# On certain time- and space-fractional evolution equations

Ahmad Fino, Mokhtar Kirane

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**Ahmad FINO** [1,2] **and Mokhtar KIRANE** [1*]

[1] *Département de Mathématiques, Pôle Sciences et technologie, Université de La Rochelle, Av. M. Crépeau, La Rochelle* 17042, *France*
[2] *LaMA-Liban, Lebanese University, P.O. Box* 826, *Tripoli, Liban*

**Abstract**

In this article, we present first a new technique to prove, in a general case, the recent result of Cazenave, Dickstein and Weissler [6] on the blowing-up solutions to a temporally nonlocal nonlinear parabolic equation.
Then, we study the blow-up rate and the global existence in time of the solutions. Furthermore, we show necessary conditions for global existence.

**Keywords:** Parabolic equation, local and global existence, blow-up, blow-up rate, critical exponent, fractional laplacian, Riemann-Liouville fractional integrals and derivatives.

## 1 Introduction

In this paper, we investigate the nonlinear parabolic equation with a nonlocal in time nonlinearity

$$\begin{cases} \partial_t u + (-\Delta)^{\beta/2} u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s)\, ds & x \in \mathbb{R}^N, t > 0, \\ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u_0 \in C_0(\mathbb{R}^N)$, $0 < \beta \leq 2$, $0 < \gamma < 1$, $1 < p$ and the nonlocal operator $(-\Delta)^{\beta/2}$ defined by

$$(-\Delta)^{\beta/2} v(x) := \mathcal{F}^{-1}\left( |\xi|^{\beta} \mathcal{F}(v)(\xi) \right)(x)$$

for every $v \in D((-\Delta)^{\beta/2}) = H^{\beta}(\mathbb{R}^N)$, where $H^{\beta}(\mathbb{R}^N)$ is the standard Sobolev space of order $\beta$, $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for its inverse, $\Gamma$ is the Euler gamma function.
When Eq. (1.1) is considered with a nonlinearity of the form $u^p$, it reads

$$u_t + (-\Delta)^{\beta/2} u = u^p.$$

*E-mail addresses: ahmadfino01@univ-lr.fr(A.Fino),mokhtar.kirane@univ-lr.fr(M.Kirane)

This equation has been considered by Nagasawa and Sirao [21], Kobayashi [15], Birkner, Lopez-Mimbela and Wakolbinger [3], and by Kirane and Qafsaoui [16], in the general case. At this stage, let us observe that Eq. (1.3) can be converted formally into:

$$D^{\alpha}_{0|t}u_t + D^{\alpha}_{0|t}(-\Delta)^{\beta/2}u = u^p, \tag{1.2}$$

where $D^{\alpha}_{0|t}$ is the left-sided Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined in (2.7) below; we have set in (1.2) $\alpha = 1-\gamma \in (0,1)$.
In [22], Sokolov and Klafter discussed modern extensions of the usual heat equation $u_t - \Delta u = 0$ to include heavy tailed particle jumps (in the case of random walk model of particle motion) and random waiting times, leading to fractional diffusion equations, the simplest of which is

$$D^{\alpha}_{0|t}u = -k(-\Delta)^{\beta/2}u.$$

($u(x,t)$ is the probability distribution, $k$ is a constant of diffusion).
The fractional Laplacian reflects heavy tailed waiting times, where the probability of waiting longer than $t$ falls like $t^{-\alpha}$. Heavy tails in space lead to super-diffusion, where a plume of particles spreads faster that the classical $\sqrt{t}$ rate associated with Brownian motion. Heavy tails in time lead to sub-diffusion, since long waiting times retard particle motion. Much more can be said; see the valuable contributions of Zaslavsky [28], Balescu [2], Sokolov, Klafter and Bluman [23].
Let us dwell a moment on some literature concerning equations of the form

$$u_t = \Delta u^m + f(u(x,t), \int_0^t g(x,t,s,u(x,s))\,ds)$$

$t > 0, x \in \Omega \subseteq \mathbb{R}^N$, which serve as models for nonlinear diffusion phenomena with memory effects. In population dynamics, for example, one encounters

$$g(x,t,s,u(x,s)) = \mathcal{K}(t-s)h(u(x,s))$$

but with regular kernels $\mathcal{K}(t-s)$, and so usually

$$f(u, \int_0^t g(x,t,s,u(x,s))\,ds) = (a-bu)u - \int_0^t \mathcal{K}(t-s)u(x,s)\,ds.$$

Other works concern the nonlocal in time equation

$$u_t - \Delta u = u^q \int_0^t u^p(x,s)\,ds$$

which for $p = q = 1$ plays a role in the theory of nuclear reactor kinetics as reported by Kastenberg in [12].

Our article is motivated by the recent and very interesting one of Cazenave, Dickstein and Weissler [6] which deals with the global existence and blow-up for the parabolic equation with nonlocal in time nonlinearity

$$\partial_t u - \Delta u = \int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,ds \quad x \in \mathbb{R}^N, t > 0, \tag{1.3}$$

where $0 \leq \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^N)$, which is (1.1) with $\beta = 2$.
They proved that, if we set

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma)_+}$$

and

$$p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0, +\infty],$$

**(i)** If $\gamma \neq 0$, $p \leq p_*$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then $u$ blows up in finite time.

**(ii)** If $\gamma \neq 0$, $p > p_*$, and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) with $||u_0||_{L^{q_{sc}}}$ sufficiently small, then $u$ exists globally.

If $\gamma = 0$ then all nontrivial positive solutions blow-up as proved by Souplet in [25]. Their study reveals the surprising fact that for equation (1.3) the critical exponent in Fujita's sense $p_*$ is not the one predicted by scaling.
Needless to say that the equation considered by Cazenave, Dickstein and Weissler [6] is a genuine extension of the one considered by Fujita in his pioneering work [8].
In this article, concerning blowing-up solutions, we present a different proof from the one presented in [6], and for the more general equation (1.1). Our proof is more versatile and can be applied to more nonlinear equations.
Our analysis is based on the observation that the nonlinear differential Equation (1.1) can be written in the form:

$$u_t + (-\Delta)^{\beta/2} u = J^{\alpha}_{0|t}\left(|u|^{p-1}u\right), \tag{1.4}$$

where $\alpha := 1 - \gamma \in (0,1)$ and $J^{\alpha}_{0|t}$ is the Riemann-Liouville fractional integral defined in (2.9).
we will show that:

**(1)** For $u_0 \geq 0$, $u_0 \not\equiv 0$, and $u_0 \in C_0(\mathbb{R}^N) \cap H^{\beta}(\mathbb{R}^N)$. If

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} \quad \text{or} \quad p < \frac{1}{\gamma},$$

then all solutions of problem (1.1) blow-up in finite time.

**(2)** For $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, where $p_{sc} := \frac{(p-1)N}{\beta(2-\gamma)}$. If

$$p > \max\left\{1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+}; \frac{1}{\gamma}\right\},$$

and $||u_0||_{L^{p_{sc}}}$ sufficiently small, then $u$ exists globally.

The method used to prove the blow-up theorem is the test function method considered by Mitidieri and Pohozaev [18, 19] and Kirane et al. [9, 14].

Furthermore, in the case $\beta = 2$, we derive the blow-up rate estimates for the parabolic equation (1.1). We shall prove that, if $u$ is the blow-up solution of (1.1) at the finite time $T^* > 0$,

then there are constants $c, C > 0$ such that $c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(., t) \leq C(T^* - t)^{-\alpha_1}$ for $1 < p \leq 1 + \frac{2(2-\gamma)}{(N-\beta+\beta\gamma)_+}$ or $1 < p < \frac{1}{\gamma}$ and all $t \in (0, T^*)$, where $\alpha_1 := \frac{2-\gamma}{p-1}$. We use a scaling argument to reduce the problems of blow-up rate to Fujita-type theorems (it is similar to "blow-up analysis" in elliptic problems to reduce the problems of a priori bounds to Liouville-type theorems). As far as we know, this method was first applied to parabolic problems by Hu [10], and sequently, was used in various parabolic equations and system (see [4, 7, 26]) where in the case of $\gamma = 0$, the rate constant $\frac{2}{p-1}$ found by P. Souplet [26] is a particular case of our one.

The organization of this paper is as follows. In section 2, we give the sufficient properties that will be used in the sequel. In Section 3, we derive the local existence of solutions for the parabolic equation (1.1). Section 4 contains the blow-up of solutions for (1.1). Section 5 is dedicated to the blow-up rate of solutions. Next, global existence is studied in the section 6. Finally, we give a necessary conditions for local and global existence in section 7.

## 2 Preliminaries

In this section, we present some definitions and results concerning fractional laplacians, fractional integrals and fractional derivatives that will be used hereafter.
First, if we take the usual linear fractional equation

$$u_t + (-\Delta)^{\beta/2} = 0, \quad \beta \in (0, 2],\ t \in (0, \infty),\ x \in \mathbb{R}^N, \tag{2.1}$$

then, the fundamental solution $S_\beta$ of Eq. (2.1) can be represented via the Fourier transform

$$S_\beta(t) := S_\beta(x, t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix.\xi - t|\xi|^\beta}\, d\xi. \tag{2.2}$$

It is well-known that this function satisfies

$$S_\beta(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad S_\beta(x, t) \geq 0, \quad \int_{\mathbb{R}^N} S_\beta(x, t)\, dx = 1, \tag{2.3}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using the Young inequality for the convolution and the following self-similar form $S_\beta(x, t) = t^{-N/\beta} S_\beta(xt^{-1/\beta}, 1)$, we have

$$\|S_\beta(t) * v\|_q \leq Ct^{-\frac{N}{\beta}(\frac{1}{r} - \frac{1}{q})} \|v\|_r, \tag{2.4}$$

$$\|S_\beta(t) * v\|_r \leq \|v\|_r, \tag{2.5}$$

for all $v \in L^r(\mathbb{R}^N)$ and all $1 \leq r < q \leq \infty$, $t > 0$.
Moreover, as $(-\Delta)^{\beta/2}$ is a self-adjoint operator with $D(-\Delta)^{\beta/2} = H^\beta(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^{\beta/2} v(x)\, dx = \int_{\mathbb{R}^N} v(x)(-\Delta)^{\beta/2} u(x)\, dx, \tag{2.6}$$

for all $u, v \in H^\beta(\mathbb{R}^N)$.
Next, if $AC[0,T]$ is the space of all functions which are absolutely continuous on $[0,T]$ with $0 < T < \infty$, then, for $f \in AC[0,T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^\alpha_{0|t} f(t)$ and $D^\alpha_{t|T} f(t)$ of order $\alpha \in (0,1)$ are defined by (see [13])

$$D^\alpha_{0|t} f(t) := D J^{1-\alpha}_{0|t} f(t), \tag{2.7}$$

$$D^\alpha_{t|T} f(t) := -\frac{1}{\Gamma(1-\alpha)} D \int_t^T (s-t)^{-\alpha} f(s)\, ds, \tag{2.8}$$

for all $t \in [0,T]$, where $D := \frac{d}{dt}$ and

$$J^\alpha_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{2.9}$$

is the Riemann-Liouville fractional integral defined in [13].
Now, for every $f, g \in C([0,T])$, such that $D^\alpha_{0|t} f(t), D^\alpha_{t|T} g(t)$ exist and are continuous, for all $t \in [0,T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [24])

$$\int_0^T \left(D^\alpha_{0|t} f\right)(s) g(s)\, ds = \int_0^T f(s) \left(D^\alpha_{t|T} g\right)(s)\, ds. \tag{2.10}$$

Note also that, for all $f \in AC^2[0,T]$, we have (see 2.2.30 in [13])

$$-D.D^\alpha_{t|T} f = D^{1+\alpha}_{t|T} f, \tag{2.11}$$

where

$$AC^2[0,T] := \{f : [0,T] \to \mathbb{R} \text{ and } Df \in AC[0,T].\}$$

Moreover, for all $1 \le q \le \infty$, the following equalities (see [Lemma 2.4 p.74][13])

$$D^\alpha_{0|t} J^\alpha_{0|t} = Id_{L^q(0,T)} \tag{2.12}$$

hold almost everywhere on $[0,T]$. Later on, we will use the following results.
• If $w_1(t) = \left(1 - \frac{t}{T}\right)^\sigma_+$, $t \ge 0$, $T > 0$, $\sigma >> 1$, where $(.)_+$ is the positive part, then

$$D^\alpha_{t|T} w_1(t) = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\sigma} (T-t)^{\sigma-\alpha}_+, \tag{2.13}$$

$$D^{\alpha+1}_{t|T} w_1(t) = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\sigma} (T-t)^{\sigma-\alpha-1}_+, \tag{2.14}$$

for all $\alpha \in (0,1)$; so

$$\left(D^\alpha_{t|T} w_1\right)(T) = 0 \quad ; \quad \left(D^\alpha_{t|T} w_1\right)(0) = C\, T^{-\alpha}, \tag{2.15}$$

where $C = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}$; indeed, using the Euler change of variable $y = \frac{s-t}{T-t}$, we get

$$\begin{aligned} D^{\alpha}_{t|T} w_1(t) &:= -\frac{1}{\Gamma(1-\alpha)} D\left[\int_t^T (s-t)^{-\alpha}\left(1-\frac{s}{T}\right)^{\sigma} ds\right] \\ &= -\frac{T^{-\sigma}}{\Gamma(1-\alpha)} D\left[(T-t)^{1-\alpha+\sigma}\int_0^1 (y)^{-\alpha}(1-y)^{\sigma} ds\right] \\ &= +\frac{(1-\alpha+\sigma)B(1-\alpha;\sigma+1)}{\Gamma(1-\alpha)} T^{-\sigma}(T-t)^{\sigma-\alpha}, \end{aligned}$$

where $B(.;.)$ stands for the beta function. Then, (2.13) follows using the relation

$$B(1-\alpha;\sigma+1) = \frac{\Gamma(1-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

• If $w_2(t) = \left(1-\frac{t^2}{T^2}\right)_+^{\ell}$, $T>0$, $\ell >> 1$, then, using the change of variable $y = \frac{s-t}{T-t}$, we have

$$D^{\alpha}_{t|T} w_2(t) = \frac{T^{-2\ell}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} C_1(\ell,k,\alpha) T^{\ell-k}(T-t)^{\ell+k-\alpha}, \tag{2.16}$$

$$D^{1+\alpha}_{t|T} w_2(t) = \frac{T^{-2\ell}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} C_2(\ell,k,\alpha) T^{\ell-k}(T-t)^{\ell+k-\alpha-1}, \tag{2.17}$$

for all $-T \le t \le T$, $\alpha \in (0,1)$, where

$$\begin{cases} C_1(\ell,k,\alpha) := c_\ell^k (1-\alpha+\ell+k) 2^{\ell-k}(-1)^k \frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)}, \\ C_2(\ell,k,\alpha) := (\ell+k-\alpha) C_1(\ell,k,\alpha), \\ c_\ell^k := \frac{\ell!}{(\ell-k)!k!}; \end{cases}$$

so

$$\left(D^{\alpha}_{t|T} w_2\right)(T) = 0 \quad ; \quad \left(D^{\alpha}_{t|T} w_2\right)(-T) = C_3(\ell,k,\alpha)\, T^{-\alpha}, \tag{2.18}$$

where

$$C_3(\ell,k,\alpha) := \frac{2^{2\ell-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} c_\ell^k (1-\alpha+\ell+k) 2^{\ell-k}(-1)^k \frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)}.$$

We should mention that the investigation of properties of nonlinear fractional kinetic equations is at its beginning stage and a number of important questions are not answered yet.

## 3 Local existence

In order to prove the existence of a local mild solution for the problem (1.1), we consider the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s) J^{\alpha}_{0|s} \left( |u|^{p-1}u \right) \, ds, \quad t \in [0,T], \tag{3.1}$$

where $T(t) := e^{-t(-\Delta)^{\beta/2}}$. As, for every real number $\beta$, $(-\Delta)^{\beta/2}$ is a positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$, hence $T(t)$ is a strongly continuous semigroup on $L^2(\mathbb{R}^N)$ of (2.1), for $t > 0$, generated by the fractional power $-(-\Delta)^{\beta/2}$ (see Yosida [27]). In fact, $T(t)$ is an integral operator for all $t > 0$. It is represented as follows $T(t)v = S_\beta(t) * v$ (see [Prop. 3.3][20]), for all $v \in L^2(\mathbb{R}^N), t > 0$, where $S_\beta$ is given by (2.2) and $u * v$ is the convolution of $u$ and $v$.

**Theorem 1** (*Mild solution*)
*Given $u_0 \in C_0(\mathbb{R}^N)$ and $p > 1$, there exist a maximal time $T_{\max} > 0$ and a unique mild solution $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$ to the problem* (1.1). *In addition, if $u_0 \geq 0$, $u_0 \not\equiv 0$, then $u(t) > 0$ for all $0 < t < T_{\max}$.*
*Moreover, if $u_0 \in L^r(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)$ for $1 \leq r < \infty$, $0 < \beta \leq 2$, then $u \in C([0, T_{\max}), L^r(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N))$.*

**Proof** For arbitrary $T > 0$, we define a Banach space $E_T$ such that

$$E_T := \left\{ u \in L^\infty((0,T), C_0(\mathbb{R}^N)) : ||u||_1 \leq 2||u_0||_{L^\infty} \right\},$$

where $||.||_1 := ||.||_{L^\infty((0,T),L^\infty(\mathbb{R}^N))}$. Next, for every $u \in E_T$, we define

$$\Psi(u) := T(t)u_0 + \int_0^t T(t-s) J^{\alpha}_{0|s} \left( |u|^{p-1}u \right) \, ds.$$

As usual, we prove the local existence by the Banach fixed point theorem.
Let $u \in E_T$, by using (2.5), we obtain, if $||.||_\infty := ||.||_{L^\infty(\mathbb{R}^N)}$,

$$\begin{aligned}
||\Psi(u)||_1 &\leq ||u_0||_\infty + \frac{1}{\Gamma(1-\gamma)} || \int_0^t \int_0^s (s-\sigma)^{-\gamma} ||u(\sigma)||^p_\infty \, d\sigma \, ds ||_{L^\infty(0,T)} \\
&= ||u_0||_\infty + \frac{1}{\Gamma(1-\gamma)} || \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} ||u(\sigma)||^p_\infty \, ds \, d\sigma ||_{L^\infty(0,T)} \\
&\leq ||u_0||_\infty + \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} ||u||^p_1 \\
&\leq ||u_0||_\infty + \frac{T^{2-\gamma} 2^p ||u_0||^{p-1}_{L^\infty}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} ||u_0||_\infty.
\end{aligned}$$

Now, if we choose $T$ such that

$$\frac{T^{2-\gamma} 2^p ||u_0||^{p-1}_\infty}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} \leq 1, \tag{3.2}$$

we conclude that $||\Psi(u)||_1 \leq 2||u_0||_\infty$, and then $\Psi(u) \in E_T$.

Now, for $u, v \in E_T$, we have

$$\begin{aligned}
||\Psi(u) - \Psi(v)||_1 &\leq \frac{1}{\Gamma(1-\gamma)}|| \int_0^t \int_0^s (s-\sigma)^{-\gamma} |||u|^{p-1}u(\sigma) - |v|^{p-1}v(\sigma)||_\infty \, d\sigma \, ds||_{L^\infty(0,T)} \\
&= \frac{1}{\Gamma(1-\gamma)}|| \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} |||u|^{p-1}u(\sigma) - |v|^{p-1}v(\sigma)||_\infty \, ds \, d\sigma||_{L^\infty(0,T)} \\
&\leq \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)}|||u|^{p-1}u - |v|^{p-1}v||_1 \\
&\leq \frac{p2^{p-1}||u_0||_\infty^{p-1}T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)}||u-v||_1 \\
&\leq \frac{1}{2}||u-v||_1,
\end{aligned}$$

thanks to the choice of $T$. Then, by the Banach fixed point theorem, there exists a unique mild solution $u \in \Pi_T$, where $\Pi_T := L^\infty((0,T), C_0(\mathbb{R}^N))$, to the problem (1.1).
Next, using the uniqueness of solutions, we conclude the existence of a solution on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup \{T > 0 : \ u \text{ is a solution to (1.1) in } \Pi_T\}.$$

Note that, using usual tools, we can prove that $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$.
Finally, if $u_0 \in L^r(\mathbb{R}^N)$, for $1 \leq r < \infty$, then by repeating the fixed point argument in the space

$$\begin{aligned}
E_{T,\beta,r} &:= \{u \in L^\infty((0,T), C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)) : \\
&\quad ||u||_1 \leq 2||u_0||_{L^\infty}, ||u||_{\infty,\beta} \leq 2||u_0||_{H^\beta}, ||u||_{\infty,r} \leq 2||u_0||_{L^r}\},
\end{aligned}$$

instead of $E_T$, where

$$||.||_{\infty,\beta} := ||.||_{L^\infty((0,T),H^\beta(\mathbb{R}^N))} \quad , \quad ||.||_{\infty,r} := ||.||_{L^\infty((0,T),L^r(\mathbb{R}^N))},$$

and by estimating $||u^p||_{H^\beta(\mathbb{R}^N)}$ by $C||u||^{p-1}_{L^\infty(\mathbb{R}^N)}||u||_{H^\beta(\mathbb{R}^N)}$ (see Appendix.B) and $||u^p||_{L^r(\mathbb{R}^N)}$ by $||u||^{p-1}_{L^\infty(\mathbb{R}^N)}||u||_{L^r(\mathbb{R}^N)}$ in the contraction mapping argument, using (2.5), we obtain a unique solution in $E_{T,\beta,r}$; we conclude then that

$$u \in C([0, T_{\max}), C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)).$$

Moreover, if $u_0 \geq 0$, then we can construct a nonnegative solution on some interval $[0, T]$ by applying the fixed-point argument in the set $E_T^+ = \{u \in E_T; \ u \geq 0\}$. In particular, it follows from (3.1) that $u(t) \geq T(t)u_0 > 0$ on $(0, T]$. It is not difficult to deduce that $u$ stays positive on $(0, T_{\max})$. □

**Corollary 1** (*Strong solution*)
*If $u_0 \in C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)$, then, the mild solution $u$ is a strong solution, i.e. $u \in C([0,T], H^\beta(\mathbb{R}^N)) \cap C^1([0,T], L^2(\mathbb{R}^N))$, for every $0 < T < T_{\max}$, and satisfies (1.1) with $u(t=0) = u_0$.*

**Proof** It suffices to see that as $u \in C([0,T], C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N))$, for $0 < T < T_{\max}$, then $J^\alpha_{0|t}(|u|^{p-1}u) \in C([0,T], L^2(\mathbb{R}^N)) \cap L^1((0,T), H^\beta)$ and we conclude by [Prop. 4.1.6 p.52][5]. □

When $T_{\max} = \infty$ we say that $u$ is a global solution, and when $T_{\max} < \infty$ we say that $u$ blows up in a finite time or that $u$ is non-global and in this case we have $\lim\limits_{t \to T_{\max}} ||u(.,t)||_{L^\infty(\mathbb{R}^N)} = \infty$.

# 4 Blow-up of solutions

Now, we want to derive a blow-up result for Eq. (1.1).
Hereafter, $\int_{Q_T} = \int_0^T \int_{\mathbb{R}^N}$ for all $0 < T \leq \infty$.

**Theorem 2** *Let $u_0 \in C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)$ be such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. If*

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} := p^*, \quad or \quad p < \frac{1}{\gamma}, \tag{4.1}$$

*then any solution to (1.1), with initial condition $u_0$, blows-up in a finite time.*
*Note that in the case where $p = p^*, \beta \in (0,2)$ we take $p > \frac{N}{N-\beta}$ with $N > \beta$.*

**Proof** The proof is by contradiction. Suppose that $u$ is a global solution to (1.1), then $u$ is a solution of (1.1) in $C([0,T], C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)) \cap C^1([0,T], L^2)$ for all $T > 0$ such that $u(t) > 0$ for all $t \in [0,T]$.
Now, we take $\varphi(x,t) = (\varphi_1(x))^\ell \varphi_2(t)$ with $\varphi_1(x) := \Phi\left(\frac{|x|}{B^{-1/\beta}T^{1/\beta}}\right)$, $\varphi_2(t) := \Psi\left(\frac{t}{T}\right)$, where $\ell \geq \frac{p}{p-1}$, $1 \leq B < T$ large enough such as $T \to \infty \not\Rightarrow B \to \infty$, and $\Phi$ a smooth non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $|\Phi'(r)| \leq C_1/r$, for all $r > 0$, and

$$\Psi(s) = \begin{cases} (1-s)^\eta & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 1, \end{cases}$$

with $\eta \geq \frac{\alpha p+1}{p-1}$.
Here $B \geq 1$ is a fixed positive constant which plays some role in the case $p = p^*$, while in the case $p < p^*$ we take $B = 1$. Then, multiplying, scalarly in $L^2$, Eq. (1.4) by $D^\alpha_{t|T}\varphi$, we obtain

$$\int_{Q_T} u_t \, D^\alpha_{t|T}\varphi + \int_{Q_T} (-\Delta)^{\beta/2} u \, D^\alpha_{t|T}\varphi = \int_{Q_T} J^\alpha_{0|t}(u^p) D^\alpha_{t|T}\varphi. \tag{4.2}$$

Using (2.6) and integrating by parts the left hand side of (4.2) while in the right hand side we use formula (2.10), we conclude that

$$-\int_{Q_T} u\, DD^{\alpha}_{t|T}\varphi + \left[\int_{\mathbb{R}^N} u\, D^{\alpha}_{t|T}\varphi\right]_0^T + \int_{Q_T} u(-\Delta)^{\beta/2} D^{\alpha}_{t|T}\varphi = \int_{Q_T} D^{\alpha}_{0|T} J^{\alpha}_{0|t}(u^p)\varphi. \quad (4.3)$$

Moreover, from (2.11), (2.12) and (2.15), we may write with $\Sigma_B = (0,T)\times\Omega_B$, where $\Omega_B := \left\{x \in \mathbb{R}^N \ ; \ |x| \le 2B^{-1/\beta}T^{1/\beta}\right\}$,

$$\begin{aligned}
&\int_{\Sigma_B} u^p\ \varphi\, dx\, dt \ +\ C\ T^{-\alpha} \int_{\Omega_B} (\varphi_1(x))^{\ell}\, u_0(x)\, dx \\
&= \int_{\Sigma_B} u\, (\varphi_1(x))^{\ell}\, D^{1+\alpha}_{t|T}\varphi_2(t)\, dx\, dt \\
&+ \int_{\Sigma_B} u(-\Delta_x)^{\beta/2}\, (\varphi_1(x))^{\ell}\, D^{\alpha}_{t|T}\varphi_2(t)\, dx\, dt. \qquad (4.4)
\end{aligned}$$

So, Ju's inequality $(-\Delta)^{\beta/2}\left(\varphi_1^{\ell}\right) \le \ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}(\varphi_1)$ (see Appendix.A) will allow us to write:

$$\begin{aligned}
&\int_{\Sigma_B} u^p\ \varphi\, dx\, dt \ +\ C\ T^{-\alpha} \int_{\Omega_B} (\varphi_1(x))^{\ell}\, u_0(x)\, dx \\
&\le\ C \int_{\Sigma_B} u\, (\varphi_1(x))^{\ell} \left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|\, dx\, dt \\
&+ C \int_{\Sigma_B} u\, (\varphi_1(x))^{\ell-1} \left|(-\Delta_x)^{\beta/2}\varphi_1(x)\ D^{\alpha}_{t|T}\varphi_2(t)\right|\, dx\, dt \\
&= C \int_{\Sigma_B} u\ \varphi^{1/p}\varphi^{-1/p}\, (\varphi_1(x))^{\ell} \left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|\, dx\, dt \\
&+ C \int_{\Sigma_B} u\ \varphi^{1/p}\varphi^{-1/p}\ (\varphi_1(x))^{\ell-1} \left|(-\Delta_x)^{\beta/2}\varphi_1(x)\ D^{\alpha}_{t|T}\varphi_2(t)\right|\, dx\, dt. \qquad (4.5)
\end{aligned}$$

Therefore, as $u_0 \ge 0$, then, using Young's inequality

$$ab \le\ \frac{1}{2p}a^{\,p}\ +\ \frac{2^{\widetilde{q}-1}}{\widetilde{p}}\ b^{\,\widetilde{p}} \qquad \text{where } \ a>0, b>0,\ p\widetilde{p} = p+\widetilde{p}, \quad (4.6)$$

with

$$\begin{cases} a = u\ \varphi^{1/p}, \\ b = \varphi^{-1/p}\, (\varphi_1(x))^{\ell} \left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the first integral of the right hand side of (4.5) and with

$$\begin{cases} a = u\ \varphi^{1/p}, \\ b = \varphi^{-1/p}\ (\varphi_1(x))^{\ell-1} \left|(-\Delta_x)^{\beta/2}\varphi_1(x)\ D^{\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the second integral of the right hand side of (4.5), we obtain

$$\begin{aligned}
&\int_{\Sigma_B} u^p\ \varphi(x,t)\,dx\,dt \\
&\leq C\int_{\Sigma_B} (\varphi_1(x))^{\ell}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|D_{t|T}^{1+\alpha}\varphi_2(t)\right|^{\widetilde{p}}\,dx\,dt \\
&+C\int_{\Sigma_B} (\varphi_1(x))^{\ell-\widetilde{p}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|(-\Delta_x)^{\beta/2}\varphi_1(x)D_{t|T}^{\alpha}\varphi_2(t)\right|^{\widetilde{p}}\,dx\,dt. \qquad (4.7)
\end{aligned}$$

At this stage, we introduce the scaled variables: $\tau = T^{-1}t$, $\xi = (T/B)^{-1/\beta}\,x$; using formula (2.13) and (2.14) in the right hand-side of (4.7), we obtain:

$$\int_{\Sigma_B} u^p(x,t)\ \varphi(x,t)\,dx\,dt \leq\ C\ T^{-\delta}B^{-N/\beta}\ +\ C\ T^{-\delta}B^{-N/\beta+\widetilde{p}}, \qquad (4.8)$$

where $\delta := (1+\alpha)\widetilde{p} - 1 - (N/\beta)$, $C = C(|\Omega_2|\ ,\ |\Omega_3|)$, ($|\Omega_i|$ stands for the measure of $\Omega_i$, for $i = 2, 3$), with

$$\Omega_2 := \left\{\xi\in\mathbb{R}^N\ ;\ |\xi|\leq 2\right\}\quad ,\quad \Omega_3 := \left\{\tau\geq 0\ ;\ \tau\leq 1\right\}.$$

Now, noting that as

$$p\leq p^*\quad \text{or}\quad p<\frac{1}{\gamma}\quad\Longleftrightarrow\quad \delta\geq 0\quad\text{or}\quad p<\frac{1}{\gamma}, \qquad (4.9)$$

we have to distinguish three cases:

• The case $p < p^*$ ($\delta > 0$): We take $B = 1$ and pass to the limit in (4.8), as $T$ goes to $\infty$, we get

$$\lim_{T\to\infty}\int_0^T\int_{|x|\leq 2T^{1/\beta}} u^p\ \varphi(x,t)\,dx\,dt = 0.$$

Using the dominated convergence theorem and the continuity in time and space of $u$, we infer that

$$\int_{Q_\infty} u^p(x,t)\,dx\,dt = 0\qquad\Longrightarrow\qquad u\equiv 0.$$

Contradiction.

• The case $p = p^*$ ($\delta = 0$): Let's recall here that $B > 1$ is sufficiently large. Then, from inequality (4.8), using the the fact that $p = p^*$, we have

$$\int_{\Sigma_B} u^p(x,t)\ \varphi(x,t)\,dx\,dt \leq\ C\ B^{-N/\beta}\ +\ C\ B^{-N/\beta+\widetilde{p}}. \qquad (4.10)$$

Using $p > \frac{N}{N-\beta}$ and taking the limit when $T\longrightarrow\infty$ and then $B\longrightarrow\infty$, we get:

$$\int_{Q_\infty} u^p\,dx\,dt = 0\ \Longrightarrow\ u\equiv 0, \qquad (4.11)$$

which is a contradiction.
Note that, in the case $\beta = 2$, it is not necessary to take the condition $p > \frac{N}{N-\beta}$ with $N > \beta$. Indeed, from (4.5), we may write

$$\begin{aligned}
&\int_{\Sigma_B} u^p \, \varphi \, dx \, dt \\
&\leq C \int_{\Sigma_B} u \, \varphi^{1/p} \varphi^{-1/p} \left(\varphi_1(x)\right)^{\ell} \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \, dx \, dt \\
&+ C \int_{\Delta_B} u \, \varphi^{1/p} \varphi^{-1/p} \, \left(\varphi_1(x)\right)^{\ell-1} \left| \Delta_x \varphi_1(x) \, D_{t|T}^{\alpha} \varphi_2(t) \right| \, dx \, dt, \qquad (4.12)
\end{aligned}$$

where

$$\Delta_B = (0,T) \times \left\{ x \in \mathbb{R}^N \ ; \ B^{-1/\beta} T^{1/\beta} \leq |x| \leq 2B^{-1/\beta} T^{1/\beta} \right\}.$$

Moreover, using Young's inequality

$$ab \leq \frac{1}{p} a^{\,p} + \frac{1}{\widetilde{p}} \, b^{\,\widetilde{p}} \qquad \text{where } a > 0, b > 0, \ p\widetilde{p} = p + \widetilde{p}, \qquad (4.13)$$

with

$$\begin{cases} a = u \, \varphi^{1/p}, \\ b = \varphi^{-1/p} \left(\varphi_1(x)\right)^{\ell} \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|, \end{cases}$$

in the first integral of the right hand side of (4.12), and using Hölder's inequality

$$\int_{\Delta_B} ab \, dx \, dt \leq \left( \int_{\Delta_B} a^{\,p} \, dx \, dt \right)^{1/p} \left( \int_{\Delta_B} b^{\,\widetilde{p}} \, dx \, dt \right)^{1/\widetilde{p}},$$

with

$$\begin{cases} a = u \, \varphi^{1/p}, \\ b = \varphi^{-1/p} \, \left(\varphi_1(x)\right)^{\ell-1} \left| \Delta_x \varphi_1(x) \, D_{t|T}^{\alpha} \varphi_2(t) \right|, \end{cases}$$

in the second integral of the right hand side of (4.12), we obtain

$$\begin{aligned}
&\int_{\Sigma_B} u^p \, \varphi(x,t) \, dx \, dt \\
&\leq C \int_{\Sigma_B} \left(\varphi_1(x)\right)^{\ell} \left(\varphi_2(t)\right)^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right|^{\widetilde{p}} \\
&+ C \left( \int_{\Delta_B} u^p \, \varphi \right)^{1/p} \left( \int_{\Delta_B} \left(\varphi_1(x)\right)^{\ell-\widetilde{p}} \left(\varphi_2(t)\right)^{-\frac{1}{p-1}} \left| \Delta_x \varphi_1(x) D_{t|T}^{\alpha} \varphi_2(t) \right|^{\widetilde{p}} \right)^{1/\widetilde{p}}. \qquad (4.14)
\end{aligned}$$

Taking account of the scaled variables: $\tau = T^{-1}t$, $\xi = (T/B)^{-1/2} \, x$, and the fact that $\delta = 0$, we get

$$\int_{\Sigma_B} u^p(x,t) \, \varphi(x,t) \, dx \, dt \leq C \, B^{-N/2} + C \, B^{-\frac{N}{\widetilde{p}2}+1} \left( \int_{\Delta_B} u^p \, \varphi \right)^{1/p}. \qquad (4.15)$$

Now, as

$$\lim_{T\to\infty}\left(\int_{\Delta_B} u^p\,\varphi\right)^{1/p}=0,$$

then, passing to the limit in (4.15), as $T\to\infty$, we get

$$\int_0^\infty\int_{\mathbb{R}^{\mathbb{N}}} u^p(x,t)\,dx\,dt\le\ C\ B^{-N/2}.$$

Then, taking the limit when $B$ goes to infinity, we obtain $u\equiv 0$, contradiction.

• For the case $p\ <(1/\gamma)$, we repeat the same argument as in the case $p<p^*$ by choosing the following function $\varphi(x,t)=(\varphi_1(x))^\ell\,\varphi_2(t)$ where $\varphi_1(x)=\Phi\left(|x|/R\right),\ \varphi_2(t)=\Psi\left(t/T\right)$ and $R\in(0,T)$ large enough such that $T\to\infty\nRightarrow R\to\infty$, with the same functions $\Phi$ and $\Psi$ as above. We then obtain

$$\begin{aligned}
&\int_0^T\int_{\mathcal{C}} u^p\,\varphi\,dx\,dt\ +\ C\,T^{-\alpha}\int_{\mathcal{C}}(\varphi_1(x))^\ell\,u_0(x)\,dx\\
&\le\ C\int_0^T\int_{\mathcal{C}} u\,\varphi^{1/p}\varphi^{-1/p}\,(\varphi_1(x))^\ell\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|\,dx\,dt\\
&+\ C\int_0^T\int_{\mathcal{C}} u\,\varphi^{1/p}\varphi^{-1/p}(\varphi_1(x))^{\ell-1}\left|(-\Delta_x)^{\beta/2}\varphi_1(x)\ D^{\alpha}_{t|T}\varphi_2(t)\right|\,dx\,dt, \qquad (4.16)
\end{aligned}$$

where

$$\mathcal{C}:=\left\{x\in\mathbb{R}^N\ \ ;\ \ |x|\le 2R\right\}.$$

Now, by Young's inequality (4.6), with the same $a$ and $b$, and using the fact that $u_0\ge 0$, we get

$$\begin{aligned}
\int_0^T\int_{\mathcal{C}} u^p\,\varphi\ \ &\le\ \ C\int_0^T\int_{\mathcal{C}}(\varphi_1(x))^\ell\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}}\\
&+\ \ C\int_0^T\int_{\mathcal{C}}(\varphi_1(x))^{\ell-\widetilde{p}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|(-\Delta_x)^{\beta/2}\varphi_1 D^{\alpha}_{t|T}\varphi_2\right|^{\widetilde{p}}.
\end{aligned}$$

Then, the new variables $\xi=R^{-1}x$, $\tau=T^{-1}t$ and (2.13) allow us to write:

$$\int_0^T\int_{\mathcal{C}} u^p(x,t)\ \varphi(x,t)\,dx\,dt\le\ C\,T^{1-(1+\alpha)\widetilde{p}}\,R^N\ +\ C\,T^{1-\alpha\widetilde{p}}\,R^{N-\beta\widetilde{p}}.$$

Taking the limit as $T\to\infty$, we infer, as $p<\frac{1}{\gamma}\ \iff\ 1-\alpha\widetilde{p}<0$, that

$$\int_0^\infty\int_{\mathcal{C}} u(x,t)^p\,(\varphi_1(x))^\ell\ dx\,dt=0.$$

Finally, by taking $R\to\infty$, we get a contradiction as $u(x,t)>0$ for all $x\in\mathbb{R}^N$, $t>0$. □

### 4.0.1 Remarks

(1) We can extend our analysis to the equation

$$u_t = -(-\Delta)^{\beta/2}u + \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\psi(x,s)|u(s)|^{p-1}u(s)}{(t-s)^\gamma}\,ds, \qquad x \in \mathbb{R}^N, \tag{4.17}$$

with $\psi \in L^1_{Loc}(\mathbb{R}^N \times (0,\infty))$, $\psi(t) \geq 0$ for all $t \geq 0$, $\psi \not\equiv 0$, $\psi(B^{-1/\beta}T^{1/\beta}\xi, T\tau) \geq C > 0$ in the case $p \leq p^*$ and $\psi(R\xi, T\tau) \geq C > 0$ in the case $p < \frac{1}{\gamma}$, for any $0 < R, B < T$, $\tau \in [0,1]$, $\xi \in [0,2]$, $1 < p$, $0 < \beta \leq 2$ and $0 < \gamma < 1$.

(2) If we take $\beta = 2$ and $v(x,t) = (\Gamma(1-\gamma))^{\frac{1-\gamma}{p-1}}\, u(\Gamma(1-\gamma)^{1/2}x, \Gamma(1-\gamma)t)$ where $u$ is a solution of (1.1), we recover the result in [6] as a particular case.

(3) We can take the nonlocal porous-medium spatio- fractional problem

$$\begin{cases} \partial_t u + (-\Delta)^{\beta/2}|u|^{m-1}u = \dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,ds & x \in \mathbb{R}^N, t > 0, \\ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N, \end{cases}$$

where $0 < \beta \leq 2$, $0 < \gamma < 1$, $1 \leq m < p$, $u_0 \geq 0$ and $u_0 \not\equiv 0$.
The threshold on $p$ will be

$$p \leq 1 + \frac{(2-\gamma)(N(m-1)+\beta)}{(N-\beta+\beta\gamma)_+} \qquad \text{or} \qquad p < \frac{m}{\gamma}.$$

# 5 Blow-up Rate

In this section, we study the blow-up rate for the parabolic problem (1.1) in the case of $\beta = 2$.
We take the solution of (1.1) with an initial condition satisfying

$$u(.,0) = u_0 \geq 0, \quad u_0 \not\equiv 0 \quad \text{and} \quad u_0 \in C_0(\mathbb{R}^N) \cap H^2(\mathbb{R}^N). \tag{5.1}$$

The following lemma will be used in the proof of Theorem (3) below.

**Lemma 1** *Let $u$ be a nonnegative classical global solution of*

$$(\varphi)_t = \Delta\varphi + J^\alpha_{-\infty|t}(\varphi^p) \qquad \text{in } \mathbb{R}^N \times \mathbb{R}. \tag{5.2}$$

*Then, for*

$$p \leq p^* \quad \text{or} \quad p < \frac{1}{\gamma}, \tag{5.3}$$

*we have $u \equiv 0$.*

**Proof** We repeat the same computations as in Theorem 2 with $\varphi_2(t) := \Psi\left(t^2/T^2\right)$ instead of $\varphi_2(t) := \Psi\left(t/T\right)$, using $(2.16) - (2.18)$, into account the following inequalites

$$J^{\alpha}_{-\infty|t}(\varphi^p) \geq J^{\alpha}_{0|t}(\varphi^p).$$

Moreover, we take $\varphi_1^{\ell/p}\varphi_1^{-\ell/p}$ instead of $\varphi^{1/p}\varphi^{-1/p}$ in $(4.5) - (4.12)$ and $(4.16)$ to use the Young and Hölder's inequality.
Note that here, we use the $\varepsilon-$Young inequality

$$ab \leq \frac{\varepsilon}{2}a^p + C(\varepsilon)b^{\widetilde{p},}$$

by choosing $0 < \varepsilon < 1$ small enough such that $\varphi_2(t) \geq \varepsilon$ for all $t \in \mathbb{R}$. □

**Theorem 3** *For $p \leq p^*$ or $p < \frac{1}{\gamma}$, let $\alpha_1 := (2-\gamma)/(p-1)$ and $u$ be the blowing-up solution of $(1.1), (5.1)$ in a finite time $T_{\max} := T^*$. Then there are constants $c, C > 0$ such that*

$$c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(., t) \leq C(T^* - t)^{-\alpha_1}, \qquad t \in (0, T^*). \tag{5.4}$$

**Proof** We decompose the proof into two parts:

• The upper blow-up rate estimate. Let

$$M(t) := \sup_{\mathbb{R}^N \times (0,t]} u, \qquad t \in (0, T^*).$$

Clearly, $M$ is positive, continuous and nondecreasing on $(0, T^*)$. As $\lim_{t\to T^*} M(t) = \infty$, then for all $t_0 \in (0, T^*)$ we can define

$$t_0^+ := t^+(t_0) := \max\{t \in (t_0, T^*) : \ M(t) = 2M(t_0)\}.$$

Choose $A \geq 1$ and let

$$\lambda_0 := \lambda(t_0) := \left(\frac{1}{2A}M(t_0)\right)^{-1/(2\alpha_1)}. \tag{5.5}$$

we claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq D, \qquad t_0 \in \left(\frac{T^*}{2}, T^*\right), \tag{5.6}$$

where $D$ is a positive constant which does not depend on $t_0$.
We proceed by contradiction. If (5.6) were false, then there would exist a sequence $t_n \to T^*$ such that

$$\lambda_n^{-2}(t_n^+ - t_n) \longrightarrow \infty,$$

where $\lambda_n = \lambda(t_n)$ and $t_n^+ = t^+(t_n)$. For each $t_n$ choose

$$(\widehat{x}_n, \widehat{t}_n) \in \mathbb{R}^N \times (0, t_n] \quad \text{such that} \quad u(\widehat{x}_n, \widehat{t}_n) \geq \frac{1}{2}M(t_n). \tag{5.7}$$

Obviously, $M(t_n) \to \infty$; hence, $\widehat{t}_n \to T^*$. Next, rescale the function $u$ as

$$\varphi^{\lambda_n}(y,s) := \lambda_n^{2\alpha_1} u(\lambda_n y + \widehat{x}_n, \lambda_n^2 s + \widehat{t}_n), \qquad (y,s) \in \mathbb{R}^N \times I_n(T^*), \tag{5.8}$$

where $I_n(t) := (-\lambda_n^{-2}\widehat{t}_n, \lambda_n^{-2}(t - \widehat{t}_n))$ for all $t > 0$. Then $\varphi^{\lambda_n}$ is a solution of

$$(\varphi)_s = \Delta\varphi + J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|t}(\varphi^p) \qquad \text{in } \mathbb{R}^N \times I_n(T^*), \tag{5.9}$$

such that $\varphi^{\lambda_n}(0,0) \geq A$, and

$$0 \leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M(t_n^+) = \lambda_n^{2\alpha_1} 2M(t_n) = 4A \qquad \text{in } \mathbb{R}^N \times I_n(t_n^+),$$

thanks to (5.5) and the definition of $t_n^+$.
It follows, from the parabolic interior regularity theory (cf. [17]), that there is $\mu \in (0,1)$ such that the sequence $\varphi^{\lambda_n}$ is bounded in the $C^{\mu,\mu/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$-norm by a constant that does not depend on $n$. Similar uniform estimates for $J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|t}(\varphi^p)$ follow if $\mu$ is sufficiently small. The parabolic interior Schauder's estimates imply now that the $C^{2+\mu,1+\mu/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$-norm of $\varphi^{\lambda_n}$ is uniformly bounded. Hence, we obtain a subsequence converging in $C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^N \times \mathbb{R})$ to a strong solution $\varphi$ of

$$(\varphi)_s = \Delta\varphi + J^{\alpha}_{-\infty|s}(\varphi^p) \qquad \text{in } \mathbb{R}^N \times (-\infty, +\infty).$$

Then using Lemma (1) we infer that $\varphi \equiv 0$. Contradiction with the fact that $\varphi(0,0) \geq A > 1$. This proves (5.6).
Next we use an idea from Hu [10]. From (5.5) and (5.6) it follows that

$$(t_0^+ - t_0) \leq D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \qquad \text{for any } t_0 \in \left(\frac{T^*}{2}, T^*\right).$$

Fix $t_0 \in \left(\frac{T^*}{2}, T^*\right)$ and denote $t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \ldots$. Then

$$\begin{aligned} t_{j+1} - t_j &\leq D(2A)^{1/\alpha_1} M(t_j)^{-1/\alpha_1}, \\ M(t_{j+1}) &= 2M(t_j), \end{aligned}$$

$j = 0, 1, 2, \ldots$. Consequently,

$$\begin{aligned} T^* - t_0 &= \sum_{j=0}^{\infty} (t_{j+1} - t_j) \leq D(2A)^{1/\alpha_1} \sum_{j=0}^{\infty} M(t_j)^{-1/\alpha_1} \\ &= D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \sum_{j=0}^{\infty} 2^{-j/\alpha_1}. \end{aligned}$$

And finally we conclude that

$$u(x,t_0) \leq M(t_0) \leq C(T^* - t_0)^{-\alpha_1}, \qquad \forall\, t_0 \in (0, T^*)$$

where

$$C = 2A\left(D\sum_{j=0}^{\infty} 2^{-j/\alpha_1}\right)^{\alpha_1},$$

and consequently

$$\sup_{\mathbb{R}^N} u(.,t) \leq C(T^*-t)^{-\alpha_1}, \qquad \forall\, t \in (0,T^*).$$

• The lower blow-up rate estimate. If we repeat the same proof of the local existence of Theorem 1, by taking $||u||_1 \leq \theta$ instead of $||u||_1 \leq 2||u_0||_\infty$ in the space $E_T$ for all positive constant $\theta > 0$ and all $0 < t < T$, then the condition (3.2) of $T$ will be:

$$||u_0||_\infty + CT^{2-\gamma}\theta^p \leq \theta, \tag{5.10}$$

and then, like before, we infer that $||u(t)||_\infty \leq \theta$ for (almost) all $0 < t < T$. Consequently, if $||u_0||_\infty + Ct^{2-\gamma}\theta^p \leq \theta$, then $||u(t)||_\infty \leq \theta$. Applying this to any point in the trajectory, we see that if $0 \leq s < t$ and

$$(t-s)^{2-\gamma} \leq \frac{\theta - ||u(s)||_\infty}{C\theta^p}, \tag{5.11}$$

then $||u(t)||_\infty \leq \theta$, for all $0 < t < T$.
Moreover, if $0 \leq s < T^*$ and $||u(s)||_\infty < \theta$, then:

$$(T^*-s)^{2-\gamma} > \frac{\theta - ||u(s)||_\infty}{C\theta^p}. \tag{5.12}$$

Indeed, arguing by contradiction and assume that for some $\theta > ||u(s)||_\infty$ and all $t \in (s,T^*)$ we have

$$(t-s)^{2-\gamma} \leq \frac{\theta - ||u(s)||_\infty}{C\theta^p},$$

then, using (5.11), we infer that $||u(t)||_\infty \leq \theta$ for all $t \in (s,T^*)$; this contradicts with the fact that $||u(t)||_\infty \to \infty$ as $t \to T^*$.
Next, for example, letting $\theta = 2||u(s)||_\infty$ in (5.12), we see that for $0 < s < T^*$ we have:

$$(T^*-s)^{2-\gamma} > C'||u(s)||_\infty^{1-p},$$

and by the positivity and the continuity of $u$ we get

$$c(T^*-s)^{-\alpha_1} < \sup_{x\in\mathbb{R}^N} u(x,s), \qquad \forall\, s \in (0,T^*) \tag{5.13}$$

□

## 6 Global existence

In this section, we prove the existence of global solutions of (1.1) with initial data small enough. We give a similar proof than that in [6] just for the seek of completness. In the following, we use the notation $p_{sc} := \frac{(p-1)N}{\beta(2-\gamma)}$. As $p^* > 1 + \frac{\beta(2-\gamma)}{N}$, we note that $p > p^* \Rightarrow p_{sc} > 1$.

**Theorem 4** *Let $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$. If*

$$p > \max\{\frac{1}{\gamma}; p^*\}, \tag{6.1}$$

*and $||u_0||_{L^{p_{sc}}}$ is sufficiently small, then $u$ exists globally.*
*Note that we can take $|u_0(x)| \leq C|x|^{-\frac{\beta(2-\gamma)}{p-1}}$ instead of $u_0 \in L^{p_{sc}}$.*

**Proof** As $p > \frac{1}{\gamma}$, then we have the possibility to take a positive constant $q > 0$ so that:

$$\frac{2-\gamma}{p-1} - \frac{1}{p} < \frac{N}{\beta q} < \frac{1}{p-1}, \quad q \geq p. \tag{6.2}$$

It follows, using (6.1), that

$$q > \frac{N(p-1)}{\beta} > p_{sc} > 1. \tag{6.3}$$

Let

$$b := \frac{N}{\beta p_{sc}} - \frac{N}{\beta q} = \frac{2-\gamma}{p-1} - \frac{N}{\beta q}. \tag{6.4}$$

Then, using (6.2) − (6.4), we conclude that

$$b > \frac{1-\gamma}{p-1} > 0, \quad pb < 1, \quad \frac{N(p-1)}{\beta q} + (p-1)b + \gamma = 2. \tag{6.5}$$

As $u_0 \in L^{p_{sc}}$, using (2.4) and (6.4), we get, for all $t > 0$,

$$\sup_{t>0} t^b ||e^{-t(-\Delta)^{\beta/2}} u_0||_{L^q} \leq C||u_0||_{L^p_{sc}} = \eta < \infty. \tag{6.6}$$

Set

$$\Xi := \left\{ u \in L^\infty((0,\infty), L^q(\mathbb{R}^N)); \ \sup_{t>0} t^b ||u(t)||_{L^q} \leq \delta \right\}, \tag{6.7}$$

where $\delta > 0$ is to be chosen sufficiently small. If

$$d_\Xi(u,v) = \sup_{t>0} t^b ||u(t) - v(t)||_{L^q}, \qquad \forall u, v \in \Xi, \tag{6.8}$$

then $(\Xi, d)$ is a complete metric space. Given $u \in \Xi$, let's set:

$$\Phi(u)(t) := e^{-t(-\Delta)^{\beta/2}} u_0 + \frac{1}{\Gamma(1-\gamma)} \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_0^s (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma) \, d\sigma \, ds, \tag{6.9}$$

for all $t \geq 0$. We have by (2.4), (6.6) and (6.7)

$$\begin{aligned} t^b||\Phi(u)(t)||_{L^q} &\leq \eta + Ct^b \int_0^t (t-s)^{-\frac{N}{\beta}(\frac{p}{q}-\frac{1}{q})} \int_0^s (s-\sigma)^{-\gamma} ||u^p(\sigma)||_{L^{\frac{q}{p}}} \, d\sigma \, ds \\ &\leq \eta + C\delta^p t^b \int_0^t \int_0^s (t-s)^{-\frac{N(p-1)}{\beta q}} (s-\sigma)^{-\gamma} \sigma^{-bp} \, d\sigma \, ds. \end{aligned} \tag{6.10}$$

Next, using (6.2) and $pb < 1$, we get

$$\begin{aligned} \int_0^t \int_0^s \frac{(t-s)^{-\frac{N}{\beta q}(p-1)}}{(s-\sigma)^\gamma} \sigma^{-bp} \, d\sigma \, ds &= \left( \int_0^1 (1-\sigma)^{-\gamma} \sigma^{-bp} \, d\sigma \right) \int_0^t \frac{(t-s)^{-\frac{N(p-1)}{\beta q}}}{s^{bp+\gamma-1}} \, ds \\ &= Ct^{-\frac{N(p-1)}{\beta q} - bp - \gamma + 2} = Ct^{-b}, \end{aligned} \tag{6.11}$$

for all $t \geq 0$. So, we deduce from (6.10) − (6.11) that

$$t^b||\Phi(u)(t)||_{L^q} \leq \eta + C\delta^p. \tag{6.12}$$

Therefore, if $\eta$ and $\delta$ are chosen small enough so that $\eta + C\delta^p \leq \delta$, we see that $\Phi : \Xi \to \Xi$. Similar calculations show that (assuming $\eta$ and $\delta$ small enough) $\Phi$ is a strict contraction, so it has a fixed point $u \in \Xi$ which is a solution of (1.1). Now, we show that $u \in C([0,\infty), C_0(\mathbb{R}^N))$.

First, we show that $u \in C([0,T], C_0(\mathbb{R}^N))$ if $T > 0$ is sufficiently small. Indeed, note that the above argument shows uniqueness in $\Xi_T$, where, for any $T > 0$,

$$\Xi_T := \left\{ u \in L^\infty((0,T), L^q(\mathbb{R}^N)); \ \sup_{0<t<T} t^b||u(t)||_{L^q} \leq \delta \right\}.$$

Let $\widetilde{u}$ be the local solution of (1.1) constructed in Theorem 1. Since $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, then, using (6.3), $u_0 \in L^q(\mathbb{R}^N)$, we have $\widetilde{u} \in C([0, T_{\max}), L^q(\mathbb{R}^N))$ by Theorem 1. It follows that $\sup_{0<t<T} t^b||\widetilde{u}(t)||_{L^q} \leq \delta$ if $T > 0$ is sufficiently small. Therefore, by uniqueness, $u = \widetilde{u}$ on $[0,T]$, so that $u \in C([0,T], C_0(\mathbb{R}^N))$.

Next, we show that $u \in C([T,\infty), C_0(\mathbb{R}^N))$ by a bootstrap argument. Indeed, for $t > T$, we write

$$\begin{aligned} u(t) - e^{-t(-\Delta)^{\beta/2}} u_0 &= \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_0^T (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma) \, d\sigma \, ds \\ &+ \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_T^s (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma) \, d\sigma \, ds \\ &\equiv I_1(t) + I_2(t). \end{aligned}$$

Since $u \in C([0,T], C_0(\mathbb{R}^N))$, it follows that $I_1 \in C([T,\infty), C_0(\mathbb{R}^N))$. Also, by the calculations used to construct the fixed point, using the fact that $t^{-b} \leq T^{-b} < \infty$ and $pq > q$, $I_1 \in C([T,\infty), L^q(\mathbb{R}^N))$. Next, note that $\frac{N}{\beta}(\frac{p}{q} - \frac{1}{q}) < 1$ by (6.3). Therefore, there exists $r \in (q, \infty]$ such that

$$\frac{N}{\beta}(\frac{p}{q} - \frac{1}{r}) < 1. \tag{6.13}$$

Given $T < s < t$ (the case of $s < T < t$ is obvious). Since $u \in L^\infty((0,\infty), L^q(\mathbb{R}^N))$, we have $|u|^{p-1}u \in L^\infty((T,s), L^{\frac{q}{p}}(\mathbb{R}^N))$, and it easily follows, using (2.4) and (6.13), that $I_2 \in C([T,\infty), L^r(\mathbb{R}^N))$. As the terms $e^{-\cdot(-\Delta)^{\beta/2}}u_0$ and $I_1$ both belong to $C([T,\infty), C_0(\mathbb{R}^N)) \cap C([T,\infty), L^q(\mathbb{R}^N))$, we see that $u \in C([T,\infty), L^r(\mathbb{R}^N))$. Iterating this procedure a finite number of times, we deduce that $u \in C([T,\infty), C_0(\mathbb{R}^N))$. This completes the proof.

# 7 Necessary conditions for local and global existence

In this section, we establish necessary conditions for the existence of local and global solutions to the problem (1.1); these conditions depend on the behavior of the initial condition for large $x$.

**Theorem 5** (*Necessary conditions for global existence*)
*Let $u_0 \in C_0(\mathbb{R}^N) \cap H^\beta(\mathbb{R}^N)$, $u_0 \geq 0$, $0 < \beta \leq 2$ and $p > 1$. If $u$ is a global solution to problem* (1.1), *then there is a positive constant $C > 0$ such that*

$$\lim_{|x|\to\infty} (u_0(x)|x|^{\frac{\beta(2-\gamma)}{p-1}}) \leq C. \tag{7.1}$$

**Proof** We repeat the same calculation as in the proof of Theorem 1 by taking, for $0 < R \leq \infty$ sufficiently large, $\varphi(x,t) := \varphi_1(x)\varphi_2(t) := \Phi(|x|/R)\,\Psi\left(t/R^\beta\right)$ instead of the one chosen in Theorem 1, where, for $x \in \mathbb{R}^N$ and $R = 1$, $0 \leq \varphi_1 \in H^\beta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is the first eigenfunction of the fractional laplacian operator $(-\Delta_x)^{\beta/2}$, associated to the first eigenvector $\lambda_1 := \inf\{||u||_{H^{\beta/2}};\ ||u||_{L^2} = 1 \text{ and } u = 0 \text{ in } B_2^c\}$ where $B_2^c$ is the ball of center 0 and radius 2, and, for all $s > 0$,

$$\Psi(s) = \begin{cases} (1-s)^\ell & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 1, \end{cases}$$

with $\ell$ large enough. Then, as for the estimate (4.7), we obtain, with
$\Sigma_2 := \left\{(x,t) \in \mathbb{R}^N \times (0,\infty);\ |x| \leq 2R,\ t \leq R^\beta\right\}$,

$$\begin{aligned}
&\int_{\Sigma_2} u^p\ \varphi\, dx\, dt + C\ R^{-\alpha\beta} \int_{|x|\leq 2R} u_0(x)\varphi_1(x)\, dx \\
&\leq C \int_{\Sigma_2} \varphi_1(x)\,(\varphi_2(t))^{-\frac{1}{p-1}} \left|D^{1+\alpha}_{t|R^\beta}\varphi_2(t)\right|^{\widetilde{p}}\, dx\, dt \\
&+C \int_{\Sigma_2} (\varphi_1(x))^{-\frac{1}{p-1}}\,(\varphi_2(t))^{-\frac{1}{p-1}} \left|(-\Delta_x)^{\beta/2}\varphi_1(x) D^{\alpha}_{t|R^\beta}\varphi_2(t)\right|^{\widetilde{p}}\, dx\, dt,
\end{aligned} \tag{7.2}$$

where $\alpha := 1 - \gamma$ and $\widetilde{p} := p/(p-1)$. If we take the scaled variables $\tau = t/R^\beta$, $\xi = x/R$ and using the fact that $(-\Delta_x)^{\beta/2}\varphi_1(x) = R^{-\beta}\lambda_1\varphi_1(x)$ and $u \geq 0$, then (7.2) implies

$$C\ R^{-\alpha\beta} \int_{|\xi|\leq 2} u_0(R\xi)\Phi(|\xi|)\, d\xi$$

$$
\begin{aligned}
&\leq C(R)\int_{|\xi|\leq 2}\Phi(|\xi|)\,d\xi\\
&= C(R)\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(\widetilde{p}-1)}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\Phi(|\xi|)\,d\xi\\
&\leq C(R)(2R)^{\beta(1+\alpha)(\widetilde{p}-1)}\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\Phi(|\xi|)\,d\xi \qquad (7.3)
\end{aligned}
$$

where $C(R)=R^{\beta-(1+\alpha)\beta\widetilde{p}}(C+C\lambda_1)$.
Using the estimate

$$
\begin{aligned}
\inf_{|\xi|>1}(u_0(R\xi)|R\xi|^{\beta(1+\alpha)(\widetilde{p}-1)})\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\Phi(|\xi|)\,d\xi &\leq \int_{1<|\xi|\leq 2}u_0(R\xi)\Phi(|\xi|)\,d\xi\\
&\leq \int_{|\xi|\leq 2}u_0(R\xi)\Phi(|\xi|)\,d\xi
\end{aligned}
$$

in the right-hand side of (7.3), we conclude, after dividing by the term $\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\Phi(|\xi|)\,d\xi$ that

$$
\inf_{|\xi|>1}(u_0(R\xi)|R\xi|^{\beta(1+\alpha)(\widetilde{p}-1)})\leq C(R)R^{\alpha\beta}(2R)^{\beta(1+\alpha)(\widetilde{p}-1)}\leq C. \qquad (7.4)
$$

Passing to the limit in (7.4), as $R\to\infty$, and taking account of the continuity of $u_0$, we obtain

$$
\lim_{|x|\to\infty}(u_0(x)|x|^{\beta(1+\alpha)(\widetilde{p}-1)})=\liminf_{|x|\to\infty}(u_0(x)|x|^{\beta(1+\alpha)(\widetilde{p}-1)})\leq C.
$$

□

**Corollary 2** (*sufficient conditions for the nonexistence of global solution*)
*Let $u_0\in C_0(\mathbb{R}^N)\cap H^\beta(\mathbb{R}^N)$, $u_0\geq 0$, $0<\beta\leq 2$ and $p>1$. If*

$$
\lim_{|x|\to\infty}(u_0(x)|x|^{\frac{\beta(2-\gamma)}{p-1}})=+\infty,
$$

*then the problem* (1.1) *cannot have a global solution.* □

Next, we give a necessary condition for local existence where we obtain a similar estimate of $T$ found in the proof of Theorem 1, as $|x|$ goes to infinity.

**Theorem 6** (*Necessary conditions for local existence*)
*Let $u_0\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap H^\beta(\mathbb{R}^N)$, $u_0\geq 0$, $0<\beta\leq 2$ and $p>1$. If $u$ is a local solution to problem* (1.1) *on $[0,T]$ where $0<T<+\infty$, then we have the estimates*

$$
\lim_{|x|\to\infty}(u_0(x))\leq C\,T^{-\frac{2-\gamma}{p-1}}, \qquad (7.5)
$$

*for some positive constant $C>0$.*
*Note that, if $A:=\lim_{|x|\to\infty}(u_0(x))$, then we obtain a similar estimate as that found in* (3.2),

$$
\frac{T^{2-\gamma}A^{p-1}}{C^{p-1}}\leq 1.
$$

**Proof** We take, for $0 < R \leq \infty$ sufficiently large, $\varphi(x,t) := \varphi_1(x)\varphi_2(t) := \Phi\left(\frac{|x|}{R}\right)\Psi\left(\frac{t}{T}\right)$ instead of the one chosen in Theorem 5, where, for $x \in \mathbb{R}^N$ and $R = 1$, $0 \leq \varphi_1 \in H^\beta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is the first eigenfunction, for the fractional laplacian operator $(-\Delta_x)^{\beta/2}$, relative to the first eigenvector $\lambda_1 := \inf\{||u||_{H^{\beta/2}};\ ||u||_{L^2} = 1 \text{ and } u = 0 \text{ in } B_2^c\}$. Then, as (7.2), we obtain, with $\Sigma_3 := \left\{(x,t) \in \mathbb{R}^N \times (0,\infty);\ |x| \leq 2R,\ t \leq T\right\}$,

$$\begin{aligned}
&\int_{\Sigma_3} u^p\ \varphi\, dx\, dt + C\ T^{-\alpha} \int_{|x|\leq 2R} u_0(x)\varphi_1(x)\, dx \\
&\leq C \int_{\Sigma_3} \varphi_1(x)\left(\varphi_2(t)\right)^{-\frac{1}{p-1}} \left|D_{t|T}^{1+\alpha}\varphi_2(t)\right|^{\widetilde{p}}\, dx\, dt \\
&+C \int_{\Sigma_2} \left(\varphi_1(x)\right)^{-\frac{2}{p-1}} \left(\varphi_2(t)\right)^{-\frac{1}{p-1}} \left|(-\Delta_x)^{\beta/2}\varphi_1(x) D_{t|T}^{\alpha}\varphi_2(t)\right|^{\widetilde{p}}\, dx\, dt. \qquad (7.6)
\end{aligned}$$

where $\alpha := 1 - \gamma$ and $\widetilde{p} := \frac{p}{p-1}$. If we take the scaled variables $\tau = \frac{t}{T}$, $\xi = \frac{x}{R}$ and use the fact that $(-\Delta_x)^{\beta/2}\varphi_1(x) = R^{-\beta}\lambda_1\varphi_1(x)$ and $u \geq 0$, then (7.6) implies

$$C\ T^{-\alpha} \int_{|\xi|\leq 2} u_0(R\xi)\Phi(|\xi|)\, d\xi \leq \left(C\ T^{1-(1+\alpha)\widetilde{p}} + C\lambda_1\ T^{1-\alpha\widetilde{p}} R^{-\beta\widetilde{p}}\right) \int_{|\xi|\leq 2} \Phi(|\xi|)\, d\xi;$$

so

$$\int_{|\xi|\leq 2} u_0(R\xi)\Phi(|\xi|)\, d\xi \leq C(R,T) \int_{|\xi|\leq 2} \Phi(|\xi|)\, d\xi \qquad (7.7)$$

where $C(R,T) = C\ T^{(1+\alpha)(1-\widetilde{p})} + C\ T^{1+\alpha(1-\widetilde{p})} R^{-\beta\widetilde{p}}$.
Using the estimate

$$\begin{aligned}
\inf_{|\xi|>1}(u_0(R\xi)) \int_{|\xi|\leq 2} \Phi(|\xi|)\, d\xi &\leq \int_{1<|\xi|\leq 2} u_0(R\xi)\Phi(|\xi|)\, d\xi \\
&\leq \int_{|\xi|\leq 2} u_0(R\xi)\Phi(|\xi|)\, d\xi
\end{aligned}$$

in the left-hand side of (7.7), we conclude, after dividing by the term $\int_{|\xi|\leq 2} \Phi(|\xi|)\, d\xi$, that

$$\inf_{|\xi|>1}(u_0(R\xi)) \leq C(R,T). \qquad (7.8)$$

Passing to the limit in (7.8), as $R \to \infty$, and taking account of the continuity of $u_0$, we obtain

$$\lim_{|x|\to\infty}(u_0(x)) = \liminf_{|x|\to\infty}(u_0(x)) \leq C\ T^{(1+\alpha)(1-\widetilde{p})} = C\ T^{-\frac{2-\gamma}{p-1}}.$$

□

# Appendix.A

We give a proof of Ju's inequality (see proposition 3.3 in [11]), in dimension $N$ where $\delta \in [0,2]$ and $q \geq 1$, for all nonnegative Schwartz function $\psi$ (in general case)

$$(-\Delta)^{\delta/2}\psi^q \leq q\psi^{q-1}(-\Delta)^{\delta/2}\psi. \qquad (7.9)$$

The cases $\delta = 0$, $\delta = 1$ and $\delta = 2$ are obvious.

- If $\delta \in (0,1)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)}{|z|^{N+\delta}}\,dz, \quad \forall\, x \in \mathbb{R}^N, \tag{7.10}$$

where $c_N(\delta) = \frac{\delta\Gamma(\frac{N+\delta}{2})}{2\pi^{\frac{N}{2}+\delta}\Gamma(1-\frac{\delta}{2})}$. Then

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{(\psi(x))^{q-1}\psi(x+z)-(\psi(x))^q}{|z|^{N+\delta}}\,dz.$$

The case $q = 1$ is clear. If $q > 1$, then by Young's inequality we have

$$(\psi(x))^{q-1}\psi(x+z) \le \frac{q-1}{q}(\psi(x))^q + \frac{1}{q}(\psi(x+z))^q.$$

Therefore, we infer

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) \ge \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N} \frac{(\psi(x+z))^q-(\psi(x))^q}{|z|^{N+\delta}}\,dz = \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.$$

- If $\delta \in (1,2)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)-\nabla\psi(x).z}{|z|^{N+\delta}}\,dz, \quad \forall\, x \in \mathbb{R}^N. \tag{7.11}$$

Then by the similar process, we have

$$\begin{aligned}
(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) &\ge -c_N(\delta)\int_{\mathbb{R}^N} \frac{\frac{1}{q}\left((\psi(x+z))^q-(\psi(x))^q\right)-(\nabla\psi(x).z)(\psi(x))^{q-1}}{|z|^{N+\delta}}\,dz \\
&= \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N} \frac{(\psi(x+z))^q-(\psi(x))^q-\nabla(\psi(x))^q.z}{|z|^{N+\delta}}\,dz \\
&= \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.
\end{aligned}$$

## Appendix.B

For all $\psi \in L^\infty(\mathbb{R}^N) \cap H^\delta(\mathbb{R}^N)$, we have the estimate

$$||\psi^q||_{H^\delta} \le C||\psi||_{L^\infty}^{q-1}||\psi||_{H^\delta}. \tag{7.12}$$

The case $\delta = 1$ is obvious, while for the rest we distinguish 3 cases:

- If $\delta \in (0,1)$, using the Mean value theorem and the following equivalent norm of $||.||_{H^\delta}$ (see Th. 7.48 p. 214 in [1]), we have:

$$||\psi^q||_{H^\delta}^2 = ||\psi^q||_{L^2}^2 + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left(\psi^q(x)-\psi^q(y)\right)^2}{|x-y|^{N+2\delta}}\,dx\,dy$$

$$
\begin{aligned}
&= ||\psi^q||_{L^2}^2 + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left(qz^{q-1}(\psi(x)-\psi(y))\right)^2}{|x-y|^{N+2\delta}}\,dx\,dy \\
&\leq q^2||\psi^{q-1}||_{L^\infty}^2\left[||\psi||_{L^2}^2 + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2\delta}}\,dx\,dy\right] \\
&\leq q^2||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^\delta}^2,
\end{aligned}
$$

where $\min(\psi(x),\psi(y)) < z < \max(\psi(x),\psi(y))$ for every $x, y \in \mathbb{R}^N$.

- If $\delta \in (1,2)$, by the equivalent norm of $||.||_{H^\delta}$ (see Th. 7.48 p. 214 in [1]), we obtain:

$$
\begin{aligned}
||\psi^q||_{H^\delta}^2 &= ||\psi^q||_{H^1}^2 + \sum_{|\alpha|=1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(D^\alpha\psi^q(x)-D^\alpha\psi^q(y))^2}{|x-y|^{N+2(\delta-1)}}\,dx\,dy \\
&= ||q\psi^{q-1}\nabla\psi||_{L^2}^2 + \sum_{|\alpha|=1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(D^\alpha(\psi^q(x)-\psi^q(y)))^2}{|x-y|^{N+2(\delta-1)}}\,dx\,dy \\
&\leq q^2||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^1}^2 + \sum_{|\alpha|=1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left(D^\alpha(qz^{q-1}(\psi(x)-\psi(y)))\right)^2}{|x-y|^{N+2(\delta-1)}}\,dx\,dy \\
&\leq q^2||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^\delta}^2,
\end{aligned}
$$

where $z$ is like above.

- If $\delta = 2$, using the definition of $\Delta$, we get:

$$
\begin{aligned}
||\psi^q||_{H^2}^2 &:= ||(-\Delta)\psi^q||_{L^2}^2 \\
&= ||q(q-1)\psi^{q-2}|\nabla\psi|^2 + q\psi^{q-1}(-\Delta)\psi||_{L^2}^2 \\
&\leq q^2(q-1)^2||\psi^{q-2}|\nabla\psi|^2||_{L^2}^2 + q^2||\psi^{q-1}(-\Delta)\psi||_{L^2}^2 \\
&\leq q^2(q-1)^2||\psi||_{L^\infty}^{2(q-2)}|||\nabla\psi|^2||_{L^2}^2 + q^2||\psi||_{L^\infty}^{2(q-1)}||(-\Delta)\psi||_{L^2}^2 \\
&= q^2(q-1)^2||\psi||_{L^\infty}^{2(q-2)}||\nabla\psi||_{L^4}^4 + q^2||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^2}^2 \\
&\leq q^2(q-1)^2||\psi||_{L^\infty}^{2(q-2)}\left[C\left(\sum_{|\alpha|=2}||D^\alpha\psi||_{L^2}\right)^{1/2}||\psi||_{L^\infty}^{1/2}\right]^4 + q^2||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^2}^2 \\
&\leq C\,q^2(q-1)^2||\psi||_{L^\infty}^{2(q-1)}\left(\sum_{|\alpha|=2}||D^\alpha\psi||_{L^2}\right)^2 + q^2||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^2}^2 \\
&\leq C||\psi||_{L^\infty}^{2(q-1)}||\psi||_{H^\delta}^2
\end{aligned}
$$

thanks to the Gagliardo-Nirenberg inequality see [Th. 1.3.4][5]. Finally, taking the following equivalent norm of $\|.\|_{H^2}$, to conclude that

$$
\|\psi^q\|_{H^2}^2 = \|\psi^q\|_{L^2}^2 + \|\nabla(\psi^q)\|_{L^2}^2 + \|(-\triangle)(\psi^q)\|_{L^2}^2 \leq C\|\psi\|_{L_\infty}^{2(q-1)}\|\psi\|_{H^2}^2.
$$

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