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# FORMAL GEOMETRIC QUANTIZATION II

PAUL-EMILE PARADAN

ABSTRACT. In this paper we pursue the study of formal geometric quantization of non-compact Hamiltonian manifolds. Our main result is the proof that two quantization process coincide. This fact was obtained by Ma and Zhang in the preprint Arxiv:0812.3989 by completely different means.

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In the previous article [21], we have studied some functorial properties of the “formal geometric quantization” process  $\mathcal{Q}^{-\infty}$ , which is defined on *proper Hamiltonian manifolds*, e.g. *non-compact* Hamiltonian manifolds with *proper* moment map.

There is another way, denoted  $\mathcal{Q}^\Phi$ , of quantizing proper Hamiltonian manifolds by localizing the index of the Dolbeault Dirac operator on the critical points of the square of the moment map [15, 19, 20].

The main purpose of this paper is to provide a geometric proof that the quantization process  $\mathcal{Q}^{-\infty}$  and  $\mathcal{Q}^\Phi$  coincide. This fact was proved by Ma and Zhang in the recent preprint [15] by completely different means.

*Keywords:* moment map ; symplectic reduction ; geometric quantization ; transversally elliptic symbol.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let us first recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show two way of extending the notion of geometric quantization to the case of a *non-compact* Hamiltonian manifold.

Let  $K$  be a compact connected Lie group, with Lie algebra  $\mathfrak{k}$ . In the Kostant-Souriau framework, a Hamiltonian  $K$ -manifold  $(M, \Omega, \Phi)$  is pre-quantized if there

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is an equivariant Hermitian line bundle  $L$  with an invariant Hermitian connection  $\nabla$  such that

$$(1.1) \quad \mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega,$$

for every  $X \in \mathfrak{k}$ . Here  $X_M$  is the vector field on  $M$  defined by  $X_M(m) = \frac{d}{dt}e^{-tX}m|_0$ .

The data  $(L, \nabla)$  is also called a Kostant-Souriau line bundle, and  $\Phi : M \rightarrow \mathfrak{k}^*$  is the moment map. Remark that conditions (1.1) imply via the equivariant Bianchi formula the relation

$$(1.2) \quad \iota(X_M)\Omega = -d\langle \Phi, X \rangle, \quad X \in \mathfrak{k}.$$

Let us recall the notion of geometric quantization when  $M$  is **compact**. Choose a  $K$ -invariant almost complex structure  $J$  on  $M$  which is compatible with  $\Omega$  in the sense that the symmetric bilinear form  $\Omega(\cdot, J\cdot)$  is a Riemannian metric. Let  $\bar{\partial}_L$  be the Dolbeault operator with coefficients in  $L$ , and let  $\bar{\partial}_L^*$  be its (formal) adjoint. The *Dolbeault-Dirac operator* on  $M$  with coefficients in  $L$  is  $D_L = \bar{\partial}_L + \bar{\partial}_L^*$ , considered as an elliptic operator from  $\mathcal{A}^{0,\text{even}}(M, L)$  to  $\mathcal{A}^{0,\text{odd}}(M, L)$ . Let  $R(K)$  be the representation ring of  $K$ .

**Definition 1.1.** *The geometric quantization of a compact Hamiltonian  $K$ -manifold  $(M, \Omega, \Phi)$  is the element  $\mathcal{Q}_K(M) \in R(K)$  defined as the equivariant index of the Dolbeault-Dirac operator  $D_L$ .*

Let us consider the case of a **proper** Hamiltonian  $K$ -manifold  $M$ : the manifold is (perhaps) **non-compact** but the moment map  $\Phi : M \rightarrow \mathfrak{k}^*$  is supposed to be proper. Under this properness assumption, one defines the *formal geometric quantization* of  $M$  as an element  $\mathcal{Q}_K^-(M) \in R^{-\infty}(K)$  [30, 21]. Let us recall the definition.

Let  $T$  be a maximal torus of  $K$ . Let  $\mathfrak{t}^*$  be the dual of the Lie algebra of  $T$  containing the weight lattice  $\Lambda^* : \alpha \in \Lambda^*$  if  $i\alpha : \mathfrak{t} \rightarrow i\mathbb{R}$  is the differential of a character of  $T$ . Let  $C_K \subset \mathfrak{t}^*$  be a Weyl chamber, and let  $\widehat{K} := \Lambda^* \cap C_K$  be the set of dominant weights. The ring of characters  $R(K)$  has a  $\mathbb{Z}$ -basis  $V_\mu^K, \mu \in \widehat{K} : V_\mu^K$  is the irreducible representation of  $K$  with highest weight  $\mu$ .

A representation  $E$  of  $K$  is *admissible* if it has finite  $K$ -multiplicities :  $\dim(\text{hom}_K(V_\mu^K, E)) < \infty$  for every  $\mu \in \widehat{K}$ . Let  $R^{-\infty}(K)$  be the Grothendieck group associated to the  $K$ -admissible representations. We have an inclusion map  $R(K) \hookrightarrow R^{-\infty}(K)$  and  $R^{-\infty}(K)$  is canonically identified with  $\text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$ .

For any  $\mu \in \widehat{K}$  which is a regular value of moment map  $\Phi$ , the reduced space (or symplectic quotient)  $M_\mu := \Phi^{-1}(K \cdot \mu)/K$  is a *compact* orbifold equipped with a symplectic structure  $\Omega_\mu$ . Moreover  $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$  is a Kostant-Souriau line orbibundle over  $(M_\mu, \Omega_\mu)$ . The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$  is defined. In Section 2.3, we explain how this notion of geometric quantization extends further to the case of singular symplectic quotients. So the integer  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$  is well defined for every  $\mu \in \widehat{K}$ : in particular  $\mathcal{Q}(M_\mu) = 0$  if  $\mu \notin \Phi(M)$ .

**Definition 1.2.** *Let  $(M, \Omega, \Phi)$  be a proper Hamiltonian  $K$ -manifold which is pre-quantized by a Kostant-Souriau line bundle  $L$ . The formal quantization of  $(M, \Omega, \Phi)$*

is the element of  $R^{-\infty}(K)$  defined by

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \hat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

When  $M$  is compact, the fact that

$$(1.3) \quad \mathcal{Q}_K(M) = \mathcal{Q}_K^{-\infty}(M)$$

is known as the “quantization commutes with reduction Theorem”. This was conjectured by Guillemin-Sternberg in [9] and was first proved by Meinrenken [17] and Meinrenken-Sjamaar [18]. Other proofs of (1.3) were also given by Tian-Zhang [26] and the author [19]. For complete references on the subject the reader should consult [25, 28].

One of the main feature of the formal geometric quantization  $\mathcal{Q}^{-\infty}$  is its stability relatively to the restriction to subgroups.

**Theorem 1.3** ([21]). *Let  $M$  be a pre-quantized Hamiltonian  $K$ -manifold which is proper. Let  $H \subset K$  be a closed connected Lie subgroup such that  $M$  is still proper as a Hamiltonian  $H$ -manifold. Then  $\mathcal{Q}_K^{-\infty}(M)$  is  $H$ -admissible and we have  $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$  in  $R^{-\infty}(H)$ .*

When  $M$  is a proper Hamiltonian  $K$ -manifold, we can also define another “formal geometric quantization”, denoted

$$(1.4) \quad \mathcal{Q}_K^\Phi(M) \in R^{-\infty}(K),$$

by localizing the index of the Dolbeault-Dirac operator  $D_L$  on the set  $\text{Cr}(\|\Phi\|^2)$  of critical points of the square of the moment map (see Section 2.2 for the precise definition). We proved in previous papers [20, 21, 23] that

$$(1.5) \quad \mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^\Phi(M).$$

in some situations:

- $M$  is a coadjoint orbit of a semi-simple Lie group  $S$  that parametrizes a representation of the discrete series of  $S$ ,
- $M$  is a Hermitian vector space.

In her ICM 2006 plenary lecture [29], Vergne conjectured that (1.5) holds when  $\text{Cr}(\|\Phi\|^2)$  is compact. Recently, Ma and Zhang [15] prove the following generalisation of this conjecture.

**Theorem 1.4.** *The equality (1.5) holds for **any** proper Hamiltonian  $K$ -manifold.*

This article is dedicated to the study of the quantization map  $\mathcal{Q}^\Phi$ :

- In Section 2.2, we give the precise definition of the quantization process  $\mathcal{Q}^\Phi$ . In particular, we refine the constant  $c_\gamma$  appearing in [15][Theorem 0.1].
- In Section 2.4, we explain how to compute the quantization of a point.
- We give in Section 3 another proof of Theorem 1.4 by using the technique of symplectic cutting developed in [21].
- In Section 4, we consider the case where  $K = K_1 \times K_2$  acts on  $M$  in a way that the symplectic reduction  $M //_0 K_1$  is a *smooth* proper  $K_2$ -Hamiltonian manifold. We show then that the  $K_1$ -invariant part of  $\mathcal{Q}_{K_1 \times K_2}^\Phi(M)$  is equal to  $\mathcal{Q}_{K_2}^{\Phi_2}(M //_0 K_1)$ .

In Section 5, we study the example where  $M$  is the cotangent bundle of a homogeneous space:  $M = T^*(K/H)$  where  $H$  is a closed subgroup of  $K$ . We see that  $T^*(K/H)$  is a proper Hamiltonian  $K$ -manifold prequantized by the trivial line bundle. A direct computation gives

$$(1.6) \quad \mathcal{Q}_K^\Phi(T^*(K/H)) = L^2(K/H) \quad \text{in } R^{-\infty}(K).$$

Let us denote  $[T^*(K/H)]_{\mu, K}$  the symplectic reduction at  $\mu \in \widehat{K}$  of the  $K$ -Hamiltonian manifold  $T^*(K/H)$ . Theorem 1.4 together with (1.6) give

$$\mathcal{Q}([T^*(K/H)]_{\mu, K}) = \dim[V_\mu^K]^H,$$

for any  $\mu \in \widehat{K}$ . Here  $[V_\mu^K]^H \subset V_\mu^K$  is the subspace of  $H$ -invariant vectors.

Then we consider the action of a closed connected subgroup  $G \subset K$  on  $T^*(K/H)$ . We first check that  $T^*(K/H)$  is a *proper* Hamiltonian  $G$ -manifold if and only if the restriction  $L^2(K/H)|_G$  is an admissible  $G$ -representation. Then, using Theorem 1.3, we get that

$$(1.7) \quad \mathcal{Q}_G^{-\infty}(T^*(K/H)) = L^2(K/H)|_G \quad \text{in } R^{-\infty}(G).$$

In other words, the multiplicity of  $V_\lambda^G$  in  $L^2(K/H)$  is equal to the quantization of the reduced space  $[T^*(K/H)]_{\lambda, G}$ .

## 2. QUANTIZATIONS OF NON-COMPACT MANIFOLDS

In this section we define the quantization process  $\mathcal{Q}^\Phi$ , and we give another definition of the quantization process  $\mathcal{Q}^{-\infty}$  that uses the notion of symplectic cutting [21].

**2.1. Transversally elliptic symbols.** Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [6, 7] and Paradan-Vergne [22]. For a short introduction see [19].

Let  $\mathcal{X}$  be a *compact*  $K$ -manifold. Let  $p : T\mathcal{X} \rightarrow \mathcal{X}$  be the projection, and let  $(-, -)_\mathcal{X}$  be a  $K$ -invariant Riemannian metric. If  $E^0, E^1$  are  $K$ -equivariant complex vector bundles over  $\mathcal{X}$ , a  $K$ -equivariant morphism  $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$  is called a *symbol* on  $\mathcal{X}$ . The subset of all  $(x, v) \in T\mathcal{X}$  where<sup>1</sup>  $\sigma(x, v) : E_x^0 \rightarrow E_x^1$  is not invertible is called the *characteristic set* of  $\sigma$ , and is denoted by  $\text{Char}(\sigma)$ .

In the following, the product of a symbol  $\sigma$  by a complex vector bundle  $F \rightarrow M$ , is the symbol

$$\sigma \otimes F$$

defined by  $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \text{Id}_{F_x}$  from  $E_x^0 \otimes F_x$  to  $E_x^1 \otimes F_x$ . Note that  $\text{Char}(\sigma \otimes F) = \text{Char}(\sigma)$ .

Let  $T_K\mathcal{X}$  be the following subset of  $T\mathcal{X}$  :

$$T_K\mathcal{X} = \{(x, v) \in T\mathcal{X}, (v, X_\mathcal{X}(x))_x = 0 \quad \text{for all } X \in \mathfrak{k}\}.$$

A symbol  $\sigma$  is *elliptic* if  $\sigma$  is invertible outside a compact subset of  $T\mathcal{X}$  (i.e.  $\text{Char}(\sigma)$  is compact), and is  *$K$ -transversally elliptic* if the restriction of  $\sigma$  to  $T_K\mathcal{X}$  is invertible outside a compact subset of  $T_K\mathcal{X}$  (i.e.  $\text{Char}(\sigma) \cap T_{K_2}\mathcal{X}$  is compact). An elliptic symbol  $\sigma$  defines an element in the equivariant  $\mathbf{K}$ -theory of  $T\mathcal{X}$  with compact support, which is denoted by  $\mathbf{K}_K(T\mathcal{X})$ , and the index of  $\sigma$  is a virtual finite dimensional representation of  $K$ , that we denote  $\text{Index}_\mathcal{X}^K(\sigma) \in R(K)$  [2, 3, 4, 5].

<sup>1</sup>The map  $\sigma(x, v)$  will be also denote  $\sigma|_x(v)$

Let

$$R_{tc}^{-\infty}(K) \subset R^{-\infty}(K)$$

be the  $R(K)$ -submodule formed by all the infinite sum  $\sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$  where the map  $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$  has at most a *polynomial* growth. The  $R(K)$ -module  $R_{tc}^{-\infty}(K)$  is the Grothendieck group associated to the *trace class* virtual  $K$ -representations: we can associate to any  $V \in R_{tc}^{-\infty}(K)$ , its trace  $k \rightarrow \text{Tr}(k, V)$  which is a generalized function on  $K$  invariant by conjugation. Then the trace defines a morphism of  $R(K)$ -module

$$(2.8) \quad R_{tc}^{-\infty}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^K.$$

A  $K$ -transversally elliptic symbol  $\sigma$  defines an element of  $\mathbf{K}_K(\text{T}_K \mathcal{X})$ , and the index of  $\sigma$  is defined as a trace class virtual representation of  $K$ , that we still denote  $\text{Index}_{\mathcal{X}}^K(\sigma) \in R_{tc}^{-\infty}(K)$ .

Remark that any elliptic symbol of  $\text{T}\mathcal{X}$  is  $K$ -transversally elliptic, hence we have a restriction map  $\mathbf{K}_K(\text{T}\mathcal{X}) \rightarrow \mathbf{K}_K(\text{T}_K \mathcal{X})$ , and a commutative diagram

$$(2.9) \quad \begin{array}{ccc} \mathbf{K}_K(\text{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_K(\text{T}_K \mathcal{X}) \\ \text{Index}_{\mathcal{X}}^K \downarrow & & \downarrow \text{Index}_{\mathcal{X}}^K \\ R(K) & \longrightarrow & R_{tc}^{-\infty}(K) . \end{array}$$

Using the *excision property*, one can easily show that the index map  $\text{Index}_{\mathcal{U}}^K : \mathbf{K}_K(\text{T}_K \mathcal{U}) \rightarrow R_{tc}^{-\infty}(K)$  is still defined when  $\mathcal{U}$  is a  $K$ -invariant relatively compact open subset of a  $K$ -manifold (see [19][section 3.1]).

Suppose now that the group  $K$  is equal to the product  $K_1 \times K_2$ . When a symbol  $\sigma$  is  $K_1 \times K_2$ -transversally elliptic we will be interested in the  $K_1$ -invariant part of its index, that we denote

$$\left[ \text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) \right]^{K_1} \in R_{tc}^{-\infty}(K_2).$$

An intermediate notion between the “ellipticity” and “ $K_1 \times K_2$ -transversal ellipticity” is the “ $K_1$ -transversal ellipticity”. When a  $K_1 \times K_2$ -equivariant morphism  $\sigma$  is  $K_1$ -transversally elliptic, its index  $\text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) \in R_{tc}^{-\infty}(K_1 \times K_2)$ , viewed as a generalized function on  $K_1 \times K_2$ , is *smooth* relatively to the variable in  $K_2$ . It implies that  $\text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) = \sum_{\lambda} \theta(\lambda) \otimes V_{\lambda}^{K_1}$  with

$$\theta(\lambda) \in R(K_2), \quad \forall \lambda \in \widehat{K_1}.$$

In particular, we know that

$$\left[ \text{Index}_{\mathcal{X}}^{K_1 \times K_2}(\sigma) \right]^{K_1} = \theta(0)$$

belongs to  $R(K_2)$ .

Let us recall the multiplicative property of the index map for the product of manifolds that was proved by Atiyah-Singer in [1]. Consider a compact Lie group  $K_2$  acting on two manifolds  $\mathcal{X}_1$  and  $\mathcal{X}_2$ , and assume that another compact Lie group  $K_1$  acts on  $\mathcal{X}_1$  commuting with the action of  $K_2$ .

The external product of complexes on  $\text{T}\mathcal{X}_1$  and  $\text{T}\mathcal{X}_2$  induces a multiplication (see [1, 22]):

$$\odot : K_{K_1 \times K_2}(\text{T}_{K_1} \mathcal{X}_1) \times K_{K_2}(\text{T}_{K_2} \mathcal{X}_2) \longrightarrow K_{K_1 \times K_2}(\text{T}_{K_1 \times K_2}(\mathcal{X}_1 \times \mathcal{X}_2)).$$

The following property will be used frequently in the paper.

**Theorem 2.1** (Multiplicative property). *For any  $[\sigma_1] \in K_{K_1 \times K_2}(\mathbb{T}_{K_1} \mathcal{X}_1)$  and any  $[\sigma_2] \in K_{K_2}(\mathbb{T}_{K_2} \mathcal{X}_2)$  we have*

$$\text{Index}_{\mathcal{X}_1 \times \mathcal{X}_2}^{K_1 \times K_2}([\sigma_1] \odot [\sigma_2]) = \text{Index}_{\mathcal{X}_1}^{K_1 \times K_2}([\sigma_1]) \otimes \text{Index}_{\mathcal{X}_2}^{K_2}([\sigma_2]).$$

We will use in this article the notion of *support* of a generalized character.

**Definition 2.2.** *The support of  $\chi := \sum_{\mu \in \widehat{K}} a_\mu V_\mu^K \in R^{-\infty}(K)$  is the set of  $\mu \in \widehat{K}$  such that  $a_\mu \neq 0$ .*

We will say that  $\chi \in R^{-\infty}(K)$  is supported outside  $B \subset \mathfrak{t}^*$  if the support of  $\chi$  does not intersect  $B$ . Note that an infinite sum  $\sum_{i \in I} \chi_i$  converges in  $R^{-\infty}(K)$  if for each ball

$$B_r = \{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}$$

the set  $\{i \in I \mid \text{support}(\chi_i) \cap B_r \neq \emptyset\}$  is finite.

**Definition 2.3.** *We denote by  $O(r)$  any character of  $R^{-\infty}(K)$  which is supported outside the ball  $B_r$ .*

**2.2. Definition and first properties of  $\mathcal{Q}^\Phi$ .** Let  $(M, \Omega, \Phi)$  be a proper Hamiltonian  $K$ -manifold prequantized by an equivariant line bundle  $L$ . Let  $J$  be an invariant almost complex structure compatible with  $\Omega$ . Let  $p : TM \rightarrow M$  be the projection.

Let us first describe the principal symbol of the Dolbeault-Dirac operator  $\overline{\partial}_L + \overline{\partial}_L^*$ . The complex vector bundle  $(\mathbb{T}^*M)^{0,1}$  is  $K$ -equivariantly identified with the tangent bundle  $TM$  equipped with the complex structure  $J$ . Let  $h$  be the Hermitian structure on  $(TM, J)$  defined by  $h(v, w) = \Omega(v, Jw) - i\Omega(v, w)$  for  $v, w \in TM$ . The symbol

$$\text{Thom}(M, J) \in \Gamma(M, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} TM), p^*(\wedge_{\mathbb{C}}^{\text{odd}} TM)))$$

at  $(m, v) \in TM$  is equal to the Clifford map

$$(2.10) \quad \mathbf{c}_m(v) : \wedge_{\mathbb{C}}^{\text{even}} \mathbb{T}_m M \longrightarrow \wedge_{\mathbb{C}}^{\text{odd}} \mathbb{T}_m M,$$

where  $\mathbf{c}_m(v).w = v \wedge w - \iota(v)w$  for  $w \in \wedge_{\mathbb{C}}^{\bullet} \mathbb{T}_m M$ . Here  $\iota(v) : \wedge_{\mathbb{C}}^{\bullet} \mathbb{T}_m M \rightarrow \wedge_{\mathbb{C}}^{\bullet-1} \mathbb{T}_m M$  denotes the contraction map relative to  $h$ . Since  $\mathbf{c}_m(v)^2 = -\|v\|^2 \text{Id}$ , the map  $\mathbf{c}_m(v)$  is invertible for all  $v \neq 0$ . Hence the characteristic set of  $\text{Thom}(M, J)$  corresponds to the 0-section of  $TM$ .

It is a classical fact that the principal symbol of the Dolbeault-Dirac operator  $\overline{\partial}_L + \overline{\partial}_L^*$  is equal to<sup>2</sup>

$$(2.11) \quad \text{Thom}(M, J) \otimes L,$$

see [8]. Here also we have  $\text{Char}(\text{Thom}(M, J) \otimes L) = 0$  – section of  $TM$ .

**Remark 2.4.** *When the manifold  $M$  is a product  $M_1 \times M_2$  the symbol  $\text{Thom}(M, J) \otimes L$  is equal to the product  $\sigma_1 \odot \sigma_2$  where  $\sigma_k = \text{Thom}(M_k, J_k) \otimes L_k$ .*

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<sup>2</sup>Here we use an identification  $\mathbb{T}^*M \simeq TM$  given by an invariant Riemannian metric.

When  $M$  is compact, the symbol  $\text{Thom}(M, J) \otimes L$  is elliptic and then defines an element of the equivariant  $\mathbf{K}$ -group of  $\text{TM}$ . The topological index of  $\text{Thom}(M, J) \otimes L \in \mathbf{K}_K(\text{TM})$  is equal to the analytical index of the Dolbeault-Dirac operator  $\bar{\partial}_L + \bar{\partial}_L^*$ :

$$(2.12) \quad \mathcal{Q}_K(M) = \text{Index}_M^K(\text{Thom}(M, J) \otimes L) \quad \text{in} \quad R(K).$$

When  $M$  is not compact the topological index of  $\text{Thom}(M, J) \otimes L$  is not defined. In order to extend the notion of geometric quantization to this setting we deform the symbol  $\text{Thom}(M, J) \otimes L$  in the ‘‘Witten’’ way [19, 20]. Consider the identification  $\xi \mapsto \tilde{\xi}, \mathfrak{k}^* \rightarrow \mathfrak{k}$  defined by a  $K$ -invariant scalar product on  $\mathfrak{k}^*$ . We define the *Kirwan vector field* on  $M$ :

$$(2.13) \quad \kappa_m = \left( \widetilde{\Phi(m)} \right)_M(m), \quad m \in M.$$

**Definition 2.5.** *The symbol  $\text{Thom}(M, J) \otimes L$  pushed by the vector field  $\kappa$  is the symbol  $\mathbf{c}^\kappa$  defined by the relation*

$$\mathbf{c}^\kappa|_m(v) = \text{Thom}(M, J) \otimes L|_m(v - \kappa_m)$$

for any  $(m, v) \in \text{TM}$ .

Note that  $\mathbf{c}^\kappa|_m(v)$  is invertible except if  $v = \kappa_m$ . If furthermore  $v$  belongs to the subset  $\text{T}_K M$  of tangent vectors orthogonal to the  $K$ -orbits, then  $v = 0$  and  $\kappa_m = 0$ . Indeed  $\kappa_m$  is tangent to  $K \cdot m$  while  $v$  is orthogonal.

Since  $\kappa$  is the Hamiltonian vector field of the function  $\frac{1}{2}\|\Phi\|^2$ , the set of zeros of  $\kappa$  coincides with the set  $\text{Cr}(\|\Phi\|^2)$  of critical points of  $\|\Phi\|^2$ . Finally we have

$$\text{Char}(\mathbf{c}^\kappa) \cap \text{T}_K M \simeq \text{Cr}(\|\Phi\|^2).$$

In general  $\text{Cr}(\|\Phi\|^2)$  is not compact, so  $\mathbf{c}^\kappa$  does not define a transversally elliptic symbol on  $M$ . In order to define a kind of index of  $\mathbf{c}^\kappa$ , we proceed as follows. For any invariant open relatively compact subset  $U \subset M$  the set  $\text{Char}(\mathbf{c}^\kappa|_U) \cap \text{T}_K U \simeq \text{Cr}(\|\Phi\|^2) \cap U$  is compact when

$$(2.14) \quad \partial U \cap \text{Cr}(\|\Phi\|^2) = \emptyset.$$

When (2.14) holds we denote

$$(2.15) \quad \mathcal{Q}_K^\Phi(U) := \text{Index}_U^K(\mathbf{c}^\kappa|_U) \in R_{tc}^{-\infty}(K)$$

the equivariant index of the transversally elliptic symbol  $\mathbf{c}^\kappa|_U$ .

It will be useful to understand the dependance of the generalized character  $\mathcal{Q}_K^\Phi(U)$  relatively to the data  $(U, \Omega, L)$ . So we consider two proper Hamiltonian  $K$ -manifolds  $(M, \Omega, \Phi)$  and  $(M', \Omega', \Phi')$  respectively prequantized by the line bundles  $L$  and  $L'$ . Let  $V \subset M$  and  $V' \subset M'$  two invariant open subsets.

**Proposition 2.6.** • *The generalized character  $\mathcal{Q}_K^\Phi(U)$  does not depend of the choice of an invariant almost complex structure on  $U$  which is compatible with  $\Omega|_U$ .*

• *Suppose that there exists an equivariant diffeomorphism  $\Psi : V \rightarrow V'$  such that*

- (1)  $\Psi^*(\Phi') = \Phi$ ,
- (2)  $\Psi^*(L') = L$ ,
- (3) *there exists an homotopy of symplectic forms taking  $\Psi^*(\Omega'|_{V'})$  to  $\Omega|_V$ .*



Let  $U' \subset \overline{U'} \subset V'$  be an invariant open relatively compact subset such that  $\partial U'$  satisfies (2.14). Take  $U = \Psi^{-1}(U')$ . Then  $\partial U$  satisfies (2.14) and

$$\mathcal{Q}_K^{\Phi'}(U') = \mathcal{Q}_K^{\Phi}(U) \in R^{-\infty}(K).$$

*Proof.* Let us prove the first point. Let  $\mathbf{c}_i^{\kappa}|_U, i = 0, 1$  be the transversally elliptic symbols defined with the compatible almost complex structure  $J_i, i = 0, 1$ . Since the space of compatible almost complex structure is contractible, there exist an homotopy  $J_t, t \in [0, 1]$  of almost complex structures linking  $J_0$  and  $J_1$ . If we use Lemma 2.2 in [19], we know that there exists an invertible bundle map  $A \in \Gamma(U, \text{End}(TU))$ , homotopic to the identity, such that  $A \circ J_0 = J_1 \circ A$ . With the help of  $A$  we prove then that the symbols  $\mathbf{c}_0^{\kappa}|_U$  and  $\mathbf{c}_1^{\kappa}|_U$  define the same class in  $\mathbf{K}_K(\text{TK}U)$  (see [19][Lemma 2.2]). Hence their equivariant index coincide.

Let us prove the second point. The characters  $\mathcal{Q}_K^{\Phi}(U)$  and  $\mathcal{Q}_K^{\Phi'}(U')$  are computed as the equivariant index of the symbols  $\mathbf{c}^{\kappa}|_U$  and  $\mathbf{c}^{\kappa'}|_{U'}$ . Let  $\tilde{\mathbf{c}}^{\kappa}|_U$  the pull back of  $\mathbf{c}^{\kappa'}|_{U'}$  by  $\Psi$ . Thanks to the point (1) and (2), the only thing which differs in the definitions of the symbols  $\mathbf{c}^{\kappa}|_U$  and  $\tilde{\mathbf{c}}^{\kappa}|_U$  are the almost complex structures  $J$  and  $\tilde{J} = \Psi^*(J')$ : the first one is comptible with  $\Omega$  and the second one with  $\Psi^*(\Omega'|_{V'})$ . Since these two symplectic structure are homotopic, one sees that the almost complex structures  $J$  and  $\tilde{J}$  are also homotopic. So we can conclude like in the first point.  $\square$

Let us recall the basic fact concerning the singular values of  $\|\Phi\|^2$ .

**Lemma 2.7.** *The set of singular values of  $\|\Phi\|^2 : M \rightarrow \mathbb{R}$  forms a sequence  $0 \leq r_1 < r_2 < \dots < r_k < \dots$  which is finite iff  $\text{Cr}(\|\Phi\|^2)$  is compact. In the other case  $\lim_{k \rightarrow \infty} r_k = \infty$ .*

At each regular value  $R$  of  $\text{Cr}(\|\Phi\|^2)$ , we associate the invariant open subset  $M_{<R} := \{\|\Phi\|^2 < R\}$  which satisfies (2.14). The restriction  $\mathbf{c}^{\kappa}|_{M_{<R}}$  defines then a transversally elliptic symbol on  $M_{<R}$ : let  $\mathcal{Q}_K^{\Phi}(M_{<R})$  be its equivariant index.

Let us show that  $\mathcal{Q}_K^{\Phi}(M_{<R})$  has a limit when  $R \rightarrow \infty$ . The set  $\text{Cr}(\|\Phi\|^2)$  has the following decomposition

$$(2.16) \quad \text{Cr}(\|\Phi\|^2) = \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (M^{\tilde{\beta}} \cap \Phi^{-1}(\beta))}_{Z_{\beta}}$$

where the  $\mathcal{B}$  is a subset of the Weyl chamber  $\mathfrak{t}_+^*$ . Note that each part  $Z_{\beta}$  is compact, hence  $\mathcal{B}$  is finite only if  $\text{Cr}(\|\Phi\|^2)$  is compact. When  $\text{Cr}(\|\Phi\|^2)$  is non-compact, the set  $\mathcal{B}$  is infinite, but it is easy to see that  $\mathcal{B} \cap B_r$  is finite for any  $r \geq 0$ . For any  $\beta \in \mathcal{B}$ , we consider a relatively compact open invariant neighborhood  $\mathcal{U}_{\beta}$  of  $Z_{\beta}$  such that  $\text{Cr}(\|\Phi\|^2) \cap \overline{\mathcal{U}_{\beta}} = Z_{\beta}$ .

**Definition 2.8.** *We denote*

$$\mathcal{Q}_K^{\beta}(M) \in R_{tc}^{-\infty}(K)$$

*the index<sup>3</sup> of the transversally elliptic symbol  $\mathbf{c}^{\kappa}|_{\mathcal{U}_{\beta}}$ .*

A simple application of the excision property [19] gives that

$$(2.17) \quad \mathcal{Q}_K^{\Phi}(M_{<R}) := \sum_{\|\beta\|^2 < R} \mathcal{Q}_K^{\beta}(M).$$

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<sup>3</sup>The index of  $\mathbf{c}^{\kappa}|_{\mathcal{U}_{\beta}}$  was denoted  $RR_{\beta}^K(M, L)$  in [19].

We have now the key fact

**Theorem 2.9.** *The generalized character  $\mathcal{Q}_K^\beta(M)$  is supported outside the open ball  $B_{\|\beta\|}$ .*

*Proof.* Proposition 2.9 follows directly from the computations done in [19]. First consider the case where  $\beta \neq 0$  is a  $K$ -invariant element of  $\mathcal{B}$ . Let  $i : \mathbb{T}_\beta \hookrightarrow T$  be the compact torus generated by  $\beta$ . If  $F$  is  $\mathbb{Z}$ -module we denote  $F \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$  the  $\mathbb{Z}$ -module formed by the infinite formal sums  $\sum_a E_a h^a$  taken over the set of weights of  $\mathbb{T}_\beta$ , where  $E_a \in F$  for every  $a$ .

Since  $\mathbb{T}_\beta$  lies in the center of  $K$ , the morphism  $\pi : (k, t) \in K \times \mathbb{T}_\beta \mapsto kt \in K$  induces a map  $\pi^* : R^{-\infty}(K) \rightarrow R^{-\infty}(K) \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$ .

The normal bundle  $\mathcal{N}$  of  $M^{\tilde{\beta}}$  in  $M$  inherits a canonical complex structure  $J_{\mathcal{N}}$  on the fibers. We denote by  $\overline{\mathcal{N}} \rightarrow M^{\tilde{\beta}}$  the complex vector bundle with the opposite complex structure. The torus  $\mathbb{T}_\beta$  is included in the center of  $K$ , so the bundle  $\overline{\mathcal{N}}$  and the virtual bundle  $\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}} := \wedge_{\mathbb{C}}^{\text{even}} \overline{\mathcal{N}} \xrightarrow{0} \wedge_{\mathbb{C}}^{\text{odd}} \overline{\mathcal{N}}$  carry a  $K \times \mathbb{T}_\beta$ -action: they can be considered as elements of  $K_{K \times \mathbb{T}_\beta}(M^{\tilde{\beta}}) = K_K(M^{\tilde{\beta}}) \otimes R(\mathbb{T}_\beta)$ .

In [19], we have defined an inverse of  $\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}$ ,  $[\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}]_{\beta}^{-1} \in K_K(M^{\tilde{\beta}}) \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$ , which is polarized by  $\beta$ . It means that  $[\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}]_{\beta}^{-1} = \sum_a N_a h^a$  with  $N_a \neq 0$  only if  $\langle a, \beta \rangle \geq 0$ .

We prove in [19] the following localization formula :

$$(2.18) \quad \pi^* \left[ \mathcal{Q}_K^\beta(M) \right] = RR_{\beta}^{K \times \mathbb{T}_\beta} \left( M^{\tilde{\beta}}, L|_{M^{\tilde{\beta}}} \otimes [\wedge_{\mathbb{C}}^{\bullet} \overline{\mathcal{N}}]_{\beta}^{-1} \right),$$

as an equality in  $R^{-\infty}(K) \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$ . With (2.18) in hand, it is easy to see that  $V_{\mu}^K$  occurs in the character  $\mathcal{Q}_K^\beta(M)$  only if  $(\mu, \beta) \geq \|\beta\|^2$  (See Lemma 9.4 in [19]).

Now we consider the case where  $\beta \in \mathcal{B}$  is not a  $K$ -invariant element. Let  $\sigma$  be the unique open face of the Weyl chamber  $\mathfrak{t}_+^*$  which contains  $\beta$ . Let  $K_\sigma$  be the corresponding stabilizer subgroup. We consider the symplectic slice  $\mathcal{Y}_\sigma \subset M$ : it is a  $K_\sigma$  invariant Hamiltonian submanifold of  $M$  which is prequantized by the line bundle  $L|_{\mathcal{Y}_\sigma}$ . The restriction of  $\Phi$  to  $\mathcal{Y}_\sigma$  is a moment map  $\Phi_\sigma : \mathcal{Y}_\sigma \rightarrow \mathfrak{k}_\sigma^*$  which is proper in a neighborhood of  $\beta \in \mathfrak{k}_\sigma^*$ . The set

$$K_\sigma \cdot (\mathcal{Y}_\sigma^{\tilde{\beta}} \cap \Phi_\sigma^{-1}(\beta)) = M^{\tilde{\beta}} \cap \Phi^{-1}(\beta)$$

is a component of  $\text{Cr}(\|\Phi_\sigma\|^2)$ . Let  $\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) \in R_{tc}^{-\infty}(K_\sigma)$  be the corresponding character (see Definition 2.8).

We prove in [19][Section 7], the following induction formula:

$$(2.19) \quad \mathcal{Q}_K^\beta(M) = \text{Hol}_{K_\sigma}^K \left( \mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) \right)$$

where  $\text{Hol}_{K_\sigma}^K : R^{-\infty}(K_\sigma) \rightarrow R^{-\infty}(K)$  is the holomorphic induction map. See the Appendix in [19] for the definition and properties of these induction maps.

We know from the previous case that

$$\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) = \sum_{\mu \in \widehat{K_\sigma}} m_\mu V_\mu^{K_\sigma}$$

where  $m_\mu \neq 0 \implies (\mu, \beta) \geq \|\beta\|^2$ . Then, with (2.19), we get

$$\begin{aligned} \mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) &= \sum_{(\mu, \beta) \geq \|\beta\|^2} m_\mu \text{Hol}_{K_\sigma}^K(V_\mu^{K_\sigma}) \\ &= \sum_{(\mu, \beta) \geq \|\beta\|^2} m_\mu \text{Hol}_T^K(t^\mu), \end{aligned}$$

where  $\text{Hol}_T^K : R^{-\infty}(T) \rightarrow R^{-\infty}(K)$  is the holomorphic induction map.

Let  $\rho$  be half the sum of the positive roots. The term  $\text{Hol}_T^K(t^\mu)$  is equal to 0 when  $\mu + \rho$  is not a regular element of  $\mathfrak{t}^*$ . When  $\mu + \rho$  is a regular element of  $\mathfrak{t}^*$ , we have  $\text{Hol}_T^K(t^\mu) = (-1)^{|\omega|} V_{\mu_\omega}^K$  where

$$\mu_\omega = \omega(\mu + \rho) - \rho$$

is dominant for a unique  $\omega \in W$ .

Finally, a representation  $V_\lambda^K$  appears in the character  $\mathcal{Q}_K^\beta(M)$  only if  $\lambda = \mu_\omega$  for a weight  $\mu$  satisfying  $(\mu, \beta) \geq \|\beta\|^2$ . Hence, for such  $\lambda$ , we have

$$\begin{aligned} \|\lambda\| &= \|\mu + \rho - \omega^{-1}\rho\| \\ &\geq (\mu + \rho - \omega^{-1}\rho, \frac{\beta}{\|\beta\|}) \\ &\geq \|\beta\|. \end{aligned}$$

In the last inequality we use that  $(\rho - \omega^{-1}\rho, \beta) \geq 0$  since  $\rho - \omega^{-1}\rho$  is a sum of positive roots, and  $\beta \in \mathfrak{t}_+^*$ . □

With the help of Theorem 2.9 and decomposition (2.17), we see that the multiplicity of  $V_\gamma^K$  in  $\mathcal{Q}_K^\Phi(M_{<R})$  does not depend on the regular value  $R > \|\gamma\|^2$ . We can refine the constant  $c_\gamma$  appearing in [15][Theorem 0.1]: take  $c_\gamma$  equal to  $\|\gamma\|^2$  instead of<sup>4</sup>  $\|\gamma + \rho\|^2 - \|\rho\|^2 \geq \|\gamma\|^2$ .

**Definition 2.10.** *The generalized character  $\mathcal{Q}_K^\Phi(M)$  is defined as the limit in  $R^{-\infty}(K)$  of  $\mathcal{Q}_K^\Phi(M_{<R})$  when  $R$  goes to infinity. In other words*

$$(2.20) \quad \mathcal{Q}_K^\Phi(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}_K^\beta(M).$$

Note that for any regular value  $R$  of  $\|\Phi\|^2$  we have the useful relation

$$(2.21) \quad \mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^\Phi(M_{<R}) + O(\sqrt{R}).$$

**2.3. Quantization of a symplectic quotient.** We will now explain how we define the geometric quantization of *singular* compact Hamiltonian manifolds : here “singular” means that the manifold is obtained by symplectic reduction.

Let  $(N, \Omega)$  be a smooth symplectic manifold equipped with a Hamiltonian action of  $K_1 \times K_2$  : we denote  $(\Phi_1, \Phi_2) : N \rightarrow \mathfrak{k}_1^* \times \mathfrak{k}_2^*$  the corresponding moment map. We assume that  $N$  is pre-quantized by a  $K_1 \times K_2$ -equivariant line bundle  $L$  and we

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<sup>4</sup>Here  $\rho$  is half the sum of the positive roots. Hence  $\|\gamma + \rho\|^2 - \|\rho\|^2 - \|\gamma\|^2 = 2(\rho, \gamma) \geq 0$  and  $(\rho, \gamma) = 0$  only if the weight  $\gamma$  belongs to the center of  $\mathfrak{k} \simeq \mathfrak{k}^*$ .

suppose that the map  $\Phi_1$  is **proper**. One wants to define the geometric quantization of the (compact) symplectic quotient

$$N//_0K_1 := \Phi_1^{-1}(0)/K_1.$$

Let  $\kappa_1$  be the Kirwan vector field attached to the moment map  $\Phi_1$ . We denote by  $\mathbf{c}^{\kappa_1}$  the symbol  $\text{Thom}(N, J) \otimes L$  pushed by the vector field  $\kappa_1$ . For any regular value  $R_1$  of  $\|\Phi_1\|^2$ , we consider the restriction  $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$  to the invariant, open subset  $N_{<R_1} := \{\|\Phi_1\|^2 < R_1\}$ . The symbol  $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$  is  $K_1 \times K_2$ -equivariant and  $K_1$ -transversally elliptic, hence we can consider its index

$$\text{Index}_{N_{<R_1}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}}) \in R^{-\infty}(K_1 \times K_2).$$

which is smooth relatively to the parameter in  $K_2$ . We consider the following extension of Definition 2.10.

**Definition 2.11.** *The generalized character  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$  is defined as the limit in  $R^{-\infty}(K_1 \times K_2)$  of  $\text{Index}_{N_{<R_1}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}})$  when  $R_1$  goes to infinity.*

Here  $\text{Cr}(\|\Phi_1\|^2)$  is equal to the disjoint union of the compact  $K_1 \times K_2$ -invariant subsets  $Z_{\beta_1} := K_1 \cdot (M^{\beta_1} \cap \Phi_1^{-1}(\beta_1))$ ,  $\beta_1 \in \mathcal{B}_1$ . For  $\beta_1 \in \mathcal{B}_1$ , we consider an invariant relatively compact open subset  $\mathcal{U}_{\beta_1}$  such that:  $Z_{\beta_1} \subset \mathcal{U}_{\beta_1}$  and  $Z_{\beta_1} = \text{Cr}(\|\Phi_1\|^2) \cap \overline{\mathcal{U}_{\beta_1}}$ . Let  $\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) \in R^{-\infty}(K_1 \times K_2)$  be the equivariant index of the  $K_1$ -transversally elliptic symbol  $\mathbf{c}_{\kappa_1}^L|_{\mathcal{U}_{\beta_1}}$ . The  $K_1$ -transversality condition imposes that  $\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda} \theta^{\beta_1}(\lambda) \otimes V_{\lambda}^{K_1}$  with

$$\theta^{\beta_1}(\lambda) \in R(K_2), \quad \forall \lambda \in \widehat{K_1}.$$

We have the following extension of Theorem 2.9

**Theorem 2.12.** *We have  $\mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda \in \widehat{K_1}} \theta^{\beta_1}(\lambda) \otimes V_{\lambda}^{K_1}$  where  $\theta^{\beta_1}(\lambda) \neq 0$  only if  $\|\lambda\| \geq \|\beta_1\|$ .*

*Proof.* The proof works exactly like the one of Theorem 2.9.  $\square$

Let us explain the “quantization commutes with reduction theorem”, or why we can consider the geometric quantization of

$$N//_0K_1 := \Phi_1^{-1}(0)/K_1$$

as the  $K_1$ -invariant part of  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$ .

Let us first suppose that 0 is a regular value of  $\Phi_1$ . Then  $N//_0K_1$  is a compact symplectic orbifold equipped with a Hamiltonian action of  $K_2$ : the corresponding moment map is induced by the restriction of  $\Phi_2$  to  $\Phi_1^{-1}(0)$ . The symplectic quotient  $N//_0K_1$  is pre-quantized by the line orbibundle

$$L_0 := \left( L|_{\Phi_1^{-1}(0)} \right) / K_1.$$

Definition 1.1 extends to the orbifold case. We can still define the geometric quantization of  $N//_0K_1$  as the index of an elliptic operator: we denote it by  $\mathcal{Q}_{K_2}(N//_0K_1) \in R(K_2)$ . We have

**Theorem 2.13.** *If 0 is a regular value of  $\Phi_1$ , the  $K_1$ -invariant part of  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$  is equal to  $\mathcal{Q}_{K_2}(N//_0K_1) \in R(K_2)$ .*

Suppose now that 0 is not a regular value of  $\Phi_1$ . Let  $T_1$  be a maximal torus of  $K_1$ , and let  $C_1 \subset \mathfrak{t}_1^*$  be a Weyl chamber. Since  $\Phi_1$  is proper, the convexity Theorem says that the image of  $\Phi_1$  intersects  $C_1$  in a closed locally polyhedral convex set, that we denote  $\Delta_{K_1}(N)$  [14].

We consider an element  $a \in \Delta_{K_1}(N)$  which is generic and sufficiently close to  $0 \in \Delta_{K_1}(N)$  : we denote  $(K_1)_a$  the subgroup of  $K_1$  which stabilizes  $a$ . When  $a \in \Delta_{K_1}(N)$  is generic, one can show (see [18]) that

$$N//_a K_1 := \Phi_{K_1}^{-1}(a)/(K_1)_a$$

is a compact Hamiltonian  $K_2$ -orbifold, and that

$$L_a := \left( L|_{\Phi_{K_1}^{-1}(a)} \right) / (K_1)_a.$$

is a  $K_2$ -equivariant line orbibundle over  $N//_a K_1$  : we can then define, like in Definition 1.1, the element  $\mathcal{Q}_{K_2}(N//_a K_1) \in R(K_2)$  as the equivariant index of the Dolbeault-Dirac operator on  $N//_a K_1$  (with coefficients in  $L_a$ ).

**Theorem 2.14.** *The  $K_1$ -invariant part of  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)$  is equal to  $\mathcal{Q}_{K_2}(N//_a K_1) \in R(K_2)$ . In particular, the elements  $\mathcal{Q}_{K_2}(N//_a K_1)$  do not depend on the choice of the generic element  $a \in \Delta_H(N)$ , when  $a$  is sufficiently close to 0.*

PROOFS OF THEOREM 2.13 AND THEOREM 2.14 . When  $N$  is compact and  $K_2 = \{e\}$ , the proofs can be found in [18] and in [19]. Let us explain briefly how the  $\mathbf{K}$ -theoretic proof of [19] extends naturally to our case. Like in Definition 2.10, we have the following decomposition

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N) = \sum_{\beta \in \mathcal{B}_1} \mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N),$$

And Theorem 2.12 tells us that  $\left[ \mathcal{Q}_{K_1 \times K_2}^{\beta_1}(N) \right]^{K_1} = 0$  if  $\beta_1 \neq 0$ . We have proved the first step:

$$\left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N) \right]^{K_1} = \left[ \mathcal{Q}_{K_1 \times K_2}^0(N) \right]^{K_1}.$$

The analysis of the term  $\left[ \mathcal{Q}_{K_1 \times K_2}^0(N) \right]^{K_1}$  is undertaken in [19] when  $K_2 = \{e\}$ : we explain that this term is equal either to  $\mathcal{Q}(N//_0 K_1)$  when 0 is a regular value, or to  $\mathcal{Q}(N//_a K_1)$  with  $a$  generic. It work similarly with an action of a compact Lie group  $K_2$ .  $\square$

**Definition 2.15.** *The geometric quantization of  $N//_0 K_1 := \Phi_1^{-1}(0)/K_1$  is taken as the  $K_1$ -invariant part of  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(N)$ . We denote it  $\mathcal{Q}_{K_2}(N//_0 K_1)$ .*

**2.4. Quantization of points.** Let  $(M, \Omega, \Phi)$  be a proper Hamiltonian  $K$ -manifold prequantized by a Kostant-Souriau line bundle  $L$ . Let  $\mu \in \widehat{K}$  be dominant weight such that  $\Phi^{-1}(K \cdot \mu)$  is a  $K$ -orbit in  $M$ . Let  $m^\circ \in \Phi^{-1}(\mu)$  so that

$$\Phi^{-1}(K \cdot \mu) = K \cdot m^\circ$$

Then the reduced space  $M_\mu := \Phi^{-1}(K \cdot \mu)/K$  is a point. The aim of this section is to compute the quantization of  $M_\mu$ :  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ .

Let  $H$  be the stabilizer subgroup of  $m^\circ$ . We have a linear action of  $H$  on the 1-dimensional vector space  $L_{m^\circ} \subset L$ . We have  $H \subset K_\mu$  where  $K_\mu$  is the connected

subgroup of  $K$  that fixes  $\mu \in \mathfrak{k}^*$ . Let  $\mathbb{C}_{-\mu}$  be the 1-dimensional representation of  $K_\mu$  associated to the infinitesimal character  $-i\mu$ .

Let us denote  $\chi$  be the character of  $H$  defined by the 1-dimensional representation  $\mathbb{C}_\chi := L_{m^\circ} \otimes \mathbb{C}_{-\mu}$ . We know from the Kostant formula (1.1) that  $\chi = 1$  on the identity component  $H^\circ \subset H$ .

**Theorem 2.16.** *We have*

$$(2.22) \quad \mathcal{Q}(M_\mu) = \begin{cases} 1 & \text{if } \chi = 1 \text{ on } H \\ 0 & \text{in the other case.} \end{cases}$$

This Theorem tells us in particular that  $\mathcal{Q}(M_\mu) = 1$  when the stabiliser subgroup  $H \subset K$  of a point  $m^\circ \in \Phi^{-1}(\mu)$  is *connected*.

*Proof.* Let  $N = M \times \overline{K \cdot \mu}$  be the proper Hamiltonian  $K$ -manifold which is pre-quantized by the line bundle  $L_N := L \otimes [\mathbb{C}_{-\mu}]$ . Let us denote  $\Phi_N$  the moment map on  $N$ . Since  $\Phi^{-1}(K \cdot \mu)$  is a  $K$ -orbit in  $M$ , we see that  $\Phi_N^{-1}(0)$  is the  $K$ -orbit through  $n^\circ := (m^\circ, \mu)$  where  $m^\circ \in \Phi^{-1}(\mu)$ . Note that  $H$  is the stabilizer subgroups of  $n^\circ$ .

Let  $\mathcal{Q}_K^{\Phi_N}(N) \in R^{-\infty}(K)$  be the formal quantization of  $N$  through the proper map  $\Phi_N$ . By definition

$$\begin{aligned} \mathcal{Q}(M_\mu) &= \left[ \mathcal{Q}_K^{\Phi_N}(N) \right]^K \\ &= \left[ \mathcal{Q}_K^0(N) \right]^K. \end{aligned}$$

where  $\mathcal{Q}_K^0(N)$  depends only of a neighborhood of  $\Phi_N^{-1}(0)$ .

The orbit  $K \cdot n^\circ \hookrightarrow N$  is an isotropic embedding since it is the 0-level of the moment map  $\Phi_N$ . Then to describe a  $K$ -invariant neighborhood of  $K \cdot n^\circ$  in  $N$  we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First we consider, following Weinstein (see [11, 31]), the symplectic normal bundle

$$(2.23) \quad \mathcal{V} := \mathrm{T}(K \cdot n^\circ)^{\perp, \Omega} / \mathrm{T}(K \cdot n^\circ),$$

where the orthogonal  $(\perp, \Omega)$  is taken relatively to the symplectic 2-form. We have

$$\mathcal{V} = K \times_H V$$

where the vector space  $V := \mathrm{T}_{n^\circ}(K \cdot n^\circ)^{\perp, \Omega} / \mathrm{T}_{n^\circ}(K \cdot n^\circ)$  inherits a symplectic structure and an Hamiltonian action of the group  $H$ : we denote  $\Phi_H : V \rightarrow \mathfrak{h}^*$  the corresponding moment map.

Consider now the following symplectic manifold

$$(2.24) \quad \tilde{N} := \mathcal{V} \oplus \mathrm{T}^*(K/H) = K \times_H \left( (\mathfrak{k}/\mathfrak{h})^* \oplus V \right).$$

The action of  $H$  on  $\tilde{N}$  is Hamiltonian and the moment map  $\Phi_{\tilde{N}} : \tilde{N} \rightarrow \mathfrak{k}^*$  is given by the equation

$$(2.25) \quad \Phi_{\tilde{N}}([k; \xi, v]) = k \cdot (\xi + \Phi_H(v)) \quad k \in K, \xi \in (\mathfrak{k}/\mathfrak{h})^*, v \in V.$$

The Hamiltonian  $K$ -manifold  $\tilde{N}$  is prequantized by the line bundle  $L_{\tilde{N}} := K \times_H \mathbb{C}_\chi$ .

The *local normal form* Theorem (see [10], [24] Proposition 2.5 ) tells us that there exists a  $K$ -Hamiltonian isomorphism  $\Upsilon : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$  between a  $K$ -invariant neighborhood  $\mathcal{U}_1$  of  $K \cdot n^o$  in  $N$ , and a  $K$ -invariant neighborhood  $\mathcal{U}_2$  of  $K/H$  in  $\tilde{N}$ . This isomorphism  $\Upsilon$ , when restricted to  $K \cdot n^o$ , corresponds to the natural isomorphism  $K \cdot n^o \xrightarrow{\sim} K/H$ .

Thanks to  $\Upsilon$ , we know that the fiber  $\Phi_H^{-1}(0) \subset V$  is reduced to  $\{0\}$ . This last point is equivalent to the fact that  $\Phi_H$  (and then  $\Phi_{\tilde{N}}$ ) is proper map (see [21]). We check easily that the set of critical points of  $\|\Phi_{\tilde{N}}\|^2$  is reduced to  $\Phi_{\tilde{N}}^{-1}(0) = K/H$ . Then, thank to the isomorphism  $\Upsilon$ , we have that

$$(2.26) \quad \mathcal{Q}_K^0(N) = \mathcal{Q}_K^0(\tilde{N}) = \mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}).$$

Let  $\text{Ind}_H^K : R^{-\infty}(H) \rightarrow R^{-\infty}(K)$  be the induction map that is defined by the relation  $\langle \text{Ind}_H^K(\phi), E \rangle = \langle \phi, E|_H \rangle$  for any  $\phi \in R^{-\infty}(H)$  and  $E \in R(K)$ . Note that

$$[\text{Ind}_H^K(\phi)]^K = \langle \text{Ind}_H^K(\phi), \mathbb{C} \rangle = \langle \phi, \mathbb{C} \rangle = [\phi]^H.$$

Since  $\Phi_H : V \rightarrow \mathfrak{h}^*$  is proper one can consider the quantization of the vector space  $V$  through the map  $\Phi_H$ :  $\mathcal{Q}_H^{\Phi_H}(V) \in R^{-\infty}(H)$ .

**Proposition 2.17.** • *We have*

$$(2.27) \quad \mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}) = \text{Ind}_H^K \left( \mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi \right)$$

• *The formal quantization  $\mathcal{Q}_H^{\Phi_H}(V)$  coincides, as a generalized  $H$ -module, to the  $H$ -module  $S(V^*)$  of polynomial function on  $V$ .*

• *The set  $[S(V^*)]^{H^o}$  of polynomials invariant by the connected component  $H^o$  is reduced to the scalars.*

With the last Proposition we can finish the proof of Theorem 2.16 as follows. We have

$$\begin{aligned} \mathcal{Q}(M_\mu) &= [\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N})]^K \\ &= [\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N})]^K \\ &= [\mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi]^H \\ &= [S(V^*) \otimes \mathbb{C}_\chi]^H = [\mathbb{C}_\chi]^H. \end{aligned}$$

*Proof.* The first point of Proposition 2.17 follows from the property of induction defined by Atiyah (see Section 3.4 in [19]). Let us explain the arguments. We work with the  $H$ -manifold  $\mathcal{Y} = (\mathfrak{k}/\mathfrak{h})^* \oplus V$  and the  $H$ -equivariant map  $j : \mathcal{Y} \hookrightarrow \tilde{N} := K \times_H \mathcal{Y}, y \mapsto [e, y]$ .

We notice<sup>5</sup> that  $T\tilde{N} \simeq K \times_H (\mathfrak{k}/\mathfrak{h} \oplus T\mathcal{Y})$ , and that  $T_K \tilde{N} \simeq K \times_H (T_H \mathcal{Y})$ . Hence the map  $j$  induces an isomorphism  $j_* : K_H(T_H \mathcal{Y}) \rightarrow K_K(T_K \tilde{N})$ . Theorem 4.1 of

<sup>5</sup> These identities come from the following  $K \times H$ -equivariant isomorphism of vector bundles over  $K \times \mathcal{Y}$ :  $T_H(\tilde{N}) \rightarrow K \times (\mathfrak{k}/\mathfrak{h} \oplus T\mathcal{Y}), (k, m; \frac{d}{dt}|_{t=0}(ke^{tX}) + v_m) \mapsto (k, m; pr_{\mathfrak{k}/\mathfrak{h}}(X) + v_m)$ . Here  $pr_{\mathfrak{k}/\mathfrak{h}} : \mathfrak{k} \rightarrow \mathfrak{k}/\mathfrak{h}$  is the orthogonal projection.

Atiyah [1] tells us that the following diagram

$$(2.28) \quad \begin{array}{ccc} K_H(\mathbb{T}_H \mathcal{Y}) & \xrightarrow{j_*} & K_K(\mathbb{T}_K \tilde{N}) \\ \text{Index}_y^H \downarrow & & \downarrow \text{Index}_N^K \\ R^{-\infty}(H) & \xrightarrow{\text{Ind}_H^K} & R^{-\infty}(K) . \end{array}$$

is commutative.

The tangent bundle  $\mathbb{T}\tilde{N}$  is equivariantly diffeomorphic to

$$K \times_H [\mathfrak{k}/\mathfrak{h} \oplus (\mathfrak{k}/\mathfrak{h})^* \oplus \text{TV}] \simeq K \times_H [(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus \text{TV}]$$

where  $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$  is the complexification of the real vector space  $\mathfrak{k}/\mathfrak{h}$ . We consider on  $\tilde{N}$  the almost complex structure  $J_{\tilde{N}} = (i, J_V)$  where  $i$  is the complex structure on  $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$  and  $J_V$  is a compatible (constant) complex structure on the symplectic vector space  $V$ . Note that  $J_{\tilde{N}}$  is compatible with the symplectic structure on a neighborhood  $U$  of the 0-section of the bundle  $\tilde{N} \rightarrow K/H$ .

Let  $\kappa_{\tilde{N}}$  be the Kirwan vector field on  $\tilde{N}$ :

$$\kappa_{\tilde{N}}([k; \xi, v]) = -\xi + i[\xi, \Phi_H(v)] \oplus \kappa_V(v) \in (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus V.$$

Here  $\kappa_V$  is the Kirwan vector field relative to the Hamiltonian action of  $H$  on the symplectic vector space  $V$ . Note that  $\kappa_{\tilde{N}}$  vanishes exactly on the 0-section of the bundle  $\tilde{N} \rightarrow K/H$ .

Let  $\mathbf{c}^{\kappa_{\tilde{N}}}$  be the symbol  $\text{Thom}(\tilde{N}, J_{\tilde{N}}) \otimes L_{\tilde{N}}$  pushed by the vector field  $\kappa_{\tilde{N}}$ . The generalized character  $\mathcal{Q}_{K/\tilde{N}}^{\Phi_{\tilde{N}}}(\tilde{N})$  is either computed as the equivariant index of the symbols  $\mathbf{c}^{\kappa_{\tilde{N}}}$  or  $\mathbf{c}^{\kappa_{\tilde{N}}}|_U$ .

**Remark 2.18.** *The fact that  $J_{\tilde{N}}$  is not compatible on the entire manifold  $\tilde{N}$  is not problematic, since  $J_{\tilde{N}}$  is compatible in a neighborhood  $U$  of the set where  $\kappa_{\tilde{N}}$  vanishes. See the first point of Lemma 2.6.*

For  $X + i\eta \oplus w \in \mathbb{T}_{[k; \xi, v]}\tilde{N} \simeq (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus V$ , the map

$$(2.29) \quad \mathbf{c}^{\kappa_{\tilde{N}}}(X + i\eta \oplus w) = \mathbf{c}\left(X + \xi + i(\eta - [\xi, \Phi_H(v)])\right) \odot \mathbf{c}\left(w - \kappa_V(v)\right)$$

acts on the vector space  $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \otimes \wedge_{J_V} V \otimes \mathbb{C}_{\mathcal{X}}$ .

Let  $\text{Bott}(\mathfrak{k}/\mathfrak{h})$  be the Bott morphism of the vector space  $\mathfrak{k}/\mathfrak{h}$ . It is an elliptic morphism defined by

$$\text{Bott}(\mathfrak{k}/\mathfrak{h})|_{\xi}(\eta) = \mathbf{c}(\xi + i\eta) \quad \text{acting on} \quad \wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}},$$

for  $\eta \in \mathbb{T}_{\xi}(\mathfrak{k}/\mathfrak{h})$ . Let  $\mathbf{c}^{\kappa_V}$  be the symbol  $\text{Thom}(V, J_V)$  pushed by the vector field  $\kappa_V$ .

**Lemma 2.19.** *We have*

$$\mathbf{c}^{\kappa_{\tilde{N}}} = j_*\left(\text{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_{\mathcal{X}}\right).$$

*Proof.* We work with the symbol

$$\sigma^T|_{(\xi, v)}(\eta) = \mathbf{c}(\xi + i\eta - iT[\xi, \Phi_H(v)])$$

acting on  $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$ . Note that  $\text{Bott}(\mathfrak{k}/\mathfrak{h}) = \sigma^0$ . From (2.29), we see that  $\mathbf{c}^{\kappa_{\tilde{N}}} = j_*\left(\sigma^1 \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_{\mathcal{X}}\right)$ . It is now easy to check that  $\sigma^T \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_{\mathcal{X}}$ ,  $T \in [0, 1]$  is an homotopy of transversally elliptic symbols on  $\mathfrak{k}/\mathfrak{h} \times V$ .  $\square$



The commutative diagram (2.28) and the last Lemma gives

$$\begin{aligned}
\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}) &= \text{Index}_{\tilde{N}}^K(\mathbf{c}^{\kappa_{\tilde{N}}}) \\
&= \text{Ind}_H^K \left( \text{Index}_{\mathfrak{k}/\mathfrak{h} \times V}^K \left( \text{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \right) \otimes \mathbb{C}_\chi \right) \\
&= \text{Ind}_H^K \left( \text{Index}_{\mathfrak{k}/\mathfrak{h}}^K(\text{Bott}(\mathfrak{k}/\mathfrak{h})) \otimes \text{Index}_V^K(\mathbf{c}^{\kappa_V}) \otimes \mathbb{C}_\chi \right) \\
&= \text{Ind}_H^K \left( \mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi \right).
\end{aligned}$$

We have used here that the equivariant index of  $\text{Bott}(\mathfrak{k}/\mathfrak{h})$  is equal to 1 (e.g. the trivial representation).

Let us prove now the second point of Proposition 2.17. The Kirwan vector field  $\kappa^V$  satisfies the simple rule:

$$(2.30) \quad (\kappa^V(v), J_V v) = -\Omega(\kappa^V(v), v) = \frac{1}{2} \|\Phi_H(v)\|^2, \quad v \in V.$$

It shows in particular that  $\kappa^V(v) = 0 \Leftrightarrow \Phi_H(v) = 0$ . Since the moment map  $\Phi_H : V \rightarrow \mathfrak{h}^*$  is quadratic, the fact that  $\Phi_H$  is proper is equivalent to the fact that  $\Phi_H^{-1}(0) = 0$ .

We consider on  $V$  the family of symbol  $\sigma^s$  :

$$\sigma^s|_v(w) = \mathbf{c} \left( w - s\kappa^V(v) - (1-s)J_V v \right)$$

viewed as a map from  $\Lambda_{\mathbb{C}}^{\text{even}} V$  to  $\Lambda_{\mathbb{C}}^{\text{odd}} V$ . Thanks to (2.30), one sees that  $\sigma^s$  is a family of  $K$ -transversally elliptic symbol on  $V$ . Hence  $\sigma^1 = \mathbf{c}^{\kappa^V}$  and  $\sigma^0 = \mathbf{c}(w - J_V v)$  defines the same class in the group  $K_K(\mathbb{T}_K V)$ . The symbol  $\sigma^0$  was first studied by Atiyah [1] when  $\dim_{\mathbb{C}} V = 1$ . The author considered the general case in [19]. We have

$$\text{Index}_V^K(\sigma^0) = S(V^*) \quad \text{in } R^{-\infty}(K).$$

The last point of Proposition 2.17 is a consequence of the properness of the moment map  $\Phi_H$  (see Section 5 of [21]).

□

□

**Example 2.20** ([21]). *We consider the action of the unitary group  $U_n$  on  $\mathbb{C}^n$ . The symplectic form on  $\mathbb{C}^n$  is defined by  $\Omega(v, w) = \frac{i}{2} \sum_k v_k \bar{w}_k - \bar{v}_k w_k$ . Let us identify the Lie algebra  $\mathfrak{u}_n$  with its dual through the trace map. The moment map  $\Phi : \mathbb{C}^n \rightarrow \mathfrak{u}_n$  is defined by  $\Phi(v) = \frac{1}{2i} v \otimes v^*$  where  $v \otimes v^* : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is the linear map  $w \mapsto (\sum_k \bar{v}_k w_k) v$ . One checks easily that the pull-back by  $\Phi$  of a  $U_n$ -orbit in  $\mathfrak{u}_n$  is either empty or a  $U_n$ -orbit in  $\mathbb{C}^n$ . We know also that the stabiliser subgroup of a non-zero vector of  $\mathbb{C}^n$  is connected since it is diffeomorphic to  $U_{n-1}$ . Finally we have*

$$(2.31) \quad \mathcal{Q}((\mathbb{C}^n)_\mu) = \begin{cases} 1 & \text{if } \mu \in \widehat{U}_n \text{ belongs to the image of } \Phi \\ 0 & \text{if } \mu \in \widehat{U}_n \text{ does not belong to the image of } \Phi. \end{cases}$$

*Then one checks that  $\mathcal{Q}_{U_n}^{-\infty}(\mathbb{C}^n)$  coincides in  $R^{-\infty}(U_n)$  with the algebra  $S((\mathbb{C}^n)^*)$  of polynomial function on  $\mathbb{C}^n$ .*

**Example 2.21** ([23]). We consider the Lie group  $\mathrm{SL}_2(\mathbb{R})$  and its compact torus of dimension 1 denoted by  $T$ . The Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$  is identified with its dual through the trace map, and the Lie algebra  $\mathfrak{t}$  is naturally identified with  $\mathfrak{sl}_2(\mathbb{R})^T$ . For  $l \in \mathbb{Z} \setminus \{0\}$ , we consider the character  $\chi_l$  of  $T$  defined by

$$\chi_l \left( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right) = e^{il\theta}.$$

Its differential  $\frac{1}{2}d\chi_l \in \mathfrak{t}^*$  correspond (through the trace map) to the matrix

$$X_l = \begin{pmatrix} 0 & l/2 \\ -l/2 & 0 \end{pmatrix}.$$

Let  $\mathcal{O}_l$  be the coadjoint orbit of the group  $\mathrm{SL}_2(\mathbb{R})$  through the matrix  $X_l$ . It is a Hamiltonian  $\mathrm{SL}_2(\mathbb{R})$ -manifold prequantized by the  $\mathrm{SL}_2(\mathbb{R})$ -equivariant line bundle  $L_l \simeq \mathrm{SL}_2(\mathbb{R}) \times_T \mathbb{C}_l$ , where  $\mathbb{C}_l$  is the  $T$ -module associated to the character  $\chi_l$ . We look at the Hamiltonian action of  $T$  on  $\mathcal{O}_l$ . Let  $\Phi_T : \mathcal{O}_l \rightarrow \mathfrak{t}^*$  be the corresponding moment map. One checks that the moment map  $\Phi_T$  is proper and that its image is equal to the half-line  $\{aX_l, a \geq 1\} \subset \mathfrak{t}^*$ .

We check that for each  $\xi \in \{aX_l, a \geq 1\}$  the fiber  $\Phi_T^{-1}(\xi)$  is equal to a  $T$ -orbit in  $\mathcal{O}_l$ . For  $k \in \mathbb{Z}$ , let us denote  $(\mathcal{O}_l)_k$  the symplectic reduction of  $\mathcal{O}_l$  at the level  $X_k$ . We know that  $(\mathcal{O}_l)_k = \emptyset$  if  $k \notin \{al, a \geq 1\}$ , and that  $(\mathcal{O}_l)_k$  is a point if  $k \in \{al, a \geq 1\}$ .

In order to compute  $\mathcal{Q}((\mathcal{O}_l)_k)$  we look at the stabilizer subgroup  $T_m := \{t \in T \mid t \cdot m = m\}$  for each point  $m \in \mathcal{O}_l$ . One sees that  $T_m = T$  if  $m = X_l$  and  $T_m$  is equal to the center  $\{\pm Id\}$  of  $\mathrm{SL}_2(\mathbb{R})$ , when  $m \neq X_l$ .

Theorem 2.16 gives in this setting that, for  $k \in \{al, a \geq 1\}$ ,

$$(2.32) \quad \mathcal{Q}((\mathcal{O}_l)_k) = \begin{cases} 1 & \text{if } l - k \text{ is even} \\ 0 & \text{if } l - k \text{ is odd.} \end{cases}$$

Hence the formal geometric quantization of the proper  $T$ -manifold  $\mathcal{O}_l$  is

$$(2.33) \quad \mathcal{Q}_T^{-\infty}(\mathcal{O}_l) = \begin{cases} \mathbb{C}_l \cdot \sum_{p \geq 0} \mathbb{C}_{2p} & \text{if } l > 0 \\ \mathbb{C}_l \cdot \sum_{p \geq 0} \mathbb{C}_{-2p} & \text{if } l < 0. \end{cases}$$

Here we recognize that  $\mathcal{Q}_T^{-\infty}(\mathcal{O}_l)$  coincides with the restriction of the holomorphic (resp. anti-holomorphic) discrete series representation  $\Theta_l$  to the group  $T$  when  $l > 0$  (resp.  $l < 0$ ).

**2.5. Wonderful compactifications and symplectic cuts.** Another equivalent definition of the quantization  $\mathcal{Q}^{-\infty}$  uses a generalisation of the technique of symplectic cutting (originally due to Lerman [13]) that was introduced in [21] and was motivated by the wonderful compactifications of De Concini and Procesi. Let us recall the method.

We recall that  $T$  is a maximal torus in the compact connected Lie group  $K$ , and  $W$  is the Weyl group. We define a  $K$ -adapted polytope in  $\mathfrak{t}^*$  to be a  $W$ -invariant Delzant polytope  $P$  in  $\mathfrak{t}^*$  whose vertices are regular elements of the weight lattice  $\Lambda^*$ . If  $\{\lambda_1, \dots, \lambda_N\}$  are the dominant weights lying in the union of all the closed one-dimensional faces of  $P$ , then there is a  $G \times G$ -equivariant embedding of  $G = K_{\mathbb{C}}$  into

$$\mathbb{P} \left( \bigoplus_{i=1}^N V_{\lambda_i}^* \otimes V_{\lambda_i} \right)$$

associating to  $g \in G$  its representation on  $\bigoplus_{i=1}^N V_{\lambda_i}$ . The closure  $\mathcal{X}_P$  of the image of  $G$  in this projective space is smooth and is equipped with a  $K \times K$  that we denote:

$$(k_1, k_2) \cdot x = k_2 \cdot x \cdot k_1^{-1}.$$

Let  $\Omega_{\mathcal{X}_P}$  be the symplectic 2-form on  $\mathcal{X}_P$  which given by the Kahler structure. We recall briefly the different properties of  $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$ : all the details can be found in [21].

- (1)  $\mathcal{X}_P$  is equipped with an Hamiltonian action of  $K \times K$ . Let  $\Phi = (\Phi_l, \Phi_r) : M \rightarrow \mathfrak{k}^* \times \mathfrak{k}^*$  be the corresponding moment map.
- (2) The image of  $\Phi := (\Phi_l, \Phi_r)$  is equal to  $\{(k \cdot \xi, -k' \cdot \xi) \mid \xi \in P \text{ and } k, k' \in K\}$ .
- (3) The Hamiltonian manifold  $(\mathcal{X}_P, K \times K)$  has no multiplicities: the pull-back by  $\Phi$  of a  $K \times K$ -orbit in the image is a  $K \times K$ -orbit in  $\mathcal{X}_P$ .

Let  $\mathcal{U}_P := K \cdot P^\circ$  where  $P^\circ$  is the interior of  $P$ . We define

$$\mathcal{X}_P^\circ := \Phi_l^{-1}(\mathcal{U}_P)$$

which is an invariant, open and dense subset of  $\mathcal{X}_P$ . We have the following important property concerning  $\mathcal{X}_P^\circ$ .

- (4) There exists an equivariant diffeomorphism  $\Upsilon : K \times \mathcal{U}_P \rightarrow \mathcal{X}_P^\circ$  such that  $\Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi$  and  $\Upsilon^*(\Phi_r)(k, \xi) = -\xi$ .
- (5) This diffeomorphism  $\Upsilon$  is a quasi-symplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on the open subset  $K \times \mathcal{U}_P$  of the cotangent bundle  $T^*K$  to the pullback of the symplectic form  $\Omega_{\mathcal{X}_P}$  on  $\mathcal{X}_P^\circ$ .
- (6) The symplectic manifold  $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$  is prequantized by the restriction of the hyperplane line bundle  $\mathcal{O}(1) \rightarrow \mathbb{P}(\bigoplus_{i=1}^N V_{\lambda_i}^* \otimes V_{\lambda_i})$  to  $\mathcal{X}_P$ : let us denoted  $L_P$  the corresponding  $K \times K$ -equivariant line bundle.
- (7) The pull-back of the line bundle  $L_P$  by the map  $\Upsilon : K \times \mathcal{U}_P \hookrightarrow \mathcal{X}_P$  is trivial.

Let  $(M, \Omega_M, \Phi_M)$  be a proper Hamiltonian  $K$ -manifold. We also consider the Hamiltonian  $K \times K$ -manifold  $\mathcal{X}_P$  associated to a  $K$ -adapted polytope  $P$ . We consider now the product  $M \times \mathcal{X}_P$  with the following  $K \times K$  action:

- the action  $k \cdot_1 (m, x) = (k \cdot m, x \cdot k^{-1})$ : the corresponding moment map is  $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$ ,
- the action  $k \cdot_2 (m, x) = (m, k \cdot x)$ : the corresponding moment map is  $\Phi_2(m, x) = \Phi_l(x)$ .

**Definition 2.22.** We denote  $M_P$  the symplectic reduction at 0 of  $M \times \mathcal{X}_P$  for the action  $\cdot_1 : M_P := (\Phi_1)^{-1}(0)/(K, \cdot_1)$ .

Then  $M_P$  inherits a Hamiltonian  $K$ -action with moment map  $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$  whose image is  $\Phi(M) \cap K \cdot P$ .

One checks that  $M_P$  contains an open and dense subset of smooth points which quasi-symplectomorphic to the open subset  $(\Phi_M)^{-1}(\mathcal{U}_P)$ . If the polytope  $P$  is fixed, we can work with the dilated polytopes  $nP$  for  $n \geq 1$ . We have then the family of compact, perhaps singular,  $K$ -hamiltonian manifolds  $M_{nP}$ ,  $n \geq 1$ : in Section 2.3, we have explained how was defined their geometric quantization  $\mathcal{Q}_K(M_{nP}) \in R(K)$ .

We have a convenient definition for  $\mathcal{Q}^{-\infty}$ .

**Proposition 2.23** ([21]). *We have the following equality in  $R^{-\infty}(K)$ :*

$$(2.34) \quad \mathcal{Q}_K^{-\infty}(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP}).$$

### 3. PROOF OF THEOREM 1.4

The main result of this section is

**Theorem 3.1.** *Let  $r_P := \inf_{\xi \in \partial P} \|\xi\|$ . The generalized character*

$$\mathcal{Q}_K^\Phi(M) - \mathcal{Q}_K(M_P) \in R^{-\infty}(K)$$

*is supported outside the ball  $B_{r_P}$ .*

Then, for the dilated polytope  $nP, n \geq 1$ , the character  $\mathcal{Q}_K^\Phi(M) - \mathcal{Q}^K(M_{nP})$  is supported outside the ball  $B_{nr_P}$ . Taking the limit when  $n$  goes to infinity gives

$$(3.35) \quad \mathcal{Q}_K^\Phi(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP}).$$

Finally, the identity of Theorem 1.4,

$$\mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^{-\infty}(M),$$

is a direct consequence of (2.34) and (3.35).

Recall that  $O(r) \in R^{-\infty}(K)$  denoted any generalized character supported outside the ball  $B_r$ .

Theorem 3.1 follows from the comparison of three different geometrical situations. All of them concern Hamiltonian actions of  $K_1 \times K_2$ , where  $K_1$  and  $K_2$  are two copies of  $K$ .

**First setting.** We work with the Hamiltonian  $K_1 \times K_2$ -manifold  $M \times \mathcal{X}_P$ : here  $K_1$  acts both on  $M$  and on  $\mathcal{X}_P$ . Since the moment map  $\Phi_1$  (relative to the  $K_1$ -action) is proper we may “quantize”  $M \times \mathcal{X}_P$  via the map  $\|\Phi_1\|^2$ : let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character. Recall that  $\mathcal{Q}_{K_2}(M_P)$  is equal to  $[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P)]^{K_1}$ .

**Second setting.** We consider the same setting than before: the Hamiltonian action of  $K_1 \times K_2$  on  $M \times \mathcal{X}_P$ . But we “quantize”  $M \times \mathcal{X}_P$  through the global moment map  $\Phi = (\Phi_1, \Phi_2)$ . Here we have some liberty in the choice of the scalar product on  $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ . If  $\|\xi\|^2$  is an invariant Euclidean norm on  $\mathfrak{k}^*$ , we take on  $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$  the Euclidean norm

$$(3.36) \quad \|(\xi_1, \xi_2)\|_\rho^2 = \|\xi_1\|^2 + \rho\|\xi_2\|^2$$

depending on a parameter  $\rho > 0$ . Let us consider the quantization of  $M \times \mathcal{X}_P$  via the map  $\|\Phi\|_\rho^2$ :

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2).$$

**Third setting.** We consider the cotangent bundle  $T^*K$  with the Hamiltonian action of  $K_1 \times K_2$ :  $K_1$  acts by *right* translations, and  $K_2$  by *left* translations. We consider the Hamiltonian action of  $K_1 \times K_2$  on  $M \times T^*K$ : here  $K_1$  acts both on

$M$  and on  $T^*K$ . Let  $\Phi = (\Phi_1, \Phi_2)$  be the global moment map on  $M \times T^*K$ . Since the moment map  $\Phi$  is proper we can “quantize”  $M \times T^*K$  via the map  $\|\Phi\|_\rho^2$ : let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 3.1 is a consequence of the following propositions.

First we compare  $\mathcal{Q}_{K_2}^{\Phi}(M)$  with the  $K_1$ -invariant part of  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ .

**Proposition 3.2.** *For any  $\rho \in ]0, 1]$ , we have*

$$(3.37) \quad \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = \mathcal{Q}_{K_2}^{\Phi}(M) \quad \text{in } R^{-\infty}(K_2).$$

Then we compare the  $K_1$ -invariant part of the generalized characters  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$  and  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$ .

**Proposition 3.3.** *For any  $\rho \in ]0, 1]$ , we have the following relation in  $R^{-\infty}(K_2)$*

$$(3.38) \quad \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} - \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = O(r_P)$$

Finally we compare the  $K_1$ -invariant part of the generalized characters  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$  and  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P)$ .

**Proposition 3.4.** *There exists  $\epsilon > 0$  such that*

$$(3.39) \quad \mathcal{Q}_{K_2}(M_P) - \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} = O((\epsilon/\rho)^{1/2}) \quad \text{in } R^{-\infty}(K_2)$$

if  $\rho > 0$  is small enough.

If we sum the relations (3.37), (3.38) and (3.39) we get

$$\mathcal{Q}_{K_2}^{\Phi}(M) = \mathcal{Q}_{K_2}(M_P) + O(r_P) + O((\epsilon/\rho)^{1/2})$$

if  $\rho$  is small enough. So Theorem 3.1 follows by taking  $(\epsilon/\rho)^{1/2} \geq r_P$ .

**3.1. Proof of Proposition 3.2.** The cotangent bundle  $T^*K$  is identified with  $K \times \mathfrak{k}^*$ . The data is then (see Section 5.1):

- the Liouville 1-form  $\lambda = \sum_j \omega_j \otimes E_j$ . Here  $(E_j)$  is a basis of  $\mathfrak{k}$  with dual basis  $(E_j^*)$ , and  $\omega_j$  is the left invariant 1-form on  $K$  defined by  $\omega_j(\frac{d}{dt}a e^{tX}|_0) = \langle E_j^*, X \rangle$ .
- the symplectic form  $\Omega := -d\lambda$ ,
- the action of  $K_1 \times K_2$  on  $K \times \mathfrak{k}^*$  is  $(k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi)$ ,
- the moment map relative to the  $K_1$ -action is  $\Phi_r(a, \xi) = -\xi$ ,
- the moment map relative to the  $K_2$ -action is  $\Phi_l(a, \xi) = a \cdot \xi$ .

We work now with the Hamiltonian action of  $K_1 \times K_2$  on  $M \times T^*K$  given by

$$(k_1, k_2) \cdot (m, a, \xi) = (k_1 \cdot m, k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The corresponding moment map is  $\Phi = (\Phi_1, \Phi_2)$ :  $\Phi_1(m, a, \xi) = \Phi_M(m) - \xi$  and  $\Phi_2(m, a, \xi) = a \cdot \xi$ .

Let  $\mathfrak{c}_1$  be a symbol  $\text{Thom}(M, J_1) \otimes L$  attached to the prequantized Hamiltonian  $K_1$ -manifold  $(M, \Omega)$ . The cotangent bundle  $T^*K$  is prequantized by the trivial line bundle: let  $\mathfrak{c}_2$  be the symbol  $\text{Thom}(T^*K, J_2)$  attached to the prequantized

Hamiltonian  $K_1 \times K_2$ -manifold  $T^*K$ . The product  $\mathbf{c} = \mathbf{c}_1 \odot \mathbf{c}_2$  corresponds to the symbol  $\text{Thom}(N, J) \otimes L$  on  $N = M \times T^*K$ .

Let  $\kappa_\rho$  be the Kirwan vector field associated to the map  $\|\Phi\|_\rho^2 : M \times T^*K \rightarrow \mathbb{R}$ . We check that  $\|\Phi\|_\rho^2(m, k, \xi) = \|\Phi_M(m) - \xi\|^2 + \rho\|\xi\|^2$ , and

$$\kappa_\rho(m, k, \xi) = \left( \underbrace{(\Phi_M(m) - \xi) \cdot m}_{\kappa_I}; \underbrace{\tilde{\Phi}_M(m) - (1 + \rho)\tilde{\xi}}_{\kappa_{II, \rho}}; \underbrace{-[\tilde{\Phi}_M(m), \tilde{\xi}]}_{\kappa_{III}} \right).$$

Here  $\mathbb{T}_{(m, k, \xi)}(M \times T^*K) \simeq \mathbb{T}_m M \times \mathfrak{k} \times \mathfrak{k}$ . We have

$$\begin{aligned} \text{Cr}(\|\Phi\|_\rho^2) &= \{\kappa_\rho = 0\} \\ &= \bigcup_{\beta \in \mathcal{B}} K_1 \times K_2 \cdot \left[ M^{\tilde{\beta}} \cap \Phi_M^{-1}(\beta) \times \{1\} \times \left\{ \frac{\beta}{\rho + 1} \right\} \right] \end{aligned}$$

where  $\mathcal{B}$  parametrizes  $\text{Cr}(\|\Phi_M\|^2)$ . Hence one checks that the critical values of  $\|\Phi\|_\rho^2$  are  $\frac{\rho}{\rho+1}\|\beta\|^2, \beta \in \mathcal{B}$ .

Let  $\mathbf{c}^{\kappa_\rho}$  be the symbol  $\mathbf{c}$  pushed by the vector field  $\kappa_\rho$ : we have

$$\mathbf{c}^{\kappa_\rho}(v; X; Y) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_2(X - \kappa_{II, \rho}; Y - \kappa_{III})$$

for  $(v; X; Y) \in \mathbb{T}_{(m, k, \xi)}(M \times T^*K) \simeq \mathbb{T}_m M \times \mathfrak{k} \times \mathfrak{k}$ .

For a real  $R > 0$  we define the open invariant subsets of  $M \times T^*K$

$$\begin{aligned} U_R &:= \{\|\Phi\|_\rho^2 < R\} \\ V_R &:= \{\|\Phi_M\|^2 < R\} \times T^*K. \end{aligned}$$

By definition the generalized index  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$  is defined as the limit of the equivariant index

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(U_R) := \text{Index}_{U_R}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{U_R}),$$

when  $R$  goes to infinity (and stays outside the critical values of  $\|\Phi\|_\rho^2$ ).

In the other hand, when  $R'$  is a regular value of  $\|\Phi_M\|^2$ , we see that the symbol  $\mathbf{c}_\rho|_{V_{R'}}$  is  $K_1 \times K_2$ -transversally elliptic. Let

$$(3.40) \quad \text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})$$

be its equivariant index. Notice that the index map is well-defined on  $V_R = \{\|\Phi_M\|^2 < R\} \times T^*K$  since  $T^*K$  can be seen as a open subset of a compact manifold.

It is easy to check that for any  $R > 0$  there exists  $R' > R$  such that  $U_R \subset V_{R'}$ . It implies that  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$  is also defined as the limit of (3.40) when  $R'$  goes to infinity.

We look now to the deformation  $\kappa_\rho(s) = (\kappa_I^s; \kappa_{II, \rho}^s; s\kappa_{III})$ ,  $s \in [0, 1]$  where

$$\kappa_I^s(m, \xi) = (\Phi_M(m) - s\xi) \cdot m \quad \text{and} \quad \kappa_{II, \rho}^s(m, \xi) = s\tilde{\Phi}_M(m) - (1 + s\rho)\tilde{\xi}.$$

Let  $\mathbf{c}^{\kappa_\rho(s)}$  be the symbol  $\mathbf{c}$  pushed by the vector field  $\kappa_\rho(s)$ .

**Lemma 3.5.** *Let  $R'$  be a regular value of  $\|\Phi_M\|^2$ .*

- *The family  $\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}$ ,  $s \in [0, 1]$  defines an homotopy of  $K_1 \times K_2$ -transversally elliptic symbols on  $V_{R'}$ .*
- *The  $K_1$ -invariant part of  $\text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}})$  is equal to  $\mathcal{Q}_{K_2}^\Phi(M_{< R'})$ .*

*Proof.* The first point follows from the fact that  $\text{Char}(\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}) \cap \mathbb{T}_{K_1 \times K_2}(V_{R'})$ , which is equal to

$$\left\{ \left( m, k, \frac{s}{1+s\rho} \Phi_M(m) \right), k \in K \text{ and } m \in \text{Cr}(\|\Phi_M\|^2) \cap \{\|\Phi_M\|^2 < R'\} \right\},$$

stays in a compact set when  $s \in [0, 1]$ .

The symbol  $\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}$  is equal to the product of the symbol  $\mathbf{c}_1^\kappa|_{M < R'}$ , which is  $K_1$ -transversally elliptic, with the symbol

$$\mathbf{c}_2^\kappa(X; Y) = \mathbf{c}_2(X + \xi; Y)$$

which is a  $K_2$ -transversally elliptic on  $\mathbb{T}^*K$ . A basic computation done in section 5.1.2 gives that

$$\begin{aligned} \text{Index}_{\mathbb{T}^*K}^{K_1 \times K_2}(\mathbf{c}_2^\kappa) &= L^2(K) \\ &= \sum_{\mu \in \widehat{K}} (V_\mu^{K_1})^* \otimes V_\mu^{K_2} \end{aligned}$$

in  $R^{-\infty}(K_1 \times K_2)$ . Finally the ‘‘multiplicative property’’ (see Theorem 2.1) gives

$$\begin{aligned} \text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}) &= \text{Index}_{M < R'}^{K_1}(\mathbf{c}_1^\kappa|_{M < R'}) \otimes \text{Index}_{\mathbb{T}^*K}^{K_1 \times K_2}(\mathbf{c}_2^\kappa) \\ &= \sum_{\mu \in \widehat{K}} \mathcal{Q}_{K_1}^\Phi(M < R') \otimes (V_\mu^{K_1})^* \otimes V_\mu^{K_2} \end{aligned}$$

Taking the  $K_1$ -invariant completes the proof of the second point.  $\square$

Finally we have proved that the generalized character  $[\text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})]^{K_1}$  is equal to  $\mathcal{Q}_{K_2}^\Phi(M < R')$ . Taking the limit  $R' \rightarrow \infty$  gives

$$\begin{aligned} \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathbb{T}^*K) \right]^{K_1} &= \lim_{R' \rightarrow \infty} \left[ \text{Index}_{V_{R'}}^{K_1 \times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}}) \right]^{K_1} \\ &= \lim_{R' \rightarrow \infty} \mathcal{Q}_{K_2}^\Phi(M < R') = \mathcal{Q}_{K_2}^\Phi(M). \end{aligned}$$

**3.2. Proof of Proposition 3.3.** We work here with the Hamiltonian action of  $K_1 \times K_2$  on  $M \times \mathcal{X}_P$ . The action is  $(k_1, k_2) \cdot (m, x) = (k_1 \cdot m, k_2 \cdot x \cdot k_1^{-1})$  and the corresponding moment map is  $\Phi = (\Phi_1, \Phi_2)$  with  $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$  and  $\Phi_2(m, x) = \Phi_l(x)$ . Let  $\|(\xi_1, \xi_2)\|_\rho^2 = \|\xi_1\|^2 + \rho\|\xi_2\|^2$  be the Euclidean norm  $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$  attached to  $\rho > 0$ .

Let us consider the quantization of  $M \times \mathcal{X}_P$  via the map  $\|\Phi\|_\rho^2$ :

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2)$$

The critical set  $\text{Cr}(\|\Phi\|_\rho^2)$  admits the decomposition

$$(3.41) \quad \text{Cr}(\|\Phi\|_\rho^2) = \bigcup_{\gamma \in \mathcal{B}_\rho} K_1 \times K_2 \cdot \mathcal{C}_\gamma$$

where  $(m, x) \in \mathcal{C}_\gamma$  if and only if  $\gamma = (\gamma_1, \gamma_2)$  with

$$(3.42) \quad \begin{cases} \Phi_M(m) + \Phi_r(x) = \gamma_1 \\ \Phi_l(x) = \gamma_2 \\ \tilde{\gamma}_1 \cdot m = 0 \\ \tilde{\gamma}_1 \cdot_r x + \rho \tilde{\gamma}_2 \cdot_l x = 0. \end{cases}$$

We have

$$(3.43) \quad \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) = \sum_{\gamma \in \mathcal{B}_\rho} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)$$

where the generalized character  $\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)$  is computed as an index of a transversally elliptic symbol in a neighborhood of

$$K_1 \times K_2 \cdot \mathcal{C}_\gamma \subset M \times \Phi_l^{-1}(K_2 \cdot \gamma_2).$$

Thanks to Theorem 2.9 we know that the support of the generalized character  $\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)$  is contained in  $\{(a, b) \in \widehat{K_1} \times \widehat{K_2} \mid \|a\|^2 + \rho\|b\|^2 \geq \|\gamma\|_\rho^2\}$ . Hence

$$\text{support} \left( \left[ \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P) \right]^{K_1} \right) \subset \left\{ b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \|\gamma\|_\rho^2 \right\}$$

Let  $r_P = \inf_{\xi \in \partial P} \|\xi\|$ . We know then that

$$\left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} = \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma\|_\rho^2 < \rho r_P^2}} \left[ \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P) \right]^{K_1} + O(r_P).$$

Let  $R_P < \rho r_P^2$  be a regular value of  $\|\Phi\|_\rho^2 : M \times \mathcal{X}_P \rightarrow \mathbb{R}$  such that for all  $\gamma \in \mathcal{B}_\rho$  we have  $\|\gamma\|_\rho^2 < \rho r_P^2 \iff \|\gamma\|_\rho^2 < R_P$ . Then

$$(3.44) \quad \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} = \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times \mathcal{X}_P)_{< R_P}) \right]^{K_1} + O(r_P).$$

For the generalized index  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$  we have also a decomposition

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) = \sum_{\gamma \in \mathcal{B}'_\rho} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times T^*K)$$

where  $\mathcal{B}'_\rho$  parametrizes the critical set of  $\|\Phi\|_\rho^2 : M \times T^*K \rightarrow \mathbb{R}$ . Like before we get

$$(3.45) \quad \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times T^*K)_{< R'_P}) \right]^{K_1} + O(r_P).$$

Here  $R'_P < \rho r_P^2$  is a regular value of  $\|\Phi\|_\rho^2 : M \times T^*K \rightarrow \mathbb{R}$  such that for all  $\gamma \in \mathcal{B}'_\rho$  we have  $\|\gamma\|_\rho^2 < \rho r_P^2 \iff \|\gamma\|_\rho^2 < R'_P$ .

**Lemma 3.6.** *We have*

$$(3.46) \quad \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times \mathcal{X}_P)_{< R_P}) = \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}((M \times T^*K)_{< R'_P}).$$

*Proof.* The Lemma will follow from Proposition 2.6. We take here  $V' = M \times \mathcal{X}_P^o$ ,  $V = M \times K \times \mathcal{U}_P \subset M \times T^*K$  and the equivariant diffeomorphism  $\Psi : V \rightarrow V'$  is equal to  $\text{Id} \times \Upsilon$  where  $\Upsilon$  was introduced in Section 2.5. Note that  $\Psi$  satisfies points (1) – (3) of Proposition 2.6.

Note that  $\|\Phi(m, x)\|_\rho^2 < \rho r_P^2$  implies that  $\|\Phi_l(x)\| < r_P$  and then  $x \in \mathcal{X}_P^o$ . Hence the open subset  $U' := (M \times \mathcal{X}_P)_{< R_P}$  is contained in  $V' = M \times \mathcal{X}_P^o$ . In the same way the open subset  $U := (M \times T^*K)_{< R'_P}$  is contained in  $V$ . We have  $\Psi(U) = U'$  if  $R_P = R'_P$ .

We have proved that (3.46) is a consequence of Proposition 2.6.  $\square$

Finally, if we take the difference between (3.44) and (3.45), we get

$$\left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) \right]^{K_1} - \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K) \right]^{K_1} = O(r_P).$$

which is the relation of Proposition 3.3.



**3.3. Proof of Proposition 3.4.** Here we want to compare the  $K_1$ -invariant part of the characters  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$  and  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P)$ .

We know after Theorem 2.14 that

$$\begin{aligned} \mathcal{Q}_{K_2}(M_P) &= \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M \times \mathcal{X}_P) \right]^{K_1} \\ &= \left[ \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(U_\epsilon) \right]^{K_1} \end{aligned}$$

when  $\epsilon > 0$  is any regular value of  $\|\Phi_1\|^2$ , and  $U_\epsilon := \{\|\Phi_1\|^2 < \epsilon\} \subset M \times \mathcal{X}_P$ .

In this section we fix once for all  $\epsilon > 0$  small enough so that

$$(3.47) \quad \text{Cr}(\|\Phi_1\|^2) \cap \{\|\Phi_1\|^2 \leq \epsilon\} = \{\Phi_1 = 0\}.$$

Let  $\mathbf{c}_1$  be the symbol  $\text{Thom}(M, J_1) \otimes L$  attached to the prequantized Hamiltonian  $K_1$ -manifold  $(M, \Omega)$ . Let  $\mathbf{c}_3$  be the symbol  $\text{Thom}(\mathcal{X}_P, J_3) \otimes L_P$  attached to the prequantized Hamiltonian  $K_1 \times K_2$ -manifold  $\mathcal{X}_P$ . The product  $\mathbf{c} = \mathbf{c}_1 \odot \mathbf{c}_3$  corresponds to the symbol  $\text{Thom}(N, J) \otimes L$  on  $N = M \times \mathcal{X}_P$ .

Let  $\kappa_0$  and  $\kappa_\rho$  be the Kirwan vector fields associated to the functions  $\|\Phi_1\|^2$  and  $\|\Phi\|_\rho^2$  on  $M \times \mathcal{X}_P$ :

$$\kappa_0(m, x) = \left( \underbrace{\Phi_1(m, x) \cdot m}_{\kappa_I}; \underbrace{\Phi_1(m, x) \cdot_T x}_{\kappa_{II}} \right), \quad \kappa_\rho(m, x) = \kappa^0(m, x) + \rho \left( 0, \underbrace{\Phi_I(x) \cdot_I x}_{\kappa_{III}} \right).$$

Let  $\mathbf{c}^{\kappa_\rho}$  be the symbol  $\mathbf{c}$  pushed by the vector field  $\kappa_\rho$ : we have

$$\mathbf{c}^{\kappa_\rho}(v; \eta) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_3(\eta - \kappa_{II} - \rho \kappa_{III})$$

for  $(v; \eta) \in \mathbb{T}_{(m, x)}(M \times \mathcal{X}_P)$ .

The character  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(U_\epsilon)$  is given by the index of the  $K_1$ -transversally elliptic symbol  $\mathbf{c}^{\kappa_0}|_{U_\epsilon}$ . The character  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$  is given by the index of the  $K_1 \times K_2$ -transversally elliptic symbol  $\mathbf{c}^{\kappa_\rho}$ .

**Lemma 3.7.** • *There exists  $\rho(\epsilon) > 0$  such that*

$$\text{Cr}(\|\Phi\|_\rho^2) \cap \{\|\Phi_1\|^2 \leq \epsilon\} \subset \left\{ \|\Phi_1\|^2 \leq \frac{\epsilon}{2} \right\}$$

for any  $0 \leq \rho \leq \rho(\epsilon)$ .

*Proof.* With the help of Riemannian metrics on  $M$  and  $\mathcal{X}_P$  we define

$$\begin{aligned} a(\epsilon) &:= \inf_{\epsilon/2 \leq \|\Phi_1(m, x)\| \leq \epsilon} \|\kappa^0(m, x)\| \\ b &:= \sup_{x \in \mathcal{X}_P} \|\Phi_I(x) \cdot_I x\|. \end{aligned}$$

We have  $a(\epsilon) > 0$  thanks to (3.47), and  $b < \infty$  since  $\mathcal{X}_P$  is compact. It is now easy to check that  $\{\kappa_\rho = 0\} \cap \{\epsilon/2 \leq \|\Phi_1\|^2 \leq \epsilon\} = \emptyset$  if  $0 \leq \rho < \frac{a(\epsilon)}{b}$ .  $\square$

The symbols  $\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}$ ,  $\rho \in [0, \rho(\epsilon)]$  are  $K_1 \times K_2$ -transversally elliptic, and they define the same class in  $\mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2} U_\epsilon)$ . Hence  $\mathcal{Q}_{K_2}(M_P)$  can be computed as the  $K_1$ -invariant part of

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(U_\epsilon) := \text{Index}_{K_1 \times K_2}^{U_\epsilon}(\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}) \in R^{-\infty}(K_1 \times K_2)$$

for  $\rho \in [0, \rho(\epsilon)]$ .

A component  $K_1 \times K_2 \cdot \mathcal{C}_\gamma$  of  $\text{Cr}(\|\Phi\|_\rho^2)$  is contained in  $U_\epsilon$  if and only if  $\|\gamma_1\| < \epsilon$ : hence the decomposition (3.43) for the character  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)$  gives

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P) = \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(U_\epsilon) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 \geq \epsilon}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P).$$

where

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(U_\epsilon) = \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 < \epsilon}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P).$$

Taking the  $K_1$ -invariant gives

$$(3.48) \quad [\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 \geq \epsilon}} [\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1}.$$

In general we know that the support of the generalized character  $[\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1}$  is included in  $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \|\gamma_1\|^2 + \rho\|\gamma_2\|^2\}$ . When  $\|\gamma_1\|^2 \geq \epsilon$  we have then that the support of  $[\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M \times \mathcal{X}_P)]^{K_1}$  is contained in  $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \epsilon\}$ .

Finally (3.48) imposes that

$$[\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + O((\epsilon/\rho)^{1/2}).$$

when  $0 < \rho \leq \rho(\epsilon)$ , which is the precise content of Proposition 3.4.

#### 4. OTHER PROPERTIES OF $\mathcal{Q}^\Phi$

Let  $(M, \omega, \Phi)$  be a proper Hamiltonian  $K$ -manifold which is prequantized by a line bundle  $L$ . The character  $\mathcal{Q}_K^\Phi(M)$  is computed by means of a scalar product on  $\mathfrak{k}^*$ . The fact that  $\mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^{-\infty}(M)$  gives the following

**Proposition 4.1.** *The character  $\mathcal{Q}_K^\Phi(M)$  does not depend of the choice of a scalar product on  $\mathfrak{k}^*$*

In this section we work in the setting where  $K = K_1 \times K_2$ . Let  $\Phi_1$  be the moment map relative to the  $K_1$ -action.

**4.1.  $\Phi_1$  is proper.** In this subsection we suppose that the moment map  $\Phi_1$  relative to the  $K_1$ -action is *proper*. We fix an invariant Euclidean norm  $\|\bullet\|^2$  on  $\mathfrak{k}$  in such a way that  $\mathfrak{k}_1 = \mathfrak{k}_2^\perp$ .

Let us “quantize”  $(M, \Omega)$  via the invariant proper function  $\|\Phi_1\|^2$ : let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

**Theorem 4.2.** *We have*

$$(4.49) \quad \mathcal{Q}_{K_1 \times K_2}^\Phi(M) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) \quad \text{in } R^{-\infty}(K_1 \times K_2).$$

*Proof.* On  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$  we may consider the family of invariant Euclidean norms:  $\|X_1 \oplus X_2\|_\rho^2 = \|X_1\|^2 + \rho\|X_2\|^2$  for  $X_j \in \mathfrak{k}_j$ . Let

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the quantization of  $M$  computed via the map  $\|\Phi\|_\rho^2 = \|\Phi_1\|^2 + \rho\|\Phi_2\|^2$ . By definition,  $\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)$  is equal to  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, 0}(M)$ , and we know after Proposition 4.1 that  $\mathcal{Q}_{K_1 \times K_2}^{\Phi}(M)$  coincides with the generalized character  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) \in R^{-\infty}(K)$  for any  $\rho > 0$ .

Let us prove that  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)$ . We denote  $O(r) \in R^{-\infty}(K_1 \times K_2)$  any generalized character supported outside the ball

$$\{\xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < r^2\}.$$

And we denote  $O_1(r) \in R^{-\infty}(K_1 \times K_2)$  any generalized character supported outside the

$$\{\xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\| < r\}.$$

Let  $R_1 > 0$  be a regular value of  $\|\Phi_1\|^2$ : the open subset  $\{\|\Phi_1\|^2 < R_1\}$  is denoted  $M_{<R_1}$ . We know that

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M_{<R_1}) + O_1(\sqrt{R_1}).$$

Like in the Lemma 3.7, we know that

$$(4.50) \quad \text{Cr}(\|\Phi\|_\rho^2) \cap \{\|\Phi_1\|^2 = R_1\} = \emptyset.$$

for  $\rho \geq 0$  small enough. The identity (4.50) first implies that

$$\begin{aligned} \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) &= \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 < R_1}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M) + \sum_{\substack{\gamma \in \mathcal{B}_\rho \\ \|\gamma_1\|^2 > R_1}} \mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M) \\ &= \mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M_{<R_1}) + O(\sqrt{R_1}). \end{aligned}$$

In the second equality we have used that  $\mathcal{Q}_{K_1 \times K_2}^{\gamma, \rho}(M) = O(\sqrt{R_1})$  when  $\|\gamma_1\|^2 > R_1$  since the ball  $\{(\xi_1, \xi_2) \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < R_1\}$  is contained in

$$\{(\xi_1, \xi_2) \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|(\xi_1, \xi_2)\|_\rho^2 < \|(\gamma_1, \gamma_2)\|_\rho^2\}.$$

The identity (4.50) shows also that the symbol  $\mathbf{c}^{\kappa, \rho}|_{M_{<R_1}}$  are homotopic for  $\rho \geq 0$  small enough. Hence

$$\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M_{<R_1}) = \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M_{<R_1})$$

We get finally that  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) - \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) = O(\sqrt{R_1}) + O_1(\sqrt{R_1})$  for any regular value  $R_1$  of  $\|\Phi_1\|^2$ . We have proved that  $\mathcal{Q}_{K_1 \times K_2}^{\Phi, \rho}(M) - \mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M) = 0$ .  $\square$

Let us explain how Theorem 4.2 contains the identity that we called “*quantization commutes with reduction in the singular setting*” in [21]. By definition the  $K_1$ -invariant part of the right hand side of (4.49) is equal to the geometric quantization of the (possibly singular) compact Hamiltonian  $K_2$ -manifold

$$M//_0 K_1 := \Phi_1^{-1}(0)/K_1.$$

Using now the fact that the left hand side of (4.49) is equal to  $\mathcal{Q}_{K_1 \times K_2}^{-\infty}(M)$ , we see that the multiplicity of  $V_\mu^{K_2}$  in  $\mathcal{Q}_{K_2}(M//_0 K_1)$  is equal to the geometric quantization of the (possibly singular) compact manifold

$$M \times \overline{K_2 \cdot \mu} //_{(0, \mu)} K_1 \times K_2.$$

**4.2. The symplectic reduction  $M//_0 K_1$  is smooth.** Let  $(M, \Omega)$  be an Hamiltonian  $K_1 \times K_2$ -manifold with a proper moment map  $\Phi = (\Phi_1, \Phi_2)$ . In this section we suppose that 0 is a regular value of  $\Phi_1$  and that  $K_1$  acts freely on  $\Phi_1^{-1}(0)$ . We work then with the (smooth) Hamiltonian  $K_2$ -manifold

$$N := \Phi_1^{-1}(0)/K_1.$$

We still denote by  $\Phi_2 : N \rightarrow \mathfrak{k}_2^*$  the moment map relative to the  $K_2$ -action: note that this map is proper. Hence we can quantize the  $K_2$ -action on  $N$  via the map  $\Phi_2$ . Let  $\mathcal{Q}_{K_2}^{\Phi_2}(N) \in R^{-\infty}(K_2)$  be the corresponding character.

**Proposition 4.3.** *We have*

$$(4.51) \quad [\mathcal{Q}_{K_1 \times K_2}^{\Phi}(M)]^{K_1} = \mathcal{Q}_{K_2}^{\Phi_2}(N) \quad \text{in } R^{-\infty}(K_2).$$

*Proof.* When  $\Phi_1$  is proper, the manifold  $N$  is compact. Then the right hand side of (4.51) is equal to  $\mathcal{Q}_{K_2}(N)$ , and we know from Theorem 4.2 that the left hand side of (4.51) is equal to  $[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)]^{K_1}$ . In this case (4.51) becomes  $[\mathcal{Q}_{K_1 \times K_2}^{\Phi_1}(M)]^{K_1} = \mathcal{Q}_{K_2}(M//_0 K_1)$  which is the content of Theorem 2.13.

Let us consider the general case where  $\Phi_1$  is not proper. Thanks to Theorem 1.4 one knows that the multiplicities of  $V_\mu^{K_2}$  in  $[\mathcal{Q}_{K_1 \times K_2}^{\Phi}(M)]^{K_1}$  and  $\mathcal{Q}_{K_2}^{\Phi_2}(N)$  are respectively equal to the quantization of the (possibly singular) symplectic reductions

$$\mathcal{M}_\mu := M \times \overline{K_2 \cdot \mu} //_{(0,0)} K_1 \times K_2.$$

and

$$\mathcal{M}'_\mu := N \times \overline{K_2 \cdot \mu} //_0 K_2, \quad \text{with } N = M//_0 K_1.$$

Note that  $\mathcal{M}_\mu$  and  $\mathcal{M}'_\mu$  coincide as symplectic reduced space. Let us prove that their geometric quantization are identical also. The proof will be done for  $\mu = 0$ : the other case follows from the shifting trick.

Let  $\mathbf{c}$  be the  $K_1 \times K_2$ -equivariant symbol  $\text{Thom}(M, J) \otimes L_M$ . Let  $\kappa$  be the Kirwan vector field attached to the moment map  $\Phi = (\Phi_1, \Phi_2)$ . Let  $\mathbf{c}^\kappa$  be the symbol  $\mathbf{c}$  pushed by  $\kappa$ . Let us denote  $M_{<\epsilon}$  the open subset  $\{\|\Phi\|^2 < \epsilon\}$ . For  $\epsilon > 0$  small enough, the symbol  $\mathbf{c}^\kappa|_{M_{<\epsilon}}$  is  $K_1 \times K_2$ -transversally elliptic, and  $\mathcal{Q}(\mathcal{M}_0)$  is the  $K_1 \times K_2$ -invariant part of  $\text{Index}_{M_{<\epsilon}}^{K_1 \times K_2}(\mathbf{c}^\kappa|_{M_{<\epsilon}})$ .

Let  $\mathbf{c}_2$  be the  $K_2$ -equivariant symbol  $\text{Thom}(N, J) \otimes L_N$ . Let  $\kappa_2$  be the Kirwan vector field attached to the moment map  $\Phi_2$ . Let  $\mathbf{c}_2^{\kappa_2}$  be the symbol  $\mathbf{c}_2$  pushed by  $\kappa_2$ . Let us denote  $N_{<\epsilon}$  the open subset  $\{\|\Phi_2\|^2 < \epsilon\}$ . For  $\epsilon > 0$  small enough, the symbol  $\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}$  is  $K_2$ -transversally elliptic, and  $\mathcal{Q}(\mathcal{M}'_0)$  is the  $K_2$ -invariant part of  $\text{Index}_{N_{<\epsilon}}^{K_2}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}})$ .

Our proof follows from the comparison of the classes

$$[\mathbf{c}^\kappa|_{M_{<\epsilon}}] \in \mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2} M_{<\epsilon})$$

and

$$[\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}] \in \mathbf{K}_{K_2}(\mathbb{T}_{K_2} N_{<\epsilon})$$

A neighborhood of the smooth submanifold  $Z := \Phi_1^{-1}(0)$  in  $M$  is diffeomorphic to a neighborhood of the 0-section of the bundle  $Z \times \mathfrak{k}_1^* \rightarrow Z$ . Let  $Z_{<\epsilon} = Z \cap M_{<\epsilon}$  so that  $N_{<\epsilon} = Z_{<\epsilon}/K_1$ . Hence  $[\mathbf{c}^\kappa|_{M_{<\epsilon}}]$  can be seen naturally a class in the  $\mathbf{K}$ -group  $\mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*))$ .

Following Atiyah [1][Theorem 4.3], the inclusion map  $j : Z_{<\epsilon} \hookrightarrow Z_{<\epsilon} \times \mathfrak{k}_1^*$  induces the Thom isomorphism

$$j! : \mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2} Z_{<\epsilon}) \longrightarrow \mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*)),$$

with the commutative diagram

$$(4.52) \quad \begin{array}{ccc} \mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2} Z_{<\epsilon}) & \xrightarrow{j!} & \mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*)) \\ & \searrow \text{Index}_{Z_{<\epsilon}}^{K_1 \times K_2} & \downarrow \text{Index}_{Z_{<\epsilon} \times \mathfrak{k}_1^*}^{K_1 \times K_2} \\ & & R^{-\infty}(K_1 \times K_2) \end{array}$$

Let  $\pi_1 : Z_{<\epsilon} \rightarrow N_{<\epsilon}$  be the quotient relative to the free action of  $K_1$ . The corresponding isomorphism

$$\pi_1^* : \mathbf{K}_{K_2}(\mathbb{T}_{K_2} N_{<\epsilon}) \longrightarrow \mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2} Z_{<\epsilon})$$

satisfies the following rule :

$$(4.53) \quad \left[ \text{Index}_{K_1 \times K_2}^{Z_{<\epsilon}}(\pi_1^* \theta) \right]^{K_1} = \text{Index}_{K_2}^{N_{<\epsilon}}(\theta)$$

for any  $\theta \in \mathbf{K}_{K_2}(\mathbb{T}_{K_2} N_{<\epsilon})$ .

**Lemma 4.4** ([19]). *We have*

$$j! \circ \pi_1^* \left( [\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}] \right) = [\mathbf{c}^\kappa|_{M_{<\epsilon}}]$$

in  $\mathbf{K}_{K_1 \times K_2}(\mathbb{T}_{K_1 \times K_2}(Z_{<\epsilon} \times \mathfrak{k}_1^*))$ .

*Proof.* This Lemma is proven in [19][Section 6.2] when the group  $K_2$  is trivial. It is easy to check that the proof extends naturally to our setting.  $\square$

If one uses Lemma 4.4 together with (4.52) and (4.53), we get that

$$\begin{aligned} \mathcal{Q}(\mathcal{M}_0) &= \left[ \text{Index}_{Z_{<\epsilon} \times \mathfrak{k}_1^*}^{K_1 \times K_2}(\mathbf{c}^\kappa|_{M_{<\epsilon}}) \right]^{K_1 \times K_2} \\ &= \left[ \text{Index}_{N_{<\epsilon}}^{K_2}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}) \right]^{K_2} = \mathcal{Q}(\mathcal{M}'_0). \end{aligned}$$

$\square$

## 5. EXAMPLE: THE COTANGENT BUNDLE OF AN ORBIT

**5.1. The formal quantization of  $\mathbb{T}^*K$ .** Let  $K$  be a compact connected Lie group equipped with the action of two copies of  $K$ :  $(k_1, k_2) \cdot a = k_2 a k_1^{-1}$ . Then we have a Hamiltonian action of  $K_1 \times K_2$  on the cotangent bundle  $\mathbb{T}^*K$ . In this section, we check that each formal geometric quantization of  $\mathbb{T}^*K$ ,  $\mathcal{Q}_{K_1 \times K_2}^{-\infty}(\mathbb{T}^*K)$  and  $\mathcal{Q}_{K_1 \times K_2}^\Phi(\mathbb{T}^*K)$ , are both equal to the  $K_1 \times K_2$ -module  $L^2(K)$ .

The tangent bundle  $TK$  is identified with  $K \times \mathfrak{k}$  through the right translations: to  $(a, X) \in K \times \mathfrak{k}$  we associate  $\frac{d}{dt}ae^{tX}|_0$ . The action of  $K_1 \times K_2$  on the cotangent bundle  $T^*K \simeq K \times \mathfrak{k}^*$  is then

$$(k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The symplectic form on  $T^*K$  is  $\Omega := -d\lambda$ , where  $\lambda$  is the Liouville 1-form. Let us compute these two form in coordinates. The tangent bundle of  $T^*K \simeq K \times \mathfrak{k}^*$  is identified with  $T^*K \times \mathfrak{k} \times \mathfrak{k}^*$ : for each  $(a, \xi) \in T^*K$ , we have a two form  $\Omega_{(a, \xi)}$  on  $\mathfrak{k} \times \mathfrak{k}^*$ . A direct computation gives

$$\Omega_{(a, \xi)}(X, Y) = \langle \xi, [X, Y] \rangle, \quad \Omega_{(a, \xi)}(\eta, \eta') = 0, \quad \Omega_{(a, \xi)}(X, \eta) = \langle \eta, X \rangle$$

for  $X, Y \in \mathfrak{k}$  and  $\eta, \eta' \in \mathfrak{k}^*$ . So  $\Omega_{(a, \xi)} = \Omega_0 + \pi_\xi$  where  $\Omega_0$  is the canonical (constant) symplectic form on  $\mathfrak{k} \times \mathfrak{k}^*$  and  $\pi_\xi$  is the closed two form on  $\mathfrak{k}$  defined by  $\pi_\xi(X, Y) = \langle \xi, [X, Y] \rangle$ .

If we identify  $\mathfrak{k} \simeq \mathfrak{k}^*$  through an invariant Euclidean norm, the symplectic structure on  $T_{(a, \xi)}(T^*K) \simeq \mathfrak{k} \times \mathfrak{k}^*$  is given by a skew-symmetric matrix

$$A_\xi := \begin{pmatrix} \text{ad}(\xi) & I_n \\ -I_n & 0 \end{pmatrix}.$$

We will work with the following compatible almost complex structure on the tangent bundle of  $T^*K$  :  $J_\xi = -A_\xi(-A_\xi^2)^{-1/2}$ . When  $\xi = 0$ , the complex structure  $J_0$  on  $\mathfrak{k} \times \mathfrak{k}^*$  is defined by the matrix

$$J_0 := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Hence the complex  $K$ -module  $(\mathfrak{k} \times \mathfrak{k}^*, J_0)$  is naturally identified with the complexification  $\mathfrak{k}_\mathbb{C}$  of  $\mathfrak{k}$ .

One checks easily that the moment map relative to the  $K_1 \times K_2$ -action is the *proper* map  $\Phi : T^*K \rightarrow \mathfrak{k}_1^* \times \mathfrak{k}_2^*$  defined by  $\Phi(a, \xi) = (-\xi, a \cdot \xi)$ .

Here the symplectic manifold  $T^*K$  is prequantized by the trivial line bundle.

5.1.1. *Computation of  $\mathcal{Q}_{K_1 \times K_2}^{-\infty}(T^*K)$ .* Let  $\mathcal{O}_1 \times \mathcal{O}_2$  be a coadjoint orbit of  $K_1 \times K_2$  in  $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ . One checks that

$$(5.54) \quad \Phi^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \emptyset & \text{if } \mathcal{O}_1 \neq -\mathcal{O}_2 \\ \text{a } K_1 \times K_2 \text{-orbit} & \text{if } \mathcal{O}_1 = -\mathcal{O}_2. \end{cases}$$

We know that the stabiliser subgroup  $K_\xi$  of an element  $\xi \in \mathfrak{k}^*$  is connected. Then the stabilizer subgroup  $(K_1 \times K_2)_{(a, \xi)} = \{(k_1, a k_1 a^{-1}), k_1 \in K_\xi\}$  is also connected.

Let  $(T^*K)_{(\mu, \lambda)}$  be the symplectic reduction of  $T^*K$  at the level  $(\mu, \lambda) \in \widehat{K}^2$ . For any  $\mu \in \widehat{K}$ , we define  $\mu^* \in \widehat{K}$  by the relation  $-K \cdot \mu = K \cdot \mu^*$ : note that  $V_{\mu^*}^K \simeq (V_\mu^K)^*$ . If one uses Theorem 2.16, one has

$$(5.55) \quad \mathcal{Q}((T^*K)_{(\mu, \lambda)}) = \begin{cases} 0 & \text{if } \lambda \neq \mu^* \\ 1 & \text{if } \lambda = \mu^*. \end{cases}$$

Finally

$$\begin{aligned} \mathcal{Q}_{K_1 \times K_2}^{-\infty}(\mathbb{T}^*K) &= \sum_{(\mu, \lambda) \in \widehat{K} \times \widehat{K}} \mathcal{Q}((\mathbb{T}^*K)_{(\mu, \lambda)}) V_{\mu}^{K_1} \otimes V_{\lambda}^{K_2} \\ &= \sum_{\mu \in \widehat{K}} V_{\mu}^{K_1} \otimes (V_{\mu}^{K_2})^* = L^2(K). \end{aligned}$$

5.1.2. *Computation of  $\mathcal{Q}_{K_1 \times K_2}^{\Phi}(\mathbb{T}^*K)$ .* The Kirwan vector field on  $\mathbb{T}^*K$  is

$$\kappa(a, \xi) = -2\xi \in \mathfrak{k}_{\mathbb{C}}.$$

Let  $\mathbf{c}^{\kappa}$  be the symbol  $\text{Thom}(\mathbb{T}^*K, J)$  pushed by the vector field  $\frac{1}{2}\kappa$ . At each  $(a, \xi) \in \mathbb{T}^*K$ , the map  $\mathbf{c}_{(a, \xi)}^{\kappa}(X \oplus \eta)$  from  $\wedge_{J_{\xi}}^{\text{even}}(\mathfrak{k} \times \mathfrak{k}^*)$  to  $\wedge_{J_{\xi}}^{\text{odd}}(\mathfrak{k} \times \mathfrak{k}^*)$  is equal to the clifford map  $\mathbf{c}(X + \xi \oplus \eta)$ . Note that  $\mathbf{c}^{\kappa}$  is a  $K_2$ -transversally elliptic symbol on  $\mathbb{T}^*K$ : we have  $\text{Char}(\mathbf{c}^{\kappa}) \cap \mathbb{T}_{K_2}(\mathbb{T}^*K) = \{(1, 0)\}$ . We will now compute the equivariant index of  $\mathbf{c}^{\kappa}$ .

First we consider the homotopy  $t \in [0, 1] \rightarrow J_{t\xi}$  of symplectic structure on  $\mathbb{T}^*K$ . Let  $\tilde{\mathbf{c}}^{\kappa}$  be the symbol acting on  $\wedge_{J_0}^{\bullet}(\mathfrak{k} \times \mathfrak{k}^*) = \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}_{\mathbb{C}}$ . Proposition 2.6 shows that the symbols  $\mathbf{c}^{\kappa}$  and  $\tilde{\mathbf{c}}^{\kappa}$  define the same class in  $K_{K_1 \times K_2}(\mathbb{T}_{K_2}(\mathbb{T}^*K))$ .

The projection  $\pi : \mathbb{T}^*K \rightarrow \mathfrak{k}^*$  corresponds to the quotient map relative to the free action of  $K_2$ . At the level of  $K$ -groups we get an isomorphism

$$\pi_* : K_{K_1 \times K_2}(\mathbb{T}_{K_2}(\mathbb{T}^*K)) \rightarrow K_{K_1}(\mathbb{T}\mathfrak{k}^*).$$

Atiyah [1] proves that

$$\text{Index}_{K_1 \times K_2}^{\mathbb{T}^*K}(\sigma) = \sum_{\mu \in \widehat{K}} \text{Index}_{K_1}^{\mathfrak{k}^*}(\pi_*(\sigma \otimes V_{\mu}^{K_2})) \otimes (V_{\mu}^{K_2})^*$$

for any class  $\sigma \in K_{K_1 \times K_2}(\mathbb{T}_{K_2}(\mathbb{T}^*K))$ . In our case the symbol  $\pi^*(\tilde{\mathbf{c}}^{\kappa})$  is equal to the Bott symbol  $\text{Bott}(\mathfrak{k}^*)$ , and for any  $K_2$ -module  $E_2$  we have

$$\pi_*(\tilde{\mathbf{c}}^{\kappa} \otimes E_2) = \text{Bott}(\mathfrak{k}^*) \otimes E_1$$

where  $E_1$  is the module  $E_2$  with the action of  $K_1$ . Then

$$\begin{aligned} \mathcal{Q}_{K_1 \times K_2}^{\Phi}(\mathbb{T}^*K) &= \text{Index}_{K_1 \times K_2}^{\mathbb{T}^*K}(\tilde{\mathbf{c}}^{\kappa}) \\ &= \sum_{\mu \in \widehat{K}} \text{Index}_{K_1}^{\mathfrak{k}^*}(\text{Bott}(\mathfrak{k}^*) \otimes V_{\mu}^{K_1}) \otimes (V_{\mu}^{K_2})^* \\ &= \sum_{\mu \in \widehat{K}} V_{\mu}^{K_1} \otimes (V_{\mu}^{K_2})^* = L^2(K), \end{aligned}$$

since  $\text{Index}_{K_1}^{\mathfrak{k}^*}(\text{Bott}(\mathfrak{k}^*)) = 1$ .

**5.2. The formal quantization of  $\mathbb{T}^*(K/H)$ .** Let  $H$  be a closed connected subgroup of  $K$ . Look at  $\mathbb{T}^*K$  as a Hamiltonian manifold relatively to the action of  $H \times K \subset K_1 \times K_2$ . The moment map  $\Phi = (\Phi_H, \Phi_K)$  is defined by :  $\Phi_H(a, \xi) = -\text{pr}(\xi)$  and  $\Phi_K(a, \xi) = a \cdot \xi$ , where  $\text{pr} : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$  is the projection. Note that  $\Phi$  is a proper map.

The cotangent bundle  $\mathbb{T}^*(K/H)$ , viewed as  $K$ -manifold, is equal to the symplectic reduction of  $\mathbb{T}^*K$  relatively to the  $H$ -action: if the kernel of the projection  $\text{pr}$  is denoted  $\mathfrak{h}^{\perp}$ , we have

$$\Phi_H^{-1}(0)/H = K \times_H \mathfrak{h}^{\perp} = \mathbb{T}^*(K/H).$$

We are here in the setting of section 4.2. The reduction of the  $H \times K$  proper Hamiltonian manifold  $T^*K$  relatively to the  $H$ -action is smooth, then its formal quantization is computed as follows

$$\begin{aligned}
 \mathcal{Q}_K^\Phi(T^*(K/H)) &= [\mathcal{Q}_{H \times K}^\Phi(T^*K)]^H = [\mathcal{Q}_{K_1 \times K_2}^\Phi(T^*K)|_{H \times K}]^H \\
 (5.56) \qquad \qquad \qquad &= [L^2(K)]^H \\
 &= L^2(K/H).
 \end{aligned}$$

Here the fact that  $\mathcal{Q}_{H \times K}^\Phi(T^*K)$  is equal to the restriction of  $\mathcal{Q}_{K_1 \times K_2}^\Phi(T^*K) = L^2(K)$  to  $H \times K$  is a consequence of Theorem 1.3.

Let us denote  $\left[ T^*(K/H) \right]_\mu$  the symplectic reduction at  $\mu \in \widehat{K}$  of the K-Hamiltonian manifold  $T^*(K/H)$ . Theorem 1.4 together with (1.6) gives

**Corollary 5.1.** *For any  $\mu \in \widehat{K}$ , we have*

$$\mathcal{Q} \left( \left[ T^*(K/H) \right]_\mu \right) = \dim [V_\mu^K]^H,$$

where  $[V_\mu^K]^H$  is the subspace of  $H$ -invariant vector.

### 5.3. The formal quantization of $T^*(K/H)$ relatively to the action of $G$ .

Let  $G$  be a closed connected subgroup of  $K$ . We look at the hamiltonian action of  $G$  on  $T^*(K/H)$ . Let  $\Phi_G : T^*(K/H) \rightarrow \mathfrak{g}^*$  be the moment map. We consider also the restriction of the K-module  $L^2(K/H)$  to  $G$ .

We have

**Proposition 5.2.** *The following statements are equivalent*

- (1) *The moment map  $\Phi_G : T^*(K/H) \rightarrow \mathfrak{g}^*$  is proper.*
- (2)  *$\Phi_G^{-1}(0)$  is equal to the zero section.*
- (3)  *$k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$ , for any  $k \in K$ .*
- (4)  *$\mathfrak{g} + \mathfrak{h} = \mathfrak{k}$*
- (5)  *$G$  acts transitively on  $K/H$ .*
- (6)  *$L^2(K/H)]^G \simeq \mathbb{C}$*
- (7)  *$L^2(K/H)|_G$  is an admissible  $G$ -representation.*

*Proof.* (1)  $\implies$  (7) is a consequence of Theorem 1.3. Let us prove that (7)  $\implies$  (6). Suppose now that

$$L^2(K/H)|_G = \sum_{\mu \in \widehat{K}} [V_\mu^K]^H \otimes (V_\mu^K)^*|_G$$

is an admissible  $G$ -representation. It means that for any  $\lambda \in \widehat{G}$  the set

$$A_\lambda := \left\{ \mu \in \widehat{K} \mid [V_\mu^K]^H \neq \{0\} \text{ and } [(V_\lambda^G)^* \otimes (V_\mu^K)^*|_G]^G \neq \{0\} \right\}$$

is finite. Then the vector space  $L^2(K/H)]^G$  is equal to the finite dimensional vector space  $\sum_{\mu \in A_0} [V_\mu^K]^H \otimes [(V_\mu^K)^*|_G]^G$ . It is not difficult to check that if  $\mu \in A_0$ , then  $k\mu \in A_0$  for  $k \gg 1$ . Finally the fact that  $A_0$  is finite implies that  $A_0$  is reduced to  $\mu = 0$ . Hence the only  $G$ -invariant functions on  $K/H$  are the scalars.

(6)  $\iff$  (5)  $\iff$  (4)  $\iff$  (3) is a general fact concerning smooth actions of a compact connected Lie group  $G$  on a compact connected manifold  $M$ . The manifold  $M$  does not have  $G$ -invariant functions which are not scalar if and only if the action of  $G$  on  $M$  is transitive. And given a point  $m \in M$ , the orbit  $G \cdot m$  is all of  $M$  if



and only if tangent spaces  $T_m(G \cdot m)$  and  $T_m M$  are equal. If we take  $m = \overline{k^{-1}}$  in  $M = K/H$ , the condition  $T_m(G \cdot m) = T_m M$  is equivalent to  $k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$ .

Let us check (3)  $\implies$  (2). Let  $[k, \xi] \in K \times_H \mathfrak{h}^\perp = T^*(K/H)$ . We have  $\Phi_G([k, \xi]) = 0$  if and only if  $k \cdot \xi \in \mathfrak{g}^\perp$ . Hence the vector  $\xi$  belongs to

$$k^{-1} \cdot \mathfrak{g}^\perp \cap \mathfrak{h}^\perp = (k^{-1} \cdot \mathfrak{g} + \mathfrak{h})^\perp.$$

Hence condition (3) imposes that  $\xi = 0$ .

(2)  $\iff$  (1) comes from the fact that  $\Phi_G$  is a homogeneous map of degree one between the vector bundle  $T^*(K/H)$  and the vector space  $\mathfrak{g}^*$ .  $\square$

Suppose now that the cotangent bundle  $T^*(K/H)$  is a *proper* Hamiltonian  $G$ -manifold. Let us denote  $[T^*(K/H)]_{\mu, G}$  the (compact) symplectic reduction at  $\mu \in \widehat{G}$  of the  $G$ -Hamiltonian manifold  $T^*(K/H)$ . Then,

**Corollary 5.3.** *The multiplicity of  $V_\mu^G$  in  $L^2(K/H)$  is equal to the quantization of the reduced space  $[T^*(K/H)]_{\mu, G}$ .*

*Proof.* Using Theorem 1.3, equality (5.56) gives then

$$\begin{aligned} \mathcal{Q}_G^{-\infty}(T^*(K/H)) &= \mathcal{Q}_K^{-\infty}(T^*(K/H))|_G \\ &= L^2(K/H)|_G. \end{aligned}$$

In other words, the multiplicity of  $V_\mu^G$  in  $L^2(K/H)$  is equal to the quantization of the reduced space  $[T^*(K/H)]_{\mu, G}$ .  $\square$

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