Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit

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Abstract

Self-stabilizing diffusions are stochastic processes, solutions of nonlinear stochastic differential equation, which are attracted by their own law. This specific self-interaction leads to singular phenomenons like non uniqueness of associated stationary measures when the diffusion leaves in some non convex environment (see [5]). The aim of this paper is to describe these invariant measures and especially their asymptotic behavior as the noise intensity in the nonlinear SDE becomes small. We prove in particular that the limit measures are discrete measures and point out some properties of their support which permit in several situations to describe explicitly the whole set of limit measures. This study requires essentially generalized Laplace's method approximations.

Key words and phrases: self-interacting diffusion; stationary measures; double well potential; perturbed dynamical system; Laplace's method

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1 Introduction

Historically self-stabilizing processes were obtained as McKean-Vlasov limit in particle systems and were associated with nonlinear partial differential equations [6, 7]. The description of the huge system is classical: it suffices to consider N

particles which form the solution of the stochastic differential system:

$$dX_t^{i,N} = \sqrt{\varepsilon} dW_t^i - V'(X_t^{i,N})dt - \frac{1}{N}\sum_{j=1}^N F'(X_t^{i,N} - X_t^{j,N})dt, \qquad (1.1)$$
$$X_0^{i,N} = x_0 \in \mathbb{R}, \quad 1 \le i \le N,$$

where $(W_t^i)_i$ is a family of independent one-dimensional Brownian motions, ε some positive parameter. In (1.1), the function V represents roughly the environment the Brownian particles move in and the interaction function F describes the attraction between one particle and the whole ensemble. As N becomes large, the law of each particle converges and the limit is the distribution $u_t^\varepsilon(dx)$ of the so-called *self-stabilizing diffusion* $(X_t^\varepsilon,\ t \ge 0)$. This particular phenomenon is well-described in a survey written by A.S. Sznitman [8]. The process $(X_t^\varepsilon,\ t \ge 0)$ is given by

$$dX_t^\varepsilon = \sqrt{\varepsilon} dW_t - V'(X_t^\varepsilon)dt - \int_{\mathbb{R}} F'(X_t^\varepsilon - x)du_t^\varepsilon(x)dt. \qquad (1.2)$$

This process is of course nonlinear since solving the preceding SDE (1.2) consists in pointing out the couple $(X_t^\varepsilon, u_t^\varepsilon)$. By the way, let us note in order to emphasize the nonlinearity of the study that $u_t^\varepsilon(x)$ satisfies:

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{\partial}{\partial x}\Big(u^\varepsilon(V' + F' * u^\varepsilon)\Big). \qquad (1.3)$$

Here $*$ stands for the convolution product. There exists a relative numerous literature dealing with the questions of existence and uniqueness of solutions for (1.2) and (1.3), the existence and uniqueness of stationary measures, the propagation of chaos (convergence in the large system of particles)... The results depend of course on the assumptions concerning both the environment function V and the interaction function F. Let us just cite some key works: [4], [6], [7], [10], [9], [1] and [2], [3].
The aim of this paper is to describe the ε-dependence of the stationary measures for self-stabilizing diffusions. S. Benachour, B. Roynette, D. Talay and P. Vallois [1] proved the existence and uniqueness of the invariant measure for self-stabilizing diffusions without the environment function V. Their study increased our motivation to analyze the general equation (1.2), that's why our assumptions concerning the interaction function F are close to theirs.
In some preceding paper [5], the authors considered some symmetric double-well potential function V and pointed out some particular phenomenon which is directly related to the nonlinearity of the dynamical system: under suitable conditions, there exist at least three invariant measures for the self-stabilizing diffusion (1.2). In particular, there exists a symmetric invariant measure and several so-called *outlying* measures which are concentrated around one bottom of the double-well potential V. Moreover, if V'' is some convex function and if the interaction is linear, that is $F'(x) = \alpha x$ with $\alpha > 0$, then there exist exactly

three stationary measures as ε is small enough, one of them being symmetric. What's about the ε-dependence of these measures ? In the classical diffusion case, i.e. without interaction, the invariant measure converges in the small noise limit ($\varepsilon \to 0$) to $\frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$ where δ represents the Dirac measure and, both a and $-a$ stand for the localization of the double-well V bottoms. The aim of this work is to point out how strong the interaction function F shall influence the asymptotic behavior of the stationary measures. In [5], under some moment condition: the $8q^2$-th moment has to be bounded, the analysis of the stationary measures permits to prove that their density satisfies the following exponential expression:

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + (F * u_\epsilon)(x)\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + (F * u_\epsilon)(y)\right)\right] dy}. \tag{1.4}$$

To prove the hypothetical convergence of u_ϵ towards some u_0, the natural framework is Laplace's method, already used in [5]. Nevertheless, the nonlinearity of our situation doesn't allow us to use directly these classical results.

Main results: We shall describe all possible limit measures for the stationary laws. Under a weak moment condition satisfied for instance by symmetric invariant measures (Lemma 5.2) or in the particular situation when V is a polynomial function satisfying $\deg(V) > \deg(F)$ (Proposition 3.1), a precise description of each limit measure u_0 is pointed out: u_0 is a discrete measure $u_0 = \sum_{i=1}^r p_i \delta_{A_i}$ (Theorem 3.6). The support of the measure is directly related to the global minima of some potential W_0 which permits to obtain the following properties (Proposition 3.7): for any $1 \le i, j \le r$, we get

$$\begin{aligned} V'(A_i) + \sum_{l=1}^r p_l F'(A_i - A_l) &= 0, \\ V(A_i) - V(A_j) + \sum_{l=1}^r p_l \left(F(A_i - A_l) - F(A_j - A_l)\right) &= 0 \\ \text{and} \quad V''(A_i) + \sum_{l=1}^r p_l F''(A_i - A_l) &\ge 0. \end{aligned}$$

We shall especially construct families of invariant measures which converge to δ_a and δ_{-a} where a and $-a$ represents the bottom locations of the potential V (Proposition 4.1). For suitable functions F and V, these measures are the only possible asymmetric limit measures (Proposition 4.4 and 4.5). Concerning families of symmetric invariant measures, we prove the convergence, as $\varepsilon \to 0$, under weak convexity conditions, towards the unique symmetric limit measure $\frac{1}{2}\delta_{-x_0} + \frac{1}{2}\delta_{x_0}$ where $0 \le x_0 < a$ (Theorem 5.4). A natural bifurcation appears then for $F''(0) = \sup_{z\in\mathbb{R}} -V''(z) =: \vartheta$. Indeed, the support of the limiting measure contains two different points if $x_0 > 0$ which is equivalent to the inequality $-\vartheta + F''(0) < 0$ stating some competition behavior between the functions V and F. We shall finally emphasize two examples of functions V and F which lead to the convergence of any sequence of symmetric self-stabilizing invariant measures towards a limit measure whose support contains at least three points (Proposition 5.6 and 5.7): the initial system is strongly perturbed by the interaction function F.

The material is organized as follows. After presenting the essential assumptions concerning the environment function V and the interaction function F, we start the asymptotic study by the simple linear case $F'(x) = \alpha x$ with $\alpha > 0$ (Section 2) which permits explicit computation for both the symmetric measure (Section 2.1) and the outlying ones (Section 2.2). In Section 3 the authors handle with the general interaction case proving the convergence of invariant measures subsequences towards finite combination of Dirac measures. The attention shall be focused on these limit measures (Section 3.2). To end the study, it suffices to consider assumptions which permit to deduce that there exist exactly three limit measures: one symmetric, δ_{-a} and δ_a. This essential result implies the convergence of both any asymmetric invariant measure (Section 4) and any symmetric one (Section 5). Some examples are presented.

Main assumptions

Let us first describe different assumptions concerning the environment function V and the interaction function F. The context is similar to Herrmann and Tugaut's previous study [5] and is also weakly related to the work [1].
We assume the following properties for the function V:

(V-1) *Regularity:* $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. $\mathcal{C}^\infty$ denotes the Banach space of infinitely bounded continuously differentiable function.

(V-2) *Symmetry:* V is an even function.

(V-3) V is a double-well potential. The equation $V'(x) = 0$ admits exactly three solutions : a, $-a$ and 0 with $a > 0$; $V''(a) > 0$ and $V''(0) < 0$. The bottoms of the wells are reached for $x = a$ and $x = -a$.

(V-4) There exist two constants $C_4, C_2 > 0$ such that $\forall x \in \mathbb{R}$, $V(x) \geq C_4 x^4 - C_2 x^2$.

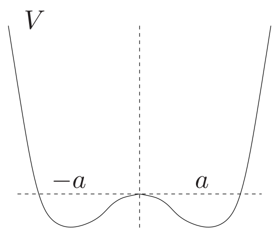

Figure 1: Potential V

(V-5) $\lim_{x \to \pm\infty} V''(x) = +\infty$ and $\forall x \geq a$, $V''(x) > 0$.

(V-6) *Analyticity:* There exists an analytic function $\mathcal{V}$ such that $V(x) = \mathcal{V}(x)$ for all $x \in [-a; a]$.

(V-7) The growth of the potential V is at most polynomial: there exist $q \in \mathbb{N}^*$ and $C_q > 0$ such that $|V'(x)| \leq C_q \left(1 + x^{2q}\right)$.

(V-8) *Initialization:* $V(0) = 0$.

Typically, V is a double-well polynomial function. We introduce the parameter ϑ which plays some important role in the following:

$$\vartheta = \sup_{x\in\mathbb{R}} -V''(x). \tag{1.5}$$

Let us note that the simplest example (most famous in the literature) is $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ which bottoms are localized in -1 and 1 and with parameter $\vartheta = 1$. Let us now present the assumptions concerning the attraction function F.

(F-1) F is an even polynomial function of degree $2n$ with $F(0) = 0$. Indeed we consider some classical situation: the attraction between two points x and y only depends on the distance $F(x-y) = F(y-x)$.

(F-2) F is a convex function.

(F-3) F' is a convex function on $\mathbb{R}_+$ therefore for any $x \geq 0$ and $y \geq 0$ such that $x \geq y$ we get $F'(x) - F'(y) \geq F''(0)(x-y)$.

(F-4) The polynomial growth of the attraction function F is related to the growth condition (V-7): $|F'(x) - F'(y)| \leq C_q|x-y|(1+|x|^{2q-2}+|y|^{2q-2})$.

Let us define the parameter $\alpha \geq 0$:

$$F'(x) = \alpha x + F_0'(x) \quad \text{with } \alpha = F''(0) \geq 0. \tag{1.6}$$

2 The linear interaction case

First, we shall analyze the convergence of different invariant measures when the interaction function F' is linear: $F(x) = \frac{\alpha}{2}x^2$ with $\alpha > 0$. In [5], the authors proved that any invariant density satisfies some exponential expression given by (1.4) provided that its $8q^2$-th moment is finite. This expression can be easily simplified, the convolution product is determined in relation with the mean of the stationary law. The symmetric invariant measure denoted by u_ϵ^0 becomes

$$u_\epsilon^0(x) = \frac{\exp\left[-\frac{2}{\epsilon}W_0(x)\right]}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}W_0(y)\right]dy} \quad \text{with} \quad W_0(x) = V(x) + \frac{\alpha}{2}x^2,\ \forall x\in\mathbb{R}. \tag{2.1}$$

The asymptotic behavior of the preceding expression is directly related to classical Laplace's method for estimating integrals and is presented in Section 2.1. If ε is small, Proposition 3.1 in [5] emphasizes the existence of at least two asymmetric invariant densities $u_\epsilon^\pm$ defined by

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \alpha\frac{x^2}{2} - \alpha m_\epsilon^\pm x\right)\right]}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}\left(V(y) + \alpha\frac{y^2}{2} - \alpha m_\epsilon^\pm y\right)\right]dy}. \tag{2.2}$$

Here $m_\epsilon^\pm$ represents the average of the measure $u_\epsilon^\pm(dx)$ which satisfies: for any $\delta \in]0,1[$ there exists $\varepsilon_0 > 0$ such that

$$\left| m_\epsilon^\pm - (\pm a) + \frac{V^{(3)}(\pm a)}{4V''(a)\left(\alpha + V''(a)\right)}\,\epsilon \right| \leq \delta\,\epsilon, \quad \forall \varepsilon \leq \varepsilon_0. \tag{2.3}$$

Equation (2.3) permits to develop, in Section 2.2, the asymptotic analysis of the invariant law in the asymmetric case.

2.1 Convergence of the symmetric invariant measure.

First of all, let us determine the asymptotic behavior of the measure $u_\varepsilon^0(dx)$ as $\varepsilon \to 0$. By (2.1), the density is directly related to the function W_0 which admits a finite number of global minima. Indeed due to conditions (V-5) and (V-6), we know that $W_0'' \geq 0$ on $[-a;a]^c$ and V is equal to an analytic function $\mathcal{V}$ on $[-a;a]$. Hence W_0' admits a finite number of zeros on $\mathbb{R}$ or $W_0' = 0$ on the whole interval $[-a,a]$ whcih implies immediately $W_0'' = 0$. In the second case, we get in particular that $W_0''(a) = 0$ which contradicts $W_0''(a) = V''(a) + \alpha > 0$. We deduce that W_0' admits a finite number of zeros on $\mathbb{R}$.
The measure u_ε^0 can therefore be developed with respect to the minima of W_0.

Theorem 2.1. *Let $A_1 < \ldots < A_r$ the r global minima of W_0 and $\omega_0 = \min_{z\in\mathbb{R}} W_0(z)$. For any i, we introduce $k_0(i) = \min\{k \in \mathbb{N} \mid W_0^{(2k)}(A_i) > 0\}$. Let us define $k_0 = \max\{k_0(i), i \in [1;r]\}$ and $I = \{i \in [1;r] \mid k_0(i) = k_0\}$. As $\varepsilon \to 0$, the measure u_ϵ^0 defined by (2.1) converges weakly to the following discrete measure*

$$u_0^0 = \frac{\sum_{i\in I}\left(W_0^{(2k_0)}(A_i)\right)^{-\frac{1}{2k_0}}\delta_{A_i}}{\sum_{j\in I}\left(W_0^{(2k_0)}(A_j)\right)^{-\frac{1}{2k_0}}} \tag{2.4}$$

Proof. Let f be a continuous and bounded function on $\mathbb{R}$. We define $A_0 = -\infty$ and $A_{r+1} = +\infty$. Then, for $i \in]1;r[$, we apply Lemma A.1 to the function $U = W_0$ and to each integration support $J_i = [\frac{A_{i-1}+A_i}{2}; \frac{A_i+A_{i+1}}{2}]$. We obtain the asymptotic equivalence as $\varepsilon \to 0$:

$$e^{\frac{2}{\epsilon}\omega_0}\int_{J_i} f(t)e^{-\frac{2W_0(t)}{\epsilon}}dt = \frac{f(A_i)}{k_0(i)}\Gamma\left(\frac{1}{2k_0(i)}\right)\left(\frac{\epsilon(2k_0(i))!}{2W_0^{(2k_0(i))}(A_i)}\right)^{\frac{1}{2k_0(i)}}(1+o(1)).$$

This equivalence is also true for the supports J_1 and J_r. Indeed, we can restrict the semi-infinite support of the integral to a compact one since f is bounded and since the function W_0 admits some particular growth property: $W_0(x) \geq x^2$

for $|x|$ large enough. Hence denoting $I = e^{\frac{2}{\epsilon}\omega_0}\int_{\mathbb{R}} f(t)\, e^{-\frac{2W_0(t)}{\epsilon}}dt$, we get

$$\begin{aligned}
I &= \sum_{i=1}^{r}\frac{f(A_i)}{k_0(i)}\Gamma\left(\frac{1}{2k_0(i)}\right)\left(\frac{\epsilon(2k_0(i))!}{2W_0^{(2k_0(i))}(A_i)}\right)^{\frac{1}{2k_0(i)}}(1+o(1))\\
&= \sum_{i\in I}\frac{f(A_i)}{k_0}\Gamma\left(\frac{1}{2k_0}\right)\left(\frac{\epsilon(2k_0)!}{2W_0^{(2k_0)}(A_i)}\right)^{\frac{1}{2k_0}}(1+o(1))\\
&= C(k_0)\epsilon^{\frac{1}{2k_0}}\sum_{i\in I}\frac{f(A_i)}{\left(W_0^{(2k_0)}(A_i)\right)^{\frac{1}{2k_0}}}(1+o(1)) \qquad (2.5)
\end{aligned}$$

with $C(k_0) = \frac{1}{k_0}\Gamma\left(\frac{1}{2k_0}\right)\left(\frac{(2k_0)!}{2}\right)^{\frac{1}{2k_0}}$. Applying the preceding asymptotic result (2.5) on one hand to the function f and on the other hand to the constant function 1, we estimate the ratio

$$\frac{\int_{\mathbb{R}} f(t)\exp\left[-\frac{2W_0(t)}{\epsilon}\right]dt}{\int_{\mathbb{R}}\exp\left[-\frac{2W_0(t)}{\epsilon}\right]dt} = \frac{\sum_{i\in I}\left(W_0^{(2k_0)}(A_i)\right)^{-\frac{1}{2k_0}}f(A_i)}{\sum_{j\in I}\left(W_0^{(2k_0)}(A_j)\right)^{-\frac{1}{2k_0}}}(1+o(1)). \qquad (2.6)$$

We deduce the announced weak convergence of u_ϵ^0 towards u_0^0. □

By definition, V is a symmetric double-well potential: it admits exactly two global minima. We focus now our attention on W_0 and particularly on the number of global minima which represents the support cardinal (denoted by r in the preceding statement) of the limiting measure.

Proposition 2.2. *If V'' is a convex function, then u_0^0 is concentrated on either one or two points, that is $r = 1$ or $r = 2$.*

Proof. We shall proceed using *reductio ad absurdum*. We assume that the support of u_0^0 contains at least three elements. According to the Theorem 2.1, they correspond to minima of W_0. Therefore there exist at least two local maxima: W_0' admits then at least five zeros. Applying Rolle's Theorem, we deduce that $W_0''(x)$ is vanishing at four distinct locations. This leads to some contradiction since V'' is a convex function so is W_0''. □

Let us note that the condition of convexity for the function V'' has already appeared in [5] (Theorem 3.2). In that paper, the authors proved the existence of exactly three invariant measures for (1.2) as the interaction function F' is linear. What happens if V'' is not convex ? In particular, we can wonder if there exists some potential V whose associated measure u_0^0 is supported by three points or more.

Proposition 2.3. *Let $p_0 \in [0;1[$ and $r \geq 1$. We introduce*

- *a probability measure $(p_i)_{1\leq i\leq r} \in]0;1[^r$ satisfying $p_1 + \cdots + p_r = 1 - p_0$,*

- *some family $(A_i)_{1\leq i\leq r}$ with $0<A_1<\cdots<A_r$.*

There exists a potential function V which verifies all assumptions (V-1)–(V-8) and a positive constant α such that the measure u_0^0, associated to V and the linear interaction $F'(x)=\alpha x$, is given by

$$u_0^0=p_0\delta_0+\sum_{i=1}^r\frac{p_i}{2}\left(\delta_{A_i}+\delta_{-A_i}\right).$$

Proof. **Step 1.** Let us define some function W as follows:

$$W(x)=\Big(C+\xi(x^2)^2\Big)^2\prod_{i=1}^r\left(x^2-A_i^2\right)^2\quad\text{if}\quad p_0=0,\tag{2.7}$$

$$W(x)=x^2\Big(C+\xi(x^2)^2\Big)^2\prod_{i=1}^r\left(x^2-A_i^2\right)^2\quad\text{if}\quad p_0\neq0,\tag{2.8}$$

where C is a positive constant and ξ is a polynomial function. C and ξ will be specified in the following. Using (2.1), we introduce $V(x)=W(x)-W(0)-\frac{\alpha}{2}x^2$. According to Theorem 2.1, the symmetric invariant measure associated to the functions V and F converges to a discrete measure whose support is either the set $\{\pm A_i, 1\leq i\leq r\}$, if $p_0=0$, either the set $\{0\}\cup\{\pm A_i,1\leq i\leq r\}$ if $p_0\neq0$. Theorem 2.1 permits also to evaluate the weights $(p_i)_i$. An obvious analysis leads to $k_0=1$. Moreover, the second derivative of W satisfies

$$W''(A_i)=8A_i^2\Big(C+\xi(A_i^2)^2\Big)^2\prod_{j=1,j\neq i}^r\left(A_i^2-A_j^2\right)^2\quad\text{if}\quad p_0=0,\tag{2.9}$$

$$W''(A_i)=8A_i^4\Big(C+\xi(A_i^2)^2\Big)^2\prod_{j=1,j\neq i}^r\left(A_i^2-A_j^2\right)^2\quad\text{if}\quad p_0\neq0,\tag{2.10}$$

$$\text{and}\quad W''(0)=2\Big(C+\xi(0)^2\Big)^2\prod_{j=1}^r A_j^4\quad\text{if}\quad p_0\neq0.\tag{2.11}$$

By (2.4), we know that $\sqrt{\frac{W''(A_k)}{W''(A_i)}}=\frac{p_i}{p_k}$ if $i\neq0$ and $\frac{2p_0}{p_i}=\sqrt{\frac{W''(A_i)}{W''(0)}}$ if $p_0\neq0$. Hence, if $p_0=0$,

$$\frac{p_i}{p_k}=\sqrt{\frac{W''(A_k)}{W''(A_i)}}=\frac{A_k\Big(C+\xi(A_k^2)^2\Big)\prod_{j=1,j\neq k}^r\left|A_k^2-A_j^2\right|}{A_i\Big(C+\xi(A_i^2)^2\Big)\prod_{j=1,j\neq i}^r\left|A_i^2-A_j^2\right|},\tag{2.12}$$

$$\text{and}\quad\frac{2p_0}{p_i}=\sqrt{\frac{W''(A_i)}{W''(0)}}=2\frac{C+\xi(A_i^2)^2}{C+\xi(0)^2}\prod_{j=1,j\neq i}^r\left|1-\frac{A_i^2}{A_j^2}\right|\quad\text{if}\quad p_0\neq0.\tag{2.13}$$

Step 2. Let us determine the polynomial function ξ.
Step 2.1. First case: $p_0=0$. We choose ξ such that $\xi(A_r^2)=1$. Then (2.12)

leads to the equation

$$\frac{C+\xi(A_k^2)^2}{C+1}=\eta_k=\frac{A_rp_r}{A_kp_k}\prod_{j=1,j\neq k}^{r-1}\left|\frac{A_r^2-A_j^2}{A_k^2-A_j^2}\right| \tag{2.14}$$

Let us fix $C=\inf\{\eta_k,\ k\in[1;r]\}>0$ so that $\eta_k(C+1)-C\geq C^2>0$. Therefore the preceding equality becomes

$$\xi(A_k^2)=\sqrt{(C+1)\eta_k-C},\ \text{for all } 1\leq k\leq r.$$

Finally it suffices to choose the following polynomial function which solves in particular (2.14):

$$\xi(x)=\sum_{k=1}^r\prod_{j=1,j\neq k}^r\frac{x-A_j^2}{A_k^2-A_j^2}\sqrt{(C+1)\eta_k-C}.$$

Step 2.2. Second case: $p_0\neq 0$. Using similar arguments as these presented in Step 2.1, we construct some polynomial function ξ satisfying (2.13). First we set $\xi(0)=1$ and define

$$\eta_i=\frac{p_0}{p_i}\prod_{j=1,j\neq i}^r\left|\frac{A_j^2}{A_j^2-A_i^2}\right|.$$

For $C=\inf\{\eta_i,\ i\in[1;r]\}$, we get $\eta_i(C+1)-C\geq C^2>0$. We choose the following function

$$\xi(x)=\prod_{j=1}^r\left(1-\frac{x}{A_j^2}\right)+\sum_{i=1}^r\frac{x}{A_i^2}\prod_{j=1,j\neq i}^r\frac{x-A_j^2}{A_i^2-A_j^2}\sqrt{(C+1)\eta_i-C}.$$

In Step 2.1 and 2.2, the stationary symmetric measures associated to ξ converge to $p_0\delta_0+\sum_{i=1}^r\frac{p_i}{2}\left(\delta_{A_i}+\delta_{-A_i}\right)$.
Step 3. It remains to prove that all conditions (V-1)–(V-8) are satisfied by the function V defined in Step 1. The only one which really needs to be carefully analyzed is the existence of three solutions to the equation $V'(x)=0$. Since W defined by (2.7) and (2.8) is an even function, $\rho(x)=\frac{W'(x)}{x}$ is well defined and represents some even polynomial function which tends to $+\infty$ as $|x|$ becomes large. Hence, there exists some $R>0$ large enough such that ρ is strictly decreasing on the interval $]-\infty;-R]$ and strictly increasing on $[R;+\infty[$. Let us now define $\alpha'=\sup_{z\in[-R;R]}\rho(z)$. Then, for any $\alpha>\alpha'$, the equation $\rho(z)=\alpha$ admits exactly two solutions. This implies the existence of exactly three solutions to $V'(x)=0$ i.e. condition (V-3). □

2.2 Convergence of the outlying measures

In the preceding section, we analyzed the convergence of the unique symmetric invariant measure u_ε^0 as $\varepsilon\to 0$. In [5], the authors proved that the set of

invariant measures doesn't only contain u_ε^0. In particular, for ε small enough, there exist asymmetric ones. We suppose therefore that ϵ is less than the critical threshold below which the measures u_ϵ^+ and u_ϵ^- defined by (2.2) and (2.3) exist. We shall focus our attention to their asymptotic behavior.
Let us recall the main property concerning the mean $m^\pm(\varepsilon)$ of these measures: for all $\delta > 0$, there exists $\epsilon_0 > 0$ small enough such that

$$\left| m^\pm(\epsilon) - (\pm a) + \frac{V^{(3)}(\pm a)}{4V''(a)\left(\alpha + V''(a)\right)}\,\epsilon \right| \le \delta\epsilon, \quad \varepsilon \le \varepsilon_0. \tag{2.15}$$

Here a and $-a$ are defined by (V-3). Let us note that we don't assume V'' to be a convex function nor u_ϵ^+ (resp. u_ϵ^-) to be unique.

Theorem 2.4. *The invariant measure u_ϵ^+ (resp. u_ϵ^-) defined by* (2.2) *and* (2.3) *converges weakly to δ_a (resp. δ_{-a}) as ϵ tends to* 0.

Proof. We just present the proof for u_ϵ^+ since the arguments used for u_ϵ^- are similar. Let us define $W_\epsilon^+(x) = V(x) + \frac{\alpha}{2}x^2 - \alpha m^+(\epsilon)x$.
Let f a continuous non-negative bounded function on $\mathbb{R}$ whose maximum is denoted by $M = \sup_{z\in\mathbb{R}} f(z)$. According to (2.2), we have

$$\int_{\mathbb{R}} f(x)u_\epsilon^+(x)dx = \frac{\int_{\mathbb{R}} f(x)\exp\left[-\frac{2}{\epsilon}W_\epsilon^+(x)\right]dx}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}W_\epsilon^+(x)\right]dx}. \tag{2.16}$$

We introduce $U(y) = V(y) + \frac{\alpha}{2}y^2 - \alpha a y$. By (2.15), for ε small enough, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}W_\epsilon^+(y)\right]dy &\le \int_{\mathbb{R}}\xi^+(y)\exp\left[-\frac{2}{\epsilon}U(y)\right]dy \\ \text{where}\quad \xi^+(y) &= \exp\left\{-\frac{\alpha V^{(3)}(a)}{2V''(a)\left(\alpha+V''(a)\right)}y + 2\alpha\delta\,|y|\right\}.\end{aligned}$$

By Lemma A.4 in [5] (in fact a slightly modification of the result: the function f_m appearing in the statement needs just to be $\mathcal{C}^{(3)}$-continuous in a small neighborhood of the particular point x_μ), the following asymptotic result (as $\varepsilon \to 0$) yields:

$$\int_{\mathbb{R}}\xi^+(y)\exp\left[-\frac{2}{\epsilon}U(y)\right]dy = \sqrt{\frac{\pi\epsilon}{\alpha+V''(a)}}\exp\left[-\frac{2}{\epsilon}U(a)\right]\xi^+(a)(1+o(1)).$$

We can obtain the lower-bound by similar arguments, just replacing $\xi^+(y)$ by $\xi^-(y)$ where

$$\xi^-(y) = \exp\left\{-\frac{\alpha V^{(3)}(a)}{2V''(a)\left(\alpha+V''(a)\right)}y - 2\alpha\delta\,|y|\right\}.$$

Therefore, for any $\eta > 1$, there exists some $\varepsilon_1 > 0$, such that

$$\frac{1}{\eta}\xi^-(a) \le \sqrt{\frac{U''(a)}{\pi\epsilon}}\; e^{\frac{2U(a)}{\epsilon}}\int_{\mathbb{R}} e^{-\frac{2}{\epsilon}W_\epsilon^+(x)}dx \le \eta\,\xi^+(a), \tag{2.17}$$

for $\varepsilon \leq \varepsilon_1$. In the same way, we obtain some $\varepsilon_2 > 0$, such that

$$\frac{1}{\eta} f(a)\xi^-(a) \leq \sqrt{\frac{U''(a)}{\pi\epsilon}}\, e^{\frac{2U(a)}{\epsilon}} \int_{\mathbb{R}} f(x) e^{-\frac{2}{\epsilon}W_\epsilon^+(x)} dx \leq \eta f(a)\xi^+(a), \qquad (2.18)$$

for $\varepsilon \leq \varepsilon_2$. Taking the ratio of (2.18) and (2.17), we immediately obtain:

$$\frac{1}{\eta^2} f(a) \exp\left[-4\alpha a\delta\right] \leq \int_{\mathbb{R}} f(x) u_\epsilon^+(x) dx \leq \eta^2 f(a) \exp\left[4\alpha a\delta\right],$$

for $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$. δ is arbitrarily small and η is arbitrary close to 1, so we deduce the convergence of $\int_{\mathbb{R}} f(x) u_\epsilon^+(x) dx$ towards $f(a)$. $\square$

2.3 The set of limit measures

In the particular case where V'' is a convex function and ε is fixed, we can describe exactly the set of invariant measures associated with (1.2). The statement of Theorem 3.2 in [5] precises that this set contains exactly three elements. What happens as $\varepsilon \to 0$? We shall describe in this section the set of all measures defined as limit of stationary measures as $\varepsilon \to 0$. We start with some preliminary result:

Lemma 2.5. *If V'' is a convex function then there exists a unique $x_0 \geq 0$ such that $V'(x_0) = -\alpha x_0$ and $\alpha + V''(x_0) \geq 0$. Moreover, if $\alpha + V''(0) \geq 0$ then $x_0 = 0$ otherwise $x_0 > 0$.*

Proof. Since V'' is a symmetric convex function, we get $\vartheta = -V''(0)$ where ϑ is defined by (1.5). Let χ the function defined by $\chi(x) = V'(x) + \alpha x$. We distinguish two different cases:
1) If $\alpha \geq -V''(0)$, then the convexity of V'' implies $\alpha + V''(x) > 0$ for all $x \neq 0$. Therefore 0 is the unique non negative solution to the equation $V'(x) + \alpha x = 0$. Moreover we get $\alpha + V''(0) \geq 0$.
2) If $\alpha < -V''(0)$, then χ admits at most three zeros due to the convexity of its derivative. Since $\chi(0) = 0$ and since χ is an odd function, there exists at most one positive zero. The inequality $\alpha + V''(0) < 0$ implies that χ is strictly decreasing at the right side of the origin. Moreover $\lim_{x\to+\infty} \chi(x) = +\infty$ which permits to conclude the announced existence of one positive zero x_0. We easily verify that $\alpha + V''(x_0) \geq 0$. $\square$

Proposition 2.6. *If V'' is a convex function then the family of invariant measures admits exactly three limit points, as $\varepsilon \to 0$. Two of them are asymmetric: δ_a and δ_{-a} and the third one is symmetric $\frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$; x_0 has been introduced in Lemma 2.5.*

Proof. Since V'' a is convex function, there exist exactly three invariant measures for (1.2) provided that ε is small enough (see Proposition 3.2 in [5]). These measures correspond to u_ϵ^0, u_ϵ^+ and u_ϵ^- defined by (2.1) and (2.2).
1) Theorem 2.1 emphasizes the convergence of u_ϵ^0 towards the discrete probability measure u_0^0 defined by (2.4). Due to the convexity of V'', the support of

this limit measure contains one or two reals (Proposition 2.2) which correspond to the global minima of $U(x) = V(x) + \frac{\alpha}{2}x^2$. According to the Lemma 2.5, we know that U admits a unique global minimum on $\mathbb{R}_+$ denoted by x_0. If $\alpha \geq -V''(0)$ then $x_0 = 0$ and consequently $r = 1$ i.e. $u_0^0 = \delta_0$. If $\alpha < -V''(0)$ then $x_0 > 0$ and $r = 2$ i.e. $u_0^0 = \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$ since u_0^0 is symmetric.
2) According to the Theorem 2.4, $u_\epsilon^\pm$ converges to $\delta_{\pm a}$. $\square$

3 The general interaction case

In this section we shall analyze the asymptotic behavior of invariant measures for the self-stabilizing diffusion as $\varepsilon \to 0$. Let us consider some stationary measure u_ϵ. According to Lemma 2.2 of [5], the following exponential expression holds:

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}W_\epsilon(x)\right]}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\epsilon}W_\epsilon(y)\right]dy} \quad \text{with} \quad W_\epsilon := V + F * u_\epsilon - F * u_\epsilon(0). \tag{3.1}$$

Since F is a polynomial function of degree $2n$, the function W_ε just introduced can be developed as follows

$$W_\epsilon(x) = V(x) + \sum_{k=1}^{\infty}\frac{x^k}{k!}\omega_k(\epsilon) \quad \text{with} \quad \omega_k(\epsilon) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}F^{(l+k)}(0)\mu_l(\epsilon), \tag{3.2}$$

$\mu_l(\epsilon)$ being the l-order moment of the measure u_ϵ.
W_ϵ is called the *pseudo-potential*. Since F is a polynomial function of degree $2n$, the preceding sums in (3.2) are just composed with a finite number of terms. In order to study the behavior of u_ε for small ϵ, we need to estimate precisely the pseudo-potential W_ϵ.
Let us note that, for some specific $p \in \mathbb{N}$, the L^p-convergence of u_ϵ towards some measure u_0 implies the convergence of the associated pseudo-potential W_ϵ towards a limit pseudo-potential W_0.
The study of the asymptotic behavior of u_ε shall be organized as follows:

- Step 1. First we'll prove that, under the boundedness of the family $\{\mu_{2n-1}(\epsilon),\ \varepsilon > 0\}$ with $2n = \deg(F)$, we can find a sequence $(\epsilon_k)_{k\geq 0}$ satisfying $\lim_{k\to\infty}\varepsilon_k = 0$ such that W_{ϵ_k} converges to a limit function W_0 associated to some measure u_0.

- Step 2. We shall describe the measure u_0: it is a discrete measure and its support and the corresponding weights satisfy particular conditions.

- Step 3. We analyze the behavior of the outlying measures concentrated around a and $-a$ and prove that these measures converge towards δ_a and δ_{-a} respectively. We show secondly that these Dirac measures are the only asymmetric limit measures. We present some example associated to the potential function $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$.

- Step 4. Finally we focus our attention to symmetric measures. After proving the boundedness of the moments, we discuss about non trivial examples (i.e. nonlinear interaction function F') where there exist at least three limit points.

3.1 Weak convergence for a subsequence of invariant measures

Let $(u_\epsilon)_{\epsilon>0}$ be a family of stationary measures. The main assumption in the subsequent developments is:

(H) We assume that the family $\{\mu_{2n-1}(\epsilon), \epsilon > 0\}$ is bounded for $2n = \deg(F)$.

This assumption is for instance satisfied if the degree of the environment potential V is larger than the degree of the interaction potential:

Proposition 3.1. *Let $(u_\epsilon)_{\epsilon>0}$ a family of invariant measures for the diffusion* (1.2)*. We assume that V is a polynomial function whose degree satisfies $2m_0 := \deg(V) > \deg(F) = 2n$. Then the family $\left(\int_{\mathbb{R}} x^{2m_0} u_\epsilon(x)dx\right)_{\epsilon>0}$ is bounded.*

Proof. Let us assume the existence of some decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ which tends to 0 and such that the sequence of moments $\mu_{2m_0}(k) := \int_{\mathbb{R}} x^{2m_0} u_{\epsilon_k}(x)dx$ tends to $+\infty$. By (3.1) and (3.2) we obtain:

$$\mu_{2m_0}(k) = \frac{\int_{\mathbb{R}} x^{2m_0} \exp\left[-\frac{2}{\epsilon_k}\left(\sum_{r=1}^{2m_0-1} \mathcal{M}_r(k)x^r + C_{2m_0}x^{2m_0}\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon_k}\left(\sum_{r=1}^{2m_0-1} \mathcal{M}_r(k)x^r + C_{2m_0}x^{2m_0}\right)\right] dx},$$

where $\mathcal{M}_r(k)$ is a combination of the moments $\mu_j(k)$, with $0 \le j \le \max(0, 2n-r)$, and $C_{2m_0} = V^{(2m_0)}(0)/(2m_0)!$. Let us note that the coefficient of degree 0 in the polynomial expression disappears since the numerator and the denominator of the ratio contain the same expression: that leads to cancellation. Moreover $\mathcal{M}_r(k)$ doesn't depend on ε_k for all r s.t. $2n \le r \le 2m_0$ and that there exists some r such that $\mathcal{M}_r(k)$ tends to $+\infty$ or $-\infty$ since $\mu_{2m_0}(k)$ is unbounded.
Let us define

$$\phi_k := \sup_{1\le r\le 2m_0-1} |\mathcal{M}_r(k)|^{\frac{1}{2m_0-r}}.$$

Then the sequences $\eta_r(k) := \frac{\mathcal{M}_r(k)}{\phi_k^{2m_0-r}}$, for $1 \le r \le 2m_0-1$, are bounded. Hence we extract a subsequence $(\epsilon_{\Psi(k)})_{k\in\mathbb{N}}$ such that $\eta_r(\Psi(k))$ converges towards some η_r as $k \to \infty$, for any $1 \le r \le 2m_0-1$. For simplicity, we shall conserve all notations: $\mu_{2m_0}(k)$, $\eta_r(k)$, ϕ_k... even for sub-sequences. The change of variable $x := \phi_k y$ provides

$$\frac{\mu_{2m_0}(k)}{\phi_k^{2m_0}} = \frac{\int_{\mathbb{R}} y^{2m_0} \exp\left[-\frac{2\phi_k^{2m_0}}{\epsilon_k}\left(\sum_{r=1}^{2n-1} \eta_r(k)y^r + C_{2m_0}y^{2m_0}\right)\right] dy}{\int_{\mathbb{R}} \exp\left[-\frac{2\phi_k^{2m_0}}{\epsilon_k}\left(\sum_{r=1}^{2n-1} \eta_r(k)y^r + C_{2m_0}y^{2m_0}\right)\right] dy}. \tag{3.3}$$

The highest moment appearing in the expression $\mathcal{M}_k$ is the moment of order $2n-k$. By Jensen's inequality, there exists a constant $C>0$ such that $|\mathcal{M}_r| \leq C\mu_{2m_0}(k)^{\frac{2n-r}{2m_0}}$ for all $1 \leq r \leq 2n-1$. The following upper-bound holds:

$$\phi_k^{2m_0} \leq \sup_{1\leq r\leq 2m_0-1} \left\{ C^{\frac{2m_0}{2m_0-r}} \mu_{2m_0}(k)^{\frac{2n-r}{2m_0-r}} \right\} = o\left\{\mu_{2m_0}(k)\right\}$$

We deduce from the previous estimate that the left hand side of (3.3) tends to $+\infty$. Let us focus our attention to the right term. In order to estimate the ratio of integrals, we use asymptotic results developed in the annex of [5], typically generalizations of Laplace's method. An adaptation of Lemma A.4 permits to prove that the right hand side of (3.3) is bounded: in fact it suffices to adapt the asymptotic result to the particular function $U(y) := \sum_{r=1}^{2n-1} \eta_r y^r + C_{2m_0} y^{2m_0}$ which doesn't satisfy *a priori* $U''(y_0) > 0$. This generalization is obvious since the result we need is just the boundedness of the limit, so we don't need precise developments for the asymptotic estimation. In the lemma the small parameter ε will be the ratio $\xi_k := \epsilon_k/\phi_k^{2m_0}$ and $f(y) := y^{2m_0}$.
Since the right hand side of (3.3) is bounded and the left hand side tends to infinity, we obtain some contradiction. Finally we get that $\{\mu_{2m_0}(\varepsilon),\, \varepsilon > 0\}$ is a bounded family. □

From now on, we assume that (H) is satisfied. Therefore applying Bolzano-Weierstrass's theorem we obtain the following result:

Lemma 3.2. *Under the assumption (H), there exists a sequence $(\epsilon_k)_{k\geq 0}$ satisfying $\lim_{k\to\infty}\varepsilon_k = 0$ such that, for any $1 \leq l \leq \deg(F)-1$, $\mu_l(\epsilon_k)$ converges towards some limit value denoted by $\mu_l(0)$ with $|\mu_l(0)| < \infty$.*

As presented in (3.2), the moments $\mu_l(\varepsilon)$ characterize the pseudo-potential W_ϵ. We obtained a sequence of measures which the moments are convergent so we can extract a subsequence such that the pseudo-potential converges. For $r \in \mathbb{N}^*$, we write :

$$\omega_k(0) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} F^{(l+k)}(0)\mu_l(0) \quad \text{and} \quad W_0(x) = V(x) + \sum_{k=0}^{\infty} \frac{x^k}{k!}\omega_k(0). \quad (3.4)$$

Like in (3.2), there is a finite number of terms non equal to 0 in the two sums so $W_0(x)$ is defined for all $x \in \mathbb{R}$.

Proposition 3.3. *Under condition (H), there exists a sequence $(\epsilon_k)_{k\geq 0}$ satisfying $\lim_{k\to\infty}\varepsilon_k = 0$ such that, for all $j \in \mathbb{N}$, $(W_{\epsilon_k}^{(j)})_{k\geq 1}$ converges towards $W_0^{(j)}$, uniformly on each compact subset of $\mathbb{R}$, where the* limit pseudo-potential *W_0 is defined by* (3.4)*, and $(u_{\epsilon_k})_{k\geq 1}$ converges weakly towards some probability measure u_0.*

Proof. By (3.4), we obtain, for any $p \geq 1$,

$$|\omega_p(\epsilon) - \omega_p(0)| \leq \sum_{l=1}^{\infty} \frac{\left|F^{(l+p)}(0)\right|}{l!} \left|\mu_l(\epsilon) - \mu_l(0)\right|.$$

Let us note that the sum in the right hand side of the previous inequality is finite. Using Lemma 3.2, we obtain the existence of some subsequence which permits the convergence towards 0 of each term. Therefore, for all $p \in \mathbb{N}$, $\omega_p(\epsilon_k)$ tends to $\omega_p(0)$ when k tends to infinity. Let $x \in \mathbb{R}$, since V is $\mathcal{C}^\infty$-continuous, both derivatives $W_{\epsilon_k}^{(j)}(x)$ and $W_0^{(j)}(x)$ are well defined. Moreover we get

$$\begin{aligned}
\left|W_{\epsilon_k}^{(j)}(x) - W_0^{(j)}(x)\right| &= \left|\sum_{p=j}^{\infty} \frac{x^{p-j}}{(p-j)!}\omega_p(\epsilon_k) - \sum_{p=j}^{\infty} \frac{x^{p-j}}{(p-j)!}\omega_p(0)\right| \\
&\leq \sum_{p=j}^{\infty} \frac{|x|^{p-j}}{(p-j)!} \left|\omega_p(\epsilon_k) - \omega_p(0)\right|.
\end{aligned}$$

Since there is just a finite number of terms, and since each term tends to 0, we obtain the pointwise convergence of $W_{\epsilon_k}^{(j)}$ towards $W_0^{(j)}$. In order to get the uniform convergence, we introduce some compact K: it suffices then to prove that, for $j \geq 1$, $\sup_{y \in K,\, k \geq 0} \left|W_{\varepsilon_k}^{(j)}(y)\right|$ is bounded. This is just an obvious consequence of the regularity of V and of the following bound:

$$|W_{\varepsilon_k}^{(j)}(y)| \leq |V^{(j)}(y)| + \sum_{p=j}^{\infty} \frac{|y|^{p-j}}{(p-j)!} |w_p(\varepsilon_k)|.$$

Let us prove now the existence of a subsequence of $(\varepsilon_k)_{k\geq 0}$ for which we can prove the weak convergence of the corresponding sequence of invariant measures. According to condition (H), $\{\mu_2(\epsilon_k); k \geq 1\}$ is bounded; we denote m_2 the upper-bound. Using the Bienayme-Tchebychev's inequality, the following bound holds for all $R > 0$ and $k \geq 1$: $u_{\epsilon_k}([-R; R]) \geq 1 - \frac{m_2}{R^2}$. Consequently the family of probability measures $\{u_{\epsilon_k}; k \in \mathbb{N}^*\}$ is tight. Prohorov's Theorem permits to conclude that $\{u_{\epsilon_k}; k \in \mathbb{N}^*\}$ is relatively compact with respect to the weak convergence. □

Lemma 3.4. *Each limit function W_0 admits a finite number $r \geq 1$ of global minima.*

Proof. According to (3.4), $W_0'(x) = V'(x) + P'(x)$ where P is a polynomial function. Since V is equal to an analytic function $\mathcal{V}$ on $[-a; a]$ (see Condition (V-6)), we deduce that W_0' has a finite number of zeros on this interval. Indeed, if this assertion is false, then $W_0''(a) = 0$ which contradicts the following limit $W_0''(a) = V''(a) + \lim_{k\to\infty} F'' * u_{\varepsilon_k}(a) > 0$ due to the convexity property of F (F-2). By (3.1) and condition (V-4) and since F is even and F' is convex on $\mathbb{R}^+$, we deduce that $W_\epsilon(x) \geq V(x) \geq C_4x^4 - C_2x^2$ for any $\varepsilon > 0$. Taking the limit with respect to the parameter ε, we get $W_0(x) \geq C_4x^4 - C_2x^2$ for all $x \in \mathbb{R}$. This lower-bound permits to conclude that there exists some large interval $[-R; R]$ which contains all the global minima of W_0. It remains then to study the minima of W_0 on the compact $K := [-R; -a] \bigcup [a; R]$. According to the property (V-5), $V''(x) > 0$ for all $a \in K$. Moreover F is a convex function (condition (F-2)) therefore the definition (3.1) implies: $W''_\epsilon(x) \geq \min_{z\in[-R;-a]} V''(z) > 0$ for all $x \in K$. By Proposition 3.3, there exists some sequence $(\varepsilon_k)_{k\geq 0}$ such that W''_{ϵ_k} converges towards W''_0 uniformly on K. Consequently $W''_0(x) > 0$ for all $x \in K$ and so W'_0 admits a finite number of zeros on K. $\square$

Since W_0 admits r global minima, we define $A_1 < \cdots < A_r$ their location:

$$W_0(A_1) = \cdots = W_0(A_r) = \inf_{x\in\mathbb{R}} W_0(x) =: w_0. \tag{3.5}$$

We introduce a covering U^δ of these points:

$$U^\delta = \bigcup_{i=1}^{r} [A_i - \delta; A_i + \delta] \quad \text{with } \delta \in \left]0; \frac{1}{2}\min_{i\neq j} |A_i - A_j|\right[. \tag{3.6}$$

The set $\{A_1, \ldots, A_r\}$ plays a central role in the asymptotic analysis of the measures $(u_\varepsilon)_\varepsilon$. In particular we can prove that u_0 defined in Proposition 3.3 is concentrated around these points.

Proposition 3.5. *Let W_0 and $(\varepsilon_k)_{k\geq 0}$ be defined by Proposition 3.3. Then the following convergence results hold:*
1) For all $x \notin U^\delta$, $\lim_{k\to\infty} u_{\epsilon_k}(x) = 0$.
2) For $\delta > 0$ small enough, $\lim_{k\to\infty} \int_{(U^\delta)^c} u_{\epsilon_k}(x)dx = 0$.

Proof. 1) The limit pseudo-potential W_0 is $\mathcal{C}^\infty$ and its r global minima are in U^δ. Since $\lim_{x\to\pm\infty} W_0(x) = +\infty$, there exists $\eta > 0$ such that $W_0(x) \geq w_0 + \eta$ for all $x \notin U^\delta$.
Let us now point out some lower bound for W_{ε_k}. Using (V-5), (F-3), (1.6) and the definition (3.1), we prove that

$$W''_{\epsilon_k}(x) \geq V''(x) + \alpha > \alpha \quad \text{for all } |x| \geq a \text{ and } k \geq 0. \tag{3.7}$$

Moreover, due to the boundedness of the moments (condition (H)), $W'_{\varepsilon_k}(0)$ and $W''_{\varepsilon_k}(0)$ are bounded uniformly with respect to $k \geq 0$. This property combined with (3.7) leads to the existence of some $R > 0$ (independent of k) such that $W_{\epsilon_k}(x) \geq w_0 + \eta$ for $|x| \geq R$. What happens on the compact set $[-R, R] \cap (U^\delta)^c$? Proposition 3.3 emphasizes the uniform convergence of W_{ϵ_k} on this compact set. As a consequence, there exists some $k_0 \in \mathbb{N}$ such that $W_{\epsilon_k}(x) \geq w_0 + \frac{\eta}{2}$ for $k \geq k_0$ and $x \in [-R, R] \cap (U^\delta)^c$. To sum up, the lower bound $W_{\epsilon_k}(x) \geq w_0 + \frac{\eta}{2}$ holds for any $x \in (U^\delta)^c$ provided that $k \geq k_0$. Finally we obtain:

$$\exp\left[-\frac{2}{\epsilon_k} W_{\epsilon_k}(x)\right] \leq \exp\left[-\frac{2}{\epsilon_k} w_0\right] \exp\left[-\frac{\eta}{\epsilon_k}\right], \quad \forall x \notin U^\delta,\ k \geq k_0. \tag{3.8}$$

In order to estimate $u_{\varepsilon_k}(x)$, we need a lower bound for the denominator in the ratio (3.1). We denote this denominator D_ε. Since the continuous function W_0 admits a finite number of minima, there exists γ with $\delta > \gamma > 0$ such that, for all $x \in U^\gamma$, we have $W_0(x) \le w_0 + \frac{\eta}{8}$. By Proposition 3.3, W_{ϵ_k} converges uniformly on each compact subset of $\mathbb{R}$ to W_0. Therefore there exists k_1 such that for all $k \ge k_1$, for all $x \in U^\gamma$, we have $W_{\epsilon_k}(x) \le w_0 + \frac{\eta}{4}$. Finally the denominator in (3.1) can be lower bounded:

$$D_{\varepsilon_k} \ge \int_{U^\gamma} \exp\left[-\frac{2}{\epsilon_k} W_{\epsilon_k}(x)\right] dx \ge 2r\gamma \exp\left[-\frac{2}{\epsilon_k}\left(w_0 + \frac{\eta}{4}\right)\right], \quad k \ge k_1. \tag{3.9}$$

For $k \ge k_0 \vee k_1$, according to (3.8) and (3.9), the following bound holds

$$u_{\epsilon_k}(x) \le \frac{1}{2r\gamma} \exp\left[-\frac{\eta}{2\epsilon_k}\right]. \tag{3.10}$$

The right hand side in (3.10) tends to 0 as $k \to \infty$ which proves the first part of the statement.
2) Let $\delta > 0$ and $K > 0$. We split the integral support into two parts: $U^\delta = I_1 \cup I_2$ with $I_1 = [-K; K]^c$ respectively $I_2 = [-K; K] \bigcap (U^\delta)^c$. Bienayme-Tchebychev's inequality permits to bound the integral of u_{ε_k} on I_1 by $\frac{m_2}{K^2}$, where m_2 satisfies $\sup_{k \ge 0} \mu_2(\epsilon_k) \le m_2$. By (3.10), the integral on the second support I_2 is bounded by $\frac{K}{r\gamma} \exp\left[-\frac{\eta}{2\epsilon_k}\right]$. It suffices then to choose K and k large enough in order to prove the convergence of $\int_{(U^\delta)^c} u_{\epsilon_k}(x)dx$. □

The sequence of measures $(u_{\epsilon_k})_{k \in \mathbb{N}^*}$ converges to a measure u_0. Furthermore the open set $\left(U^\delta\right)^c$ is less and less weighted by u_{ϵ_k} as k becomes large. Intuitively u_0 should be a discrete measure whose support corresponds to the set $\{A_1, \ldots, A_r\}$.

Theorem 3.6. *Let $(\varepsilon_k)_{k \ge 1}$, W_0, u_0 and $A_1, \ldots, A_r$ be defined in the statement of Proposition 3.3 and by (3.5). Then the sequence of measures $(u_{\epsilon_k})_{k \ge 1}$ converges weakly, as k becomes large, to the discrete probability measure $u_0 = \sum_{i=1}^r p_i \delta_{A_i}$ where*

$$p_i = \lim_{k \to +\infty} \int_{A_i - \delta}^{A_i + \delta} u_{\epsilon_k}(x)dx, \quad 1 \le i \le r, \quad \delta > 0 \textit{ small enough.}$$

Moreover p_i is independent of the parameter δ.

Proof. 1) First we shall prove that the coefficients p_i are well defined. Let us fix some small positive constant δ. We define $p_i(\delta)$ the limit of $\int_{A_i-\delta}^{A_i+\delta} u_{\epsilon_k}(x)dx$ as $k \to \infty$. Of course this limit exists since, by Proposition 3.3, $(u_{\varepsilon_k})_{k \ge 1}$ converges weakly. Furthermore this limit is independent of δ. Indeed let us choose $\delta' < \delta$. By definition, we obtain

$$p_i(\delta) - p_i(\delta') = \lim_{k \to \infty} \left\{ \int_{A_i - \delta}^{A_i - \delta'} u_{\epsilon_k}(x)dx + \int_{A_i + \delta'}^{A_i + \delta} u_{\epsilon_k}(x)dx \right\}.$$

An obvious application of the statement 2) in Proposition 3.5 permits to obtain $p_i(\delta') = p_i(\delta) =: p_i$.

2) Let us prove now that u_0 is a discrete probability measure. Let f be a continuous and bounded function on $\mathbb{R}$. Let $\delta > 0$ small enough such that the intervals $U_i(\delta) := [A_i - \delta; A_i + \delta]$ are disjoint. The weak convergence is based on the following difference:

$$\int_{\mathbb{R}} f(x) u_{\epsilon_k}(x) dx - \sum_{i=1}^{r} p_i f(A_i) = R + \sum_{i=1}^{r} \Delta_i(f),$$

with $\Delta_i(f) = \int_{A_i-\delta}^{A_i+\delta} f(x) u_{\epsilon_k}(x) dx - p_i f(A_i)$ and $R = \int_{(U^\delta)^c} f(x) u_{\epsilon_k}(x) dx$. The boundedness of the function f and the statement 2) of Proposition 3.5 imply that R tends to 0 as $k \to \infty$. Let us now estimate each term $\Delta_i(f)$:

$$\begin{aligned} |\Delta_i(f)| &\leq \int_{A_i-\delta}^{A_i+\delta} \Big| f(x) - f(A_i) \Big| du_{\epsilon_k}(x) + |f(A_i)| \Big| u_{\epsilon_k}\Big(U_i(\delta)\Big) - p_i \Big| \\ &\leq \sup_{z \in U_i(\delta)} |f(z) - f(A_i)| \, u_{\epsilon_k}\Big(U_i(\delta)\Big) + |f(A_i)| \Big| u_{\epsilon_k}\Big(U_i(\delta)\Big) - p_i \Big| . \end{aligned}$$

Due to the continuity of f, $\sup_{x \in U_i(\delta)} |f(x) - f(A_i)|$ is small as soon as δ is small enough. Moreover for some fixed δ, the definition of p_i leads to the convergence of $u_{\epsilon_k}(U_i(\delta)) - p_i$ towards 0 as $k \to \infty$. Combining these two arguments permits to obtain the weak convergence of u_{ε_k} towards the discrete measure $\sum_{i=1}^r p_i \delta_{A_i}$ which can finally be identified with u_0. □

It is important to note that we didn't prove the convergence of any sequence of stationary measures u_ϵ as $\epsilon \to 0$ because we didn't prove the convergence of the moments. However, we know in advance that each limit value is a discrete probability measure.

3.2 Description of the limit measures

We have just pointed out in the previous section that all limit measures are discrete probability measures. Each limit measure shall be denoted in a generic way u_0 and is associated to some limit function W_0 defined by (3.4). Therefore we have the following expression $u_0 = \sum_{i=1}^r p_i \delta_{A_i}$ where the A_i are the global minima of W_0, rearranged in increasing order, and $\sum_{i=1}^r p_i = 1$, $p_i > 0$. We'll now refine this result by exhibiting properties of the points A_i and the weights p_i.

Proposition 3.7. *1) For all $1 \leq i \leq r$ and $1 \leq j \leq r$, we have :*

$$V'(A_i) + \sum_{l=1}^{r} p_l F'(A_i - A_l) = 0, \tag{3.11}$$

$$V(A_i) - V(A_j) + \sum_{l=1}^{r} p_l \left(F(A_i - A_l) - F(A_j - A_l) \right) = 0 \tag{3.12}$$

$$\textit{and} \quad V''(A_i) + \sum_{l=1}^{r} p_l F''(A_i - A_l) \geq 0 \tag{3.13}$$

2) $A_i \in [-a;a]$ for any $1 \leq i \leq r$. Besides, if $r \geq 2$, $A_1 \in]-a;0[$ and $A_r \in]0;a[$. In particular, if $r = 2$ then $A_1 A_2 < 0$.
3) If $\alpha \geq \vartheta$, where α (resp. ϑ) has been defined by (1.6) *(resp. by* (1.5)*), the support of any limit measure contains some unique point.*
4) If V'' and F'' are convex, the support of any limit measure contains at most two points.

Proof. **1)** In the proof of Proposition 3.3, the tightness of the family $(u_\varepsilon)_\varepsilon$ was presented. This argument combined with the convergence of the moments $(\mu_l(\varepsilon_k))_{k\geq 0}$ stated in Lemma 3.2 and equation (3.1), permits to express W_0' as follows:

$$W_0'(x) = V'(x) + F' * u_0(x), \quad x \in \mathbb{R}.$$

It suffices then to precise that u_0 is a discrete measure (Theorem 3.6). By definition, A_i is a local minimum of W_0 therefore it vanishes W_0' and besides verifies $W_0''(A_i) \geq 0$: we directly deduce (3.11) and (3.13). Furthermore, by definition (3.5) of A_i, $W_0(A_i) = W_0(A_j)$ for all i and j which implies (3.12).
2) By definition, A_r is the largest location of the global minimum of W_0 and A_1 is the smallest one. According to (3.11), they verify

$$V'(A_r) = -\sum_{j\neq r} p_j F'(A_r - A_j) \quad \text{and} \quad V'(A_1) = -\sum_{j\neq 1} p_j F'(A_1 - A_j).$$

Since $A_r - A_j > 0$ for $j < r$ and $A_1 - A_j < 0$ for $j > 1$ and since F' is non decreasing, we deduce the inequalities $V'(A_1) \geq 0$ and $V'(A_r) \leq 0$. Consequently we get $A_1 \geq -a$ and $A_r \leq a$.
Besides, if $r \geq 2$ then $V'(A_1) > 0$ and $V'(A_r) < 0$. However, V' is positive on $]-a;0[$ and negative on $]0;a[$, therefore $-a < A_1 < 0 < A_r < a$.
3) We assume that $A_r > A_1$. By (V-6) the equation $V''(x) = -\vartheta$ admits just a finite number of solutions, therefore we get $-(V'(A_r) - V'(A_1)) < \vartheta\,(A_r - A_1)$. According to (3.11) applied to A_1 and to A_r, we obtain:

$$-(V'(A_r) - V'(A_1)) = \sum_{i=1}^{r} p_i \Big(F'(A_r - A_i) - F'(A_1 - A_i) \Big).$$

Using the following properties: F is an even function, see (F-1), and moreover F' is a convex function on $\mathbb{R}_+$, see (F-3), we get $F'(A_r - A_i) \geq \alpha(A_r - A_i)$ and $-F'(A_1 - A_i) = F'(A_i - A_1) \geq \alpha(A_i - A_1)$. Finally the following lower bound holds $-(V'(A_r) - V'(A_1)) \geq \alpha\,(A_r - A_1)$. We conclude that $\vartheta\,(A_r - A_1) > \alpha\,(A_r - A_1)$. This inequality contradicts the hypothesis $A_r > A_1$ since $\alpha \geq \vartheta$.
4) We can compute the second derivative of the limit potential $W_0'' = V'' + \sum_{j=1}^{r} p_j F''(. - A_j)$. Moreover V'' and F'' are convex functions so is W_0''. Let us assume that W_0 admits at least three global minima then it also admits two local maxima: at least five critical points in all. The application of Rolle's theorem implies at least four zeros for the function W_0'' which is in fact nonsense since W_0'' is a convex function. $\square$

Proposition 3.7 permits in suitable situation to describe precisely the set of limit measures.

Remark 3.8. *1) In the particular case when $F(x) = 0$, for all $x \in \mathbb{R}$, that is without self-stabilization, the equations* (3.11), (3.12) *and* (3.13) *are satisfied if and only if the locations A_i are the minima of V. Hence the limit stationary measure weightes the points $-a$ and a.*
2) Let us assume $\alpha \geq \vartheta$. If $(u_{\varepsilon_k})_{k\geq 1}$ represents some family of symmetric stationary measures with bounded moments of order $1, \ldots, \deg(F) - 1$ then the sequence converges weakly to the Dirac measure $u_0 = \delta_0$. Indeed, according to (3.12) *and Proposition 3.7, the single point corresponding to the support of u_0 is a zero of V' that is $-a$, 0 or a (see hypothese (V-3)). Besides u_0 is symmetric, so we get $u_0 = \delta_0$.*
3) Let us assume that $\alpha \geq \vartheta$. If we consider a sequence of outlying stationary measures which moments of order $1, \ldots, \deg(F) - 1$ are bounded, Proposition 3.7 implies that there exist at most two limit measures δ_a and δ_{-a}.

4 Convergence of the outlying measures

In this section we shall precise the convergence results obtained for subsequences of invariant measures. We shall in particular study the case of asymmetric invariant measures. First let us recall that F is a polynomial function of degree $2n$.

Proposition 4.1. *Under the condition:*

$$\sum_{p=0}^{2n-2} \frac{\left|F^{p+2}(a)\right|}{p!} a^p < \alpha + V''(a), \qquad (2n = \deg(F)) \tag{4.1}$$

there exists a family of invariant measures $(u_\varepsilon^+)_{\varepsilon>0}$ (respectively $(u_\varepsilon^-)_{\varepsilon>0}$) which converges weakly as $\varepsilon \to 0$ towards the Dirac measure δ_a (resp. δ_{-a}). We recall that a is defined by (V-3).

Proof. Let us choose some sequence $(\eta_\epsilon)_\epsilon$ satisfying $\lim_{\epsilon\to 0} \eta_\epsilon = 0$ and $\lim_{\epsilon\to 0} \epsilon/\eta_\epsilon = 0$. Using Theorem 4.6 in [5], we know that (4.1) implies the existence of some family of outlying invariant measures $(u_\varepsilon^\pm)_{\varepsilon>0}$ which verifies the following asymptotic estimate

$$\left| \int_{\mathbb{R}} x^l u_\epsilon^\pm(dx) - (\pm a)^l \right| \leq \eta_\epsilon, \quad 1 \leq l \leq 2n-1, \text{ for } \varepsilon \text{ small enough.} \tag{4.2}$$

Using the binomial coefficients and equation (4.2), it is straightforward to prove that $u_\varepsilon^\pm$ converges in L^{2n-2} towards $\delta_{\pm a}$, and by the way converges weakly. □

Remark 4.2. *The statement 3) in Proposition 3.7 and Remark 3.8 emphasize that, under the assumption $\alpha \geq \vartheta$, δ_a and δ_{-a} are the only possible asymmetric limit measures for families of invariant selfstabilizing measures whose $(2n-1)$-th moments are uniformly bounded.*

Let us observe now the situation where δ_a and δ_{-a} are not the only asymmetric limit measures.

Proposition 4.3. *Let us assume that V'' and F'' are convex functions. If there exists some sequence of stationary measures $(u_{\varepsilon_k})_{k\in\mathbb{N}^*}$, whose $(2n-1)$-th moments are bounded uniformly w.r.t. k, and which converges weakly to an asymmetric non-extremal measure u_0 (i.e. u_0 is different from $\delta_{\pm a}$), then the limit measure satisfies $u_0 = p\delta_{A_1} + (1-p)\delta_{A_2}$ with $p(1-p)\left|p-\frac{1}{2}\right| > 0$ and $A_1A_2\left|A_2^2 - A_1^2\right| < 0$. Besides, the following properties hold:*

$$V(A_2) - V(A_1) - \frac{V'(A_1)+V'(A_2)}{F'(A_2-A_1)}F(A_2-A_1) = 0 \tag{4.3}$$

$$V'(A_1) - V'(A_2) = F'(A_2-A_1) \quad \text{and } p = \frac{-V'(A_2)}{V'(A_1)-V'(A_2)}. \tag{4.4}$$

Proof. Since $(u_{\varepsilon_k})_{k\in\mathbb{N}^*}$ admits uniformly bounded $(2n-1)$-th moments and since this sequence converges weakly, we obtain the convergence of the moments. Therefore, we can apply the statement of Proposition 3.7: the convexity of F'' and V'' implies that the support of u_0 contains at most two elements.
1) Let us first consider the case when $u_0 = \delta_{A_1}$. Equation (3.11) implies that $V'(A_1) = 0$. Therefore $A_1 = -a$, a, or 0. Since we assume that u_0 is asymmetric and non extremal, no solution of the preceeding equation leads to the construction of some u_0.
2) If $u_0 = p\delta_{A_1} + (1-p)\delta_{A_2}$ with $p(1-p) > 0$, then Proposition 3.7 gives some information on the parameters. First of all $A_1A_2 < 0$. Moreover (3.11), (3.12) et (3.13) immediatly lead to (4.3) and (4.4). If $p = 1/2$ then (4.3) and (4.4) imply that $V(A_1) = V(A_2)$. Due to the symmetry of V and since $A_1 \in]-a;0[$ and $A_2 \in]0;a[$, we obtain $A_1 = -A_2$ and therefore u_0 is symmetric which contradicts the hypothesis of the proposition. Finally $\left|p-\frac{1}{2}\right| > 0$.
Furthermore, if $A_1 = -A_2$, then we observe that $p = 1/2$. This case was just studied: we obtain finally that $A_1A_2\left|A_2^2 - A_1^2\right| < 0$ □

Let us now study particular potential functions V and F which permits to obtain that the only asymmetric limit measures are the Dirac measures δ_a and δ_{-a}.

Proposition 4.4. *Let us consider some sequence $(u_{\varepsilon_k})_{k\geq 0}$ of invariant measures, which admits uniformly bounded $(2n-1)$-th moments and which is associated to both the potential function $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and the interaction $F(x) = \beta\frac{x^{2n}}{2n}$, with $n \geq 1$ and $\beta > 0$. Then the only possible asymmetric limit measures are δ_1 and δ_{-1}.*

Proof. Since $V^{(4)}(x) = 6 > 0$, V'' is a convex function so is F''.
Step 1. If $n = 1$, Theorem 3.2 in [5] states that the system admits exactly three invariant measures: one is symmetric and the others are the outlying measures u_ϵ^+ and u_ϵ^- which tend respectively to δ_a and δ_{-a} according to Theorem 2.4. Let us just note that $a = 1$ in this particular case.
From now on we assume: $n \geq 2$.

Step 2. Let us consider an asymmetric limit measure of the sequence $(u_{\varepsilon_k})_{k\geq 1}$ denoted by u_0. Due to the convexity of both F'' and V'', we can apply Proposition 4.3. We deduce that $u_0 = p\delta_{A_1} + (1-p)\delta_{A_2}$ with $p(1-p)\left|p-\frac{1}{2}\right| > 0$ and $A_1A_2\left|A_2^2 - A_1^2\right| < 0$ so that $|A_1| < 1$ and $|A_2| < 1$. Besides, the equations (4.3) and (4.4) are satisfied.
Step 3. Let us prove that we can't solve both equations (4.3) and (4.4). Considering the particular form of the functions V and F, (4.3) becomes, after dividing by $A_2^2 - A_1^2$:

$$\frac{A_2^2 + A_1^2}{4} - \frac{1}{2} - \frac{1}{2n}\left(A_2^2 - A_1A_2 + A_1^2 - 1\right) = 0.$$

This expression is equivalent to $(n-2)A_1^2 + (n-2)A_2^2 + 2A_1A_2 = 2n-2$.
For $n = 2$, we get $A_1A_2 = 1$ which contradicts the property developed in the statement 2) of Proposition 3.7: $A_1A_2 < 0$. Hence the support of u_0 can not contains two elements if $n = 2$.
For $n > 2$ the arguments are similar. Using the bounds $|A_i| \leq 1$, for $i = 1$ and 2, we obtain the inequality:

$$2n-2 = (n-2)A_1^2 + (n-2)A_2^2 + 2A_1A_2 \leq 2n - 4 + 2A_1A_2.$$

We deduce that $A_1A_2 \geq 1$ which also contradicts the property just mentioned $A_1A_2 < 0$. $\square$

Let us just note that the previous proof is relatively simple since the function F' is a divisor of F. We shall present some other particular case where the coefficient α defined by (1.6) is between 0 and $-V''(0) = \vartheta$ (this previous equality is related to the convexity of V''). Under these conditions, Remark 4.2 can not be applied.

Proposition 4.5. *Let us consider some sequence $(u_{\varepsilon_k})_{k\geq 0}$ of invariant measures, which admits uniformly bounded third moments and which is associated to the potential function $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and the interaction $F(x) = \beta\frac{x^4}{4} + \alpha\frac{x^2}{2}$, with $\alpha \in [0;1[$ and $\beta > 0$. Then the only possible asymmetric limit measures are δ_1 and δ_{-1}.*

Proof. The arguments developed for the proof of Proposition 4.5 are similar to those presented in Proposition 4.4: the second step can directly be applied. Therefore the equations (4.3) and (4.4) are satisfied. Let us develop equation (4.3) for the particular functions V and F. Hence (4.3) is equivalent to

$$A_2^4 - A_1^4 - 2A_2^2 + 2A_1^2 - \frac{(A_1^3 + A_2^3 - A_1 - A_2)(\beta(A_2 - A_1)^4 + 2\alpha(A_2 - A_1)^2)}{\beta(A_2 - A_1)^3 + \alpha(A_2 - A_1)} = 0.$$

Reducing to the same denominator, we obtain:

$$\beta(A_2 - A_1)^4(A_1 + A_2)(A_1A_2 - 1) - \alpha(A_2 - A_1)^4(A_1 + A_2) = 0.$$

This equation is equivalent to $A_1A_2 - 1 = \alpha/\beta$ since $A_1 \neq -A_2$. We effectively consider only asymmetric limit measures and $A_1 = -A_2$ implies that $p = 1/2$.

In order to conclude it suffices to note that $A_1A_2 = 1+\alpha/\beta$ contradicts the property $A_1A_2 < 0$ which comes from the statement 2) of Proposition 3.7. We deduce that the support of any limit measure contains some unique point which corresponds to some zero of V'. The only possible value are 1 and -1 since the value 0 corresponds to some symmetric measure. $\square$

5 Convergence for sequences of symmetric invariant measures

In this last section, we consider the limit measures for families of symmetric invariant measures associated to the self-stabilizing process and denoted by $(u_\epsilon^0)_{\epsilon>0}$. Let us introduce the notation: $\mu_l^0(\varepsilon)$ represents the l-th order moments of u_ε^0. Since the functions V and F are even, the function W_ε^0 defined by (3.1) and (3.2) satisfies

$$W_\epsilon^0(x) = V(x) + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!}\omega_{2k}^0(\epsilon) \quad \text{with} \quad \omega_{2k}^0(\epsilon) = \sum_{l=0}^{\infty} \frac{F^{(2l+2k)}(0)}{(2l)!}\mu_{2l}^0(\epsilon). \tag{5.1}$$

Let us note that the series appearing in the previous equality just contain a finite number of terms.
Introducing the following normalization coefficient $\lambda_\epsilon^0 = \int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}W_\epsilon^0(x)\right]dx$, we obtain

$$\mu_{2l}^0(\epsilon) = \frac{\int_0^\infty x^{2l}\exp\left[-\frac{2}{\epsilon}W_\epsilon^0(x)\right]dx}{\int_0^\infty \exp\left[-\frac{2}{\epsilon}W_\epsilon^0(y)\right]dy} = \frac{2}{\lambda_\epsilon^0}\int_0^\infty x^{2l}\exp\left[-\frac{2}{\epsilon}W_\epsilon^0(x)\right]dx. \tag{5.2}$$

First of all we shall prove that the family of moments $(\mu_{2l}^0(\varepsilon),\, \varepsilon>0)$ is bounded. The starting step consists in the estimation of the normalization coefficient λ_ϵ^0. This result is a refinement of several arguments introduced in the proof of Lemma 4.3 in [5] and is also linked to Lemma 4.10 in [1].

Lemma 5.1. *There exists $C>0$ such that for ϵ small enough, we have $\frac{1}{\lambda_\epsilon^0} \le \frac{C}{\epsilon}$.*

Proof. **Step 1.** Using the symmetry of the functions V and F_0 (defined in (1.6)) and the measure u_ϵ^0, we get

$$\begin{aligned}
\left|(F_0' * u_\epsilon^0)(y)\right| &= \left|\int_{\mathbb{R}^+} \left(F_0'(z+y) - F_0'(z-y)\right) u_\epsilon^0(z)dz\right| \\
&\le \int_{\mathbb{R}^+} \left|F_0'(z+y) - F_0'(z-y)\right| u_\epsilon^0(z)dz.
\end{aligned}$$

Then, by using the hypotheses $(F-4)$ and the definition of F_0, the following upper-bounds hold:

$$\begin{aligned}\left|(F_0'*u_\epsilon^0)(y)\right| &\leq 2C_q|y|\int_{\mathbb{R}^+}\left(1+|z+y|^{2q-2}+|z-y|^{2q-2}\right)u_\epsilon^0(z)dz\\ &\leq 2^{2q}C_q|y|\left(1+|y|^{2q-2}\right)\left(1+\int_{\mathbb{R}^+}z^{2q-2}u_\epsilon^0(z)dz\right)\\ &\leq C|y|\left(1+|y|^{2q}\right)\left(1+\int_{\mathbb{R}^+}z^{2q}u_\epsilon^0(z)dz\right),\end{aligned} \tag{5.3}$$

due to the boundedness of the function $y\to\frac{1+|y|^{2q-2}}{1+|y|^{2q}}$.

Step 2. Let $K>0$. We split the $2q$-th moment into two integral terms:

$$\int_0^\infty z^{2q}u_\epsilon^0(z)dz=\int_0^K z^{2q}u_\epsilon^0(z)dz+\int_K^\infty z^{2q}u_\epsilon^0(z)dz$$

By construction, u_ε^0 is directly related to $F*u_\varepsilon^0$: see (3.1). According to Lemma 4.2 in [5], we have the following inequality :

$$\left(F*u_\epsilon^0\right)(x)-\left(F*u_\epsilon^0\right)(0)=\int_0^x\left(F'*u_\epsilon^0\right)(y)dy\geq 0,\quad x\geq 0.$$

Therefore, using (V-4) and the previous inequality, we get, for all x:

$$u_\epsilon^0(x)\leq\frac{1}{\lambda_\epsilon^0}\exp\left[-\frac{2}{\epsilon}V(x)\right]\leq\frac{1}{\lambda_\epsilon^0}\exp\left[-\frac{2}{\epsilon}\left(C_4x^4-C_2x^2\right)\right].$$

Hence, if $K\geq\sqrt{\frac{2C_2}{C_4}}$ then $C_4z^4-C_2z^2\geq\frac{C_4}{2}z^4$ for $z\geq K$, and so

$$\int_K^\infty z^{2q}u_\epsilon^0(z)dz\quad\leq\quad\frac{1}{\lambda_\epsilon^0}\int_K^\infty z^{2q}\exp\left[-\frac{C_4}{\epsilon}z^4\right]dz=\frac{C(q)\epsilon^{\frac{q}{2}}}{\lambda_\varepsilon^0},$$

where $C(q)$ is a positive constant. Finally there exists some $\varepsilon_0>0$ such that $\epsilon\leq\epsilon_0$ implies:

$$\int_0^\infty z^{2q}u_\epsilon^0(z)dz\leq K^{2q}+\frac{1}{\lambda_\epsilon^0}. \tag{5.4}$$

By hypotheses (V-7) and (V-8), we get $|V'(x)|\leq C_q(1+x^{2q})$ and $|V(x)|\leq C_q|x|(1+x^{2q})$. Hence, for all $|x|\geq a$, we have $|V(x)|\leq\frac{C_q}{a}|x|^2(1+x^{2q})$. Moreover the property (V-6) permits to obtain, on the interval $[-a,a]$: $\frac{V(x)}{x^2}=V''(0)+\sum_{k=2}^\infty\frac{V^{(2k)}(0)}{(2k)!}x^{2k}$ which is bounded by some constant M. Taking $C_q'=\max\left\{M,\frac{C_q}{a}\right\}$, the following bound holds: $|V(x)|\leq C_q'|x|^2(1+x^{2q})$.

The upper-bounds (5.3) and (5.4) and the symmetry property of the density u_ε^0 immediately imply the existence of constants C' and C'' such that

$$\begin{aligned}\int_0^x\left(F_0'*u_\epsilon^0\right)(y)dy+V(x) &\leq C_q'x^2\left(1+x^{2q}\right)+C'x^2(1+x^{2q})\left(1+\frac{1}{\lambda_\epsilon^0}\right)\\ &\leq C''x^2(1+x^{2q})\left(1+\frac{1}{\lambda_\epsilon^0}\right).\end{aligned} \tag{5.5}$$

Step 3. We deduce, by (5.5) and the definition of the normalization factor λ_ϵ^0 :

$$\lambda_\epsilon^0 \geq 2\int_0^\infty \exp\left\{-\frac{2}{\epsilon}\left[\frac{\alpha}{2}x^2 + C''x^2(1+x^{2q})\left(1+\frac{1}{\lambda_\epsilon^0}\right)\right]\right\}dx.$$

Defining $\mu = \sqrt{\lambda_\epsilon^0}$ and introducing the change of variable $x := \mu\sqrt{\frac{\epsilon}{2}}\xi$ permits to point out the inequality $\mu \geq h(\mu)$ with

$$h(z) = \sqrt{2\epsilon}\int_0^\infty \exp\left\{-\frac{\alpha}{2}z^2\xi^2 - C''\xi^2(1+z^2)\left(1+\left(\frac{\epsilon}{2}\right)^q z^{2q}\xi^{2q}\right)\right\}d\xi.$$

Let us study the equation $\mu \geq h(\mu)$. Obviously $h'(z) \leq 0$. Now, we provide an upper bound for the function $-h'(z)$.

$$\begin{aligned}
-h'(z) &= \sqrt{2\epsilon}\int_0^\infty \left(\alpha z\xi^2 + C''\xi^2 A(z,\xi)\right)\exp\left[-B(z,\xi)\right]d\xi,\\
\text{with}\quad A(z,\xi) &= 2q\left(\frac{\epsilon}{2}\right)^q \xi^{2q}z^{2q-1}(1+z^2) + 2z\left(1+\left(\frac{\epsilon}{2}\right)^q z^{2q}\xi^{2q}\right)\\
\text{and}\quad B(z,\xi) &= \frac{\alpha}{2}z^2\xi^2 + C''\xi^2(1+z^2)\left(1+\left(\frac{\epsilon}{2}\right)^q z^{2q}\xi^{2q}\right).
\end{aligned}$$

For ϵ small and $z \geq 0$,

$$A(z,\xi) \leq 2z + 2q\xi^{2q}z^{2q-1} + 2\xi^{2q}z^{2q+1} \quad \text{and} \quad B(z,\xi) \geq C''\xi^2 + \frac{\alpha}{2}z^2\xi^2.$$

Let us note that $\mu \geq 1$ implies directly $\frac{1}{\lambda_\epsilon^0(u)} \leq 1$. This observation leads to study essentially the case: $z \leq 1$. After some computation, we obtain

$$\begin{aligned}
-h'(z) &\leq \sqrt{2\varepsilon}\int_0^\infty \alpha z\xi^2 \exp\left\{-\left(C'' + \frac{\alpha z^2}{2}\right)\xi^2\right\}d\xi + C_1\sqrt{\varepsilon}\\
&\leq C_1\sqrt{\epsilon} - C_2\sqrt{\epsilon}\frac{d}{dz}\left\{\left(C'' + \frac{\alpha}{2}z^2\right)^{-\frac{1}{2}}\right\}.
\end{aligned}$$

For $0 \leq z \leq 1$, the previous estimation of the derivative leads to the existence of some constant $C_2' > 0$ such that

$$h(0) - h(z) \leq C_1\sqrt{\epsilon}z + C_2\sqrt{\epsilon}\left(\frac{1}{\sqrt{C''}} - \frac{1}{\sqrt{C'' + \frac{\alpha}{2}z^2}}\right) \leq C_1\sqrt{\epsilon}z + C_2'\sqrt{\epsilon}\alpha z^2.$$

Moreover $h(0) = \sqrt{2\epsilon}\int_{\mathbb{R}_+} e^{-C''\xi^2}d\xi$. Hence $h(0)/\sqrt{\varepsilon}$ is independent of ε and shall be denoted by C_3. We get $h(z) \geq C_3\sqrt{\epsilon} - C_1\sqrt{\epsilon}z - C_2'\sqrt{\epsilon}\alpha z^2$. The inequality $\mu \geq h(\mu)$ implies $a_1\mu^2 + a_2\mu - \sqrt{\epsilon}a_3 \geq 0$ with $a_1 = C_2'\sqrt{\epsilon}\alpha \geq 0$, $a_2 = 1 + C_1\sqrt{\epsilon}$ and $a_3 = C_3 > 0$. So, for $\alpha > 0$, there exists $C > 0$ satisfying

$$\sqrt{\lambda_\epsilon^0} = \mu \geq \frac{\sqrt{a_2^2 + 4a_3C_2'\alpha\epsilon} - (1 + C_1\sqrt{\epsilon})}{2C_2'\sqrt{\epsilon}\alpha} = C\sqrt{\epsilon} + o\left(\sqrt{\epsilon}\right).$$

For $\alpha = 0$, we obtain directly the bound $\mu \geq \frac{a_3\sqrt{\epsilon}}{1+a_2\sqrt{\epsilon}}$. Finally we obtain the existence of $C > 0$ s.t. $\sqrt{\lambda_\epsilon^0} \geq \frac{C}{2}\sqrt{\epsilon}$ for ε small enough. $\square$

Using the upper bound of the normalization term λ_ε^0, we analyze the moments of symmetric invariant measures. We recall that μ_l^0 represents the l-th moment.

Lemma 5.2. *For any $l \geq 1$, the family $\left\{\mu_{2l}^0(\epsilon), \epsilon > 0\right\}$ is upper-bounded.*

Let us note that this result is a generalization of Proposition 3.1 in the symmetric context. We don't need any condition on the degree of F and V and we don't even assume that V is polynomial.

Proof. Let $K := \sqrt{\frac{C_2+1}{C_4}}$. We split the moment into two different terms:

$$\mu_{2l}^0(\epsilon) \quad = \quad \int_{-K}^{K} x^{2l} u_\epsilon^0(x) dx + \frac{2}{\lambda_\epsilon^0} \int_K^\infty x^{2l} \exp\left[-\frac{2}{\epsilon} W_\epsilon^0(x)\right] dx.$$

According to Lemma 4.2 2) in [5], $F * u_\epsilon^0(x) - F * u_\epsilon^0(0) \geq 0$. Moreover $V(x) \geq C_4 x^4 - C_2 x^2$ (see (V-4)). Hence, for any $|x| > K$, $V(x) \geq x^2$. We have therefore:

$$\mu_{2l}^0(\epsilon) \leq K^{2l} + \frac{2}{\lambda_\epsilon^0} \int_K^\infty x^{2l} e^{-\frac{2x^2}{\epsilon}} dx \leq K^{2l} + \frac{2}{\lambda_\epsilon^0} \epsilon^{l+\frac{1}{2}} \int_{\frac{K}{\sqrt{\epsilon}}}^\infty y^{2l} e^{-2y^2} dy.$$

Lemma 5.1 implies $\mu_{2l}^0(\epsilon) \leq K^{2l} + C\epsilon^{l-\frac{1}{2}}$ where C is some constant. □

Let us now present the main global result concerning sequences of symmectric invariant measures for self-stabilizing processes.

Proposition 5.3. *Let $(u_\varepsilon^0)_{\varepsilon > 0}$ be a family of symmetric invariant measures. Then there exist some sequence $(\varepsilon_k)_{k \geq 0}$ satisfying $\lim_{k\to\infty} \varepsilon_k = 0$ and a discrete limit measure u_0^0 such that $\left(u_{\epsilon_k}^0\right)_{k \geq 0}$ converges weakly towards u_0^0. Moreover u_0^0 takes the following form:*

$$u_0^0 = \sum_{i=1}^r p_i \left(\frac{1}{2}\delta_{A_i} + \frac{1}{2}\delta_{-A_i}\right) \tag{5.6}$$

with $r \geq 1$, $p_i > 0$ for all $1 \leq i \leq r$, $0 \leq A_1 < \cdots < A_r \leq a$ and

$$V'(A_i) + \sum_{j=1}^r p_j \mathcal{S}(F')(A_i, A_j) \quad = \quad 0 \tag{5.7}$$

$$V(A_i) - V(A_j) + \sum_{l=1}^r p_l \Big(\mathcal{S}(F)(A_i, A_l) - \mathcal{S}(F)(A_j, A_l)\Big) \quad = \quad 0 \tag{5.8}$$

$$V''(A_i) + \sum_{j=1}^r p_j \mathcal{S}(F'')(A_i, A_j) \quad \geq \quad 0 \tag{5.9}$$

for all $1 \leq i \leq r$, $1 \leq j \leq r$. Here $\mathcal{S}(G)(x, y) = \frac{1}{2}\left(G(x+y) + G(x-y)\right)$ for any function G.

Proof. By Lemma 5.2, the moments are bounded. Therefore Theorem 3.6 and Proposition 3.7 can be directly applied. In order to conclude it suffices to note that u_0^0 is symmetric and both functions V and F are even. $\square$

This convergence result shall be precised in the following particular case:

Theorem 5.4. *If the functions V'' and F'' are convex, then any sequence of symmetric invariant measures converges weakly towards the discrete measure $\frac{1}{2}\,\delta_{x_0} + \frac{1}{2}\,\delta_{-x_0}$ where x_0 is the unique non-negative solution of the system:*

$$\begin{cases} V'(x_0) + \frac{1}{2}F'(2x_0) = 0 \\ V''(x_0) + \frac{\alpha}{2} + \frac{1}{2}F''(2x_0) \geq 0 \end{cases} \tag{5.10}$$

Besides, if $\alpha \geq -V''(0)$ then $x_0 = 0$, otherwise $x_0 > 0$. We recall that α is defined by (1.6).

Proof. **Step 1.** According to Proposition 5.3, any limit measure of the family $\{u_\varepsilon^0,\, \varepsilon > 0\}$ denoted by u_0^0 is a discrete measure. Moreover since V'' and F'' are convex functions the support of the limit measure contains at most two elements, see Proposition 3.7. We deduce immediately that $r = 1$ in (5.6) moreover if $A_1 = 0$ then the support is reduced to one point.
Step 2. Furthermore the point A_1 which characterizes the limit measure satisfies (5.7) and (5.9) that is (5.10). Since V'' is a convex even function, the coefficient ϑ defined by (1.5) takes the following value $\vartheta = -V''(0)$. Let $\chi(x) = V'(x) + \frac{1}{2}F'(2x)$. We shall solve $\chi(x) = 0$ on $\mathbb{R}_+$. Obviously 0 is a solution. Moreover $\chi'(x) = V''(x) + F''(2x)$ which implies that χ' is a convex and symmetric function which tends to infinity as x becomes large. Therefore the minimum of χ' on $\mathbb{R}_+$ is $\chi'(0) = V''(0) + F''(0) = \alpha - \vartheta$. We distinguish two different cases:

- If $\alpha < \vartheta$, we get the following description of χ on $\mathbb{R}_+$: the function is first decreasing and then increasing. Since $\lim_{x\to\infty} \chi(x) = +\infty$, there exists a unique $x_0 > 0$ such that $\chi(x_0) = 0$. We have therefore two nonnegative zeros of χ : 0 and x_0. Since $V''(0) + \frac{\alpha}{2} + \frac{1}{2}F''(0) = \alpha - \vartheta < 0$, the unique solution of (5.10) is $x_0 > 0$.

- If $\alpha \geq -V''(0)$, the function χ' reaches its minimum for $x = 0$ and $\chi'(0) \geq 0$. Hence $\chi'(x) = 0$ implies $x = 0$. We can also verify that $V''(0) + \frac{\alpha}{2} + \frac{1}{2}F''(0) = \alpha - \vartheta \geq 0$.

Since any limit measure is characterized by the unique solution of (5.10), there exists some unique limit measure which leads to the weak convergence of any family of symmetric invariant measures $\{u_\varepsilon^0,\, \varepsilon > 0\}$ due essentially to the relative compactness of this family. $\square$

Remark 5.5. *The particular case $\alpha = -V''(0)$ corresponds to a bifurcation point: the support of the limit measure changes from some two elements set to a singleton.*

We can wonder what happens if we don't consider that V'' and F'' are convex functions. We can observe limit measures whose support contains three limit points or more as prove in Proposition 2.3 for $\deg(F) = 2$. We'll now study two examples with $\deg(F) > 2$.

Proposition 5.6. *Let $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ (V'' being some convex function). There exists some polynomial function F of degree 8, satisfying the properties (F-1)–(F-4), such that the support of any limit measures for symmetric invariant measure families $\{u_\varepsilon^0,\, \varepsilon > 0\}$ contains at least three points.*

Proof. As we have already seen in the previous proof, if the support of the limit measure is reduced to one or two points then these points are associated to the solutions of (5.10). It suffices then to prove that this system can't be solved for some well chosen interaction function F.
First of all, let us note that it is sufficient to choose some function F satisfying $F''(0) < -V''(0) = 1$ in order to assure that δ_0 is not a limit measure: let us fix $F''(0) = 0$.
Let us assume that F is an even polynomial function of degree 8.
Since $\lim_{x\to\infty} F(x) = +\infty$, we obtain $F^{(8)}(0) \geq 0$. Moreover F has to satisfy (F-3), that is F' is a convex function on $\mathbb{R}_+$ which implies $F^{(4)}(0) \geq 0$. Finally since V'' is a convex function, F'' is not convex that is to say that $F^{(6)}(0)$ is necessary negative. We can therefore choose F of the following kind:

$$F(x) = \alpha_4 \frac{x^4}{4} - \alpha_6 \frac{x^6}{6} + \alpha_8 \frac{x^8}{8} \quad \text{with} \quad \alpha_4,\ \alpha_6 \text{ and } \alpha_8 > 0. \tag{5.11}$$

F' has to be convex on $\mathbb{R}_+$ (F-3) that is to say $F^{(3)}(x)/x =: Q_2(x^2) \geq 0$, for any $x \in \mathbb{R}$, where Q_2 is a polynomial function of degree 2. Moreover F'' is not convex on $\mathbb{R}$ i.e. $F^{(4)}(x) =: R_2(x^2) < 0$ for some $x \in \mathbb{R}$ where R_2 is an other polynomial function of degree 2. In other words we choose some function F such that the discriminant of Q_2 is negative and the discriminant of R_2 is positive. We deduce the following bounds:

$$\sqrt{\frac{35}{25}\alpha_4\alpha_8} < \alpha_6 < \sqrt{\frac{63}{25}\alpha_4\alpha_8}. \tag{5.12}$$

Let us now prove that there exists such a function F which does not satisfy the system (5.10). It suffices to prove the following implication

$$P_3(x^2) := 64\alpha_8 x^6 - 16\alpha_6 x^4 + (4\alpha_4 + 1)x^2 - 1 = 0 \tag{5.13}$$

$$\implies \quad 224\alpha_8 x^6 - 40\alpha_6 x^4 + (6\alpha_4 + 3)x^2 - 1 < 0. \tag{5.14}$$

We replace (5.14) by $2\times$(5.14)$-7\times$(5.13). So (5.13) and (5.14) are equivalent to $P_3(x) = 0 \Rightarrow P_2(x) < 0$ for $x \geq 0$ and $P_2(x) = 32\,\alpha_6 x^2 - (16\,\alpha_4 + 1)x + 5$. Using (5.12) we can prove that the discriminant of P_3' is negative. We deduce that there exists some unique x_0 satisfying $P_3(x_0) = 0$. We observe moreover that $x_0 > 0$.
To conclude the proof, it suffices to point out some coefficients α_4, α_6 and α_8

satisfying (5.12) and such that $x_0 \in]x_-, x_+[$ where $x_\pm$ are the possible roots of P_2. In other words, we prove that $P_3(x_-)P_3(x_+) < 0$. Let us introduce some parameter $\eta \in]35/25, 63/25[$ such that $\alpha_6 = \sqrt{\eta\alpha_4\alpha_8}$. Therefore we can express $x_\pm$, $P_3(x_-)$ and $P_3(x_+)$ with respect to η, α_4 and α_8. More precisely, if the discriminant of P_2 satisfies

$$\Delta := (16\alpha_4 + 1)^2 - 640\sqrt{\eta\alpha_4\alpha_8} > 0 \tag{5.15}$$

we get

$$x_\pm = \frac{16\alpha_4 + 1 \pm \sqrt{(16\alpha_4+1)^2 - 640\sqrt{\eta\alpha_4\alpha_8}}}{64\sqrt{\eta\alpha_4\alpha_8}}.$$

We shall verify at the end of the proof that $\Delta > 0$ is effectively satisfied. After some tedious computation we obtain that $P_3(x_-)P_3(x_+) < 0$ is equivalent to $\Psi_{\sqrt{\eta},\sqrt{\alpha_4}}(\sqrt{\alpha_8}) < 0$ where $\Psi_{u,s}(t)$ is the following polynomial function:

$$\Psi_{u,s}(t) \quad = \quad 2000t^2 - 4us\left\{320s^2 + 95 - 72u^2s^2\right\}t + s^6(256 - 64u^2) + R_u(s^2),$$

where R_u is a polynomial function of order 2 whose coefficients depend on the u-variable. By (5.12), the variable u satisfies $u^2 = \eta < \frac{320}{72}$ which implies the inequality $320s^2 + 95 - 72u^2s^2 > 0$ for all $s \in \mathbb{R}$. Let us then choose α_8 in order to minimize $\Psi_{\sqrt{\eta},\sqrt{\alpha_4}}(\sqrt{\alpha_8})$. Hence

$$\alpha_8 = \eta\alpha_4\frac{(320\alpha_4 + 95 - 72\eta\alpha_4)^2}{10^6} \quad \text{and} \quad \alpha_6 = \eta\alpha_4\frac{320\alpha_4 + 95 - 72\eta\alpha_4}{10^3}. \tag{5.16}$$

It remains to prove that $\Psi_{\sqrt{\eta},\sqrt{\alpha_4}}(\sqrt{\alpha_8}) < 0$. We obtain the following estimation of this minimum:

$$\Psi_{\sqrt{\eta},\sqrt{\alpha_4}}(\sqrt{\alpha_8}) = -\frac{16\alpha_4^3}{125}\left\{81\eta^3 - 720\eta^2 + 2100\eta - 2000\right\} + \overline{R}_\eta(\alpha_4),$$

where $\overline{R}_\eta$ is a polynomial function of degree 2. Noting that $f(\eta) = 81\eta^3 - 720\eta^2 + 2100\eta - 2000$ is a non-decreasing function with $f(63/25) > 0$, we emphasize the existence of some $\eta_0 \in]7/5, 63/25[$ such that $f(\eta) > 0$ for any $\eta \in]\eta_0, 63/25[=: I$. To conclude it suffices to first choose η in I and secondly α_4 large enough ($P_3(x_-)P_3(x_+)$ is then negative) in order to determine the parameter α_4, α_6 and α_8 which leads to (5.13) and (5.14). This procedure is successful provided that (5.15) is true. In fact, using the particular choice of α_8, (5.15) is equivalent to

$$(16\alpha_4 + 1)^4 > 640^2\eta^{3/2}\alpha_4^{3/2}\,\frac{320\alpha_4 + 95 - 72\eta\alpha_4}{1000},$$

which is satisfied when α_4 is large. $\square$

We have studied the case when V'' is a convex function and pointed out the existence of some polynomial interaction function F with non convex second derivative such that the support of any limit measure contains at least three elements. Let us observe now what happens if the interaction function F'' is convex. Is it possible to find some environment function V in order to obtain the same conclusion ?

Proposition 5.7. *Let $F(x) = \frac{\beta}{4}x^4 + \frac{\alpha}{2}x^2$ with $\alpha \geq 0$ and $\beta > 0$. There exists some function V, satisfying the properties (V-1)–(V-8), such that the support of any limit measures for symmetric invariant measure families $\{u_\varepsilon^0,\, \varepsilon > 0\}$ contains at least three points.*

Proof. The arguments developed for the proof of Proposition 5.7 are similar to those presented in Proposition 5.6. Let us first choose V such that δ_0 cannot be a limit measure of $\{u_\varepsilon^0,\, \varepsilon > 0\}$. By (5.9), applied to $r = 1$ and $A_1 = 0$, it suffices to assume that $-V''(0) > \alpha$.
Let us now focus our attention to limit measures whose support contains two elements A_1 and $-A_1$. Due to Proposition 5.3, A_1 satisfies (5.7) and (5.9). Let us choose V some polynomial function of degree 6. In order to get the limit $\lim_{x\to\infty} V(x) = +\infty$ which is part of the condition (V-5), we choose the coefficient of degree 6 positive. Moreover the coefficient of degree 4 is negative since we need that V'' is not convex. We deduce that V has the following form

$$V(x) = \frac{\alpha_6}{6}x^6 - \frac{\alpha_4}{4}x^4 - \frac{\alpha_2}{2}x^2 \quad \text{with} \quad \alpha_4,\, \alpha_6 > 0 \quad \text{and}\, \alpha_2 > \alpha. \tag{5.17}$$

The equations (5.7) and (5.9) become

$$\begin{aligned} A_1\left(\alpha_6 A_1^4 + (4\beta - \alpha_4)A_1^2 + (\alpha - \alpha_2)\right) &= 0 &\quad (5.18)\\ 5\alpha_6 A_1^4 + (6\beta - 3\alpha_4)A_1^2 + (\alpha - \alpha_2) &\geq 0 &\quad (5.19)\end{aligned}$$

Dividing the first equality by A_1, replacing (5.19) by $\frac{1}{2}$((5.19)$-5\times$(5.18)) and introducing $X := A_1^2$, we get :

$$\begin{aligned} \alpha_6 X^2 + (4\beta - \alpha_4)X + (\alpha - \alpha_2) &= 0 &\quad (5.20)\\ (\alpha_4 - 7\beta)\, X - 2\,(\alpha - \alpha_2) &\geq 0 &\quad (5.21)\end{aligned}$$

Since $\alpha < \alpha_2$, there is a unique positive solution to (5.20) X_0:

$$X_0 = \frac{\alpha_4 - 4\beta + \sqrt{(\alpha_4 - 4\beta)^2 + 4\alpha_6(\alpha_2 - \alpha)}}{2\alpha_6}.$$

The aim of the proof is to find some parameters α_2, α_4 and α_6 such that (5.18) and (5.19) are incompatible that is $(\alpha_4 - 7\beta)\, X_0 - 2\,(\alpha - \alpha_2) < 0$. In other words

$$(\alpha_4 - 7\beta)\,\frac{\alpha_4 - 4\beta + \sqrt{(\alpha_4 - 4\beta)^2 + 4\alpha_6(\alpha_2 - \alpha)}}{2\alpha_6} - 2\,(\alpha - \alpha_2) < 0.$$

We set $\alpha_4 = 6\beta$ and $\alpha_2 = \alpha + \frac{u^2-1}{\alpha_6}\beta^2$ with $u > 1$ (we need that $\alpha_2 > \alpha$), the previous inequality becomes $2u^2 - u - 3 < 0$. Therefore for any parameter u such that $1 < u < \frac{3}{2}$ and $\alpha_6 > 0$ we compute by the procedure just described α_2 and α_4 which permit to define the polynomial function V with the following properties:

- there is no solution to the system of equations (5.7) and (5.9) with $r = 1$

- the function V satisfies the properties (V-1)–(V-8).

We conclude that the support of any limit measure corresponding to some sequence of invariant symmetric self-stabilizing measures contains at least three points. $\square$

In order to conclude this study, we shall present some particular example of self-stabilizing diffusion which presents the following property: any invariant symmetric measure converges as $\varepsilon \to 0$ to a limit measure whose support contains exactly three elements.

Example. *Let* $V(x) = \frac{x^6}{6} - \frac{3}{2}x^4 - \frac{17}{32}x^2$ *and* $F(x) = \frac{x^4}{4} + \frac{x^2}{2}$ *the functions defining the self-stabilizing diffusion* (1.2). *Then any family of symmetric invariant measures* $\{u_\varepsilon^0,\, \varepsilon > 0\}$ *satisfies*

$$\lim_{\epsilon \to 0} u_\epsilon^0 = \frac{26}{45}\,\delta_0 + \frac{19}{90}\left(\delta_{\frac{\sqrt{15}}{2}} + \delta_{-\frac{\sqrt{15}}{2}}\right). \tag{5.22}$$

Proof. Using the proof of Proposition 5.7 with the following parameters $\alpha = \beta = 1$, $\alpha_2 = \alpha + \frac{1}{16\alpha_6}\beta^2$, $\alpha_4 = 6\beta$ and $\alpha_6 = 1$, we obtain immediately that the support of any limit measure of the family $\{u_\varepsilon^0,\, \varepsilon > 0\}$ contains at least three points. Let us prove now that it can't contain more than three points. As presented in the proof of Proposition 3.7, the support of any symmetric limit measure u_0^0 is contained in the set of points which minimize the function $W_0 = V + F * u_0^0$. Since the particular functions $V^{(4)}$ and $F^{(4)}$ are convex functions. $W_0^{(4)}$ is convex too. Moreover, if W_0 has at least four local minima, W_0' vanishes at least seven times. Applying Rolle's theorem three times, $W_0^{(4)}$ vanishes at least four times which contradicts the main property of convexity just mentioned. Hence the symmetric limit measure u_0^0 is a sum of exactly three Dirac measures: $u_0^0 = p_0\delta_0 + \frac{1-p_0}{2}\left(\delta_{x_1} + \delta_{-x_1}\right)$ with $p_0(1-p_0) > 0$ and $x_1 > 0$. Let us estimate the variables p_0 and x_1. Proposition 5.3 implies

$$\begin{aligned} V'(x_1) + p_0 F'(x_1) + \frac{1-p_0}{2}F'(2x_1) &= 0 \\ V(x_1) + (2p_0 - 1)F(x_1) + \frac{1-p_0}{2}F(2x_1) &= 0 \end{aligned}$$

which admits the unique solution $p_0 = \frac{26}{45}$ and $x_1 = \frac{\sqrt{15}}{2}$. $\square$

A Annex

We shall present here some useful asymptotic result which is close to the classical Laplace's method and which contributes to the proof of Theorem 2.1. These result and its proof are slightly modifications of those appearing in the annex of [5] that's why we shall omit the proof.

Lemma A.1. *Set $\epsilon > 0$. Let U a $\mathcal{C}^\infty(\mathbb{R})$-continuous functions. Let us introduce some interval $[a,b]$ satisfying: $U'(a) \neq 0$, $U'(b) \neq 0$ and $U(x)$ admits some unique global minimum on the interval $[a,b]$ reached at $x_0 \in]a,b[$. We assume that there exists some exponent k_0 such that $2k_0 = \min_{r\in\mathbb{N}^*}\left\{U^{(r)}(x_0) \neq 0\right\}$. Let f be a $\mathcal{C}^4(\mathbb{R})$-continuous function. Then taking the limit $\epsilon \to 0$ we get*

$$\int_a^b f(t)e^{-\frac{U(t)}{\epsilon}}dt = \frac{f(x_0)}{k_0}\left(\frac{\epsilon(2k_0)!}{U^{2k_0}(x_0)}\right)^{\frac{1}{2k_0}}\Gamma\left(\frac{1}{2k_0}\right)e^{-\frac{U(x_0)}{\epsilon}}(1+o(1)), \quad \text{(A.1)}$$

where Γ represents the Euler function.

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