Formal geometric quantization II

Paul-Emile Paradan

FORMAL GEOMETRIC QUANTIZATION II

PAUL-EMILE PARADAN

Abstract. In this paper we pursue the study of formal geometric quantization of non-compact Hamiltonian manifolds. Our main result is the proof that two quantization process coincide. This fact was obtained by Ma and Zhang in the preprint Arxiv:0812.3989 by completely different means.

Contents

1. Introduction and statement of results 1
2. Quantizations of non-compact manifolds 4
3. Proof of Theorem 1.4 19
4. Other properties of $\mathcal{Q}^{\Phi}$ 25
5. Example: the cotangent bundle of an orbit 28
References 32

In the previous article [21], we have studied some functorial properties of the "formal geometric quantization" process $\mathcal{Q}^{-\infty}$, which is defined on *proper Hamiltonian manifolds*, e.g. *non-compact* Hamiltonian manifolds with *proper* moment map.

There is another way, denoted $\mathcal{Q}^{\Phi}$, of quantizing proper Hamiltonian manifolds by localizing the index of the Dolbeault Dirac operator on the critical points of the square of the moment map [15, 19, 20].

The main purpose of this paper is to provide a geometric proof that the quantization process $\mathcal{Q}^{-\infty}$ and $\mathcal{Q}^{\Phi}$ coincide. This fact was proved by Ma and Zhang in the recent preprint [15] by completely different means.

Keywords: moment map ; symplectic reduction ; geometric quantization ; transversally elliptic symbol.

1. Introduction and statement of results

Let us first recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show two way of extending the notion of geometric quantization to the case of a *non-compact* Hamiltonian manifold.

Let K be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. In the Kostant-Souriau framework, a Hamiltonian K-manifold (M, Ω, Φ) is pre-quantized if there

Date: June 2009.
Keywords : moment map, reduction, geometric quantization, transversally elliptic symbol.
1991 Mathematics Subject Classification : 58F06, 57S15, 19L47.

is an equivariant Hermitian line bundle L with an invariant Hermitian connection ∇ such that

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X\rangle \quad \text{and} \quad \nabla^2 = -i\Omega, \tag{1.1}$$

for every $X \in \mathfrak{k}$. Here X_M is the vector field on M defined by $X_M(m) = \frac{d}{dt} e^{-tX} m|_0$.

The data (L, ∇) is also called a Kostant-Souriau line bundle, and $\Phi : M \to \mathfrak{k}^*$ is the moment map. Remark that conditions (1.1) imply via the equivariant Bianchi formula the relation

$$\iota(X_M)\Omega = -d\langle \Phi, X\rangle, \quad X \in \mathfrak{k}. \tag{1.2}$$

Let us recall the notion of geometric quantization when M is **compact**. Choose a K-invariant almost complex structure J on M which is compatible with Ω in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\overline{\partial}_L$ be the Dolbeault operator with coefficients in L, and let $\overline{\partial}_L^*$ be its (formal) adjoint. The *Dolbeault-Dirac operator* on M with coefficients in L is $D_L = \overline{\partial}_L + \overline{\partial}_L^*$, considered as an elliptic operator from $\mathcal{A}^{0,\mathrm{even}}(M, L)$ to $\mathcal{A}^{0,\mathrm{odd}}(M, L)$. Let $R(K)$ be the representation ring of K.

Definition 1.1. *The geometric quantization of a* compact *Hamiltonian K-manifold (M, Ω, Φ) is the element $\mathcal{Q}_K(M) \in R(K)$ defined as the equivariant index of the Dolbeault-Dirac operator D_L.*

Let us consider the case of a **proper** Hamiltonian K-manifold M: the manifold is (perhaps) **non-compact** but the moment map $\Phi : M \to \mathfrak{k}^*$ is supposed to be proper. Under this properness assumption, one define the *formal geometric quantization* of M as an element $\mathcal{Q}_K^{-\infty}(M)$ that belongs to $R^{-\infty}(K)$ [30, 21]. Let us recall the definition.

Let T be a maximal torus of K. Let $\mathfrak{t}^*$ be the dual of the Lie algebra of T containing the weight lattice $\wedge^* : \alpha \in \wedge^*$ if $i\alpha : \mathfrak{t} \to i\mathbb{R}$ is the differential of a character of T. Let $C_K \subset \mathfrak{t}^*$ be a Weyl chamber, and let $\widehat{K} := \wedge^* \cap C_K$ be the set of dominant weights. The ring of characters $R(K)$ has a $\mathbb{Z}$-basis $V_\mu^K, \mu \in \widehat{K} : V_\mu^K$ is the irreducible representation of K with highest weight μ.

A representation E of K is *admissible* if it has finite K-multiplicities : $\dim(\hom_K(V_\mu^K, E)) < \infty$ for every $\mu \in \widehat{K}$. Let $R^{-\infty}(K)$ be the Grothendieck group associated to the K-admissible representations. We have an inclusion map $R(K) \hookrightarrow R^{-\infty}(K)$ and $R^{-\infty}(K)$ is canonically identify with $\hom_{\mathbb{Z}}(R(K), \mathbb{Z})$.

For any $\mu \in \widehat{K}$ which is a regular value of moment map Φ, the reduced space (or symplectic quotient) $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is a *compact* orbifold equipped with a symplectic structure Ω_μ. Moreover $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$ is a Kostant-Souriau line orbibundle over (M_μ, Ω_μ). The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. In Section 2.3, we explain how this notion of geometric quantization extends further to the case of singular symplectic quotients. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \widehat{K}$: in particular $\mathcal{Q}(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

Definition 1.2. *Let (M, Ω, Φ) be a* proper *Hamiltonian K-manifold which is pre-quantized by a Kostant-Souriau line bundle L. The formal quantization of (M, Ω, Φ)*

is the element of $R^{-\infty}(K)$ *defined by*

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu)\, V_\mu^K \ .$$

When M is compact, the fact that

$$\mathcal{Q}_K(M) = \mathcal{Q}_K^{-\infty}(M) \tag{1.3}$$

is known as the "quantization commutes with reduction Theorem". This was conjectured by Guillemin-Sternberg in [9] and was first proved by Meinrenken [17] and Meinrenken-Sjamaar [18]. Other proofs of (1.3) were also given by Tian-Zhang [26] and the author [19]. For complete references on the subject the reader should consult [25, 28].

One of the main feature of the formal geometric quantization $\mathcal{Q}^{-\infty}$ is its stability relatively to the restriction to subgroups.

Theorem 1.3 ([21])**.** *Let M be a pre-quantized Hamiltonian K-manifold which is* proper*. Let $H \subset K$ be a closed connected Lie subgroup such that M is still* proper *as a Hamiltonian H-manifold. Then $\mathcal{Q}_K^{-\infty}(M)$ is H-admissible and we have $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$ in $R^{-\infty}(H)$.*

When M is a proper Hamiltonian K-manifold, we can also define another "formal geometric quantization", denoted

$$\mathcal{Q}_K^{\Phi}(M) \in R^{-\infty}(K), \tag{1.4}$$

by localizing the index of the Dolbeault-Dirac operator D_L on the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of the square of the moment map (see Section 2.2 for the precise definiton). We proved in previous papers [20, 21, 23] that

$$\mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^{\Phi}(M). \tag{1.5}$$

in somes situations:

- M is a coadjoint orbit of a semi-simple Lie group S that parametrizes a representation of the discrete series of S,
- M is a Hermitian vector space.

In her ICM 2006 plenary lecture [29], Vergne conjectured that (1.5) holds when $\mathrm{Cr}(\|\Phi\|^2)$ is compact. Recently, Ma and Zhang [15] prove the following generalisation of this conjecture.

Theorem 1.4. *The equality (1.5) holds for* **any** *proper Hamiltonian K-manifold.*

This article is dedicated to the study of the quantization map $\mathcal{Q}^{\Phi}$:

- In Section 2.2, we give the precise definition of the quantization process $\mathcal{Q}^{\Phi}$. In particular, we refine the constant c_γ appearing in [15][Theorem 0.1].
- In Section 2.4, we explain how to compute the quantization of a point.
- We give in Section 3 another proof of Theorem 1.4 by using the technique of symplectic cutting developped in [21].
- In Section 4, we consider the case where $K = K_1 \times K_2$ acts on M in a way that the symplectic reduction $M/\!\!/_0 K_1$ is a *smooth* proper K_2-Hamiltonian manifold. We show then that the K_1-invariant part of $\mathcal{Q}_{K_1\times K_2}^{\Phi}(M)$ is equal to $\mathcal{Q}_{K_2}^{\Phi_2}(M/\!\!/_0 K_1)$.

In Section 5, we study the example where M is the cotangent bundle of a homogeneous space: $M = \mathrm{T}^*(K/H)$ where H is a closed subgroup of K. We see that $\mathrm{T}^*(K/H)$ is a proper Hamiltonian K-manifold prequantized by the trivial line bundle. A direct computation gives

$$\mathcal{Q}_K^{\Phi}(\mathrm{T}^*(K/H)) = \mathrm{L}^2(K/H) \quad \text{in} \quad R^{-\infty}(K). \tag{1.6}$$

Let us denoted $[\mathrm{T}^*(K/H)]_{\mu,K}$ the symplectic reduction at $\mu \in \widehat{K}$ of the K-Hamiltonian manifold $\mathrm{T}^*(K/H)$. Theorem 1.4 together with (1.6) give

$$\mathcal{Q}\left([\mathrm{T}^*(K/H)]_{\mu,K}\right) = \dim[V_\mu^K]^H,$$

for any $\mu \in \widehat{K}$. Here $[V_\mu^K]^H \subset V_\mu^K$ is the subspace of H-invariant vectors.

Then we consider the action of a closed connected subgroup $G \subset K$ on $\mathrm{T}^*(K/H)$. We first check that $\mathrm{T}^*(K/H)$ is a *proper* Hamiltonian G-manifold if and only if the restriction $\mathrm{L}^2(K/H)|_G$ is an admissible G-representation. Then, using Theorem 1.3, we get that

$$\mathcal{Q}_G^{-\infty}(\mathrm{T}^*(K/H)) = \mathrm{L}^2(K/H)|_G \quad \text{in} \quad R^{-\infty}(G). \tag{1.7}$$

In other words, the multiplicity of V_λ^G in $\mathrm{L}^2(K/H)$ is equal to the quantization of the reduced space $[\mathrm{T}^*(K/H)]_{\lambda,G}$.

2. Quantizations of non-compact manifolds

In this section we define the quantization process $\mathcal{Q}^{\Phi}$, and we give another definition of the quantization process $\mathcal{Q}^{-\infty}$ that uses the notion of symplectic cutting [21].

2.1. **Transversally elliptic symbols.** Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [6, 7] and Paradan-Vergne [22]. For a short introduction see [19].

Let $\mathcal{X}$ be a *compact* K-manifold. Let $p : \mathrm{T}\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-,-)_{\mathcal{X}}$ be a K-invariant Riemannian metric. If E^0, E^1 are K-equivariant complex vector bundles over $\mathcal{X}$, a K-equivariant morphism $\sigma \in \Gamma(\mathrm{T}\mathcal{X}, \hom(p^*E^0, p^*E^1))$ is called a *symbol* on $\mathcal{X}$. The subset of all $(x,v) \in \mathrm{T}\mathcal{X}$ where[1] $\sigma(x,v) : E_x^0 \to E_x^1$ is not invertible is called the *characteristic set* of σ, and is denoted by $\mathrm{Char}(\sigma)$.

In the following, the product of a symbol σ by a complex vector bundle $F \to M$, is the symbol

$$\sigma \otimes F$$

defined by $\sigma \otimes F(x,v) = \sigma(x,v) \otimes \mathrm{Id}_{F_x}$ from $E_x^0 \otimes F_x$ to $E_x^1 \otimes F_x$. Note that $\mathrm{Char}(\sigma \otimes F) = \mathrm{Char}(\sigma)$.

Let $\mathrm{T}_K\mathcal{X}$ be the following subset of $\mathrm{T}\mathcal{X}$:

$$\mathrm{T}_K\mathcal{X} = \{(x,v) \in \mathrm{T}\mathcal{X},\ (v, X_{\mathcal{X}}(x))_{\mathcal{X}} = 0 \quad \text{for all } X \in \mathfrak{k}\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathrm{T}\mathcal{X}$ (i.e. $\mathrm{Char}(\sigma)$ is compact), and is *K-transversally elliptic* if the restriction of σ to $\mathrm{T}_K\mathcal{X}$ is invertible outside a compact subset of $\mathrm{T}_K\mathcal{X}$ (i.e. $\mathrm{Char}(\sigma) \cap \mathrm{T}_{K_2}\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant $\mathbf{K}$-theory of $\mathrm{T}\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}_K(\mathrm{T}\mathcal{X})$, and the index of σ is a virtual finite dimensional representation of K, that we denote $\mathrm{Index}_{\mathcal{X}}^K(\sigma) \in R(K)$ [2, 3, 4, 5].

[1] The map $\sigma(x,v)$ will be also denote $\sigma|_x(v)$

Let

$$R_{tc}^{-\infty}(K) \subset R^{-\infty}(K)$$

be the $R(K)$-submodule formed by all the infinite sum $\sum_{\mu\in\widehat{K}} m_\mu V_\mu^K$ where the map $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$ has at most a *polynomial* growth. The $R(K)$-module $R_{tc}^{-\infty}(K)$ is the Grothendieck group associated to the *trace class* virtual K-representations: we can associate to any $V \in R_{tc}^{-\infty}(K)$, its trace $k \to \mathrm{Tr}(k, V)$ which is a generalized function on K invariant by conjugation. Then the trace defines a morphism of $R(K)$-module

$$R_{tc}^{-\infty}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^K. \tag{2.8}$$

A *K-transversally elliptic* symbol σ defines an element of $\mathbf{K}_K(\mathrm{T}_K\mathcal{X})$, and the index of σ is defined as a trace class virtual representation of K, that we still denote $\mathrm{Index}_{\mathcal{X}}^K(\sigma) \in R_{tc}^{-\infty}(K)$.

Remark that any elliptic symbol of $\mathrm{T}\mathcal{X}$ is K-transversally elliptic, hence we have a restriction map $\mathbf{K}_K(\mathrm{T}\mathcal{X}) \to \mathbf{K}_K(\mathrm{T}_K\mathcal{X})$, and a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_K(\mathrm{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_K(\mathrm{T}_K\mathcal{X}) \\ \mathrm{Index}_{\mathcal{X}}^K \downarrow & & \downarrow \mathrm{Index}_{\mathcal{X}}^K \\ R(K) & \longrightarrow & R_{tc}^{-\infty}(K)\ . \end{array} \tag{2.9}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}_{\mathcal{U}}^K : \mathbf{K}_K(\mathrm{T}_K\mathcal{U}) \to R_{tc}^{-\infty}(K)$ is still defined when $\mathcal{U}$ is a K-invariant relatively compact open subset of a K-manifold (see [19][section 3.1]).

Suppose now that the group K is equal to the product $K_1 \times K_2$. When a symbol σ is $K_1 \times K_2$-transversaly elliptic we will be interested in the K_1-invariant part of its index, that we denote

$$\left[\mathrm{Index}_{\mathcal{X}}^{K_1\times K_2}(\sigma)\right]^{K_1} \in R_{tc}^{-\infty}(K_2).$$

An intermediate notion between the "ellipticity" and "$K_1 \times K_2$-transversal ellipticity" is the "K_1-transversal ellipticity". When a $K_1 \times K_2$-equivariant morphism σ is *K_1-transversally elliptic*, its index $\mathrm{Index}_{\mathcal{X}}^{K_1\times K_2}(\sigma) \in R_{tc}^{-\infty}(K_1 \times K_2)$, viewed as a generalized function on $K_1 \times K_2$, is *smooth* relatively to the variable in K_2. It implies that $\mathrm{Index}_{\mathcal{X}}^{K_1\times K_2}(\sigma) = \sum_\lambda \theta(\lambda) \otimes V_\lambda^{K_1}$ with

$$\theta(\lambda) \in R(K_2), \quad \forall\ \lambda \in \widehat{K_1}.$$

In particular, we know that

$$\left[\mathrm{Index}_{\mathcal{X}}^{K_1\times K_2}(\sigma)\right]^{K_1} = \theta(0)$$

belongs to $R(K_2)$.

Let us recall the multiplicative property of the index map for the product of manifolds that was proved by Atiyah-Singer in [1]. Consider a compact Lie group K_2 acting on two manifolds $\mathcal{X}_1$ and $\mathcal{X}_2$, and assume that another compact Lie group K_1 acts on $\mathcal{X}_1$ commuting with the action of K_2.

The external product of complexes on $\mathrm{T}\mathcal{X}_1$ and $\mathrm{T}\mathcal{X}_2$ induces a multiplication (see [1, 22]):

$$\odot : K_{K_1\times K_2}(\mathrm{T}_{K_1}\mathcal{X}_1) \times K_{K_2}(\mathrm{T}_{K_2}\mathcal{X}_2) \longrightarrow K_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(\mathcal{X}_1 \times \mathcal{X}_2)).$$

The following property will be used frequently in the paper.

Theorem 2.1 (Multiplicative property)**.** *For any* $[\sigma_1] \in K_{K_1\times K_2}(\mathrm{T}_{K_1}\mathcal{X}_1)$ *and any* $[\sigma_2] \in K_{K_2}(\mathrm{T}_{K_2}\mathcal{X}_2)$ *we have*

$$\mathrm{Index}^{K_1\times K_2}_{\mathcal{X}_1\times\mathcal{X}_2}([\sigma_1]\odot[\sigma_2]) = \mathrm{Index}^{K_1\times K_2}_{\mathcal{X}_1}([\sigma_1])\otimes \mathrm{Index}^{K_2}_{\mathcal{X}_2}([\sigma_2]).$$

We will use in this article the notion of *support* of a generalized character.

Definition 2.2. *The* support *of* $\chi := \sum_{\mu\widehat{K}} a_\mu V^K_\mu \in R^{-\infty}(K)$ *is the set of* $\mu \in \widehat{K}$ *such that* $a_\mu \neq 0$*.*

We will say that $\chi \in R^{-\infty}(K)$ is supported outside $B \subset \mathfrak{t}^*$ if the support of χ does not intersect B. Note that an infinite sum $\sum_{i\in I}\chi_i$ converges in $R^{-\infty}(K)$ if for each ball

$$B_r = \{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}$$

the set $\{i \in I \mid \mathrm{support}(\chi_i)\cap B_r \neq \emptyset\}$ is finite.

Definition 2.3. *We denote by* $O(r)$ *any character of* $R^{-\infty}(K)$ *which is supported outside the ball* B_r*.*

2.2. **Definition and first properties of $\mathcal{Q}^\Phi$.** Let (M,Ω,Φ) be a proper Hamiltonian K-manifold prequantized by an equivariant line bundle L. Let J be an invariant almost complex structure compatible with Ω. Let $p : \mathrm{T}M \to M$ be the projection.

Let us first describe the principal symbol of the Dolbeault-Dirac operator $\overline{\partial}_L + \overline{\partial}^*_L$. The complex vector bundle $(\mathrm{T}^*M)^{0,1}$ is K-equivariantly identified with the tangent bundle $\mathrm{T}M$ equipped with the complex structure J. Let h be the Hermitian structure on $(\mathrm{T}M, J)$ defined by : $h(v,w) = \Omega(v,Jw) - i\Omega(v,w)$ for $v,w \in \mathrm{T}M$. The symbol

$$\mathrm{Thom}(M,J) \in \Gamma\left(M, \hom(p^*(\wedge^{even}_{\mathbb{C}}\mathrm{T}M),\, p^*(\wedge^{odd}_{\mathbb{C}}\mathrm{T}M))\right)$$

at $(m,v) \in \mathrm{T}M$ is equal to the Clifford map

$$\mathbf{c}_m(v) \;:\; \wedge^{even}_{\mathbb{C}}\mathrm{T}_mM \longrightarrow \wedge^{odd}_{\mathbb{C}}\mathrm{T}_mM, \tag{2.10}$$

where $\mathbf{c}_m(v).w = v\wedge w - \iota(v)w$ for $w \in \wedge^\bullet_{\mathbb{C}}\mathrm{T}_mM$. Here $\iota(v) : \wedge^\bullet_{\mathbb{C}}\mathrm{T}_mM \to \wedge^{\bullet-1}\mathrm{T}_mM$ denotes the contraction map relative to h. Since $\mathbf{c}_m(v)^2 = -\|v\|^2\mathrm{Id}$, the map $\mathbf{c}_m(v)$ is invertible for all $v \neq 0$. Hence the characteristic set of $\mathrm{Thom}(M,J)$ corresponds to the 0-section of $\mathrm{T}M$.

It is a classical fact that the principal symbol of the Dolbeault-Dirac operator $\overline{\partial}_L + \overline{\partial}^*_L$ is equal to[2]

$$\mathrm{Thom}(M,J)\otimes L, \tag{2.11}$$

see [8]. Here also we have $\mathrm{Char}(\mathrm{Thom}(M,J)\otimes L) = 0 - \mathrm{section\ of\ T}M$.

Remark 2.4. *When the manifold* M *is a product* $M_1\times M_2$ *the symbol* $\mathrm{Thom}(M,J)\otimes L$ *is equal to the product* $\sigma_1\odot\sigma_2$ *where* $\sigma_k = \mathrm{Thom}(M_k,J_k)\otimes L_k$*.*

[2]Here we use an identification $\mathrm{T}^*M \simeq \mathrm{T}M$ given by an invariant Riemannian metric.

When M is compact, the symbol $\mathrm{Thom}(M,J)\otimes L$ is elliptic and then defines an element of the equivariant $\mathbf{K}$-group of $\mathrm{T}M$. The topological index of $\mathrm{Thom}(M,J)\otimes L \in \mathbf{K}_K(\mathrm{T}M)$ is equal to the analytical index of the Dolbeault-Dirac operator $\overline{\partial}_L + \overline{\partial}_L^*$:

$$\mathcal{Q}_K(M) = \mathrm{Index}_M^K(\mathrm{Thom}(M,J)\otimes L) \quad \text{in} \quad R(K). \tag{2.12}$$

When M is not compact the topological index of $\mathrm{Thom}(M,J)\otimes L$ is not defined. In order to extend the notion of geometric quantization to this setting we deform the symbol $\mathrm{Thom}(M,J)\otimes L$ in the "Witten" way [19, 20]. Consider the identification $\xi \mapsto \widetilde{\xi}, \mathfrak{k}^* \to \mathfrak{k}$ defined by a K-invariant scalar product on $\mathfrak{k}^*$. We define the *Kirwan vector field* on M :

$$\kappa_m = \left(\widetilde{\Phi(m)}\right)_M(m), \quad m \in M. \tag{2.13}$$

Definition 2.5. *The symbol* $\mathrm{Thom}(M,J)\otimes L$ *pushed by the vector field* κ *is the symbol* $\mathbf{c}^\kappa$ *defined by the relation*

$$\mathbf{c}^\kappa|_m(v) = \mathrm{Thom}(M,J)\otimes L|_m(v-\kappa_m)$$

for any $(m,v)\in \mathrm{T}M$.

Note that $\mathbf{c}^\kappa|_m(v)$ is invertible except if $v=\kappa_m$. If furthermore v belongs to the subset $\mathrm{T}_K M$ of tangent vectors orthogonal to the K-orbits, then $v=0$ and $\kappa_m = 0$. Indeed κ_m is tangent to $K\cdot m$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2$, the set of zeros of κ coincides with the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of $\|\Phi\|^2$. Finally we have

$$\mathrm{Char}(\mathbf{c}^\kappa)\cap \mathrm{T}_K M \simeq \mathrm{Cr}(\|\Phi\|^2).$$

In general $\mathrm{Cr}(\|\Phi\|^2)$ is not compact, so $\mathbf{c}^\kappa$ does not define a transversally elliptic symbol on M. In order to define a kind of index of $\mathbf{c}^\kappa$, we proceed as follows. For any invariant open relatively compact subset $U\subset M$ the set $\mathrm{Char}(\mathbf{c}^\kappa|_U)\cap \mathrm{T}_K U \simeq \mathrm{Cr}(\|\Phi\|^2)\cap U$ is compact when

$$\partial\mathcal{U}\cap \mathrm{Cr}(\|\Phi\|^2) = \emptyset. \tag{2.14}$$

When (2.14) holds we denote

$$\mathcal{Q}_K^\Phi(U) := \mathrm{Index}_U^K(\mathbf{c}^\kappa|_U) \quad \in \quad R_{tc}^{-\infty}(K) \tag{2.15}$$

the equivariant index of the transversally elliptic symbol $\mathbf{c}^\kappa|_U$.

It will be usefull to understand the dependance of the generalized character $\mathcal{Q}_K^\Phi(U)$ relatively to the data (U,Ω,L). So we consider two proper Hamilonian K-manifolds (M,Ω,Φ) and (M',Ω',Φ') respectively prequantized by the line bundles L and L'. Let $V\subset M$ and $V'\subset M'$ two invariant open subsets.

Proposition 2.6. • *The generalized character* $\mathcal{Q}_K^\Phi(U)$ *does not depend of the choice of an invariant almost complex structure on* U *which is compatible with* $\Omega|_U$.

• *Suppose that there exists an equivariant diffeomorphism* $\Psi : V\to V'$ *such that*

(1) $\Psi^*(\Phi') = \Phi$,
(2) $\Psi^*(L') = L$,
(3) *there exists an homotopy of symplectic forms taking* $\Psi^*(\Omega'|_{V'})$ *to* $\Omega|_V$.

Let $U' \subset \overline{U'} \subset V'$ be an invariant open relatively compact subset such that $\partial U'$ satisfies (2.14). Take $U = \Psi^{-1}(U')$. Then ∂U satisfies (2.14) and

$$\mathcal{Q}_K^{\Phi'}(U') = \mathcal{Q}_K^{\Phi}(U) \quad \in \quad R^{-\infty}(K).$$

Proof. Let us prove the first point. Let $\mathbf{c}_i^\kappa|_U, i = 0, 1$ be the transversally elliptic symbols defined with the compatible almost complex structure $J_i, i = 0, 1$. Since the space of compatible almost complex structure is contractible, there exist an homotopy $J_t, t \in [0, 1]$ of almost complex structures linking J_0 and J_1. If we use Lemma 2.2 in [19], we know that there exists an invertible bundle map $A \in \Gamma(U, \mathrm{End}(\mathrm{T}U))$, homotopic to the identity, such that $A \circ J_0 = J_1 \circ A$. With the help of A we prove then that the symbols $\mathbf{c}_0^\kappa|_U$ and $\mathbf{c}_1^\kappa|_U$ define the same class in $\mathbf{K}_K(\mathrm{T}_K U)$ (see [19][Lemma 2.2]). Hence their equivariant index coincide.

Let us prove the second point.The characters $\mathcal{Q}_K^\Phi(U)$ and $\mathcal{Q}_K^{\Phi'}(U')$ are computed as the equivariant index of the symbols $\mathbf{c}^\kappa|_U$ and $\mathbf{c}^{\kappa'}|_{U'}$. Let $\tilde{\mathbf{c}}^\kappa|_U$ the pull back of $\mathbf{c}^{\kappa'}|_{U'}$ by Ψ. Thanks to the point (1) and (2), the only thing which differs in the definitions of the symbols $\mathbf{c}^\kappa|_U$ and $\tilde{\mathbf{c}}^\kappa|_U$ are the almost complex structures J and $\tilde{J} = \Psi^*(J')$: the first one is comptible with Ω and the second one with $\Psi^*(\Omega'|_{V'})$. Since these two symplectic structure are homotopic, one sees that the almost complex structures J and $\tilde{J}$ are also homotopic. So we can conclude like in the first point. □

Let us recall the basic fact concerning the singular values of $\|\Phi\|^2$.

Lemma 2.7. *The set of singular values of $\|\Phi\|^2 : M \to \mathbb{R}$ forms a sequence $0 \leq r_1 < r_2 < \ldots < r_k < \ldots$ which is finite iff $\mathrm{Cr}(\|\Phi\|^2)$ is compact. In the other case $\lim_{k\to\infty} r_k = \infty$.*

At each regular value R of $\mathrm{Cr}(\|\Phi\|^2)$, we associate the invariant open subset $M_{<R} := \{\|\Phi\|^2 < R\}$ which satisfies (2.14). The restriction $\mathbf{c}^\kappa|_{M_{<R}}$ defines then a transversally elliptic symbol on $M_{<R}$: let $\mathcal{Q}_K^\Phi(M_{<R})$ be its equivariant index.

Let us show that $\mathcal{Q}_K^\Phi(M_{<R})$ has a limit when $R \to \infty$. The set $\mathrm{Cr}(\|\Phi\|^2)$ has the following decomposition

$$\mathrm{Cr}(\|\Phi\|^2) = \bigcup_{\beta\in\mathcal{B}} \underbrace{K \cdot (M^{\widetilde{\beta}} \cap \Phi^{-1}(\beta))}_{Z_\beta} \tag{2.16}$$

where the $\mathcal{B}$ is a subset of the Weyl chamber $\mathfrak{t}_+^*$. Note that each part Z_β is compact, hence $\mathcal{B}$ is finite only if $\mathrm{Cr}(\|\Phi\|^2)$ is compact. When $\mathrm{Cr}(\|\Phi\|^2)$ is non-compact, the set $\mathcal{B}$ is infinite, but it is easy to see that $\mathcal{B} \cap B_r$ is finite for any $r \geq 0$. For any $\beta \in \mathcal{B}$, we consider a relatively compact open invariant neighborhood $\mathcal{U}_\beta$ of Z_β such that $\mathrm{Cr}(\|\Phi\|^2) \cap \overline{\mathcal{U}_\beta} = Z_\beta$.

Definition 2.8. *We denote*

$$\mathcal{Q}_K^\beta(M) \in R_{tc}^{-\infty}(K)$$

the index[3] *of the transversally elliptic symbol $\mathbf{c}^\kappa|_{\mathcal{U}_\beta}$.*

A simple application of the excision property [19] gives that

$$\mathcal{Q}_K^\Phi(M_{<R}) := \sum_{\|\beta\|^2<R} \mathcal{Q}_K^\beta(M). \tag{2.17}$$

[3]The index of $\mathbf{c}^\kappa|_{\mathcal{U}_\beta}$ was denoted $RR_\beta^K(M, L)$ in [19].

We have now the key fact

Theorem 2.9. *The generalized character $\mathcal{Q}_K^\beta(M)$ is supported outside the open ball $B_{\|\beta\|}$.*

Proof. Proposition 2.9 follows directly from the computations done in [19]. First consider the case where $\beta \neq 0$ is a K-invariant element of $\mathcal{B}$. Let $i : \mathbb{T}_\beta \hookrightarrow T$ be the compact torus generated by β. If F is $\mathbb{Z}$-module we denote $F \widehat{\otimes} R^{-\infty}(\mathbb{T}_\beta)$ the $\mathbb{Z}$-module formed by the infinite formal sums $\sum_a E_a\, h^a$ taken over the set of weights of $\mathbb{T}_\beta$, where $E_a \in F$ for every a.

Since $\mathbb{T}_\beta$ lies in the center of K, the morphism $\pi : (k,t) \in K \times \mathbb{T}_\beta \mapsto kt \in K$ induces a map $\pi^* : R^{-\infty}(K) \to R^{-\infty}(K)\, \widehat{\otimes}\, R^{-\infty}(\mathbb{T}_\beta)$.

The normal bundle $\mathcal{N}$ of $M^{\widetilde{\beta}}$ in M inherits a canonical complex structure $J_{\mathcal{N}}$ on the fibers. We denote by $\overline{\mathcal{N}} \to M^{\widetilde{\beta}}$ the complex vector bundle with the opposite complex structure. The torus $\mathbb{T}_\beta$ is included in the center of K, so the bundle $\overline{\mathcal{N}}$ and the virtual bundle $\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}} := \wedge_{\mathbb{C}}^{even} \overline{\mathcal{N}} \xrightarrow{0} \wedge_{\mathbb{C}}^{odd} \overline{\mathcal{N}}$ carry a $K \times \mathbb{T}_\beta$-action: they can be considered as elements of $K_{K\times\mathbb{T}_\beta}(M^{\widetilde{\beta}}) = K_K(M^{\widetilde{\beta}}) \otimes R(\mathbb{T}_\beta)$.

In [19], we have defined an inverse of $\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}$, $\left[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}\right]_\beta^{-1} \in K_K(M^{\widetilde{\beta}})\, \widehat{\otimes}\, R^{-\infty}(\mathbb{T}_\beta)$, which is polarized by β. It means that $\left[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}\right]_\beta^{-1} = \sum_a N_a\, h^a$ with $N_a \neq 0$ only if $\langle a, \beta\rangle \geq 0$.

We prove in [19] the following localization formula :

$$\pi^*\left[\mathcal{Q}_K^\beta(M)\right] = RR_\beta^{K\times\mathbb{T}_\beta}\left(M^{\widetilde{\beta}}, L_{|M^\beta} \otimes \left[\wedge_{\mathbb{C}}^\bullet \overline{\mathcal{N}}\right]_\beta^{-1}\right)\ , \tag{2.18}$$

as an equality in $R^{-\infty}(K)\, \widehat{\otimes}\, R^{-\infty}(\mathbb{T}_\beta)$. With (2.18) in hand, it is easy to see that V_μ^K occurs in the character $\mathcal{Q}_K^\beta(M)$ only if $(\mu,\beta) \geq \|\beta\|^2$ (See Lemma 9.4 in [19]).

Now we consider the case were $\beta \in \mathcal{B}$ is not a K-invariant element. Let σ be the unique open face of the Weyl chamber $\mathfrak{t}_+^*$ which contains β. Let K_σ be the corresponding stabilizer subgroup. We consider the symplectic slice $\mathcal{Y}_\sigma \subset M$: it is a K_σ invariant Hamiltonian submanifold of M which is prequantized by the line bundle $L|_{\mathcal{Y}_\sigma}$. The restriction of Φ to $\mathcal{Y}_\sigma$ is a moment map $\Phi_\sigma : \mathcal{Y}_\sigma \to \mathfrak{k}_\sigma^*$ which is proper in a neighborhood of $\beta \in \mathfrak{k}_\sigma^*$. The set

$$K_\sigma \cdot (\mathcal{Y}_\sigma^{\widetilde{\beta}} \cap \Phi_\sigma^{-1}(\beta)) = M^{\widetilde{\beta}} \cap \Phi^{-1}(\beta)$$

is a component of $\mathrm{Cr}(\|\Phi_\sigma\|^2)$. Let $\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) \in R_{tc}^{-\infty}(K_\sigma)$ be the corresponding character (see Definition 2.8).

We prove in [19][Section 7], the following induction formula:

$$\mathcal{Q}_K^\beta(M) = \mathrm{Hol}_{K_\sigma}^K\left(\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma)\right) \tag{2.19}$$

where $\mathrm{Hol}_{K_\sigma}^K : R^{-\infty}(K_\sigma) \to R^{-\infty}(K)$ is the holomorphic induction map. See the Appendix in [19] for the definition and properties of these induction maps.

We know from the previous case that

$$\mathcal{Q}_{K_\sigma}^\beta(\mathcal{Y}_\sigma) = \sum_{\mu \in \widehat{K_\sigma}} m_\mu V_\mu^{K_\sigma}$$

where $m_\mu \neq 0 \Longrightarrow (\mu,\beta) \geq \|\beta\|^2$. Then, with (2.19), we get

$$\begin{aligned} \mathcal{Q}^\beta_{K_\sigma}(\mathcal{Y}_\sigma) &= \sum_{(\mu,\beta)\geq\|\beta\|^2} m_\mu \ \mathrm{Hol}^K_{K_\sigma}(V_\mu^{K_\sigma}) \\ &= \sum_{(\mu,\beta)\geq\|\beta\|^2} m_\mu \ \mathrm{Hol}^K_T(t^\mu), \end{aligned}$$

where $\mathrm{Hol}^K_T : R^{-\infty}(T) \to R^{-\infty}(K)$ is the holomorphic induction map.

Let ρ be half the sum of the positive roots. The term $\mathrm{Hol}^K_T(t^\mu)$ is equal to 0 when $\mu+\rho$ is not a regular element of $\mathfrak{t}^*$. When $\mu+\rho$ is a regular element of $\mathfrak{t}^*$, we have $\mathrm{Hol}^K_T(t^\mu) = (-1)^{|\omega|} V^K_{\mu_\omega}$ where

$$\mu_\omega = \omega(\mu+\rho)-\rho$$

is dominant for a unique $\omega \in W$.

Finally, a representation V^K_λ appears in the character $\mathcal{Q}^\beta_K(M)$ only if $\lambda = \mu_\omega$ for a weight μ satisfying $(\mu,\beta) \geq \|\beta\|^2$. Hence, for such λ, we have

$$\begin{aligned} \|\lambda\| &= \|\mu+\rho-\omega^{-1}\rho\| \\ &\geq (\mu+\rho-\omega^{-1}\rho, \frac{\beta}{\|\beta\|}) \\ &\geq \|\beta\|. \end{aligned}$$

In the last inequality we use that $(\rho-\omega^{-1}\rho,\beta) \geq 0$ since $\rho-\omega^{-1}\rho$ is a sum of positive roots, and $\beta \in \mathfrak{t}^*_+$.

□

With the help of Theorem 2.9 and decomposition (2.17), we see that the multiplicity of V^K_γ in $\mathcal{Q}^\Phi_K(M_{<R})$ does not depend on the regular value $R > \|\gamma\|^2$. We can refine the constant c_γ appearing in [15][Theorem 0.1]: take c_γ equal to $\|\gamma\|^2$ instead of[4] $\|\gamma+\rho\|^2 - \|\rho\|^2 \geq \|\gamma\|^2$.

Definition 2.10. *The generalized character $\mathcal{Q}^\Phi_K(M)$ is defined as the limit in $R^{-\infty}(K)$ of $\mathcal{Q}^\Phi_K(M_{<R})$ when R goes to infinity. In other words*

$$\mathcal{Q}^\Phi_K(M) = \sum_{\beta\in\mathcal{B}} \mathcal{Q}^\beta_K(M). \tag{2.20}$$

Note that for any regular value R of $\|\Phi\|^2$ we have the useful relation

$$\mathcal{Q}^\Phi_K(M) = \mathcal{Q}^\Phi_K(M_{<R}) + O(\sqrt{R}). \tag{2.21}$$

2.3. **Quantization of a symplectic quotient.** We will now explain how we define the geometric quantization of *singular* compact Hamiltonian manifolds : here "singular" means that the manifold is obtained by symplectic reduction.

Let (N,Ω) be a smooth symplectic manifold equipped with a Hamiltonian action of $K_1 \times K_2$: we denote $(\Phi_1,\Phi_2) : N \to \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ the corresponding moment map. We assume that N is pre-quantized by a $K_1\times K_2$-equivariant line bundle L and we

[4]Here ρ is half the sum of the positive roots. Hence $\|\gamma+\rho\|^2 - \|\rho\|^2 - \|\gamma\|^2 = 2(\rho,\gamma) \geq 0$ and $(\rho,\gamma)=0$ only if the weight γ belongs to the center of $\mathfrak{k} \simeq \mathfrak{k}^*$.

suppose that the map Φ_1 is **proper**. One wants to define the geometric quantization of the (compact) symplectic quotient

$$N/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1.$$

Let κ_1 be the Kirwan vector field attached to the moment map Φ_1. We denote by $\mathbf{c}^{\kappa_1}$ the symbol $\mathrm{Thom}(N,J)\otimes L$ pushed by the vector field κ_1. For any regular value R_1 of $\|\Phi_1\|^2$, we consider the restriction $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$ to the invariant, open subset $N_{<R_1} := \{\|\Phi_1\|^2 < R_1\}$. The symbol $\mathbf{c}^{\kappa_1}|_{N_{<R_1}}$ is $K_1\times K_2$-equivariant and K_1-transversally elliptic, hence we can consider its index

$$\mathrm{Index}^{K_1\times K_2}_{N_{<R_1}}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}}) \in R^{-\infty}(K_1\times K_2).$$

which is smooth relatively to the parameter in K_2. We consider the following extension of Definition 2.10.

Definition 2.11. *The generalized character $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$ is defined as the limit in $R^{-\infty}(K_1\times K_2)$ of $\mathrm{Index}^{K_1\times K_2}_{N_{<R_1}}(\mathbf{c}^{\kappa_1}|_{N_{<R_1}})$ when R_1 goes to infinity.*

Here $\mathrm{Cr}(\|\Phi_1\|^2)$ is equal to the disjoint union of the compact $K_1\times K_2$-invariant subsets $Z_{\beta_1} := K_1\cdot(M^{\widetilde{\beta_1}}\cap\Phi_1^{-1}(\beta_1))$, $\beta_1\in\mathcal{B}_1$. For $\beta_1\in\mathcal{B}_1$, we consider an invariant relatively compact open subset $\mathcal{U}_{\beta_1}$ such that: $Z_{\beta_1}\subset\mathcal{U}_{\beta_1}$ and $Z_{\beta_1} = \mathrm{Cr}(\|\Phi_1\|^2)\cap\overline{\mathcal{U}_{\beta_1}}$. Let $\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N)\in R^{-\infty}(K_1\times K_2)$ be the equivariant index of the K_1-transversally elliptic symbol $\mathbf{c}^L_{\kappa^1}|_{\mathcal{U}_{\beta_1}}$. The K_1-transversallity condition imposes that $\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N) = \sum_\lambda \theta^{\beta_1}(\lambda)\otimes V_\lambda^{K_1}$ with

$$\theta^{\beta_1}(\lambda)\in R(K_2),\quad \forall\lambda\in\widehat{K_1}.$$

We have the following extension of Theorem 2.9

Theorem 2.12. *We have $\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N) = \sum_{\lambda\in\widehat{K_1}}\theta^{\beta_1}(\lambda)\otimes V_\lambda^{K_1}$ where $\theta^{\beta_1}(\lambda)\neq 0$ only if $\|\lambda\|\geq\|\beta_1\|$.*

Proof. The proof works exactly like the one of Theorem 2.9. □

Let us explain the "quantization commutes with reduction theorem", or why we can consider the geometric quantization of

$$N/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1$$

as the K_1-invariant part of $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$.

Let us first suppose that 0 is a regular value of Φ_1. Then $N/\!\!/_0 K_1$ is a compact symplectic *orbifold* equipped with a Hamiltonian action of K_2 : the corresponding moment map is induced by the restriction of Φ_2 to $\Phi_1^{-1}(0)$. The symplectic quotient $N/\!\!/_0 K_1$ is pre-quantized by the line orbibundle

$$L_0 := \left(L|_{\Phi_1^{-1}(0)}\right)/K_1.$$

Definition 1.1 extends to the orbifold case. We can still define the geometric quantization of $N/\!\!/_0 K_1$ as the index of an elliptic operator : we denote it by $\mathcal{Q}_{K_2}(N/\!\!/_0 K_1)\in R(K_2)$. We have

Theorem 2.13. *If 0 is a regular value of Φ_1, the K_1-invariant part of $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$ is equal to $\mathcal{Q}_{K_2}(N/\!\!/_0 K_1)\in R(K_2)$.*

Suppose now that 0 is not a regular value of Φ_1. Let T_1 be a maximal torus of K_1, and let $C_1 \subset \mathfrak{t}_1^*$ be a Weyl chamber. Since Φ_1 is proper, the convexity Theorem says that the image of Φ_1 intersects C_1 in a closed locally polyhedral convex set, that we denote $\Delta_{K_1}(N)$ [14].

We consider an element $a \in \Delta_{K_1}(N)$ which is generic and sufficiently close to $0 \in \Delta_{K_1}(N)$: we denote $(K_1)_a$ the subgroup of K_1 which stabilizes a. When $a \in \Delta_{K_1}(N)$ is generic, one can show (see [18]) that

$$N/\!\!/_a K_1 := \Phi_{K_1}^{-1}(a)/(K_1)_a$$

is a compact Hamiltonian K_2-orbifold, and that

$$L_a := \left(L|_{\Phi_{K_1}^{-1}(a)}\right)/(K_1)_a.$$

is a K_2-equivariant line orbibundle over $N/\!\!/_a K_1$: we can then define, like in Definition 1.1, the element $\mathcal{Q}_{K_2}(N/\!\!/_a K_1) \in R(K_2)$ as the equivariant index of the Dolbeault-Dirac operator on $N/\!\!/_a K_1$ (with coefficients in L_a).

Theorem 2.14. *The K_1-invariant part of $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)$ is equal to $\mathcal{Q}_{K_2}(N/\!\!/_a K_1) \in R(K_2)$. In particular, the elements $\mathcal{Q}_{K_2}(N/\!\!/_a K_1)$ do not depend on the choice of the generic element $a \in \Delta_H(N)$, when a is sufficiently close to 0.*

Proofs of Theorem 2.13 an Theorem 2.14 . When N is compact and $K_2 = \{e\}$, the proofs can be found in [18] and in [19]. Let us explain briefly how the **K**-theoretic proof of [19] extends naturally to our case. Like in Definition 2.10, we have the following decomposition

$$\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N) = \sum_{\beta\in\mathcal{B}_1} \mathcal{Q}^{\beta_1}_{K_1\times K_2}(N),$$

And Theorem 2.12 tells us that $\left[\mathcal{Q}^{\beta_1}_{K_1\times K_2}(N)\right]^{K_1} = 0$ if $\beta_1 \neq 0$. We have proved the first step:

$$\left[\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)\right]^{K_1} = \left[\mathcal{Q}^{0}_{K_1\times K_2}(N)\right]^{K_1}.$$

The analysis of the term $\left[\mathcal{Q}^{0}_{K_1\times K_2}(N)\right]^{K_1}$ is undertaken in [19] when $K_2 = \{e\}$: we explain that this term is equal either to $\mathcal{Q}(N/\!\!/_0 K_1)$ when 0 is a regular value, or to $\mathcal{Q}(N/\!\!/_a K_1)$ with a generic. It work similarly with an action of a compact Lie group K_2. □

Definition 2.15. *The geometric quantization of $N/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1$ is taken as the K_1-invariant part of $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(N)$. We denote it $\mathcal{Q}_{K_2}(N/\!\!/_0 K_1)$.*

2.4. **Quantization of points.** Let (M, Ω, Φ) be a proper Hamiltonian K-manifold prequantized by a Kostant-Souriau line bundle L. Let $\mu \in \widehat{K}$ be dominant weight such that $\Phi^{-1}(K\cdot\mu)$ is a K-orbit in M. Let $m^o \in \Phi^{-1}(\mu)$ so that

$$\Phi^{-1}(K\cdot\mu) = K\cdot m^o$$

Then the reduced space $M_\mu := \Phi^{-1}(K\cdot\mu)/K$ is a point. The aim of this section is to compute the quantization of M_μ: $\mathcal{Q}(M_\mu) \in \mathbb{Z}$.

Let H be the stabilizer subgroup of m^o. We have a linear action of H on the 1-dimensional vector space $L_{m^o} \subset L$. We have $H \subset K_\mu$ where K_μ is the connected

subgroup of K that fixes $\mu \in \mathfrak{t}^*$. Let $\mathbb{C}_{-\mu}$ be the 1-dimensional representation of K_μ associated to the infinitesimal character $-i\mu$.

Let us denote χ be the character of H defined by the 1-dimensional representation $\mathbb{C}_\chi := L_{m^o} \otimes \mathbb{C}_{-\mu}$. We know from the Kostant formula (1.1) that $\chi = 1$ on the identity component $H^o \subset H$.

Theorem 2.16. *We have*

$$\mathcal{Q}(M_\mu) = \begin{cases} 1 & \text{if } \chi = 1 \text{ on } H \\ 0 & \text{in the other case.} \end{cases} \tag{2.22}$$

This Theorem tells us in particular that $\mathcal{Q}(M_\mu) = 1$ when the stabiliser subgroup $H \subset K$ of a point $m^o \in \Phi^{-1}(\mu)$ is *connected*.

Proof. Let $N = M \times \overline{K \cdot \mu}$ be the proper Hamiltonian K-manifold which is pre-quantized by the line bundle $L_N := L \otimes [\mathbb{C}_{-\mu}]$. Let us denote Φ_N the moment map on N. Since $\Phi^{-1}(K \cdot \mu)$ is a K-orbit in M, we see that $\Phi_N^{-1}(0)$ is the K-orbit through $n^o := (m^o, \mu)$ where $m^o \in \Phi^{-1}(\mu)$. Note that H is the stabilizer subgroups of n^o.

Let $\mathcal{Q}_K^{\Phi_N}(N) \in R^{-\infty}(K)$ be the formal quantization of N through the proper map Φ_N. By definition

$$\begin{aligned} \mathcal{Q}(M_\mu) &= \left[\mathcal{Q}_K^{\Phi_N}(N)\right]^K \\ &= \left[\mathcal{Q}_K^0(N)\right]^K . \end{aligned}$$

where $\mathcal{Q}_K^0(N)$ depends only of a neighborhood of $\Phi_N^{-1}(0)$.

The orbit $K \cdot n^o \hookrightarrow N$ is an isotropic embedding since it is the 0-level of the moment map Φ_N. Then to describe a K-invariant neighborhood of $K \cdot n^o$ in N we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First we consider, following Weinstein (see [11, 31]), the symplectic normal bundle

$$\mathcal{V} := \mathrm{T}(K \cdot n^o)^{\perp,\Omega} \Big/ \mathrm{T}(K \cdot n^o) , \tag{2.23}$$

where the orthogonal $(\,^{\perp,\Omega})$ is taken relatively to the symplectic 2-form. We have

$$\mathcal{V} = K \times_H V$$

where the vector space $V := \mathrm{T}_{n^o}(K \cdot n^o)^{\perp,\Omega} \Big/ \mathrm{T}_{n^o}(K \cdot n^o)$ inherits a symplectic structure and an Hamiltonian action of the group H: we denote $\Phi_H : V \to \mathfrak{h}^*$ the corresponding moment map.

Consider now the following symplectic manifold

$$\tilde{N} := \mathcal{V} \oplus \mathrm{T}^*(K/H) = K \times_H \Big((\mathfrak{k}/\mathfrak{h})^* \oplus V \Big). \tag{2.24}$$

The action of H on $\tilde{N}$ is Hamiltonian and the moment map $\Phi_{\tilde{N}} : \tilde{N} \to \mathfrak{k}^*$ is given by the equation

$$\Phi_{\tilde{N}}([k; \xi, v]) = k \cdot (\xi + \Phi_H(v)) \quad k \in K,\ \xi \in (\mathfrak{k}/\mathfrak{h})^*,\ v \in V\ . \tag{2.25}$$

The Hamiltonian K-manifold $\tilde{N}$ is prequantized by the line bundle $L_{\tilde{N}} := K \times_H \mathbb{C}_\chi$.

The *local normal form* Theorem (see [10], [24] Proposition 2.5) tells us that there exists a K-Hamiltonian isomorphism $\Upsilon : \mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$ between a K-invariant neighborhood $\mathcal{U}_1$ of $K \cdot n^o$ in N, and a K-invariant neighborhood $\mathcal{U}_2$ of K/H in $\tilde{N}$. This isomorphism Υ, when restricted to $K \cdot n^o$, corresponds to the natural isomorphism $K \cdot n^o \xrightarrow{\sim} K/H$.

Thanks to Υ, we know that the fiber $\Phi_H^{-1}(0) \subset V$ is reduced to $\{0\}$. This last point is equivalent to the fact that Φ_H (and then $\Phi_{\tilde{N}}$) is proper map (see [21]). We check easily that the set of critical points of $\|\Phi_{\tilde{N}}\|^2$ is reduced to $\Phi_{\tilde{N}}^{-1}(0) = K/H$. Then, thank to the isomorphism Υ, we have that

$$\mathcal{Q}_K^0(N) = \mathcal{Q}_K^0(\tilde{N}) = \mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}). \tag{2.26}$$

Let $\mathrm{Ind}_{_H}^{^K} : R^{-\infty}(H) \to R^{-\infty}(K)$ be the induction map that is defined by the relation $\langle \mathrm{Ind}_{_H}^{^K}(\phi), E\rangle = \langle \phi, E|_H\rangle$ for any $\phi \in R^{-\infty}(H)$ and $E \in R(K)$. Note that

$$[\mathrm{Ind}_{_H}^{^K}(\phi)]^K = \langle \mathrm{Ind}_{_H}^{^K}(\phi), \mathbb{C}\rangle = \langle \phi, \mathbb{C}\rangle = [\phi]^H.$$

Since $\Phi_H : V \to \mathfrak{h}^*$ is proper one can consider the quantization of the vector space V through the map Φ_H: $\mathcal{Q}_H^{\Phi_H}(V) \in R^{-\infty}(H)$.

Proposition 2.17. • *We have*

$$\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}) = \mathrm{Ind}_{_H}^{^K}\left(\mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi\right) \tag{2.27}$$

• *The formal quantization $\mathcal{Q}_H^{\Phi_H}(V)$ coincides, as a generalized H-module, to the H-module $S(V^*)$ of polynomial function on V.*

• *The set $[S(V^*)]^{H^o}$ of polynomials invariant by the connected component H^o is reduced to the scalars.*

With the last Proposition we can finish the proof of Theorem 2.16 as follows. We have

$$\begin{aligned}
\mathcal{Q}(M_\mu) &= \left[\mathcal{Q}_K^{\Phi}(N)\right]^K \\
&= \left[\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N})\right]^K \\
&= \left[\mathcal{Q}_H^{\Phi_H}(V) \otimes \mathbb{C}_\chi\right]^H \\
&= [S(V^*) \otimes \mathbb{C}_\chi]^H = [\mathbb{C}_\chi]^H.
\end{aligned}$$

Proof. The first point of Proposition 2.17 follows from the property of induction defined by Atiyah (see Section 3.4 in [19]). Let us explain the arguments. We work with the H-manifold $\mathcal{Y} = (\mathfrak{k}/\mathfrak{h})^* \oplus V$ and the H-equivariant map $j : \mathcal{Y} \hookrightarrow \tilde{N} := K \times_H \mathcal{Y}, y \mapsto [e, y]$.

We notice[5] that $\mathrm{T}\tilde{N} \simeq K \times_H (\mathfrak{k}/\mathfrak{h} \oplus \mathrm{T}\mathcal{Y})$, and that $\mathrm{T}_K\tilde{N} \simeq K \times_H (\mathrm{T}_H\mathcal{Y})$. Hence the map j induces an isomorphism $j_* : K_H(\mathrm{T}_H\mathcal{Y}) \to K_K(\mathrm{T}_K\tilde{N})$. Theorem 4.1 of

[5] These identities come from the following $K \times H$-equivariant isomorphism of vector bundles over $K \times \mathcal{Y}$: $\mathrm{T}_H(\tilde{N}) \to K \times (\mathfrak{k}/\mathfrak{h} \oplus \mathrm{T}\mathcal{Y}), (k, m; \frac{d}{dt}_{|t=0}(ke^{tX}) + v_m) \mapsto (k, m; pr_{\mathfrak{k}/\mathfrak{h}}(X) + v_m)$. Here $pr_{\mathfrak{k}/\mathfrak{h}} : \mathfrak{k} \to \mathfrak{k}/\mathfrak{h}$ is the orthogonal projection.

Atiyah [1] tells us that the following diagram

$$\begin{array}{ccc} K_H(\mathrm{T}_H\mathcal{Y}) & \xrightarrow{j_*} & K_K(\mathrm{T}_K\tilde{N}) \\ \Big\downarrow{\scriptstyle \mathrm{Index}_{\mathcal{Y}}^H} & & \Big\downarrow{\scriptstyle \mathrm{Index}_{\tilde{N}}^K} \\ R^{-\infty}(H) & \xrightarrow[\mathrm{Ind}_H^K]{} & R^{-\infty}(K)\ . \end{array} \tag{2.28}$$

is commutative.

The tangent bundle $\mathrm{T}\tilde{N}$ is equivariantly diffeomorphic to

$$K \times_H [\mathfrak{k}/\mathfrak{h} \oplus (\mathfrak{k}/\mathfrak{h})^* \oplus \mathrm{T}V] \simeq K \times_H [(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus \mathrm{T}V]$$

where $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$ is the complexification of the real vector space $\mathfrak{k}/\mathfrak{h}$. We consider on $\tilde{N}$ the almost complex structure $J_{\tilde{N}} = (i, J_V)$ where i is the complex structure on $(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$ and J_V is a compatible (constant) complex structure on the symplectic vector space V. Note that $J_{\tilde{N}}$ is compatible with the symplectic structure on a neighborhood U of the 0-section of the bundle $\tilde{N} \to K/H$.

Let $\kappa_{\tilde{N}}$ be the Kirwan vector field on $\tilde{N}$:

$$\kappa_{\tilde{N}}([k;\xi,v]) = -\xi + i\,[\xi, \Phi_H(v)] \oplus \kappa_V(v) \quad \in \quad (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus V.$$

Here κ_V is the Kirwan vector field relative to the Hamiltonian action of H on the symplectic vector space V. Note that $\kappa_{\tilde{N}}$ vanishes exactly on the 0-section of the bundle $\tilde{N} \to K/H$.

Let $\mathbf{c}^{\kappa_{\tilde{N}}}$ be the symbol $\mathrm{Thom}(\tilde{N}, J_{\tilde{N}}) \otimes L_{\tilde{N}}$ pushed by the vector field $\kappa_{\tilde{N}}$. The generalized character $\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N})$ is either computed as the equivariant index of the symbols $\mathbf{c}^{\kappa_{\tilde{N}}}$ or $\mathbf{c}^{\kappa_{\tilde{N}}}|_U$.

Remark 2.18. *The fact that $J_{\tilde{N}}$ is not compatible on the entire manifold $\tilde{N}$ is not problematic, since $J_{\tilde{N}}$ is compatible in a neighborhood U of the set where $\kappa_{\tilde{N}}$ vanishes. See the first point of Lemma 2.6.*

For $X + i\eta \oplus w \in \mathrm{T}_{[k;\xi,v]}\tilde{N} \simeq (\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \oplus V$, the map

$$\mathbf{c}^{\kappa_{\tilde{N}}}(X + i\eta \oplus w) = \mathbf{c}\Big(X + \xi + i(\eta - [\xi, \Phi_H(v)]\Big) \odot \mathbf{c}\Big(w - \kappa_V(v)\Big) \tag{2.29}$$

acts on the vector space $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}} \otimes \wedge_{J_V} V \otimes \mathbb{C}_\chi$.

Let $\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})$ be the Bott morphism of the vector space $\mathfrak{k}/\mathfrak{h}$. It is an elliptic morphism defined by

$$\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})|_\xi(\eta) = \mathbf{c}(\xi + i\eta) \quad \text{acting on} \quad \wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}},$$

for $\eta \in \mathrm{T}_\xi(\mathfrak{k}/\mathfrak{h})$. Let $\mathbf{c}^{\kappa_V}$ be the symbol $\mathrm{Thom}(V, J_V)$ pushed by the vector field κ_V.

Lemma 2.19. *We have*

$$\mathbf{c}^{\kappa_{\tilde{N}}} = j_*\Big(\mathrm{Bott}(\mathfrak{k}/\mathfrak{h}) \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_\chi\Big).$$

Proof. We work with the symbol

$$\sigma^T|_{(\xi,v)}(\eta) = \mathbf{c}(\xi + i\eta - iT\,[\xi, \Phi_H(v)])$$

acting on $\wedge_{\mathbb{C}}(\mathfrak{k}/\mathfrak{h})_{\mathbb{C}}$. Note that $\mathrm{Bott}(\mathfrak{k}/\mathfrak{h}) = \sigma^0$. From (2.29), we see that $\mathbf{c}^{\kappa_{\tilde{N}}} = j_*\Big(\sigma^1 \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_\chi\Big)$. It is now easy to check that $\sigma^T \odot \mathbf{c}^{\kappa_V} \otimes \mathbb{C}_\chi, T \in [0,1]$ is an homotopy of transversally elliptic symbols on $\mathfrak{k}/\mathfrak{h} \times V$. □

The commutative diagram (2.28) and the last Lemma gives

$$\begin{aligned}\mathcal{Q}_K^{\Phi_{\tilde{N}}}(\tilde{N}) &= \mathrm{Index}_{\tilde{N}}^K(\mathbf{c}^{\kappa_{\tilde{N}}})\\ &= \mathrm{Ind}_H^K\Big(\mathrm{Index}_{\mathfrak{k}/\mathfrak{h}\times V}^K\Big(\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})\odot\mathbf{c}^{\kappa_V}\Big)\otimes\mathbb{C}_\chi\Big)\\ &= \mathrm{Ind}_H^K\Big(\mathrm{Index}_{\mathfrak{k}/\mathfrak{h}}^K(\mathrm{Bott}(\mathfrak{k}/\mathfrak{h}))\otimes\mathrm{Index}_V^K(\mathbf{c}^{\kappa_V})\otimes\mathbb{C}_\chi\Big)\\ &= \mathrm{Ind}_H^K\Big(\mathcal{Q}_H^{\Phi_H}(V)\otimes\mathbb{C}_\chi\Big).\end{aligned}$$

We have used here that the equivariant index of $\mathrm{Bott}(\mathfrak{k}/\mathfrak{h})$ is equal to 1 (e.g. the trivial representation).

Let us proved now the second point of Proposition 2.17. The Kirwan vector field κ^V satisfies the simple rule:

$$(\kappa^V(v),J_Vv)=-\Omega(\kappa^V(v),v)=\frac{1}{2}\|\Phi_H(v)\|^2,\quad v\in V. \tag{2.30}$$

It shows in particular that $\kappa^V(v)=0\Leftrightarrow\Phi_H(v)=0$. Since the moment map $\Phi_H:V\to\mathfrak{h}^*$ is quadratic, the fact that Φ_H is proper is equivalent to the fact that $\Phi_H^{-1}(0)=0$.

We consider on V the family of symbol σ^s :

$$\sigma^s|_v(w)=\mathbf{c}\Big(w-s\kappa^V(v)-(1-s)J_Vv\Big)$$

viewed as a map from $\wedge_{\mathbb{C}}^{\mathrm{even}}V$ to $\wedge_{\mathbb{C}}^{\mathrm{odd}}V$. Thanks to (2.30), one sees that σ^s is a family of K-transversally elliptic symbol on V. Hence $\sigma^1=\mathbf{c}^{\kappa_V}$ and $\sigma^0=\mathbf{c}(w-J_Vv)$ defines the same class in the group $K_K(\mathrm{T}_KV)$. The symbol σ^0 was first studied by Atiyah [1] when $\dim_{\mathbb{C}}V=1$. The author considered the general case in [19]. We have

$$\mathrm{Index}_V^K(\sigma^0)=S(V^*)\quad\text{in } R^{-\infty}(K).$$

The last point of Proposition 2.17 is a consequence of the properness of the moment map Φ_H (see Section 5 of [21]).

□

□

Example 2.20 ([21])**.** *We consider the action of the unitary group* U_n *on* $\mathbb{C}^n$*. The symplectic form on* $\mathbb{C}^n$ *is defined by* $\Omega(v,w)=\frac{i}{2}\sum_k v_k\overline{w_k}-\overline{v_k}w_k$*. Let us identify the Lie algebra* $\mathfrak{u}_n$ *with its dual through the trace map. The moment map* $\Phi:\mathbb{C}^n\to\mathfrak{u}_n$ *is defined by* $\Phi(v)=\frac{1}{2i}v\otimes v^*$ *where* $v\otimes v^*:\mathbb{C}^n\to\mathbb{C}^n$ *is the linear map* $w\mapsto(\sum_k\overline{v_k}w_k)v$*. One checks easily that the pull-back by* Φ *of a* U_n*-orbit in* $\mathfrak{u}_n$ *is either empty or a* U_n*-orbit in* $\mathbb{C}^n$*. We knows also that the stabiliser subgroup of a non-zero vector of* $\mathbb{C}^n$ *is connected since it is diffeomorphic to* U_{n-1}*. Finally we have*

$$\mathcal{Q}((\mathbb{C}^n)_\mu)=\begin{cases}1 & \text{if } \mu\in\widehat{\mathrm{U}_n} \text{ belongs to the image of } \Phi\\ 0 & \text{if } \mu\in\widehat{\mathrm{U}_n} \text{ does not belongs to the image of } \Phi.\end{cases} \tag{2.31}$$

Then one checks that $\mathcal{Q}_{\mathrm{U}_n}^{-\infty}(\mathbb{C}^n)$ *coincides in* $R^{-\infty}(\mathrm{U}_n)$ *with the algebra* $S((\mathbb{C}^n)^*)$ *of polynomial function on* $\mathbb{C}^n$*.*

Example 2.21 ([23]). *We consider the Lie group* $\mathrm{SL}_2(\mathbb{R})$ *and its compact torus of dimension* 1 *denoted by* T. *The Lie algebra* $\mathfrak{sl}_2(\mathbb{R})$ *is identified with its dual through the trace map, and the Lie algebra* $\mathfrak{t}$ *is naturally identified with* $\mathfrak{sl}_2(\mathbb{R})^T$. *For* $l \in \mathbb{Z} \setminus \{0\}$, *we consider the character* χ_l *of* T *defined by*

$$\chi_l \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = e^{il\theta}.$$

Its differential $\frac{1}{i}d\chi_l \in \mathfrak{t}^*$ *correspond (through the trace map) to the matrix*

$$X_l = \begin{pmatrix} 0 & l/2 \\ -l/2 & 0 \end{pmatrix}.$$

Let $\mathcal{O}_l$ *be the coadjoint orbit of the group* $\mathrm{SL}_2(\mathbb{R})$ *trough the matrix* X_l. *It is a Hamiltonian* $\mathrm{SL}_2(\mathbb{R})$*-manifold prequantized by the* $\mathrm{SL}_2(\mathbb{R})$*-equivariant line bundle* $L_l \simeq \mathrm{SL}_2(\mathbb{R}) \times_T \mathbb{C}_l$, *where* $\mathbb{C}_l$ *is the* T*-module associated to the character* χ_l. *We look at the Hamiltonian action of* T *on* $\mathcal{O}_l$. *Let* $\Phi_T : \mathcal{O}_l \to \mathfrak{t}^*$ *be the corresponding moment map. One checks that the moment map* Φ_T *is* proper *and that its image is equal to the half-line* $\{aX_l, a \geq 1\} \subset \mathfrak{t}^*$.

We check that for each $\xi \in \{aX_l, a \geq 1\}$ *the fiber* $\Phi_T^{-1}(\xi)$ *is equal to a* T*-orbit in* $\mathcal{O}_l$. *For* $k \in \mathbb{Z}$, *let us denote* $(\mathcal{O}_l)_k$ *the symplectic reduction of* $\mathcal{O}_l$ *at the level* X_k. *We knows that* $(\mathcal{O}_l)_k = \emptyset$ *if* $k \notin \{al, a \geq 1\}$, *and that* $(\mathcal{O}_l)_k$ *is a point if* $k \in \{al, a \geq 1\}$.

In order to compute $\mathcal{Q}((\mathcal{O}_l)_k)$ *we look at the stabilizer subgroup* $T_m := \{t \in T \,|\, t \cdot m = m\}$ *for each point* $m \in \mathcal{O}_l$. *One sees that* $T_m = T$ *if* $m = X_l$ *and* T_m *is equal to the center* $\{\pm Id\}$ *of* $\mathrm{SL}_2(\mathbb{R})$, *when* $m \neq X_l$.

Theorem 2.16 gives in this setting that, for $k \in \{al, a \geq 1\}$,

$$\mathcal{Q}((\mathcal{O}_l)_k) = \begin{cases} 1 & \text{if } l-k \text{ is even} \\ 0 & \text{if } l-k \text{ is odd.} \end{cases} \tag{2.32}$$

Hence the formal geometric quantization of the proper T*-manifold* $\mathcal{O}_l$ *is*

$$\mathcal{Q}_T^{-\infty}(\mathcal{O}_l) = \begin{cases} \mathbb{C}_l \cdot \sum_{p \geq 0} \mathbb{C}_{2p} & \text{if } l > 0 \\ \mathbb{C}_l \cdot \sum_{p \geq 0} \mathbb{C}_{-2p} & \text{if } l < 0. \end{cases} \tag{2.33}$$

Here we recognizes that $\mathcal{Q}_T^{-\infty}(\mathcal{O}_l)$ *coincides with the restriction of the holomorphic (resp. anti-holomorphic) discrete series representation* Θ_l *to the group* T *when* $l > 0$ *(resp.* $l < 0$*).*

2.5. **Wonderful compactifications and symplectic cuts.** Another equivalent definition of the quantization $\mathcal{Q}^{-\infty}$ uses a generalisation of the technique of symplectic cutting (originally due to Lerman [13]) that was introduced in [21] and was motivated by the wonderful compactifications of De Concini and Procesi. Let us recall the method.

We recall that T is a maximal torus in the compact connected Lie group K, and W is the Weyl group. We define a K*-adapted* polytope in $\mathfrak{t}^*$ to be a W-invariant Delzant polytope P in $\mathfrak{t}^*$ whose vertices are regular elements of the weight lattice Λ^*. If $\{\lambda_1, \ldots, \lambda_N\}$ are the dominant weights lying in the union of all the closed one-dimensional faces of P, then there is a $G \times G$-equivariant embedding of $G = K_{\mathbb{C}}$ into

$$\mathbb{P}(\bigoplus_{i=1}^{N} V_{\lambda_i}^* \otimes V_{\lambda_i})$$

associating to $g \in G$ its representation on $\bigoplus_{i=1}^N V_{\lambda_i}$. The closure $\mathcal{X}_P$ of the image of G in this projective space is smooth and is equipped with a $K \times K$ that we denote:

$$(k_1, k_2) \cdot x = k_2 \cdot x \cdot k_1^{-1}.$$

Let $\Omega_{\mathcal{X}_P}$ be the symplectic 2-form on $\mathcal{X}_P$ which given by the Kahler structure. We recall briefly the different properties of $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$: all the details can be found in [21].

(1) $\mathcal{X}_P$ is equipped with an Hamiltonian action of $K \times K$. Let $\Phi = (\Phi_l, \Phi_r) : M \to \mathfrak{k}^* \times \mathfrak{k}^*$ be the corresponding moment map.
(2) The image of $\Phi := (\Phi_l, \Phi_r)$ is equal to $\{(k \cdot \xi, -k' \cdot \xi) \mid \xi \in P \text{ and } k, k' \in K\}$.
(3) The Hamiltonian manifold $(\mathcal{X}_P, K \times K)$ has no multiplicities: the pull-back by Φ of a $K \times K$-orbit in the image is a $K \times K$-orbit in $\mathcal{X}_P$.

Let $\mathcal{U}_P := K \cdot P^\circ$ where P° is the interior of P. We define

$$\mathcal{X}_P^\circ := \Phi_l^{-1}(\mathcal{U}_P)$$

which is an invariant, open and dense subset of $\mathcal{X}_P$. We have the following important property concerning $\mathcal{X}_P^\circ$.

(4) There exists an equivariant diffeomorphism $\Upsilon : K \times \mathcal{U}_P \to \mathcal{X}_P^\circ$ such that $\Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi$ and $\Upsilon^*(\Phi_r)(k, \xi) = -\xi$.
(5) This diffeomorphism Υ is a quasi-symplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on the open subset $K \times \mathcal{U}_P$ of the cotangent bundle T^*K to the pullback of the symplectic form $\Omega_{\mathcal{X}_P}$ on $\mathcal{X}_P^o$.
(6) The symplectic manifold $(\mathcal{X}_P, \Omega_{\mathcal{X}_P})$ is prequantized by the restriction of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{P}(\oplus_{i=1}^N V_{\lambda_i}^* \otimes V_{\lambda_i})$ to $\mathcal{X}_P$: let us denoted L_P the corresponding $K \times K$-equivariant line bundle.
(7) The pull-back of the line bundle L_P by the map $\Upsilon : K \times \mathcal{U}_P \hookrightarrow \mathcal{X}_P$ is trivial.

Let (M, Ω_M, Φ_M) be a proper Hamiltonian K-manifold. We also consider the Hamiltonian $K \times K$-manifold $\mathcal{X}_P$ associated to a K-adapted polytope P. We consider now the product $M \times \mathcal{X}_P$ with the following $K \times K$ action:

- the action $k \cdot_1 (m, x) = (k \cdot m, x \cdot k^{-1})$: the corresponding moment map is $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot x)$: the corresponding moment map is $\Phi_2(m, x) = \Phi_l(x)$.

Definition 2.22. *We denote M_P the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action $\cdot_1$: $M_P := (\Phi_1)^{-1}(0)/(K, \cdot_1)$.*

Then M_P inherits a Hamiltonian K-action with moment map $\Phi_{M_P} : M_P \to \mathfrak{k}^*$ whose image is $\Phi(M) \cap K \cdot P$.

One checks that M_P contains an open and dense subset of smooth points which quasi-symplectomorphic to the open subset $(\Phi_M)^{-1}(\mathcal{U}_P)$. If the polytope P is fixed, we can work with the dilated polytopes nP for $n \geq 1$. We have then the family of compact, perhaps singular, K-hamiltonian manifolds M_{nP}, $n \geq 1$: in Section 2.3, we have explained how was defined their geometric quantization $\mathcal{Q}_K(M_{nP}) \in R(K)$.

We have a convenient definition for $\mathcal{Q}^{-\infty}$.

Proposition 2.23 ([21]). *We have the following equality in $R^{-\infty}(K)$:*

$$\mathcal{Q}_K^{-\infty}(M) = \lim_{n\to\infty} \mathcal{Q}_K(M_{nP}). \tag{2.34}$$

3. Proof of Theorem 1.4

The main result of this section is

Theorem 3.1. *Let $r_P := \inf_{\xi\in\partial P} \|\xi\|$. The generalized character*

$$\mathcal{Q}_K^{\Phi}(M) - \mathcal{Q}_K(M_P) \quad \in R^{-\infty}(K)$$

is supported outside the ball B_{r_P}.

Then, for the dilated polytope $nP, n \geq 1$, the character $\mathcal{Q}_K^{\Phi}(M) - \mathcal{Q}^K(M_{nP})$ is supported outside the ball B_{nr_P}. Taking the limit when n goes to infinity gives

$$\mathcal{Q}_K^{\Phi}(M) = \lim_{n\to\infty} \mathcal{Q}_K(M_{nP}). \tag{3.35}$$

Finally, the identity of Theorem 1.4,

$$\mathcal{Q}_K^{\Phi}(M) = \mathcal{Q}_K^{-\infty}(M),$$

is a direct consequence of (2.34) and (3.35).

Recall that $O(r) \in R^{-\infty}(K)$ denoted any generalized character supported outside the ball B_r.

Theorem 3.1 follows from the comparison of three differents geometrical situation. All of them concern Hamiltonian actions of $K_1 \times K_2$, where K_1 and K_2 are two copies of K.

First setting. We work with the Hamiltonian $K_1 \times K_2$-manifold $M \times \mathcal{X}_P$: here K_1 acts both on M and on $\mathcal{X}_P$. Since the moment map Φ_1 (relative to the K_1-action) is proper we may"quantize" $M \times \mathcal{X}_P$ via the map $\|\Phi_1\|^2$: let

$$\mathcal{Q}_{K_1\times K_2}^{\Phi_1}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character. Recall that $\mathcal{Q}_{K_2}(M_P)$ is equal to $[\mathcal{Q}_{K_1\times K_2}^{\Phi_1}(M \times \mathcal{X}_P)]^{K_1}$.

Second setting. We consider the same setting than before : the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathcal{X}_P$. But we "quantize" $M \times \mathcal{X}_P$ through the global moment map $\Phi = (\Phi_1, \Phi_2)$. Here we have some liberty in the choice of the scalar product on $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$. If $\|\xi\|^2$ is an invariant Euclidean norm on $\mathfrak{k}^*$, we take on $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$ the Euclidean norm

$$\|(\xi_1, \xi_2)\|_\rho^2 = \|\xi_1\|^2 + \rho\|\xi_2\|^2 \tag{3.36}$$

depending on a parameter $\rho > 0$. Let us consider the quantization of $M \times \mathcal{X}_P$ via the map $\|\Phi\|_\rho^2$:

$$\mathcal{Q}_{K_1\times K_2}^{\Phi,\rho}(M \times \mathcal{X}_P) \in R^{-\infty}(K_1 \times K_2).$$

Third setting. We consider the cotangent bundle T^*K with the Hamiltonian action of $K_1 \times K_2$: K_1 acts by *right* translations, and K_2 by *left* translations. We consider the Hamiltonian action of $K_1 \times K_2$ on $M \times \mathrm{T}^*K$: here K_1 acts both on

M and on T^*K. Let $\Phi=(\Phi_1,\Phi_2)$ be the global moment map on $M\times\mathrm{T}^*K$. Since the moment map Φ is proper we can "quantize" $M\times\mathrm{T}^*K$ via the map $\|\Phi\|^2_\rho$: let

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)\in R^{-\infty}(K_1\times K_2)$$

be the corresponding generalized character.

Theorem 3.1 is a consequence of the following propositions.

First we compare $\mathcal{Q}^{\Phi}_{K_2}(M)$ with the K_1-invariant part of $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)$.

Proposition 3.2. *For any $\rho\in]0,1]$, we have*

$$\Big[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)\Big]^{K_1}=\mathcal{Q}^{\Phi}_{K_2}(M)\quad\text{in}\quad R^{-\infty}(K_2). \tag{3.37}$$

Then we compare the K_1-invariant part of the generalized characters $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)$ and $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$.

Proposition 3.3. *For any $\rho\in]0,1]$, we have the following relation in $R^{-\infty}(K_2)$*

$$\Big[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\Big]^{K_1}-\Big[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)\Big]^{K_1}=O(r_P) \tag{3.38}$$

Finally we compare the K_1-invariant part of the generalized characters $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ and $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M\times\mathcal{X}_P)$.

Proposition 3.4. *There exists $\epsilon>0$ such that*

$$\mathcal{Q}_{K_2}(M_P)-\Big[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\Big]^{K_1}=O((\epsilon/\rho)^{1/2})\quad\text{in}\quad R^{-\infty}(K_2) \tag{3.39}$$

if $\rho>0$ is small enough.

If we sum the relations (3.37), (3.38) and (3.39) we get

$$\mathcal{Q}^{\Phi}_{K_2}(M)=\mathcal{Q}_{K_2}(M_P)+O(r_P)+O((\epsilon/\rho)^{1/2})$$

if ρ is small enough. So Theorem 3.1 follows by taking $(\epsilon/\rho)^{1/2}\geq r_P$.

3.1. **Proof of Proposition 3.2.** The cotangent bundle T^*K is identified with $K\times\mathfrak{k}^*$. The data is then (see Section 5.1):

- the Liouville 1-form $\lambda=\sum_j\omega_j\otimes E_j$. Here (E_j) is a basis of $\mathfrak{k}$ with dual basis (E_j^*), and ω_j is the left invariant 1-form on K defined by $\omega_j(\frac{d}{dt}a\,e^{tX}|_0)=\langle E_j^*,X\rangle$.
- the symplectic form $\Omega:=-d\lambda$,
- the action of $K_1\times K_2$ on $K\times\mathfrak{k}^*$ is $(k_1,k_2)\cdot(a,\xi)=(k_2ak_1^{-1},k_1\cdot\xi)$,
- the moment map relative to the K_1-action is $\Phi_r(a,\xi)=-\xi$,
- the moment map relative to the K_2-action is $\Phi_l(a,\xi)=a\cdot\xi$.

We work now with the Hamiltonian action of $K_1\times K_2$ on $M\times\mathrm{T}^*K$ given by

$$(k_1,k_2)\cdot(m,a,\xi)=(k_1\cdot m,k_2ak_1^{-1},k_1\cdot\xi).$$

The corresponding moment map is $\Phi=(\Phi_1,\Phi_2)$: $\Phi_1(m,a,\xi)=\Phi_M(m)-\xi$ and $\Phi_2(m,a,\xi)=a\cdot\xi$.

Let $\mathbf{c}_1$ be a symbol $\mathrm{Thom}(M,J_1)\otimes L$ attached to the prequantized Hamiltonian K_1-manifold (M,Ω). The cotangent bundle T^*K is prequantized by the trivial line bundle: let $\mathbf{c}_2$ be the symbol $\mathrm{Thom}(\mathrm{T}^*K,J_2)$ attached to the prequantized

Hamiltonian $K_1 \times K_2$-manifold T^*K. The product $\mathbf{c} = \mathbf{c}_1 \odot \mathbf{c}_2$ corresponds to the symbol $\mathrm{Thom}(N, J) \otimes L$ on $N = M \times \mathrm{T}^*K$.

Let κ_ρ be the Kirwan vector field associated to the map $\|\Phi\|^2_\rho : M \times \mathrm{T}^*K \to \mathbb{R}$. We check that $\|\Phi\|^2_\rho(m, k, \xi) = \|\Phi_M(m) - \xi\|^2 + \rho\|\xi\|^2$, and

$$\kappa_\rho(m, k, \xi) = \Big(\underbrace{(\Phi_M(m) - \xi) \cdot m}_{\kappa_I}; \underbrace{\widetilde{\Phi}_M(m) - (1+\rho)\widetilde{\xi}}_{\kappa_{I\!I,\rho}}; \underbrace{-[\widetilde{\Phi}_M(m), \widetilde{\xi}]}_{\kappa_{I\!I\!I}} \Big).$$

Here $\mathrm{T}_{(m,k,\xi)}(M \times \mathrm{T}^*K) \simeq \mathrm{T}_m M \times \mathfrak{k} \times \mathfrak{k}$. We have

$$\begin{aligned} \mathrm{Cr}(\|\Phi\|^2_\rho) &= \{\kappa_\rho = 0\} \\ &= \bigcup_{\beta \in \mathcal{B}} K_1 \times K_2 \cdot \left[M^{\widetilde{\beta}} \cap \Phi_M^{-1}(\beta) \times \{1\} \times \{\frac{\beta}{\rho+1}\} \right] \end{aligned}$$

where $\mathcal{B}$ parametrizes $\mathrm{Cr}(\|\Phi_M\|^2)$. Hence one checks that the critical values of $\|\Phi\|^2_\rho$ are $\frac{\rho}{\rho+1}\|\beta\|^2, \beta \in \mathcal{B}$.

Let $\mathbf{c}^{\kappa_\rho}$ be the symbol $\mathbf{c}$ pushed by the vector field κ_ρ: we have

$$\mathbf{c}^{\kappa_\rho}(v; X; Y) = \mathbf{c}_1(v - \kappa_I) \odot \mathbf{c}_2(X - \kappa_{I\!I,\rho}\,;\, Y - \kappa_{I\!I\!I})$$

for $(v; X; Y) \in \mathrm{T}_{(m,k,\xi)}(M \times \mathrm{T}^*K) \simeq \mathrm{T}_m M \times \mathfrak{k} \times \mathfrak{k}$.

For a real $R > 0$ we define the open invariant subsets of $M \times \mathrm{T}^*K$

$$\begin{aligned} U_R &:= \{\|\Phi\|^2_\rho < R\} \\ V_R &:= \{\|\Phi_M\|^2 < R\} \times \mathrm{T}^*K. \end{aligned}$$

By definition the generalized index $\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K)$ is defined as the limit of the equivariant index

$$\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(U_R) := \mathrm{Index}^{K_1 \times K_2}_{\mathcal{U}_R}(\mathbf{c}^{\kappa_\rho}|_{U_R}),$$

when R goes to infinity (and stays outside the critical values of $\|\Phi\|^2_\rho$).

In the other hand, when R' is a regular value of $\|\Phi_M\|^2$, we see that the symbol $\mathbf{c}_\rho|_{V_{R'}}$ is $K_1 \times K_2$-transversally elliptic. Let

$$\mathrm{Index}^{K_1 \times K_2}_{V_{R'}}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}}) \tag{3.40}$$

be its equivariant index. Notice that the index map is well-defined on $V_R = \{\|\Phi_M\|^2 < R\} \times \mathrm{T}^*K$ since T^*K can be seen as a open subset of a compact manifold.

It is easy to check that for any $R > 0$ there exists $R' > R$ such that $U_R \subset V'_R$. It implies that $\mathcal{Q}^{\Phi,\rho}_{K_1 \times K_2}(M \times \mathrm{T}^*K)$ is also defined as the limit of (3.40) when R' goes to infinity.

We look now to the deformation $\kappa_\rho(s) = (\kappa^s_I; \kappa^s_{I\!I,\rho}; s\kappa_{I\!I\!I})$, $s \in [0, 1]$ where

$$\kappa^s_I(m, \xi) = (\Phi_M(m) - s\xi) \cdot m \quad \text{and} \quad \kappa^s_{I\!I,\rho}(m, \xi) = s\widetilde{\Phi}_M(m) - (1 + s\rho)\widetilde{\xi}.$$

Let $\mathbf{c}^{\kappa_\rho(s)}$ be the symbol $\mathbf{c}$ pushed by the vector field $\kappa_\rho(s)$.

Lemma 3.5. *Let R' be a regular value of $\|\Phi_M\|^2$.*

- *The familly $\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}$, $s \in [0, 1]$ defines an homotopy of $K_1 \times K_2$-transversally elliptic symbols on $V_{R'}$.*
- *The K_1-invariant part of $\mathrm{Index}^{K_1 \times K_2}_{\mathcal{V}_{R'}}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}})$ is equal to $\mathcal{Q}^{\Phi}_{K_2}(M_{<R'})$.*

Proof. The first point follows from the fact that $\mathrm{Char}(\mathbf{c}^{\kappa_\rho(s)}|_{V_{R'}}) \cap \mathrm{T}_{K_1\times K_2}(V_{R'})$, which is equal to

$$\left\{(m,k,\frac{s}{1+s\rho}\Phi_M(m)),\ k\in K \text{ and } m\in \mathrm{Cr}(\|\Phi_M\|^2)\cap\{\|\Phi_M\|^2<R'\}\right\},$$

stays in a compact set when $s\in[0,1]$.

The symbol $\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}$ is equal to the product of the symbol $\mathbf{c}_1^\kappa|_{M<R'}$, which is K_1-transversally elliptic, with the symbol

$$\mathbf{c}_2^\kappa(X;Y)=\mathbf{c}_2(X+\xi;Y)$$

which is a K_2-transversally elliptic on T^*K. A basic computation done in section 5.1.2 gives that

$$\begin{aligned}\mathrm{Index}_{\mathrm{T}^*K}^{K_1\times K_2}(\mathbf{c}_2^\kappa) &= \mathrm{L}^2(K)\\ &= \sum_{\mu\in\widehat{K}}(V_\mu^{K_1})^*\otimes V_\mu^{K_2}\end{aligned}$$

in $R^{-\infty}(K_1\times K_2)$. Finally the"multiplicative property" (see Theorem 2.1) gives

$$\begin{aligned}\mathrm{Index}_{V_{R'}}^{K_1\times K_2}(\mathbf{c}^{\kappa_\rho(0)}|_{V_{R'}}) &= \mathrm{Index}_{M<R'}^{K_1}(\mathbf{c}_1^\kappa|_{M<R'})\otimes\mathrm{Index}_{\mathrm{T}^*K}^{K_1\times K_2}(\mathbf{c}_2\kappa)\\ &= \sum_{\mu\in\widehat{K}}\mathcal{Q}_{K_1}^\Phi(M_{<R'})\otimes(V_\mu^{K_1})^*\otimes V_\mu^{K_2}\end{aligned}$$

Taking the K_1-invariant completes the proof of the second point. □

Finally we have proved that the generalized character $[\mathrm{Index}_{V_{R'}}^{K_1\times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})]^{K_1}$ is equal to $\mathcal{Q}_{K_2}^\Phi(M_{<R'})$. Taking the limit $R'\to\infty$ gives

$$\begin{aligned}\left[\mathcal{Q}_{K_1\times K_2}^{\Phi,\rho}(M\times\mathrm{T}^*K)\right]^{K_1} &= \lim_{R'\to\infty}\left[\mathrm{Index}_{V_{R'}}^{K_1\times K_2}(\mathbf{c}^{\kappa_\rho}|_{V_{R'}})\right]^{K_1}\\ &= \lim_{R'\to\infty}\mathcal{Q}_{K_2}^\Phi(M_{<R'})=\mathcal{Q}_{K_2}^\Phi(M).\end{aligned}$$

3.2. **Proof of Proposition 3.3.** We work here with the Hamiltonian action of $K_1\times K_2$ on $M\times\mathcal{X}_P$. The action is $(k_1,k_2)\cdot(m,x)=(k\cdot m,k_2\cdot x\cdot k_1^{-1})$ and the corresponding moment map is $\Phi=(\Phi_1,\Phi_2)$ with $\Phi_1(m,x)=\Phi_M(m)+\Phi_r(x)$ and $\Phi_2(m,x)=\Phi_l(x)$. Let $\|(\xi_1,\xi_2)\|_\rho^2=\|\xi_1\|^2+\rho\|\xi_2\|^2$ be the Euclidean norm $\mathfrak{k}_1^*\times\mathfrak{k}_2^*$ attached to $\rho>0$.

Let us consider the quantization of $M\times\mathcal{X}_P$ via the map $\|\Phi\|_\rho^2$:

$$\mathcal{Q}_{K_1\times K_2}^{\Phi,\rho}(M\times\mathcal{X}_P)\in R^{-\infty}(K_1\times K_2)$$

The critical set $\mathrm{Cr}(\|\Phi\|_\rho^2)$ admits the decomposition

$$\mathrm{Cr}(\|\Phi\|_\rho^2)=\bigcup_{\gamma\in\mathcal{B}_\rho}K_1\times K_2\cdot\mathcal{C}_\gamma \tag{3.41}$$

where $(m,x)\in\mathcal{C}_\gamma$ if and only if $\gamma=(\gamma_1,\gamma_2)$ with

$$\begin{cases}\Phi_M(m)+\Phi_r(x)=\gamma_1\\ \Phi_l(x)=\gamma_2\\ \widetilde{\gamma}_1\cdot m=0\\ \widetilde{\gamma}_1\cdot_r x+\rho\,\widetilde{\gamma}_2\cdot_l x=0.\end{cases} \tag{3.42}$$

We have

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)=\sum_{\gamma\in\mathcal{B}_\rho}\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P) \tag{3.43}$$

where the generalized character $\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ is computed as an index of a transversally elliptic symbol in a neighborhood of

$$K_1\times K_2\cdot\mathcal{C}_\gamma\subset M\times\Phi_l^{-1}(K_2\cdot\gamma_2).$$

Thanks to Theorem 2.9 we know that the support of the generalized character $\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ is contained in $\{(a,b)\in\widehat{K_1}\times\widehat{K_2}\mid\|a\|^2+\rho\|b\|^2\geq\|\gamma\|^2_\rho\}$. Hence

$$\text{support}\left(\left[\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P))\right]^{K_1}\right)\subset\left\{b\in\widehat{K_2}\mid\rho\|b\|^2\geq\|\gamma\|^2_\rho\right\}$$

Let $r_P=\inf_{\xi\in\partial P}\|\xi\|$. We know then that

$$\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\right]^{K_1}=\sum_{\substack{\gamma\in\mathcal{B}_\rho\\ \|\gamma\|^2_\rho<\rho r_P^2}}\left[\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\right]^{K_1}+O(r_P).$$

Let $R_P<\rho r_P^2$ be a regular value of $\|\Phi\|^2_\rho:M\times\mathcal{X}_P\to\mathbb{R}$ such that for all $\gamma\in\mathcal{B}_\rho$ we have $\|\gamma\|^2_\rho<\rho r_P^2\iff\|\gamma\|^2_\rho<R_P$. Then

$$\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\right]^{K_1}=\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathcal{X}_P)_{<R_P})\right]^{K_1}+O(r_P). \tag{3.44}$$

For the generalized index $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)$ we have also a decomposition

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)=\sum_{\gamma\in\mathcal{B}'_\rho}\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)$$

where $\mathcal{B}'_\rho$ parametrizes the critical set of $\|\Phi\|^2_\rho:M\times\mathrm{T}^*K\to\mathbb{R}$. Like before we get

$$\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)\right]^{K_1}=\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathrm{T}^*K)_{<R'_P})\right]^{K_1}+O(r_P). \tag{3.45}$$

Here $R'_P<\rho r_P^2$ is a regular value of $\|\Phi\|^2_\rho:M\times\mathrm{T}^*K\to\mathbb{R}$ such that for all $\gamma\in\mathcal{B}'_\rho$ we have $\|\gamma\|^2_\rho<\rho r_P^2\iff\|\gamma\|^2_\rho<R'_P$.

Lemma 3.6. *We have*

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathcal{X}_P)_{<R_P})=\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}((M\times\mathrm{T}^*K)_{<R'_P}). \tag{3.46}$$

Proof. The Lemma will follow from Proposition 2.6. We take here $V'=M\times\mathcal{X}_P^o$, $V=M\times K\times\mathcal{U}_P\subset M\times\mathrm{T}^*K$ and the equivariant diffeomorphism $\Psi:V\to V'$ is equal to $\mathrm{Id}\times\Upsilon$ where Υ was introduced in Section 2.5. Note that Ψ satisfies points $(1)-(3)$ of Proposition 2.6.

Note that $\|\Phi(m,x)\|^2_\rho<\rho r_P^2$ implies that $\|\Phi_l(x)\|<r_P$ and then $x\in\mathcal{X}_P^o$. Hence the open subset $U':=(M\times\mathcal{X}_P)_{<R_P}$ is contained in $V'=M\times\mathcal{X}_P^o$. In the same way the open subset $U:=(M\times\mathrm{T}^*K)_{<R'_P}$ is contained in V. We have $\Psi(U)=U'$ if $R_P=R'_P$.

We have proved that (3.46) is a consequence of Proposition 2.6. □

Finally, if we take the difference between (3.44) and (3.45), we get

$$\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)\right]^{K_1}-\left[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathrm{T}^*K)\right]^{K_1}=O(r_P).$$

which is the relation of Proposition 3.3.

3.3. **Proof of Proposition 3.4.** Here we want to compare the K_1-invariant part of the characters $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ and $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M\times\mathcal{X}_P)$.

We know after Theorem 2.14 that

$$\begin{aligned}\mathcal{Q}_{K_2}(M_P) &= \Big[\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M\times\mathcal{X}_P)\Big]^{K_1}\\ &= \Big[\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(U_\epsilon)\Big]^{K_1}\end{aligned}$$

when $\epsilon>0$ is any regular value of $\|\Phi_1\|^2$, and $U_\epsilon:=\{\|\Phi_1\|^2<\epsilon\}\subset M\times\mathcal{X}_P$.

In this section we fix once for all $\epsilon>0$ small enough so that

$$\mathrm{Cr}(\|\Phi_1\|^2)\cap\{\|\Phi_1\|^2\leq\epsilon\}=\{\Phi_1=0\}. \tag{3.47}$$

Let $\mathbf{c}_1$ be the symbol $\mathrm{Thom}(M,J_1)\otimes L$ attached to the prequantized Hamiltonian K_1-manifold (M,Ω). Let $\mathbf{c}_3$ be the symbol $\mathrm{Thom}(\mathcal{X}_P,J_3)\otimes L_P$ attached to the prequantized Hamiltonian $K_1\times K_2$-manifold $\mathcal{X}_P$. The product $\mathbf{c}=\mathbf{c}_1\odot\mathbf{c}_3$ corresponds to the symbol $\mathrm{Thom}(N,J)\otimes L$ on $N=M\times\mathcal{X}_P$.

Let κ_0 and κ_ρ be the Kirwan vector fields associated to the functions $\|\Phi_1\|^2$ and $\|\Phi\|^2_\rho$ on $M\times\mathcal{X}_P$:

$$\kappa_0(m,x)=\Big(\underbrace{\Phi_1(m,x)\cdot m}_{\kappa_I};\underbrace{\Phi_1(m,x)\cdot_r x}_{\kappa_{I\!I}}\Big),\quad \kappa_\rho(m,x)=\kappa^0(m,x)+\rho\,(0,\underbrace{\Phi_l(x)\cdot_l x}_{\kappa_{I\!I\!I}}).$$

Let $\mathbf{c}^{\kappa_\rho}$ be the symbol $\mathbf{c}$ pushed by the vector field κ_ρ: we have

$$\mathbf{c}^{\kappa_\rho}(v;\eta)=\mathbf{c}_1(v-\kappa_I)\odot\mathbf{c}_3(\eta-\kappa_{I\!I}-\rho\kappa_{I\!I\!I})$$

for $(v;\eta)\in\mathrm{T}_{(m,x)}(M\times\mathcal{X}_P)$.

The character $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(U_\epsilon)$ is given by the index of the K_1-transversally elliptic symbol $\mathbf{c}^{\kappa_0}|_{U_\epsilon}$. The character $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M\times\mathcal{X}_P)$ is given by the index of the $K_1\times K_2$-transversally elliptic symbol $\mathbf{c}^{\kappa_\rho}$.

Lemma 3.7. • *There exists $\rho(\epsilon)>0$ such that*

$$\mathrm{Cr}(\|\Phi\|^2_\rho)\bigcap\big\{\|\Phi_1\|^2\leq\epsilon\big\}\subset\Big\{\|\Phi_1\|^2\leq\frac{\epsilon}{2}\Big\}$$

for any $0\leq\rho\leq\rho(\epsilon)$.

Proof. With the help of Riemannian metrics on M and $\mathcal{X}_P$ we define

$$\begin{aligned}a(\epsilon) &:= \inf_{\epsilon/2\leq\|\Phi_1(m,x)\|\leq\epsilon}\|\kappa^0(m,x)\|\\ b &:= \sup_{x\in\mathcal{X}_P}\|\Phi_l(x)\cdot_l x\|.\end{aligned}$$

We have $a(\epsilon)>0$ thanks to (3.47), and $b<\infty$ since $\mathcal{X}_P$ is compact. It is now easy to check that $\{\kappa_\rho=0\}\cap\{\epsilon/2\leq\|\Phi_1\|^2\leq\epsilon\}=\emptyset$ if $0\leq\rho<\frac{a(\epsilon)}{b}$.

□

The symbols $\mathbf{c}^{\kappa_\rho}|_{U_\epsilon}$, $\rho\in[0,\rho(c)]$ are $K_1\times K_2$-transversally elliptic, and they define the same class in $\mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}U_\epsilon)$. Hence $\mathcal{Q}_{K_2}(M_P)$ can be computed as the K_1-invariant part of

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(U_\epsilon):=\mathrm{Index}^{U_\epsilon}_{K_1\times K_2}(\mathbf{c}^{\kappa_\rho}|_{U_\epsilon})\in R^{-\infty}(K_1\times K_2)$$

for $\rho\in[0,\rho(\epsilon)]$.

A component $K_1 \times K_2 \cdot \mathcal{C}_\gamma$ of $\mathrm{Cr}(\|\Phi\|_\rho^2)$ is contained in U_ϵ if and only $\|\gamma_1\| < \epsilon$: hence the decomposition (3.43) for the character $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)$ gives

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P) = \mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(\mathcal{U}_\epsilon) + \sum_{\substack{\gamma\in\mathcal{B}_\rho \\ \|\gamma_1\|^2\geq\epsilon}} \mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P).$$

where

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(\mathcal{U}_\epsilon) = \sum_{\substack{\gamma\in\mathcal{B}_\rho \\ \|\gamma_1\|^2<\epsilon}} \mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P).$$

Taking the K_1-invariant gives

$$[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + \sum_{\substack{\gamma\in\mathcal{B}_\rho \\ \|\gamma_1\|^2\geq\epsilon}} [\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)]^{K_1}. \tag{3.48}$$

In general we know that the support of the generalized character $[\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P))]^{K_1}$ is included in $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \|\gamma_1\|^2 + \rho\|\gamma_2\|^2\}$. When $\|\gamma_1\|^2 \geq \epsilon$ we have then that the support of $[\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P))]^{K_1}$ is contained in $\{b \in \widehat{K_2} \mid \rho\|b\|^2 \geq \epsilon\}$.

Finally (3.48) imposes that

$$[\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M \times \mathcal{X}_P)]^{K_1} = \mathcal{Q}_{K_2}(M_P) + O((\epsilon/\rho)^{1/2}).$$

when $0 < \rho \leq \rho(\epsilon)$, which is the precise content of Proposition 3.4.

4. Other properties of $\mathcal{Q}^\Phi$

Let (M, ω, Φ) be a proper Hamitlonian K-manifold which is prequantized by a line bundle L. The character $\mathcal{Q}^\Phi_K(M)$ is computed by means of a scalar product on $\mathfrak{k}^*$. The fact that $\mathcal{Q}^\Phi_K(M) = \mathcal{Q}^{-\infty}_K(M)$ gives the following

Proposition 4.1. *The character $\mathcal{Q}^\Phi_K(M)$ does not depend of the choice of a scalar product on $\mathfrak{k}^*$*

In this section we work in the setting where $K = K_1 \times K_2$. Let Φ_1 be the moment map relative to the K_1-action.

4.1. **Φ_1 is proper.** In this subsection we suppose that the moment map Φ_1 relative to the K_1-action is *proper*. We fix an invariant Euclidean norm $\| \bullet \|^2$ on $\mathfrak{k}$ in such a way that $\mathfrak{k}_1 = \mathfrak{k}_2^\perp$.

Let us "quantize" (M, Ω) via the invariant proper function $\|\Phi_1\|^2$: let

$$\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 4.2. *We have*

$$\mathcal{Q}^{\Phi}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) \quad \text{in} \quad R^{-\infty}(K_1 \times K_2). \tag{4.49}$$

Proof. On $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ we may consider the family of invariant Euclidean norms: $\|X_1 \oplus X_2\|_\rho^2 = \|X_1\|^2 + \rho\|X_2\|^2$ for $X_j \in \mathfrak{k}_j$. Let

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the quantization of M computed via the map $\|\Phi\|_\rho^2 = \|\Phi_1\|^2 + \rho\|\Phi_2\|^2$. By definition, $\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)$ is equal to $\mathcal{Q}^{\Phi,0}_{K_1\times K_2}(M)$, and we know after Proposition 4.1 that $\mathcal{Q}^{\Phi}_{K_1\times K_2}(M)$ coincides with the generalized character $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) \in R^{-\infty}(K)$ for any $\rho > 0$.

Let us prove that prove that $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)$. We denote $O(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the ball

$$\{\xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < r^2\}.$$

And we denote $O_1(r) \in R^{-\infty}(K_1\times K_2)$ any generalized character supported outside the

$$\{\xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\| < r\}.$$

Let $R_1 > 0$ be a regular value of $\|\Phi_1\|^2$: the open subset $\{\|\Phi_1\|^2 < R_1\}$ is denoted $M_{<R_1}$. We know that

$$\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M_{<R_1}) + O_1(\sqrt{R_1}).$$

Like in the Lemma 3.7, we know that

$$\mathrm{Cr}(\|\Phi\|_\rho^2) \cap \{\|\Phi_1\|^2 = R_1\} = \emptyset. \tag{4.50}$$

for $\rho \geq 0$ small enough. The identity (4.50) first implies that

$$\begin{aligned}\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) &= \sum_{\substack{\gamma\in\mathcal{B}_\rho\\ \|\gamma_1\|^2<R_1}} \mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M) + \sum_{\substack{\gamma\in\mathcal{B}_\rho\\ \|\gamma_1\|^2>R_1}} \mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M)\\ &= \mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M_{<R_1}) + O(\sqrt{R_1}).\end{aligned}$$

In the second equality we have used that $\mathcal{Q}^{\gamma,\rho}_{K_1\times K_2}(M) = O(\sqrt{R_1})$ when $\|\gamma_1\|^2 > R_1$ since the ball $\left\{(\xi_1,\xi_2) \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|\xi_1\|^2 + \|\xi_2\|^2 < R_1\right\}$ is contained in

$$\left\{(\xi_1,\xi_2) \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid \|(\xi_1,\xi_2)\|_\rho^2 < \|(\gamma_1,\gamma_2)\|_\rho^2\right\}.$$

The identity (4.50) shows also that the symbol $\mathbf{c}^{\kappa_\rho}|_{M<R_1}$ are homotopic for $\rho \geq 0$ small enough. Hence

$$\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M_{<R_1}) = \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M_{<R_1})$$

We get finally that $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) - \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) = O(\sqrt{R_1}) + O_1(\sqrt{R_1})$ for any regular value R_1 of $\|\Phi_1\|^2$. We have proved that $\mathcal{Q}^{\Phi,\rho}_{K_1\times K_2}(M) - \mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M) = 0$. □

Let us explain how Theorem 4.2 contains the identity that we called "*quantization commutes with reduction in the singular setting*" in [21]. By definition the K_1-invariant part of the right hand side of (4.49) is equal to the geometric quantization of the (possibly singular) compact Hamiltonian K_2-manifold

$$M/\!\!/_0 K_1 := \Phi_1^{-1}(0)/K_1.$$

Using now the fact that the left hand side of (4.49) is equal to $\mathcal{Q}^{-\infty}_{K_1\times K_2}(M)$, we see that the multiplicity of $V_\mu^{K_2}$ in $\mathcal{Q}_{K_2}(M/\!\!/_0 K_1)$ is equal to the geometric quantization of the (possibly singular) compact manifold

$$M\times\overline{K_2\cdot\mu}/\!\!/_{(0,\mu)}K_1\times K_2.$$

4.2. **The symplectic reduction $M/\!\!/_0 K_1$ is smooth.** Let (M,Ω) be an Hamiltonian $K_1\times K_2$-manifold with a proper moment map $\Phi=(\Phi_1,\Phi_2)$. In this section we suppose that 0 is a regular value of Φ_1 and that K_1 acts freely on $\Phi_1^{-1}(0)$. We work then with the (smooth) Hamiltonian K_2-manifold

$$N:=\Phi_1^{-1}(0)/K_1.$$

We still denote by $\Phi_2: N\to\mathfrak{k}_2^*$ the moment map relative to the K_2-action: note that this map is proper. Hence we can quantize the K_2-action on N via the map Φ_2. Let $\mathcal{Q}^{\Phi_2}_{K_2}(N)\in R^{-\infty}(K_2)$ be the corresponding character.

Proposition 4.3. *We have*

$$\left[\mathcal{Q}^{\Phi}_{K_1\times K_2}(M)\right]^{K_1}=\mathcal{Q}^{\Phi_2}_{K_2}(N)\quad\text{in}\quad R^{-\infty}(K_2). \tag{4.51}$$

Proof. When Φ_1 is proper, the manifold N is compact. Then the right hand side of (4.51) is equal to $\mathcal{Q}_{K_2}(N)$, and we know from Theorem 4.2 that the left hand side of (4.51) is equal to $[\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)]^{K_1}$. In this case (4.51) becomes $[\mathcal{Q}^{\Phi_1}_{K_1\times K_2}(M)]^{K_1}=\mathcal{Q}_{K_2}(M/\!\!/_0K_1)$ which is the content of Theorem 2.13.

Let us consider the general case where Φ_1 is not proper. Thanks to Theorem 1.4 one knows that the multiplicities of $V_\mu^{K_2}$ in $\left[\mathcal{Q}^{\Phi}_{K_1\times K_2}(M)\right]^{K_1}$ and $\mathcal{Q}^{\Phi_2}_{K_2}(N)$ are respectively equal to the quantization of the (possibly singular) symplectic reductions

$$\mathcal{M}_\mu:=M\times\overline{K_2\cdot\mu}/\!\!/_{(0,0)}K_1\times K_2.$$

and

$$\mathcal{M}'_\mu:=N\times\overline{K_2\cdot\mu}/\!\!/_0K_2,\quad\text{with}\quad N=M/\!\!/_0K_1.$$

Note that $\mathcal{M}_\mu$ and $\mathcal{M}'_\mu$ coincide as symplectic reduced space. Let us prove that their geometric quantization are identical also. The proof will be done for $\mu=0$: the other case follows from the shifting trick.

Let $\mathbf{c}$ be the $K_1\times K_2$-equivariant symbol $\mathrm{Thom}(M,J)\otimes L_M$. Let κ be the Kirwan vector field attached to the moment map $\Phi=(\Phi_1,\Phi_2)$. Let $\mathbf{c}^\kappa$ be the symbol $\mathbf{c}$ pushed by κ. Let us denote $M_{<\epsilon}$ the open subset $\{\|\Phi\|^2<\epsilon\}$. For $\epsilon>0$ small enough, the symbol $\mathbf{c}^\kappa|_{M_{<\epsilon}}$ is $K_1\times K_2$-transversally elliptic, and $\mathcal{Q}(\mathcal{M}_0)$ is the $K_1\times K_2$-invariant part of $\mathrm{Index}^{K_1\times K_2}_{M_{<\epsilon}}(\mathbf{c}^\kappa|_{M_{<\epsilon}})$.

Let $\mathbf{c}_2$ be the K_2-equivariant symbol $\mathrm{Thom}(N,J)\otimes L_N$. Let κ_2 be the Kirwan vector field attached to the moment map Φ_2. Let $\mathbf{c}_2^{\kappa_2}$ be the symbol $\mathbf{c}_2$ pushed by κ_2. Let us denote $N_{<\epsilon}$ the open subset $\{\|\Phi_2\|^2<\epsilon\}$. For $\epsilon>0$ small enough, the symbol $\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}$ is K_2-transversally elliptic, and $\mathcal{Q}(\mathcal{M}'_0)$ is the K_2-invariant part of $\mathrm{Index}^{K_2}_{N_{<\epsilon}}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}})$.

Our proof follows from the comparison of the classes

$$[\mathbf{c}^\kappa|_{M_{<\epsilon}}]\in\mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}M_{<\epsilon})$$

and

$$[\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}}]\in\mathbf{K}_{K_2}(\mathrm{T}_{K_2}N_{<\epsilon})$$

A neighborhood of the smooth submanifold $Z := \Phi_1^{-1}(0)$ in M is diffeomorphic to a neighborhood of the 0-section of the bundle $Z \times \mathfrak{k}_1^* \to Z$. Let $Z_{<\epsilon} = Z \cap M_{<\epsilon}$ so that $N_{<\epsilon} = Z_{<\epsilon}/K_1$. Hence $[\mathbf{c}^\kappa|_{M_{<\epsilon}}]$ can be seen naturally a class in the K-group $\mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon}\times\mathfrak{k}_1^*))$.

Following Atiyah [1][Theorem 4.3], the inclusion map $j : Z_{<\epsilon} \hookrightarrow Z_{<\epsilon}\times\mathfrak{k}_1^*$ induces the Thom isomorphism

$$j_! : \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} Z_{<\epsilon}) \longrightarrow \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon}\times\mathfrak{k}_1^*)),$$

with the commutative diagram

$$\begin{array}{ccc} \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} Z_{<\epsilon}) & \xrightarrow{\ j_!\ } & \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon}\times\mathfrak{k}_1^*)) \\ & \searrow^{\mathrm{Index}_{Z_{<\epsilon}}^{K_1\times K_2}} & \downarrow \mathrm{Index}_{Z_{<\epsilon}\times\mathfrak{k}_1^*}^{K_1\times K_2} \\ & & R^{-\infty}(K_1\times K_2) \end{array} \tag{4.52}$$

Let $\pi_1 : Z_{<\epsilon} \to N_{<\epsilon}$ be the quotient relative to the free action of K_1. The corresponding isomorphism

$$\pi_1^* : \mathbf{K}_{K_2}(\mathrm{T}_{K_2} N_{<\epsilon}) \longrightarrow \mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2} Z_{<\epsilon})$$

satisfies the following rule :

$$\Big[\mathrm{Index}_{K_1\times K_2}^{Z_{<\epsilon}}(\pi_1^*\theta)\Big]^{K_1} = \mathrm{Index}_{K_2}^{N_{<\epsilon}}(\theta) \tag{4.53}$$

for any $\theta \in \mathbf{K}_{K_2}(\mathrm{T}_{K_2} N_{<\epsilon})$.

Lemma 4.4 ([19])**.** *We have*

$$j_! \circ \pi_1^*\Big(\big[\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon'}}\big] \Big) = [\mathbf{c}^\kappa|_{M_{<\epsilon}}]$$

in $\mathbf{K}_{K_1\times K_2}(\mathrm{T}_{K_1\times K_2}(Z_{<\epsilon}\times\mathfrak{k}_1^*))$.

Proof. This Lemma is proven in [19][Section 6.2] when the group K_2 is trivial. It is easy to check that the proof extends naturally to our setting. □

If one uses Lemma 4.4 together with (4.52) and (4.53), we get that

$$\begin{aligned} \mathcal{Q}(\mathcal{M}_0) &= \Big[\mathrm{Index}_{Z_{<\epsilon}\times\mathfrak{k}_1^*}^{K_1\times K_2}(\mathbf{c}^\kappa|_{M_{<\epsilon}})\Big]^{K_1\times K_2} \\ &= \Big[\mathrm{Index}_{N_{<\epsilon}}^{K_2}(\mathbf{c}_2^{\kappa_2}|_{N_{<\epsilon}})\Big]^{K_2} = \mathcal{Q}(\mathcal{M}_0'). \end{aligned}$$

□

5. Example: the cotangent bundle of an orbit

5.1. **The formal quantization of** T^*K**.** Let K be a compact connected Lie group equipped with the action of two copies of K: $(k_1, k_2)\cdot a = k_2 a k_1^{-1}$. Then we have a Hamiltonian action of $K_1 \times K_2$ on the cotangent bundle T^*K. In this section, we check that each formal geometric quantization of T^*K, $\mathcal{Q}^{-\infty}_{K_1\times K_2}(\mathrm{T}^*K)$ and $\mathcal{Q}^{\Phi}_{K_1\times K_2}(\mathrm{T}^*K)$, are both equal to the $K_1 \times K_2$-module $L^2(K)$.

The tangent bundle $\mathrm{T}K$ is identified with $K \times \mathfrak{k}$ through the right translations: to $(a, X) \in K \times \mathfrak{k}$ we associate $\frac{d}{dt} a e^{tX}|_0$. The action of $K_1 \times K_2$ on the cotangent bundle $\mathrm{T}^*K \simeq K \times \mathfrak{k}^*$ is then

$$(k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The symplectic form on T^*K is $\Omega := -d\lambda$, where λ is the Liouville 1-form. Let us compute these two form in coordinates. The tangent bundle of $\mathrm{T}^*K \simeq K \times \mathfrak{k}^*$ is identified with $\mathrm{T}^*K \times \mathfrak{k} \times \mathfrak{k}^*$: for each $(a, \xi) \in \mathrm{T}^*K$, we have a two form $\Omega_{(a,\xi)}$ on $\mathfrak{k} \times \mathfrak{k}^*$. A direct computation gives

$$\Omega_{(a,\xi)}(X, Y) = \langle \xi, [X, Y] \rangle, \quad \Omega_{(a,\xi)}(\eta, \eta') = 0, \quad \Omega_{(a,\xi)}(X, \eta) = \langle \eta, X \rangle$$

for $X, Y \in \mathfrak{k}$ and $\eta, \eta' \in \mathfrak{k}^*$. So $\Omega_{(a,\xi)} = \Omega_0 + \pi_\xi$ where Ω_0 is the canonical (constant) symplectic form on $\mathfrak{k} \times \mathfrak{k}^*$ and π_ξ is the closed two form on $\mathfrak{k}$ defined by $\pi_\xi(X, Y) = \langle \xi, [X, Y] \rangle$.

If we identify $\mathfrak{k} \simeq \mathfrak{k}^*$ through an invariant Euclidean norm, the symplectic structure on $\mathrm{T}_{(a,\xi)}(\mathrm{T}^*K) \simeq \mathfrak{k} \times \mathfrak{k}^*$ is given by a skew-symmetric matrix

$$A_\xi := \begin{pmatrix} \mathrm{ad}(\xi) & I_n \\ -I_n & 0 \end{pmatrix}.$$

We will work with the following compatible almost complex structure on the tangent bundle of T^*K : $J_\xi = -A_\xi(-A_\xi^2)^{-1/2}$. When $\xi = 0$, the complex structure J_0 on $\mathfrak{k} \times \mathfrak{k}^*$ is defined by the matrix

$$J_0 := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Hence the complex K-module $(\mathfrak{k} \times \mathfrak{k}^*, J_0)$ is naturally identified with the complexification $\mathfrak{k}_{\mathbb{C}}$ of $\mathfrak{k}$.

One checks easily that the moment map relative to the $K_1 \times K_2$-action is the *proper* map $\Phi : \mathrm{T}^*K \to \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ defined by $\Phi(a, \xi) = (-\xi, a \cdot \xi)$.

Here the symplectic manifold T^*K is prequantized by the trivial line bundle.

5.1.1. *Computation of* $\mathcal{Q}^{-\infty}_{K_1 \times K_2}(\mathrm{T}^*K)$. Let $\mathcal{O}_1 \times \mathcal{O}_2$ be a coadjoint orbit of $K_1 \times K_2$ in $\mathfrak{k}_1^* \times \mathfrak{k}_2^*$. One checks that

$$\Phi^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \emptyset & \text{if } \mathcal{O}_1 \neq -\mathcal{O}_2 \\ \text{a } K_1 \times K_2 - \text{orbit} & \text{if } \mathcal{O}_1 = -\mathcal{O}_2. \end{cases} \tag{5.54}$$

We knows that the stabiliser subgroup K_ξ of an element $\xi \in \mathfrak{k}^*$ is connected. Then the stabilizer subgroup $(K_1 \times K_2)_{(a,\xi)} = \{(k_1, a k_1 a^{-1}),\ k_1 \in K_\xi\}$ is also connected.

Let $(\mathrm{T}^*K)_{(\mu,\lambda)}$ be the symplectic reduction of T^*K at the level $(\mu, \lambda) \in \widehat{K}^2$. For any $\mu \in \widehat{K}$, we define $\mu^* \in \widehat{K}$ by the relation $-K \cdot \mu = K \cdot \mu^*$: note that $V^K_{\mu^*} \simeq (V^K_\mu)^*$. If one uses Theorem 2.16, one has

$$\mathcal{Q}((\mathrm{T}^*K)_{(\mu,\lambda)}) = \begin{cases} 0 & \text{if } \lambda \neq \mu^* \\ 1 & \text{if } \lambda = \mu^*. \end{cases} \tag{5.55}$$

Finally

$$\begin{aligned}\mathcal{Q}^{-\infty}_{K_1\times K_2}(\mathrm{T}^*K) &= \sum_{(\mu,\lambda)\in\widehat{K}\times\widehat{K}} \mathcal{Q}\left((\mathrm{T}^*K)_{(\mu,\lambda)}\right) V_\mu^{K_1}\otimes V_\lambda^{K_2}\\ &= \sum_{\mu\in\widehat{K}} V_\mu^{K_1}\otimes (V_\mu^{K_2})^* = \mathrm{L}^2(K).\end{aligned}$$

5.1.2. *Computation of* $\mathcal{Q}^{\Phi}_{K_1\times K_2}(\mathrm{T}^*K)$. The Kirwan vector field on T^*K is

$$\kappa(a,\xi) = -2\xi \in \mathfrak{k}_{\mathbb{C}}.$$

Let $\mathbf{c}^\kappa$ be the symbol $\mathrm{Thom}(\mathrm{T}^*K, J)$ pushed by the vector field $\frac{1}{2}\kappa$. At each $(a,\xi)\in\mathrm{T}^*K$, the map $\mathbf{c}^\kappa_{(a,\xi)}(X\oplus\eta)$ from $\wedge^{even}_{J_\xi}(\mathfrak{k}\times\mathfrak{k}^*)$ to $\wedge^{odd}_{J_\xi}(\mathfrak{k}\times\mathfrak{k}^*)$ is equal to the clifford map $\mathbf{c}(X+\xi\oplus\eta)$. Note that $\mathbf{c}^\kappa$ is a K_2-transversally elliptic symbol on T^*K: we have $\mathrm{Char}(\mathbf{c}^\kappa)\cap\mathrm{T}_{K_2}(\mathrm{T}^*K) = \{(1,0)\}$. We will now compute the equivariant index of $\mathbf{c}^\kappa$.

First we consider the homotopy $t\in[0,1]\to J_{t\xi}$ of symplectic structure on T^*K. Let $\tilde{\mathbf{c}}^\kappa$ be the symbol acting on $\wedge^\bullet_{J_0}(\mathfrak{k}\times\mathfrak{k}^*) = \wedge^\bullet_{\mathbb{C}}\mathfrak{k}_{\mathbb{C}}$. Proposition 2.6 shows that the symbols $\mathbf{c}^\kappa$ and $\tilde{\mathbf{c}}^\kappa$ define the same class in $K_{K_1\times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))$.

The projection $\pi:\mathrm{T}^*K\to\mathfrak{k}^*$ corresponds to the quotient map relative to the free action of K_2. At the level of K-groups we get an isomorphism

$$\pi_*: K_{K_1\times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))\to K_{K_1}(\mathrm{T}\mathfrak{k}^*).$$

Atiyah [1] proves that

$$\mathrm{Index}^{\mathrm{T}^*K}_{K_1\times K_2}(\sigma) = \sum_{\mu\in\widehat{K}} \mathrm{Index}^{\mathfrak{k}^*}_{K_1}\left(\pi_*(\sigma\otimes V_\mu^{K_2})\right)\otimes(V_\mu^{K_2})^*$$

for any class $\sigma\in K_{K_1\times K_2}(\mathrm{T}_{K_2}(\mathrm{T}^*K))$. In our case the symbol $\pi^*(\tilde{\mathbf{c}}^\kappa)$ is equal to the Bott symbol $\mathrm{Bott}(\mathfrak{k}^*)$, and for any K_2-module E_2 we have

$$\pi_*(\tilde{\mathbf{c}}^\kappa\otimes E_2) = \mathrm{Bott}(\mathfrak{k}^*)\otimes E_1$$

where E_1 is the module E_2 with the action of K_1. Then

$$\begin{aligned}\mathcal{Q}^{\Phi}_{K_1\times K_2}(\mathrm{T}^*K) &= \mathrm{Index}^{\mathrm{T}^*K}_{K_1\times K_2}(\tilde{\mathbf{c}}^\kappa)\\ &= \sum_{\mu\in\widehat{K}} \mathrm{Index}^{\mathfrak{k}^*}_{K_1}\left(\mathrm{Bott}(\mathfrak{k}^*)\otimes V_\mu^{K_1}\right)\otimes(V_\mu^{K_2})^*\\ &= \sum_{\mu\in\widehat{K}} V_\mu^{K_1}\otimes(V_\mu^{K_2})^* = L^2(K),\end{aligned}$$

since $\mathrm{Index}^{\mathfrak{k}^*}_{K_1}(\mathrm{Bott}(\mathfrak{k}^*)) = 1$.

5.2. **The formal quantization of** $\mathrm{T}^*(K/H)$**.** Let H be a closed connected subgroup of K. Look at T^*K as a Hamiltonian manifold relatively to the action of $H\times K\subset K_1\times K_2$. The moment map $\Phi=(\Phi_H,\Phi_K)$ is defined by : $\Phi_H(a,\xi) = -\mathrm{pr}(\xi)$ and $\Phi_K(a,\xi) = a\cdot\xi$, where $\mathrm{pr}:\mathfrak{k}^*\to\mathfrak{h}^*$ is the projection. Note that Φ is a proper map.

The cotangent bundle $\mathrm{T}^*(K/H)$, viewed as K-manifold, is equal to the symplectic reduction of T^*K relatively to the H-action: if the kernel of the projection pr is denoted $\mathfrak{h}^\perp$, we have

$$\Phi_H^{-1}(0)/H = K\times_H\mathfrak{h}^\perp = \mathrm{T}^*(K/H).$$

We are here in the setting of section 4.2. The reduction of the $H \times K$ proper Hamiltonian manifold T^*K relatively to the H-action is smooth, then its formal quantization is computed as follows

$$\begin{aligned} \mathcal{Q}_K^\Phi(\mathrm{T}^*(K/H)) = \left[\mathcal{Q}_{H\times K}^\Phi(\mathrm{T}^*K)\right]^H &= \left[\mathcal{Q}_{K_1\times K_2}^\Phi(\mathrm{T}^*K)|_{H\times K}\right]^H \\ &= \left[\mathrm{L}^2(K)\right]^H \\ &= \mathrm{L}^2(K/H). \end{aligned} \tag{5.56}$$

Here the fact that $\mathcal{Q}_{H\times K}^\Phi(\mathrm{T}^*K)$ is equal to the restriction of $\mathcal{Q}_{K_1\times K_2}^\Phi(\mathrm{T}^*K) = \mathrm{L}^2(K)$ to $H \times K$ is a consequence of Theorem 1.3.

Let us denoted $\left[\mathrm{T}^*(K/H)\right]_\mu$ the symplectic reduction at $\mu \in \widehat{K}$ of the K-Hamiltonian manifold $\mathrm{T}^*(K/H)$. Theorem 1.4 together with (1.6) gives

Corollary 5.1. *For any $\mu \in \widehat{K}$, we have*

$$\mathcal{Q}\left(\left[\mathrm{T}^*(K/H)\right]_\mu\right) = \dim\left[V_\mu^K\right]^H,$$

where $[V_\mu^K]^H$ is the subspace of H-invariant vector.

5.3. **The formal quantization of $\mathrm{T}^*(K/H)$ relatively to the action of G.** Let G be a closed connected subgroup of K. We look at the hamiltonian action of G on $\mathrm{T}^*(K/H)$. Let $\Phi_G : \mathrm{T}^*(K/H) \to \mathfrak{g}^*$ be the moment map. We consider also the restriction of the K-module $\mathrm{L}^2(K/H)$ to G.

We have

Proposition 5.2. *The following statements are equivalent*

1. *The moment map $\Phi_G : \mathrm{T}^*(K/H) \to \mathfrak{g}^*$ is proper.*
2. *$\Phi_G^{-1}(0)$ is equal to the zero section.*
3. *$k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$, for any $k \in K$.*
4. *$\mathfrak{g} + \mathfrak{h} = \mathfrak{k}$*
5. *G acts transitively on K/H.*
6. *$L^2(K/H)]^G \simeq \mathbb{C}$*
7. *$\mathrm{L}^2(K/H)|_G$ is an admissible G-representation.*

Proof. (1) $\Longrightarrow$ (7) is a consequence of Theorem 1.3. Let us prove that (7) $\Longrightarrow$ (6). Suppose now that

$$\mathrm{L}^2(K/H)|_G = \sum_{\mu\in\widehat{K}} [V_\mu^K]^H \otimes (V_\mu^K)^*|_G$$

is an admissible G-representation. It means that for any $\lambda \in \widehat{G}$ the set

$$A_\lambda := \left\{\mu \in \widehat{K} \mid [V_\mu^K]^H \neq \{0\} \text{ and } [(V_\lambda^G)^* \otimes (V_\mu^K)^*|_G]^G \neq \{0\}\right\}$$

is finite. Then the vector space $L^2(K/H)]^G$ is equal to the finite dimensional vector space $\sum_{\mu\in A_0}[V_\mu^K]^H \otimes [(V_\mu^K)^*]^G$. It is not difficult to check that if $\mu \in A_0$, then $k\mu \in A_0$ for $k >> 1$. Finally the fact that A_0 is finite implies that A_0 is reduced to $\mu = 0$. Hence the only G-invariant functions on K/H are the scalars.

(6) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (3) is a general fact concerning smooth actions of a compact connected Lie group G on a compact connected manifold M. The manifold M does not have G-invariant functions which are not scalar if and only if the action of G on M is transitive. And given a point $m \in M$, the orbit $G \cdot m$ is all of M if

and only if tangent spaces $\mathrm{T}_m(G\cdot m)$ and $\mathrm{T}_m M$ are equal. If we take $m=\overline{k^{-1}}$ in $M=K/H$, the condition $\mathrm{T}_m(G\cdot m)=\mathrm{T}_m M$ is equivalent to $k\cdot\mathfrak{g}+\mathfrak{h}=\mathfrak{k}$.

Let us check (3) $\Longrightarrow$ (2). Let $[k,\xi]\in K\times_H\mathfrak{h}^\perp=\mathrm{T}^*(K/H)$.We have $\Phi_G([k,\xi])=0$ if and only if $k\cdot\xi\in\mathfrak{g}^\perp$. Hence the vector ξ belongs to

$$k^{-1}\cdot\mathfrak{g}^\perp\bigcap\mathfrak{h}^\perp=(k^{-1}\cdot\mathfrak{g}+\mathfrak{h})^\perp.$$

Hence condition (3) imposes that $\xi=0$.

(2) $\Longleftrightarrow$ (1) comes from the fact that Φ_G is a homogeneous map of degree one between the vector bundle $\mathrm{T}^*(K/H)$ and the vector space $\mathfrak{g}^*$. □

Suppose now that the cotangent bundle $\mathrm{T}^*(K/H)$ is a *proper* Hamiltonian G-manifold. Let us denoted $[\mathrm{T}^*(K/H)]_{\mu,G}$ the (compact) symplectic reduction at $\mu\in\widehat{G}$ of the G-Hamiltonian manifold $\mathrm{T}^*(K/H)$. Then,

Corollary 5.3. *The multiplicity of V_μ^G in $\mathrm{L}^2(K/H)$ is equal to the quantization of the reduced space $[\mathrm{T}^*(K/H)]_{\mu,G}$.*

Proof. Using Theorem 1.3, equality (5.56) gives then

$$\begin{aligned}\mathcal{Q}_G^{-\infty}(\mathrm{T}^*(K/H)) &= \mathcal{Q}_K^{-\infty}(\mathrm{T}^*(K/H))|_G\\ &= \mathrm{L}^2(K/H)|_G.\end{aligned}$$

In other words, the multiplicity of V_μ^G in $\mathrm{L}^2(K/H)$ is equal to the quantization of the reduced space $[\mathrm{T}^*(K/H)]_{\mu,G}$. □

References

[1] M.F. Atiyah, Elliptic operators and compact groups, Springer, 1974. Lecture notes in Mathematics, **401**.

[2] M.F. Atiyah, G.B. Segal, *The index of elliptic operators II*, Ann. Math. **87**, 1968, p. 531-545.

[3] M.F. Atiyah, I.M. Singer, *The index of elliptic operators I*, Ann. Math. **87**, 1968, p. 484-530.

[4] M.F. Atiyah, I.M. Singer, *The index of elliptic operators III*, Ann. Math. **87**, 1968, p. 546-604.

[5] M.F. Atiyah, I.M. Singer, *The index of elliptic operators IV*, Ann. Math. **93**, 1971, p. 139-141.

[6] N. Berline and M. Vergne, The Chern character of a transversally elliptic symbol and the equivariant index, *Invent. Math.*, **124**, 1996, p. 11-49.

[7] N. Berline and M. Vergne, L'indice équivariant des opérateurs transversalement elliptiques, *Invent. Math.*, **124**, 1996, p. 51-101.

[8] J. J. Duistermaat, *The heat equation and the Lefschetz fixed point formula for the Spin^c-Dirac operator*, Progress in Nonlinear Differential Equation and Their Applications, vol. 18, Birkhauser, Boston, 1996.

[9] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.*, **67**, 1982, p. 515-538.

[10] V. Guillemin and S. Sternberg, A normal form for the moment map, in *Differential Geometric Methods in Mathematical Physics*(S. Sternberg, ed.), Reidel Publishing Company, Dordrecht, 1984.

[11] V. Guillemin and S.Sternberg, Symplectic techniques in physics, Cambridge University Press, Cambridge, 1990.

[12] B. Kostant, Quantization and unitary representations, in *Modern Analysis and Applications*, Lecture Notes in Math., Vol. 170, Springer-Verlag, 1970, p. 87-207.

[13] E. Lerman, Symplectic cut, *Math Res. Lett.* **2**, 1995, p. 247-258.

[14] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, Non-Abelian convexity by symplectic cuts, *Topology*, **37**, 1998, p. 245-259.

[15] X. Ma, W. Zhang, Geometric quantization for proper moment map, Arxiv:0812.3989

[16] E. Meinrenken, On Riemann-Roch formulas for multiplicities, *J. Amer. Math. Soc.*, **9**, 1996, p. 373-389.
[17] E. Meinrenken, Symplectic surgery and the Spin^c-Dirac operator, *Advances in Math.*, **134**, 1998, p. 240-277.
[18] E. Meinrenken, R. Sjamaar, Singular reduction and quantization, *Topology*, **38**, 1999, p. 699-762.
[19] P-E. Paradan, Localization of the Riemann-Roch character, *J. Funct. Anal.* **187**, 2001, p. 442-509.
[20] P-E. Paradan, Spin^c quantization and the K-multiplicities of the discrete series, *Annales Scientifiques de l'E. N. S.* , **36**, 2003, p. 805-845.
[21] P-E. Paradan, Formal geometric quantization, *Ann. Inst. Fourier* **59**, 2009, p. 199-238.
[22] P-E. Paradan M. Vergne, Index of transversally elliptic operators, 40 pages, to appear in *Astérique, Soc. Math. Fr.*. Arxiv math/08041225.
[23] P-E. Paradan, Multiplicities of the discrete series, 38 pages. ArXiv:0812.0059.
[24] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, *Annals of Math.*, **134**, 1991, p. 375-422.
[25] R. Sjamaar, Symplectic reduction and Riemann-Roch formulas for multiplicities, *Bull. Amer. Math. Soc.* **33**, 1996, p. 327-338.
[26] Y. Tian, W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, *Invent. Math*, **132**, 1998, p. 229-259.
[27] M. Vergne, Multiplicity formula for geometric quantization, Part I, Part II, and Part III, *Duke Math. Journal*, **82**, 1996, p. 143-179, p 181-194, p 637-652.
[28] M. Vergne, Quantification géométrique et réduction symplectique, *Séminaire Bourbaki* **888**, 2001.
[29] M. Vergne, Applications of Equivariant Cohomology, International Congress of Mathematicians 2006, Vol. I, Eur. Math. Soc., Zürich, 2007, p. 635-664. Arxiv: math/0607389.
[30] J. Weitsman, Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds. EuroConférence Moshé Flato 2000, Part I (Dijon). *Lett. Math. Phys.*, **56**, 2001, no. 1, p. 31-40.
[31] A. Weinstein , Lecture on symplectic manifold, CBMS Regional Conf. Series in Math., **29**, 1983.

Institut de Mathématiques et de Modélisation de Montpellier (I3M), Université Montpellier 2
E-mail address: Paul-Emile.Paradan@math.univ-montp2.fr