)u_0(x)\varphi_t(x, t) \\ &= \int_{\mathbb{R}^N} T(t)u_0(x)A\varphi(x, t) + \int_{\mathbb{R}^N} T(t)u_0(x)\varphi_t(x, t), \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt} \int_0^t T(t-s)f(x, s) ds \varphi(x, t) &= \int_{\mathbb{R}^N} f(x, t)\varphi(x, t) + \int_{\mathbb{R}^N} \int_0^t A(T(t-s)f(x, s)) ds \varphi(x, t) \\ &+ \int_{\mathbb{R}^N} \int_0^t T(t-s)f(x, s) ds \varphi_t(x, t) \\ &= \int_{\mathbb{R}^N} f(x, t)\varphi(x, t) + \int_{\mathbb{R}^N} \int_0^t T(t-s)f(x, s) ds A\varphi(x, t) \\ &+ \int_{\mathbb{R}^N} \int_0^t T(t-s)f(x, s) ds \varphi_t(x, t), \end{aligned} \quad (4.4)$$

where $f := J_{0|t}^\alpha (|u|^{p-1}u) \in L^\infty([0, T]; L^2(\mathbb{R}^N))$.

Thus, using (3.1), (4.3) and (4.4), we conclude that (4.2) imply

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} u(x, t) \varphi(x, t) &= \int_{\mathbb{R}^N} u(x, t) A \varphi(x, t) + \int_{\mathbb{R}^N} u(x, t) \varphi_t(x, t) \\ &+ \int_{\mathbb{R}^N} f(x, t) \varphi(x, t). \end{aligned}$$

Finally, we conclude the result by integrating in time over $[0, T]$ and using the fact that $\varphi(x, T) = 0$ for all $x \in \mathbb{R}^N$. \square

Theorem 2 *Let $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. If*

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} := p^* \quad \text{or} \quad p < \frac{1}{\gamma}, \quad (4.5)$$

for all $\beta \in (0, 2]$, then any mild solution to (1.1) blows-up in a finite time.

Note that in the case where $p = p^*$ and $\beta \in (0, 2)$ we take $p > N/(N-\beta)$ with $N > \beta$.

Proof The proof is by contradiction. Suppose that u is a global mild solution to (1.1), then u is a solution of (1.1) in $C([0, T], C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ for all $T > 0$ such that $u(t) > 0$ for all $t \in [0, T]$.

Then, using Lemma 1, we have

$$\begin{aligned} \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0) + \int_{Q_T} J_{0|t}^\alpha (u^p)(x, t) \varphi(x, t) &= \int_{Q_T} u(x, t) (-\Delta)^{\beta/2} \varphi(x, t) \\ &- \int_{Q_T} u(x, t) \varphi_t(x, t), \end{aligned}$$

for any test function $\varphi \in C^1([0, T], H^\beta(\mathbb{R}^N))$ such that $\varphi(x, T) = 0$, $x \in \mathbb{R}^N$, where $\alpha := 1 - \gamma \in (0, 1)$.

Now we take $\varphi(x, t) = D_{t|T}^\alpha (\tilde{\varphi}(x, t)) := D_{t|T}^\alpha \left((\varphi_1(x))^\ell \varphi_2(t) \right)$ with $\varphi_1(x) := \Phi(|x|/T^{1/\beta})$, $\varphi_2(t) := (1 - t/T)_+^\eta$, where $\ell \geq p/(p-1)$, $\eta \geq (\alpha p + 1)/(p-1)$ and Φ a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $|\Phi'(r)| \leq C_1/r$, for all $r > 0$. Using (2.14), we then obtain

$$\begin{aligned} \int_{\Omega} u_0(x) D_{t|T}^\alpha \tilde{\varphi}(x, 0) + \int_{\Omega_T} J_{0|t}^\alpha (u^p)(x, t) D_{t|T}^\alpha \tilde{\varphi}(x, t) \\ = \int_{Q_T} u(x, t) (-\Delta)^{\beta/2} D_{t|T}^\alpha \tilde{\varphi}(x, t) - \int_{\Omega_T} u(x, t) D D_{t|T}^\alpha \tilde{\varphi}(x, t), \end{aligned} \quad (4.6)$$

where

$$\Omega_T := [0, T] \times \Omega \text{ for } \Omega = \left\{ x \in \mathbb{R}^N ; |x| \leq 2T^{1/\beta} \right\}, \quad \int_{\Omega} = \int_{\Omega} dx \text{ and } \int_{\Omega_T} = \int_{\Omega_T} dx dt.$$

Furthermore, using (2.9) and (2.14) in the left hand side of (4.6) while in the right hand side we use (2.10), we conclude that

$$\begin{aligned} & C T^{-\alpha} \int_{\Omega} u_0(x) \varphi_1^{\ell}(x) + \int_{\Omega_T} D_{0|t}^{\alpha} J_{0|t}^{\alpha}(u^p)(x, t) \tilde{\varphi}(x, t) \\ &= \int_{Q_T} u(x, t) (-\Delta)^{\beta/2} D_{t|T}^{\alpha} \tilde{\varphi}(x, t) + \int_{\Omega_T} u(x, t) D_{t|T}^{1+\alpha} \tilde{\varphi}(x, t). \end{aligned} \quad (4.7)$$

Moreover, using (2.11), we may write

$$\begin{aligned} & \int_{\Omega_T} u^p(x, t) \tilde{\varphi}(x, t) + C T^{-\alpha} \int_{\Omega} u_0(x) \varphi_1^{\ell}(x) \\ &= \int_{Q_T} u(x, t) (-\Delta)^{\beta/2} \varphi_1^{\ell}(x) D_{t|T}^{\alpha} \varphi_2(t) + \int_{\Omega_T} u(x, t) D_{t|T}^{1+\alpha} \tilde{\varphi}(x, t). \end{aligned} \quad (4.8)$$

So, Ju's inequality $(-\Delta)^{\beta/2} (\varphi_1^{\ell}) \leq \ell \varphi_1^{\ell-1} (-\Delta)^{\beta/2} (\varphi_1)$ (see Appendix) will allow us to write:

$$\begin{aligned} & \int_{\Omega_T} u^p(x, t) \tilde{\varphi}(x, t) + C T^{-\alpha} \int_{\Omega} u_0(x) \varphi_1^{\ell}(x) \\ & \leq C \int_{\Omega_T} u(x, t) \varphi_1^{\ell-1}(x) \left| (-\Delta)^{\beta/2} \varphi_1(x) D_{t|T}^{\alpha} \varphi_2(t) \right| \\ & \quad + \int_{\Omega_T} u(x, t) \varphi_1^{\ell}(x) \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \\ & = C \int_{\Omega_T} u(x, t) \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^{\ell-1}(x) \left| (-\Delta)^{\beta/2} \varphi_1(x) D_{t|T}^{\alpha} \varphi_2(t) \right| \\ & \quad + \int_{\Omega_T} u(x, t) \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^{\ell}(x) \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \end{aligned} \quad (4.9)$$

Therefore, as $u_0 \geq 0$, then, using Young's inequality

$$ab \leq \frac{1}{2p} a^p + \frac{2^{\tilde{p}-1}}{\tilde{p}} b^{\tilde{p}} \quad \text{where } p\tilde{p} = p + \tilde{p}, \quad a > 0, b > 0, \quad p > 1, \tilde{p} > 1, \quad (4.10)$$

with

$$\begin{cases} a = u(x, t) \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p} \varphi_1^{\ell-1}(x) \left| (-\Delta)^{\beta/2} \varphi_1(x) D_{t|T}^{\alpha} \varphi_2(t) \right|, \end{cases}$$

in the first integral of the right hand side of (4.9), and with

$$\begin{cases} a = u(x, t) \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p} \varphi_1^{\ell}(x) \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|, \end{cases}$$

in the second integral of the right hand side of (4.9), we obtain

$$\begin{aligned}
& \left(1 - \frac{1}{p}\right) \int_{\Omega_T} u^p(x, t) \tilde{\varphi}(x, t) \\
& + C \int_{\Omega_T} (\varphi_1(x))^{\ell - \tilde{p}} (\varphi_2(t))^{-\frac{1}{p-1}} \left| (-\Delta_x)^{\beta/2} \varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right|^{\tilde{p}} \\
& \leq C \int_{\Omega_T} (\varphi_1(x))^\ell (\varphi_2(t))^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|^{\tilde{p}}.
\end{aligned} \tag{4.11}$$

At this stage, we introduce the scaled variables: $\tau = T^{-1}t$, $\xi = T^{-1/\beta}x$, use formula (2.12) and (2.13) in the right hand-side of (4.11), to obtain:

$$\int_{\Omega_T} u^p(x, t) \tilde{\varphi}(x, t) \leq C T^{-\delta}, \tag{4.12}$$

where $\delta := (1 + \alpha)\tilde{p} - 1 - (N/\beta)$, $C = C(|\Omega_2|, |\Omega_3|)$, ($|\Omega_i|$ stands for the measure of Ω_i , for $i = 2, 3$), with

$$\Omega_2 := \{\xi \in \mathbb{R}^N ; |\xi| \leq 2\} \quad , \quad \Omega_3 := \{\tau \geq 0 ; \tau \leq 1\}.$$

Now, noting that as

$$p \leq p^* \quad \text{or} \quad p < \frac{1}{\gamma} \quad \iff \quad \delta \geq 0 \quad \text{or} \quad p < \frac{1}{\gamma}, \tag{4.13}$$

we have to distinguish three cases:

- The case $p < p^*$ ($\delta > 0$): We pass to the limit in (4.12), as T goes to ∞ ; we get

$$\lim_{T \rightarrow \infty} \int_0^T \int_{|x| \leq 2T^{1/\beta}} u^p(x, t) \tilde{\varphi}(x, t) dx dt = 0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of u and the fact that $\tilde{\varphi}(x, t) \rightarrow 1$ as $T \rightarrow \infty$, we infer that

$$\int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt = 0 \quad \implies \quad u \equiv 0.$$

Contradiction.

- The case $p = p^*$ ($\delta = 0$): Using inequality (4.12) with $T \rightarrow \infty$, taking into account the fact that $p = p^*$, we have on one hand

$$u \in L^p(0, \infty; L^p(\mathbb{R}^N)). \tag{4.14}$$

On the other hand, we repeat the same calculation as above by taking this time $\varphi_1(x) := \Phi(|x|/(B^{-1/\beta}T^{1/\beta}))$, where $1 \leq B < T$ is large enough such that when $T \rightarrow \infty$ we don't have $B \rightarrow \infty$ in the same time, so we arrive at

$$\int_{\Sigma_T} u^p(x, t) \tilde{\varphi}(x, t) \leq C B^{-N/\beta} + C B^{-N/\beta+\tilde{p}}, \quad (4.15)$$

thanks to the following rescaling: $\tau = T^{-1}t$, $\xi = (T/B)^{-1/\beta}x$, where

$$\Sigma_T := [0, T] \times \left\{ x \in \mathbb{R}^N ; |x| \leq 2B^{-1/\beta}T^{1/\beta} \right\} \quad \text{and} \quad \int_{\Sigma_T} = \int_{\Sigma_T} dx dt.$$

Thus, using $p > N/(N - \beta)$ and taking the limit when $T \rightarrow \infty$ and then $B \rightarrow \infty$, we get:

$$\int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt = 0 \quad \implies \quad u \equiv 0,$$

which is a contradiction.

Note that, in the case $\beta = 2$ it is not necessary to take the condition $p > N/(N - \beta)$ with $N > \beta$. Indeed, from (4.9) with the new function φ_1 , we may write

$$\begin{aligned} & \int_{\Sigma_T} u^p(x, t) \tilde{\varphi}(x, t) \\ & \leq C \int_{\Sigma_T} u(x, t) \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} (\varphi_1(x))^\ell \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \\ & + C \int_{\Delta_B} u(x, t) \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} (\varphi_1(x))^{\ell-1} \left| \Delta_x \varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right|, \end{aligned} \quad (4.16)$$

where

$$\Delta_B = [0, T] \times \left\{ x \in \mathbb{R}^N ; B^{-1/2}T^{1/2} \leq |x| \leq 2B^{-1/2}T^{1/2} \right\} \subset \Sigma_T \quad \text{and} \quad \int_{\Delta_B} = \int_{\Delta_B} dx dt.$$

Moreover, using the following Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{\tilde{p}} b^{\tilde{p}} \quad \text{where} \quad p\tilde{p} = p + \tilde{p}, \quad a > 0, b > 0, \quad p > 1, \tilde{p} > 1, \quad (4.17)$$

with

$$\begin{cases} a = u(x, t) \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p} (\varphi_1(x))^\ell \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|, \end{cases}$$

in the first integral of the right hand side of (4.16), and using Hölder's inequality

$$\int_{\Delta_B} ab \leq \left(\int_{\Delta_B} a^p \right)^{1/p} \left(\int_{\Delta_B} b^{\tilde{p}} \right)^{1/\tilde{p}},$$

with

$$\begin{cases} a = u(x, t) \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p} (\varphi_1(x))^{\ell-1} \left| \Delta_x \varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right|, \end{cases}$$

in the second integral of the right hand side of (4.16), we obtain

$$\begin{aligned}
& \left(1 - \frac{1}{p}\right) \int_{\Sigma_T} u^p(x, t) \tilde{\varphi}(x, t) \\
& \leq C \int_{\Sigma_T} (\varphi_1(x))^\ell (\varphi_2(t))^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|^{\tilde{p}} \\
& + C \left(\int_{\Delta_B} u^p \tilde{\varphi} \right)^{1/p} \left(\int_{\Delta_B} (\varphi_1(x))^{\ell-\tilde{p}} (\varphi_2(t))^{-\frac{1}{p-1}} \left| \Delta_x \varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right|^{\tilde{p}} \right)^{1/\tilde{p}}. \quad (4.18)
\end{aligned}$$

Taking account of the scaled variables: $\tau = T^{-1}t$, $\xi = (T/B)^{-1/2}x$, and the fact that $\delta = 0$, we get

$$\int_{\Sigma_T} u^p(x, t) \tilde{\varphi}(x, t) \leq C B^{-N/2} + C B^{-\frac{N}{2p}+1} \left(\int_{\Delta_B} u^p \tilde{\varphi} \right)^{1/p}. \quad (4.19)$$

Now, as

$$\lim_{T \rightarrow \infty} \left(\int_{\Delta_B} u^p \tilde{\varphi} \right)^{1/p} = 0 \quad (\text{from (4.14)}),$$

then, passing to the limit in (4.19), as $T \rightarrow \infty$, we get

$$\int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \leq C B^{-N/2}.$$

Then, taking the limit when B goes to infinity, we obtain $u \equiv 0$, contradiction.

• For the case $p < (1/\gamma)$, we repeat the same argument as in the case $p < p^*$ by choosing the test function as follows: $\varphi(x, t) = D_{t|T}^\alpha \bar{\varphi}(x, t) := D_{t|T}^\alpha (\varphi_3^\ell(x) \varphi_4(t))$ where $\varphi_3(x) = \Phi(|x|/R)$, $\varphi_4(t) = (1 - t/T)_+^\eta$ and $R \in (0, T)$ large enough such that in the case of $T \rightarrow \infty$ we don't have $R \rightarrow \infty$ in the same time, with the same functions Φ as above. We then obtain

$$\begin{aligned}
& \int_{\mathcal{C}_T} u^p(x, t) \bar{\varphi}(x, t) + C T^{-\alpha} \int_{\mathcal{C}} (\varphi_3(x))^\ell u_0(x) \\
& \leq C \int_{\mathcal{C}_T} u(x, t) \bar{\varphi}^{1/p} \bar{\varphi}^{-1/p} (\varphi_3(x))^\ell \left| D_{t|T}^{1+\alpha} \varphi_4(t) \right| \\
& + C \int_{\mathcal{C}_T} u(x, t) \bar{\varphi}^{1/p} \bar{\varphi}^{-1/p} (\varphi_3(x))^{\ell-1} \left| (-\Delta_x)^{\beta/2} \varphi_3(x) D_{t|T}^\alpha \varphi_4(t) \right|, \quad (4.20)
\end{aligned}$$

where

$$\mathcal{C}_T := [0, T] \times \mathcal{C} \quad \text{for } \mathcal{C} := \{x \in \mathbb{R}^N ; |x| \leq 2R\}, \quad \int_{\mathcal{C}} = \int_{\mathcal{C}} dx \quad \text{and} \quad \int_{\mathcal{C}_T} = \int_{\mathcal{C}_T} dx dt.$$

Now, by Young's inequality (4.10), with

$$\begin{cases} a = u(x, t) \bar{\varphi}^{1/p}, \\ b = \bar{\varphi}^{-1/p} (\varphi_3(x))^\ell \left| D_{t|T}^{1+\alpha} \varphi_4(t) \right|, \end{cases}$$

in the first integral of the right hand side of (4.20) and with

$$\begin{cases} a = u(x, t) \bar{\varphi}^{1/p}, \\ b = \bar{\varphi}^{-1/p} (\varphi_3(x))^{\ell-1} \left| (-\Delta_x)^{\beta/2} \varphi_3(x) D_{t|T}^\alpha \varphi_4(t) \right|, \end{cases}$$

in the second integral of the right hand side of (4.20) and using the fact that $u_0 \geq 0$, we get

$$\begin{aligned} \left(1 - \frac{1}{p}\right) \int_{\mathcal{C}_T} u^p(x, t) \bar{\varphi}(x, t) \leq & C \int_{\mathcal{C}_T} (\varphi_3(x))^\ell (\varphi_4(t))^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\alpha} \varphi_4(t) \right|^{\tilde{p}} \\ & + C \int_{\mathcal{C}_T} (\varphi_3(x))^{\ell-\tilde{p}} (\varphi_4(t))^{-\frac{1}{p-1}} \left| (-\Delta_x)^{\beta/2} \varphi_3 D_{t|T}^\alpha \varphi_4 \right|^{\tilde{p}}. \end{aligned}$$

Then, the new variables $\xi = R^{-1}x$, $\tau = T^{-1}t$ and (2.12) – (2.13) allow us to write:

$$\int_{\mathcal{C}_T} u^p(x, t) \bar{\varphi}(x, t) dx dt \leq C T^{1-(1+\alpha)\tilde{p}} R^N + C T^{1-\alpha\tilde{p}} R^{N-\beta\tilde{p}}.$$

Taking the limit as $T \rightarrow \infty$, we infer, as $p < 1/\gamma \iff 1 - \alpha\tilde{p} < 0$, that

$$\int_0^\infty \int_{\mathcal{C}} u(x, t)^p (\varphi_3(x))^\ell dx dt = 0.$$

Finally, by taking $R \rightarrow \infty$, we get a contradiction as $u(x, t) > 0$ for all $x \in \mathbb{R}^N$, $t > 0$. \square

Remarks 1

(1) We can extend our analysis to the equation

$$u_t = -(-\Delta)^{\beta/2} u + \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\psi(x, s) |u(s)|^{p-1} u(s)}{(t-s)^\gamma} ds, \quad x \in \mathbb{R}^N, \quad (4.21)$$

where $p > 1$, $\beta \in (0, 2]$, $0 < \gamma < 1$ and $\psi \in L^1_{Loc}(\mathbb{R}^N \times (0, \infty))$, $\psi(\cdot, t) > 0$ for all $t > 0$,

$$\begin{cases} \psi(B^{-1/\beta} T^{1/\beta} \xi, T\tau) \geq C > 0 & \text{if } p \leq p^* \\ \psi(R\xi, T\tau) \geq C > 0 & \text{if } p < 1/\gamma, \end{cases}$$

for any $0 < R, B < T$, $\tau \in [0, 1]$ and $\xi \in [0, 2]$.

(2) If we take $\beta = 2$ and $v(x, t) = (\Gamma(1-\gamma))^{(1-\gamma)/(p-1)} u(\Gamma(1-\gamma)^{1/2}x, \Gamma(1-\gamma)t)$ where u is a solution of (1.1), we recover the result in [5] as a particular case.

(3) We can take the nonlocal porous-medium spatio-fractional problem

$$\begin{cases} u_t + (-\Delta)^{\beta/2} |u|^{m-1} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s) ds & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^N, \end{cases}$$

where $\beta \in (0, 2]$, $0 < \gamma < 1$, $1 \leq m < p$, $u_0 \geq 0$ and $u_0 \not\equiv 0$.

The threshold on p will be

$$p \leq 1 + \frac{(2 - \gamma)(N(m - 1) + \beta)}{(N - \beta + \beta\gamma)_+} \quad \text{or} \quad p < \frac{m}{\gamma}.$$

5 Blow-up Rate

In this section, we present the blow-up rate for the parabolic problem (1.1) in the case of $\beta = 2$.

We take the solution of (1.1) with an initial condition satisfying

$$u(\cdot, 0) = u_0 \geq 0, \quad u_0 \not\equiv 0 \quad \text{and} \quad u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N). \quad (5.1)$$

The following lemma will be used in the proof of Theorem 3 below.

Lemma 2 *Let φ be a nonnegative classical solution of*

$$\varphi_t = \Delta\varphi + J_{-\infty|t}^\alpha(\varphi^p) \quad \text{in } \mathbb{R}^N \times \mathbb{R}. \quad (5.2)$$

Then, for

$$p \leq p^* \quad \text{or} \quad p < \frac{1}{\gamma}, \quad (5.3)$$

we have $\varphi \equiv 0$.

Proof We repeat the same computations as in Theorem 2 with $(1 - t^2/T^2)_+^\eta$ instead of $(1 - t/T)_+^\eta$ for $\eta \gg 1$, using (2.15) – (2.17), taking account of the following inequality

$$J_{-\infty|t}^\alpha(\varphi^p) \geq J_{-T|t}^\alpha(\varphi^p).$$

Moreover, we take $\varphi_1^{\ell/p} \varphi_1^{-\ell/p}$ instead of $\tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p}$ (resp. $\bar{\varphi}^{1/p} \bar{\varphi}^{-1/p}$) in (4.9), (4.16) (resp. in (4.20)) for $\ell \gg 1$ to use the Young and Hölder's inequality.

Note that here, we use the ε -Young inequality

$$ab \leq \frac{\varepsilon}{2} a^p + C(\varepsilon) b^{\tilde{p}},$$

for $0 < \varepsilon < 1$. □

Theorem 3 *For $p \leq p^*$ or $p < (1/\gamma)$, let $\alpha_1 := (2 - \gamma)/(p - 1)$ and u be the blowing-up mild solution of (1.1), under the condition (5.1), in a finite time $T_{\max} := T^*$. Then there exist constants $c, C > 0$ such that*

$$c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(\cdot, t) \leq C(T^* - t)^{-\alpha_1}, \quad t \in (0, T^*). \quad (5.4)$$

Proof We decompose the proof into two parts:

- The upper blow-up rate estimate. Let

$$M(t) := \sup_{\mathbb{R}^N \times (0, t]} u, \quad t \in (0, T^*).$$

Clearly, M is positive, continuous and nondecreasing on $(0, T^*)$. As $\lim_{t \rightarrow T^*} M(t) = \infty$, then for all $t_0 \in (0, T^*)$ we can define

$$t_0^+ := t^+(t_0) := \max\{t \in (t_0, T^*) : M(t) = 2M(t_0)\}.$$

Choose $A \geq 1$ and let

$$\lambda(t_0) := \left(\frac{1}{2A} M(t_0) \right)^{-1/(2\alpha_1)}. \quad (5.5)$$

we claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq D, \quad t_0 \in \left(\frac{T^*}{2}, T^* \right), \quad (5.6)$$

where $D > 0$ is a positive constant which does not depend on t_0 .

We proceed by contradiction. If (5.6) were false, then there would exist a sequence $t_n \rightarrow T^*$ such that

$$\lambda_n^{-2}(t_n^+ - t_n) \rightarrow \infty,$$

where $\lambda_n = \lambda(t_n)$ and $t_n^+ = t^+(t_n)$. For each t_n choose

$$(\hat{x}_n, \hat{t}_n) \in \mathbb{R}^N \times (0, t_n] \quad \text{such that} \quad u(\hat{x}_n, \hat{t}_n) \geq \frac{1}{2} M(t_n). \quad (5.7)$$

Obviously, $M(t_n) \rightarrow \infty$; hence, $\hat{t}_n \rightarrow T^*$. Next, rescale the function u as

$$\varphi^{\lambda_n}(y, s) := \lambda_n^{2\alpha_1} u(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n), \quad (y, s) \in \mathbb{R}^N \times I_n(T^*), \quad (5.8)$$

where $I_n(t) := (-\lambda_n^{-2}\hat{t}_n, \lambda_n^{-2}(t - \hat{t}_n))$ for all $t > 0$. Then φ^{λ_n} is a mild solution of

$$\varphi_s = \Delta \varphi + J_{-\lambda_n^{-2}\hat{t}_n|s}^\alpha(\varphi^p) \quad \text{in } \mathbb{R}^N \times I_n(T^*), \quad (5.9)$$

i.e., for $G(t) := G(x, t) := (4\pi t)^{-N/2} e^{-|x|^2/4t}$ and for $*$ be the space convolution, we have

$$\varphi^{\lambda_n}(s) = G(s + \lambda_n^{-2}\hat{t}_n) * \varphi^{\lambda_n}(-\lambda_n^{-2}\hat{t}_n) + \int_{-\lambda_n^{-2}\hat{t}_n}^s G(s - \sigma) * J_{-\lambda_n^{-2}\hat{t}_n|\sigma}^\alpha((\varphi^{\lambda_n})^p) d\sigma \quad (5.10)$$

in $\mathbb{R}^N \times I_n(T^*)$ such that $\varphi^{\lambda_n}(0, 0) \geq A$, and

$$0 \leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M(t_n^+) = \lambda_n^{2\alpha_1} 2M(t_n) = 4A \quad \text{in } \mathbb{R}^N \times I_n(t_n^+),$$

thanks to (5.5) and the definition of t_n^+ .

Moreover, as

$$\varphi^{\lambda_n} \in C([-\lambda_n^{-2}\hat{t}_n, T], C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)) \quad \text{for all } T \in I_n(T^*),$$

so, as in Lemma 1, φ^{λ_n} is a weak solution of (5.9).

On the other hand, if we write φ^{λ_n} as $\varphi^{\lambda_n}(s) = v(s) + w(s)$ for all $s \in I_n(T^*)$, where

$$v(s) := G(s + \lambda_n^{-2}\widehat{t}_n) * \varphi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) \quad \text{and} \quad w(s) := \int_{-\lambda_n^{-2}\widehat{t}_n}^s G(s - \sigma) * J_{-\lambda_n^{-2}\widehat{t}_n|\sigma}^\alpha((\varphi^{\lambda_n})^p) d\sigma,$$

we have, see [4][Chapter 3], for $T \in I_n(T^*)$

$$v \in C((-\lambda_n^{-2}\widehat{t}_n, T); H^2(\mathbb{R}^N)) \cap C^1((-\lambda_n^{-2}\widehat{t}_n, T); L^2(\mathbb{R}^N)) \subset L^2((-\lambda_n^{-2}\widehat{t}_n, T); H^1(\mathbb{R}^N))$$

and, using the fact that $f(s) := J_{-\lambda_n^{-2}\widehat{t}_n|s}^\alpha((\varphi^{\lambda_n})^p) \in L^2((-\lambda_n^{-2}\widehat{t}_n, T); L^2(\mathbb{R}^N))$ and the Maximal regularity theory, we have

$$w \in W^{1,2}((-\lambda_n^{-2}\widehat{t}_n, T); L^2(\mathbb{R}^N)) \cap L^2((-\lambda_n^{-2}\widehat{t}_n, T); H^2(\mathbb{R}^N)) \subset L^2((-\lambda_n^{-2}\widehat{t}_n, T); H^1(\mathbb{R}^N)).$$

It follows that

$$\varphi^{\lambda_n} \in C([-\lambda_n^{-2}\widehat{t}_n, T], L^2(\mathbb{R}^N)) \cap L^2((-\lambda_n^{-2}\widehat{t}_n, T), W^{1,2}(\mathbb{R}^N));$$

so from the parabolic interior regularity theory (cf. [18][Theorem 10.1 p. 204]) there is $\mu \in (0, 1)$ such that the sequence φ^{λ_n} is bounded in the $C_{\text{loc}}^{\mu, \mu/2}(\mathbb{R}^N \times \mathbb{R})$ -norm by a constant that does not depend on n , where $C_{\text{loc}}^{\mu, \mu/2}(\mathbb{R}^N \times \mathbb{R})$ is the locally Hölder space defined in [18]. Similar uniform estimates for $J_{-\lambda_n^{-2}\widehat{t}_n|s}^\alpha(\varphi^p)$ follow if μ is sufficiently small. The parabolic interior Schauder's estimates (see [17][Th. 8.11.1 p. 130]), using the existence theorems in Hölder's space, imply now that the $C_{\text{loc}}^{2+\mu, 1+\mu/2}(\mathbb{R}^N \times \mathbb{R})$ -norm of φ^{λ_n} is uniformly bounded. Hence, we obtain a subsequence converging in $C_{\text{loc}}^{2+\mu, 1+\mu/2}(\mathbb{R}^N \times \mathbb{R})$ to a solution φ of

$$\varphi_s = \Delta \varphi + J_{-\infty|s}^\alpha(\varphi^p) \quad \text{in } \mathbb{R}^N \times (-\infty, +\infty),$$

such that $\varphi(0, 0) \geq A$ and $0 \leq \varphi \leq 4A$ in $\mathbb{R}^N \times \mathbb{R}$. Then, using Lemma 2, we infer that $\varphi \equiv 0$ in $\mathbb{R}^N \times (-\infty, +\infty)$. Contradiction with the fact that $\varphi(0, 0) \geq A > 1$. This proves (5.6).

Next we use an idea from Hu [11]. From (5.5) and (5.6) it follows that

$$(t_0^+ - t_0) \leq D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \quad \text{for any } t_0 \in \left(\frac{T^*}{2}, T^*\right).$$

Fix $t_0 \in (T^*/2, T^*)$ and denote $t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \dots$. Then

$$\begin{aligned} t_{j+1} - t_j &\leq D(2A)^{1/\alpha_1} M(t_j)^{-1/\alpha_1}, \\ M(t_{j+1}) &= 2M(t_j), \end{aligned}$$

$j = 0, 1, 2, \dots$. Consequently,

$$\begin{aligned} T^* - t_0 &= \sum_{j=0}^{\infty} (t_{j+1} - t_j) \leq D(2A)^{1/\alpha_1} \sum_{j=0}^{\infty} M(t_j)^{-1/\alpha_1} \\ &= D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \sum_{j=0}^{\infty} 2^{-j/\alpha_1}. \end{aligned}$$

And finally we conclude that

$$u(x, t_0) \leq M(t_0) \leq C(T^* - t_0)^{-\alpha_1}, \quad \forall t_0 \in (0, T^*)$$

where

$$C = 2A \left(D \sum_{j=0}^{\infty} 2^{-j/\alpha_1} \right)^{\alpha_1};$$

consequently

$$\sup_{\mathbb{R}^N} u(., t) \leq C(T^* - t)^{-\alpha_1}, \quad \forall t \in (0, T^*).$$

• The lower blow-up rate estimate. If we repeat the same proof of the local existence of Theorem 1, by taking $\|u\|_1 \leq \theta$ instead of $\|u\|_1 \leq 2\|u_0\|_\infty$ in the space E_T for all positive constant $\theta > 0$ and all $0 < t < T$, then the condition (3.2) of T will be:

$$\|u_0\|_\infty + CT^{2-\gamma}\theta^p \leq \theta, \quad (5.11)$$

and then, like before, we infer that $\|u(t)\|_\infty \leq \theta$ for (almost) all $0 < t < T$. Consequently, if $\|u_0\|_\infty + Ct^{2-\gamma}\theta^p \leq \theta$, then $\|u(t)\|_\infty \leq \theta$. Applying this to any point in the trajectory, we see that if $0 \leq s < t$ and

$$(t - s)^{2-\gamma} \leq \frac{\theta - \|u(s)\|_\infty}{C\theta^p}, \quad (5.12)$$

then $\|u(t)\|_\infty \leq \theta$, for all $0 < t < T$.

Moreover, if $0 \leq s < T^*$ and $\|u(s)\|_\infty < \theta$, then:

$$(T^* - s)^{2-\gamma} > \frac{\theta - \|u(s)\|_\infty}{C\theta^p}. \quad (5.13)$$

Indeed, arguing by contradiction and assume that for some $\theta > \|u(s)\|_\infty$ and all $t \in (s, T^*)$ we have

$$(t - s)^{2-\gamma} \leq \frac{\theta - \|u(s)\|_\infty}{C\theta^p},$$

then, using (5.12), we infer that $\|u(t)\|_\infty \leq \theta$ for all $t \in (s, T^*)$; this contradicts with the fact that $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T^*$.

Next, for example, letting $\theta = 2\|u(s)\|_\infty$ in (5.13), we see that for $0 < s < T^*$ we have:

$$(T^* - s)^{2-\gamma} > C' \|u(s)\|_\infty^{1-p},$$

and by the positivity and the continuity of u we get

$$c(T^* - s)^{-\alpha_1} < \sup_{x \in \mathbb{R}^N} u(x, s), \quad \forall s \in (0, T^*) \quad (5.14)$$

□

6 Global existence

In this section, we prove the existence of global solutions of (1.1) with initial data small enough. We give a similar proof as that in [5] just for the seek of completeness. In the following, we use the notation $p_{sc} := N(p-1)/\beta(2-\gamma)$. As $p^* > 1 + \beta(2-\gamma)/N$, we note that $p > p^* \Rightarrow p_{sc} > 1$.

Theorem 4 *Let $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$ and $0 < \beta \leq 2$. If*

$$p > \max\left\{\frac{1}{\gamma}; p^*\right\}, \quad (6.1)$$

and $\|u_0\|_{L^{p_{sc}}}$ is sufficiently small, then u exists globally.

Note that we can take $|u_0(x)| \leq C|x|^{-\beta(2-\gamma)/(p-1)}$ instead of $u_0 \in L^{p_{sc}}$.

Proof As $p > (1/\gamma)$, then we have the possibility to take a positive constant $q > 0$ so that:

$$\frac{2-\gamma}{p-1} - \frac{1}{p} < \frac{N}{\beta q} < \frac{1}{p-1}, \quad q \geq p. \quad (6.2)$$

It follows, using (6.1), that

$$q > \frac{N(p-1)}{\beta} > p_{sc} > 1. \quad (6.3)$$

Let

$$b := \frac{N}{\beta p_{sc}} - \frac{N}{\beta q} = \frac{2-\gamma}{p-1} - \frac{N}{\beta q}. \quad (6.4)$$

Then, using (6.2) – (6.4), we conclude that

$$b > \frac{1-\gamma}{p-1} > 0, \quad pb < 1, \quad \frac{N(p-1)}{\beta q} + (p-1)b + \gamma = 2. \quad (6.5)$$

As $u_0 \in L^{p_{sc}}$, using (2.4) and (6.4), we get, for all $t > 0$,

$$\sup_{t>0} t^b \|e^{-t(-\Delta)^{\beta/2}} u_0\|_{L^q} \leq C \|u_0\|_{L^{p_{sc}}} = \eta < \infty. \quad (6.6)$$

Set

$$\Xi := \left\{ u \in L^\infty((0, \infty), L^q(\mathbb{R}^N)); \sup_{t>0} t^b \|u(t)\|_{L^q} \leq \delta \right\}, \quad (6.7)$$

where $\delta > 0$ is to be chosen sufficiently small. If

$$d_\Xi(u, v) = \sup_{t>0} t^b \|u(t) - v(t)\|_{L^q}, \quad \forall u, v \in \Xi, \quad (6.8)$$

then (Ξ, d) is a complete metric space. Given $u \in \Xi$, let's set:

$$\Phi(u)(t) := e^{-t(-\Delta)^{\beta/2}} u_0 + \frac{1}{\Gamma(1-\gamma)} \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_0^s (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma) d\sigma ds, \quad (6.9)$$

for all $t \geq 0$. We have by (2.4), (6.6) and (6.7)

$$\begin{aligned} t^b \|\Phi(u)(t)\|_{L^q} &\leq \eta + Ct^b \int_0^t (t-s)^{-\frac{N}{\beta}(\frac{p}{q}-\frac{1}{q})} \int_0^s (s-\sigma)^{-\gamma} \|u^p(\sigma)\|_{L^{\frac{q}{p}}} d\sigma ds \\ &\leq \eta + C\delta^p t^b \int_0^t \int_0^s (t-s)^{-\frac{N(p-1)}{\beta q}} (s-\sigma)^{-\gamma} \sigma^{-bp} d\sigma ds. \end{aligned} \quad (6.10)$$

Next, using (6.2) and $pb < 1$, we get

$$\begin{aligned} \int_0^t \int_0^s \frac{(t-s)^{-\frac{N}{\beta q}(p-1)}}{(s-\sigma)^\gamma} \sigma^{-bp} d\sigma ds &= \left(\int_0^1 (1-\sigma)^{-\gamma} \sigma^{-bp} d\sigma \right) \int_0^t \frac{(t-s)^{-\frac{N(p-1)}{\beta q}}}{s^{bp+\gamma-1}} ds \\ &= Ct^{-\frac{N(p-1)}{\beta q}-bp-\gamma+2} = Ct^{-b}, \end{aligned} \quad (6.11)$$

for all $t \geq 0$. So, we deduce from (6.10) – (6.11) that

$$t^b \|\Phi(u)(t)\|_{L^q} \leq \eta + C\delta^p. \quad (6.12)$$

Therefore, if η and δ are chosen small enough so that $\eta + C\delta^p \leq \delta$, we see that $\Phi : \Xi \rightarrow \Xi$. Similar calculations show that (assuming η and δ small enough) Φ is a strict contraction, so it has a fixed point $u \in \Xi$ which is a solution of (1.1). Now, we show that $u \in C([0, \infty), C_0(\mathbb{R}^N))$.

First, we show that $u \in C([0, T], C_0(\mathbb{R}^N))$ if $T > 0$ is sufficiently small. Indeed, note that the above argument shows uniqueness in Ξ_T , where, for any $T > 0$,

$$\Xi_T := \left\{ u \in L^\infty((0, T), L^q(\mathbb{R}^N)); \sup_{0 < t < T} t^b \|u(t)\|_{L^q} \leq \delta \right\}.$$

Let \tilde{u} be the local solution of (1.1) constructed in Theorem 1. Since $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, then, using the fact that $u_0 \in L^q(\mathbb{R}^N)$ and (6.3), we have $\tilde{u} \in C([0, T_{\max}), L^q(\mathbb{R}^N))$ by Theorem 1. It follows that $\sup_{0 < t < T} t^b \|\tilde{u}(t)\|_{L^q} \leq \delta$ if $T > 0$ is sufficiently small. Therefore, by uniqueness, $u = \tilde{u}$ on $[0, T]$, so that $u \in C([0, T], C_0(\mathbb{R}^N))$.

Next, we show that $u \in C([T, \infty), C_0(\mathbb{R}^N))$ by a bootstrap argument. Indeed, for $t > T$, we write

$$\begin{aligned} u(t) - e^{-t(-\Delta)^{\beta/2}} u_0 &= \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_0^T (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma) d\sigma ds \\ &\quad + \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_T^s (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma) d\sigma ds \\ &\equiv I_1(t) + I_2(t). \end{aligned}$$

Since $u \in C([0, T], C_0(\mathbb{R}^N))$, it follows that $I_1 \in C([T, \infty), C_0(\mathbb{R}^N))$. Also, by the calculations used to construct the fixed point, using the fact that $t^{-b} \leq T^{-b} < \infty$ and $pq > q$, $I_1 \in C([T, \infty), L^q(\mathbb{R}^N))$. Next, note that $N(p/q - 1/q)/\beta < 1$ by (6.3). Therefore, there exists $r \in (q, \infty]$ such that

$$\frac{N}{\beta} \left(\frac{p}{q} - \frac{1}{r} \right) < 1. \quad (6.13)$$

Given $T < s < t$ (the case of $s \leq T \leq t$ is obvious). Since $u \in L^\infty((0, \infty), L^q(\mathbb{R}^N))$, we have $|u|^{p-1}u \in L^\infty((T, s), L^{q/p}(\mathbb{R}^N))$, and it easily follows, using (2.4) and (6.13), that $I_2 \in C([T, \infty), L^r(\mathbb{R}^N))$. As the terms $e^{-(-\Delta)^{\beta/2}}u_0$ and I_1 both belong to $C([T, \infty), C_0(\mathbb{R}^N)) \cap C([T, \infty), L^q(\mathbb{R}^N))$, we see that $u \in C([T, \infty), L^r(\mathbb{R}^N))$. Iterating this procedure a finite number of times, we deduce that $u \in C([T, \infty), C_0(\mathbb{R}^N))$. This completes the proof.

7 Necessary conditions for local and global existence

In this section, we establish necessary conditions for the existence of local and global solutions to the problem (1.1); these conditions depend on the behavior of the initial condition for large x .

Theorem 5 (*Necessary conditions for global existence*)

Let $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0 \geq 0$, $0 < \beta \leq 2$ and $p > 1$. If u is a global mild solution to problem (1.1), then there is a positive constant $C > 0$ such that

$$\lim_{|x| \rightarrow \infty} (u_0(x)|x|^{\frac{\beta(2-\gamma)}{p-1}}) \leq C. \quad (7.1)$$

Proof Let u be a global mild solution to (1.1), then $u \in C([0, R^\beta], C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ for all $R > 0$ sufficiently large. So, we repeat the same calculation as in the proof of Theorem 1 by taking $\varphi(x, t) := D_{t|T}^\alpha \tilde{\varphi}(x, t) := D_{t|T}^\alpha (\varphi_1(x/R)\varphi_2(t))$ instead of the one chosen in Theorem 1, where, for $x \in \mathbb{R}^N$, $0 \leq \varphi_1 \in H^\beta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is the first eigenfunction of the fractional laplacian operator $(-\Delta_x)^{\beta/2}$, associated to the first eigenvector $\lambda_1 := \inf\{\|u\|_{H^{\beta/2}}; \|u\|_{L^2} = 1 \text{ and } u = 0 \text{ in } B_2^c\}$ where B_2 stands for the ball of center 0 and radius 2 and B_2^c for its complementary, and $\varphi_2(t) := (1 - t/R^\beta)_+^\ell$ for $\ell \gg 1$ large enough. Then, as for the estimate (4.11), we obtain, with $\Sigma := [0, R^\beta] \times \{x \in \mathbb{R}^N; |x| \leq 2R\}$,

$$\begin{aligned} & \int_\Sigma u^p \tilde{\varphi} dx dt + C R^{-\alpha\beta} \int_{|x| \leq 2R} u_0(x)\varphi_1(x/R) dx \\ & \leq C \int_\Sigma \varphi_1(x/R) (\varphi_2(t))^{-\frac{1}{p-1}} \left| D_{t|R^\beta}^{1+\alpha} \varphi_2(t) \right|^{\tilde{p}} dx dt \\ & + C \int_\Sigma (\varphi_1(x/R))^{-\frac{1}{p-1}} (\varphi_2(t))^{-\frac{1}{p-1}} \left| (-\Delta_x)^{\beta/2} \varphi_1(x/R) D_{t|R^\beta}^\alpha \varphi_2(t) \right|^{\tilde{p}} dx dt, \quad (7.2) \end{aligned}$$

where $\alpha := 1 - \gamma$ and $\tilde{p} := p/(p-1)$. If we take the scaled variables $\tau = t/R^\beta$, $\xi = x/R$ and using the fact that $(-\Delta_x)^{\beta/2} \varphi_1(x/R) = R^{-\beta} \lambda_1 \varphi_1(x/R)$ in the right-hand side of (7.2), into accounting the positivity of $u \geq 0$, we infer

$$\begin{aligned} & C R^{-\alpha\beta} \int_{|\xi| \leq 2} u_0(R\xi)\varphi_1(\xi) d\xi \\ & \leq C(R) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi \end{aligned}$$

$$\begin{aligned}
 &= C(R) \int_{|\xi| \leq 2} |R\xi|^{\beta(1+\alpha)(\tilde{p}-1)} |R\xi|^{\beta(1+\alpha)(1-\tilde{p})} \varphi_1(\xi) d\xi \\
 &\leq C(R)(2R)^{\beta(1+\alpha)(\tilde{p}-1)} \int_{|\xi| \leq 2} |R\xi|^{\beta(1+\alpha)(1-\tilde{p})} \varphi_1(\xi) d\xi
 \end{aligned}$$

where $\bar{C}(R) = R^{\beta-(1+\alpha)\beta\tilde{p}}(C + C\lambda_1)$, and so

$$\int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \leq C \int_{|\xi| \leq 2} |R\xi|^{\beta(1+\alpha)(1-\tilde{p})} \varphi_1(\xi) d\xi. \tag{7.3}$$

Using the estimate

$$\begin{aligned}
 \inf_{|\xi| > 1} (u_0(R\xi) |R\xi|^{\beta(1+\alpha)(\tilde{p}-1)}) \int_{|\xi| \leq 2} |R\xi|^{\beta(1+\alpha)(1-\tilde{p})} \varphi_1(\xi) d\xi &\leq \int_{1 < |\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \\
 &\leq \int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi
 \end{aligned}$$

in the left-hand side of (7.3), we conclude, after dividing by $\int_{|\xi| \leq 2} |R\xi|^{\beta(1+\alpha)(1-\tilde{p})} \varphi_1(\xi) d\xi$ that

$$\inf_{|\xi| > 1} (u_0(R\xi) |R\xi|^{\beta(1+\alpha)(\tilde{p}-1)}) \leq C. \tag{7.4}$$

Passing to the limit in (7.4), as $R \rightarrow \infty$, and taking account of the continuity of u_0 , we obtain

$$\lim_{|x| \rightarrow \infty} (u_0(x) |x|^{\beta(1+\alpha)(\tilde{p}-1)}) = \liminf_{|x| \rightarrow \infty} (u_0(x) |x|^{\beta(1+\alpha)(\tilde{p}-1)}) \leq C.$$

□

Corollary 1 (*sufficient conditions for the nonexistence of global solution*)

Let $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0 \geq 0$, $0 < \beta \leq 2$ and $p > 1$. If

$$\lim_{|x| \rightarrow \infty} (u_0(x) |x|^{\frac{\beta(2-\gamma)}{p-1}}) = +\infty,$$

then the problem (1.1) cannot have a global solution. □

Next, we give a necessary condition for local existence where we obtain a similar estimate of T founded in the proof of Theorem 1, as $|x|$ goes to infinity.

Theorem 6 (*Necessary conditions for local existence*)

Let $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0 \geq 0$, $\beta \in (0, 2]$ and $p > 1$. If u is a local mild solution to problem (1.1) on $[0, T]$ where $0 < T < +\infty$, then we have

$$\lim_{|x| \rightarrow \infty} u_0(x) \leq C T^{-\frac{2-\gamma}{p-1}}, \tag{7.5}$$

for some positive constant $C > 0$.

Note that, if $A := \lim_{|x| \rightarrow \infty} u_0(x)$, then we obtain a similar estimate as that found in (3.3),

$$\frac{T^{2-\gamma} A^{p-1}}{C^{p-1}} \leq 1.$$

Proof We take here, for $R > 0$ sufficiently large, $\varphi(x, t) := D_{t|T}^\alpha \tilde{\varphi}(x, t) := D_{t|T}^\alpha (\varphi_1(x/R)\varphi_2(t))$ where $\varphi_2(t) := (1 - t/T)_+^\ell$ instead of the one chosen in Theorem 5. Then, as (7.2), we obtain

$$\begin{aligned} & \int_{\Sigma_1} u^p \tilde{\varphi} dx dt + C T^{-\alpha} \int_{|x| \leq 2R} u_0(x) \varphi_1(x/R) dx \\ & \leq C \int_{\Sigma_1} \varphi_1(x/R) (\varphi_2(t))^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|^{\tilde{p}} dx dt \\ & + C \int_{\Sigma_1} (\varphi_1(x/R))^{-\frac{2}{p-1}} (\varphi_2(t))^{-\frac{1}{p-1}} \left| (-\Delta_x)^{\beta/2} \varphi_1(x/R) D_{t|T}^\alpha \varphi_2(t) \right|^{\tilde{p}} dx dt, \end{aligned} \quad (7.6)$$

where $\Sigma_1 := [0, T] \times \{x \in \mathbb{R}^N; |x| \leq 2R\}$, $\alpha := 1 - \gamma$ and $\tilde{p} := p/(p-1)$. Now, in the right-hand side of (7.6), we take the following scale of variables $\tau = T^{-1}t$, $\xi = R^{-1}x$ and we use the fact that $(-\Delta_x)^{\beta/2} \varphi_1(x/R) = R^{-\beta} \lambda_1 \varphi_1(x/R)$, while in the left-side we use the positivity ($u \geq 0$), then we get

$$C T^{-\alpha} \int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \leq \left(C T^{1-(1+\alpha)\tilde{p}} + C \lambda_1 T^{1-\alpha\tilde{p}} R^{-\beta\tilde{p}} \right) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi;$$

and so

$$\int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \leq C(R, T) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi \quad (7.7)$$

where $C(R, T) = C T^{(1+\alpha)(1-\tilde{p})} + C T^{1+\alpha(1-\tilde{p})} R^{-\beta\tilde{p}}$.

Using the estimate

$$\begin{aligned} \inf_{|\xi| > 1} (u_0(R\xi)) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi & \leq \int_{1 < |\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \\ & \leq \int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \end{aligned}$$

in the left-hand side of (7.7), we conclude, after dividing by $\int_{|\xi| \leq 2} \varphi_1(\xi) d\xi$, that

$$\inf_{|\xi| > 1} u_0(R\xi) \leq C(R, T). \quad (7.8)$$

Passing to the limit in (7.8), as $R \rightarrow \infty$, and taking account of the continuity of u_0 , we obtain

$$\lim_{|x| \rightarrow \infty} u_0(x) = \liminf_{|x| \rightarrow \infty} u_0(x) \leq C T^{(1+\alpha)(1-\tilde{p})} = C T^{-\frac{2-\gamma}{p-1}}.$$

□

Appendix

In this appendix, we give a proof of Ju's inequality (see proposition 3.3 in [12]), in dimension $N \geq 1$ where $\delta \in [0, 2]$ and $q \geq 1$, for all nonnegative Schwartz function ψ (in general case)

$$(-\Delta)^{\delta/2} \psi^q \leq q \psi^{q-1} (-\Delta)^{\delta/2} \psi. \quad (7.9)$$

The cases $\delta = 0$, $\delta = 1$ and $\delta = 2$ are obvious.

- If $\delta \in (0, 1)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta) \int_{\mathbb{R}^N} \frac{\psi(x+z) - \psi(x)}{|z|^{N+\delta}} dz, \quad \forall x \in \mathbb{R}^N, \quad (7.10)$$

where $c_N(\delta) = \delta\Gamma((N+\delta)/2)/(2\pi^{N/2+\delta}\Gamma(1-\delta/2))$. Then

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta) \int_{\mathbb{R}^N} \frac{(\psi(x))^{q-1}\psi(x+z) - (\psi(x))^q}{|z|^{N+\delta}} dz.$$

The case $q = 1$ is clear. If $q > 1$, then by Young's inequality we have

$$(\psi(x))^{q-1}\psi(x+z) \leq \frac{q-1}{q}(\psi(x))^q + \frac{1}{q}(\psi(x+z))^q.$$

Therefore, we infer

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) \geq \frac{-c_N(\delta)}{q} \int_{\mathbb{R}^N} \frac{(\psi(x+z))^q - (\psi(x))^q}{|z|^{N+\delta}} dz = \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.$$

- If $\delta \in (1, 2)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta) \int_{\mathbb{R}^N} \frac{\psi(x+z) - \psi(x) - \nabla\psi(x).z}{|z|^{N+\delta}} dz, \quad \forall x \in \mathbb{R}^N. \quad (7.11)$$

Then by the similar process, we get

$$\begin{aligned} (\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) &\geq -c_N(\delta) \int_{\mathbb{R}^N} \frac{\frac{1}{q}((\psi(x+z))^q - (\psi(x))^q) - (\nabla\psi(x).z)(\psi(x))^{q-1}}{|z|^{N+\delta}} dz \\ &= \frac{-c_N(\delta)}{q} \int_{\mathbb{R}^N} \frac{(\psi(x+z))^q - (\psi(x))^q - \nabla(\psi(x))^q.z}{|z|^{N+\delta}} dz \\ &= \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q. \end{aligned}$$

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