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► **To cite this version:**

Christian Miebach, Karl Oeljeklaus. On proper R -actions on hyperbolic Stein surfaces. *Documenta Mathematica*, 2009, 14, pp.673–689. hal-00397054

HAL Id: hal-00397054

<https://hal.science/hal-00397054>

Submitted on 19 Jun 2009

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ON PROPER \mathbb{R} -ACTIONS ON HYPERBOLIC STEIN SURFACES

by

Christian Miebach & Karl Oeljeklaus

Abstract. — In this paper we investigate proper \mathbb{R} -actions on hyperbolic Stein surfaces and prove in particular the following result: Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy which admits a proper \mathbb{R} -action by holomorphic transformations. The quotient D/\mathbb{Z} with respect to the induced proper \mathbb{Z} -action is a Stein manifold. A normal form for the domain D is deduced.

1. Introduction

Let X be a Stein manifold endowed with a real Lie transformation group G of holomorphic automorphisms. In this situation it is natural to ask whether there exists a G -invariant holomorphic map $\pi: X \rightarrow X//G$ onto a complex space $X//G$ such that $\mathcal{O}_{X//G} = (\pi_*\mathcal{O}_X)^G$ and, if yes, whether this quotient $X//G$ is again Stein. If the group G is compact, both questions have a positive answer as is shown in [Hei91].

For non-compact G even the existence of a complex quotient in the above sense of X by G cannot be guaranteed. In this paper we concentrate on the most basic and already non-trivial case $G = \mathbb{R}$. We suppose that G acts properly on X . Let $\Gamma = \mathbb{Z}$. Then X/Γ is a complex manifold and if, moreover, it is Stein, we can define $X//G := (X/\Gamma)/(G/\Gamma)$. The following was conjectured by Alan Huckleberry.

Let X be a contractible bounded domain of holomorphy in \mathbb{C}^n with a proper action of $G = \mathbb{R}$. Then the complex manifold X/Γ is Stein.

In [FI01] this conjecture is proven for the unit ball and in [Mie08] for arbitrary bounded homogeneous domains in \mathbb{C}^n . In this paper we make a first step towards a proof in the general case by showing

Theorem. — *Let D be a simply-connected bounded domain of holomorphy in \mathbb{C}^2 . Suppose that the group \mathbb{R} acts properly by holomorphic transformations on D . Then the complex manifold D/\mathbb{Z} is Stein. Moreover, D/\mathbb{Z} is biholomorphically equivalent to a domain of holomorphy in \mathbb{C}^2 .*

The authors would like to thank Peter Heinzner and Jean-Jacques Loeb for numerous discussions on the subject.

As an application of this theorem we deduce a normal form for domains of holomorphy whose identity component of the automorphism group is non-compact as well as for proper \mathbb{R} -actions on them. Notice that we make no assumption on smoothness of their boundaries.

We first discuss the following more general situation. Let X be a hyperbolic Stein manifold with a proper \mathbb{R} -action. Then there is an induced local holomorphic \mathbb{C} -action on X which can be globalized in the sense of [HI97]. The following result is central for the proof of the above theorem.

Theorem. — *Let X be a hyperbolic Stein surface with a proper \mathbb{R} -action. Suppose that either X is taut or that it admits the Bergman metric and $H^1(X, \mathbb{R}) = 0$. Then the universal globalization X^* of the induced local \mathbb{C} -action is Hausdorff and \mathbb{C} acts properly on X^* . Furthermore, for simply-connected X one has that $X^* \rightarrow X^*/\mathbb{C}$ is a holomorphically trivial \mathbb{C} -principal bundle over a simply-connected Riemann surface.*

Finally, we discuss several examples of hyperbolic Stein manifolds X with proper \mathbb{R} -actions such that X/\mathbb{Z} is not Stein. If one does not require the existence of an \mathbb{R} -action, there are bounded Reinhardt domains in \mathbb{C}^2 with proper \mathbb{Z} -actions for which the quotients are not Stein.

2. Hyperbolic Stein \mathbb{R} -manifolds

In this section we present the general set-up.

2.1. The induced local \mathbb{C} -action and its globalization. — Let X be a hyperbolic Stein manifold. It is known that the group $\text{Aut}(X)$ of holomorphic automorphisms of X is a real Lie group with respect to the compact-open topology which acts properly on X (see [Kob98]). Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be a closed one parameter subgroup of $\text{Aut}(X)$. Consequently, the action $\mathbb{R} \times X \rightarrow X$, $t \cdot x := \varphi_t(x)$, is proper. By restriction, we obtain also a proper \mathbb{Z} -action on X . Since every such action must be free, the quotient X/\mathbb{Z} is a complex manifold. This complex manifold X/\mathbb{Z} carries an action of $S^1 \cong \mathbb{R}/\mathbb{Z}$ which is induced by the \mathbb{R} -action on X .

Integrating the holomorphic vector field on X which corresponds to this \mathbb{R} -action we obtain a local \mathbb{C} -action on X in the following sense. There are an open neighborhood $\Omega \subset \mathbb{C} \times X$ of $\{0\} \times X$ and a holomorphic map $\Phi: \Omega \rightarrow X$, $\Phi(t, x) =: t \cdot x$, such that the following holds:

- (1) For every $x \in X$ the set $\Omega(x) := \{t \in \mathbb{C}; (t, x) \in \Omega\} \subset \mathbb{C}$ is connected;
- (2) for all $x \in X$ we have $0 \cdot x = x$;
- (3) we have $(t + t') \cdot x = t \cdot (t' \cdot x)$ whenever both sides are defined.

Following [Pal57] (compare [HI97] for the holomorphic setting) we say that a globalization of the local \mathbb{C} -action on X is an open \mathbb{R} -equivariant holomorphic embedding $\iota: X \hookrightarrow X^*$ into a (not necessarily Hausdorff) complex manifold X^* endowed with a holomorphic \mathbb{C} -action such that $\mathbb{C} \cdot \iota(X) = X^*$. A globalization $\iota: X \hookrightarrow X^*$ is called *universal* if for every \mathbb{R} -equivariant holomorphic map $f: X \rightarrow Y$ into a holomorphic \mathbb{C} -manifold Y there exists a holomorphic \mathbb{C} -equivariant map $F: X^* \rightarrow Y$ such that

the diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^* \\ & \searrow f & \swarrow F \\ & & Y \end{array}$$

commutes. It follows that a universal globalization is unique up to isomorphism if it exists.

Since X is Stein, the universal globalization X^* of the induced local \mathbb{C} -action exists as is proven in [HI97]. We will always identify X with its image $\iota(X) \subset X^*$. Then the local \mathbb{C} -action on X coincides with the restriction of the global \mathbb{C} -action on X^* to X .

Recall that X is said to be orbit-connected in X^* if for every $x \in X^*$ the set $\Sigma(x) := \{t \in \mathbb{C}; t \cdot x \in X\}$ is connected. The following criterion for a globalization to be universal is proven in [CTIT00].

Lemma 2.1. — *Let X^* be any globalization of the induced local \mathbb{C} -action on X . Then X^* is universal if and only if X is orbit-connected in X^* .*

Remark. — The results about (universal) globalizations hold for a bigger class of groups ([CTIT00]). However, we will need it only for the groups \mathbb{C} and \mathbb{C}^* and thus will not give the most general formulation.

For later use we also note the following

Lemma 2.2. — *The \mathbb{C} -action on X^* is free.*

Proof. — Suppose that there exists a point $x \in X^*$ such that \mathbb{C}_x is non-trivial. Because of $\mathbb{C} \cdot X = X^*$ we can assume that $x \in X$ holds. Since \mathbb{C}_x is a non-trivial closed subgroup of \mathbb{C} , it is either a lattice of rank 1 or 2, or \mathbb{C} . The last possibility means that x is a fixed point under \mathbb{C} which is not possible since \mathbb{R} acts freely on X .

We observe that the lattice \mathbb{C}_x is contained in the connected \mathbb{R} -invariant set $\Sigma(x) = \{t \in \mathbb{C}; t \cdot x \in X\}$. By \mathbb{R} -invariance $\Sigma(x)$ is a strip. Since X is hyperbolic, this strip cannot coincide with \mathbb{C} . The only lattice in \mathbb{C} which can possibly be contained in such a strip is of the form $\mathbb{Z}r$ for some $r \in \mathbb{R}$. Since this contradicts the fact that \mathbb{R} acts freely on X , the lemma is proven. \square

Note that we do not know whether X^* is Hausdorff. In order to guarantee the Hausdorff property of X^* , we make further assumptions on X . The following result is proven in [Ian03] and [IST04].

Theorem 2.3. — *Let X be a hyperbolic Stein manifold with a proper \mathbb{R} -action. Suppose in addition that X is taut or admits the Bergman metric. Then X^* is Hausdorff. If X is simply-connected, then the same is true for X^* .*

We refer the reader to Chapter 4.10 and Chapter 5 in [Kob98] for the definitions and examples of tautness and the Bergman metric.

Remark. — Every bounded domain in \mathbb{C}^n admits the Bergman metric.

2.2. The quotient X/\mathbb{Z} . — We assume from now on that X fulfills the hypothesis of Theorem 2.3. Since X^* is covered by the translates $t \cdot X$ for $t \in \mathbb{C}$ and since the action of \mathbb{Z} on each domain $t \cdot X$ is proper, we conclude that the quotient X^*/\mathbb{Z} fulfills all axioms of a complex manifold except for possibly not being Hausdorff.

We have the following commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & X^* \\ \downarrow & & \downarrow \\ X/\mathbb{Z} & \longrightarrow & X^*/\mathbb{Z}. \end{array}$$

Note that the group $\mathbb{C}^* = (S^1)^\mathbb{C} \cong \mathbb{C}/\mathbb{Z}$ acts on X^*/\mathbb{Z} . Concretely, if we identify \mathbb{C}/\mathbb{Z} with \mathbb{C}^* via $\mathbb{C} \rightarrow \mathbb{C}^*$, $t \mapsto e^{2\pi it}$, the quotient map $p: X^* \rightarrow X^*/\mathbb{Z}$ fulfills $p(t \cdot x) = e^{2\pi it} \cdot p(x)$.

Lemma 2.4. — *The induced map $X/\mathbb{Z} \hookrightarrow X^*/\mathbb{Z}$ is the universal globalization of the local \mathbb{C}^* -action on X/\mathbb{Z} .*

Proof. — The open embedding $X \hookrightarrow X^*$ induces an open embedding $X/\mathbb{Z} \hookrightarrow X^*/\mathbb{Z}$. This embedding is S^1 -equivariant and we have $\mathbb{C}^* \cdot X/\mathbb{Z} = X^*/\mathbb{Z}$. This implies that X^*/\mathbb{Z} is a globalization of the local \mathbb{C}^* -action on X/\mathbb{Z} .

In order to prove that this globalization is universal, by the globalization theorem in [CTIT00] it is enough to show that X/\mathbb{Z} is orbit-connected in X^*/\mathbb{Z} . Hence, we must show that for every $[x] \in X/\mathbb{Z}$ the set $\Sigma([x]) := \{t \in \mathbb{C}^*; t \cdot [x] \in X/\mathbb{Z}\}$ is connected in \mathbb{C}^* . For this we consider the set $\Sigma(x) = \{t \in \mathbb{C}; t \cdot x \in X\}$. Since the map $X \rightarrow X/\mathbb{Z}$ intertwines the local \mathbb{C} - and \mathbb{C}^* -actions, we conclude that $t \in \Sigma(x)$ holds if and only if $e^{2\pi it} \in \Sigma([x])$ holds. Since X^* is universal, $\Sigma(x)$ is connected which implies that $\Sigma([x])$ is likewise connected. Thus X^*/\mathbb{Z} is universal. \square

Remark. — The globalization X^*/\mathbb{Z} is Hausdorff if and only if \mathbb{Z} or, equivalently, \mathbb{R} act properly on X^* . As we shall see in Lemma 3.3, this is the case if X is taut.

2.3. A sufficient condition for X/\mathbb{Z} to be Stein. — If $\dim X = 2$, we have the following sufficient condition for X/\mathbb{Z} to be a Stein surface.

Proposition 2.5. — *If the \mathbb{C} -action on X^* is proper and if the Riemann surface X^*/\mathbb{C} is not compact, then X/\mathbb{Z} is Stein.*

Proof. — Under the above hypothesis we have the \mathbb{C} -principal bundle $X^* \rightarrow X^*/\mathbb{C}$. If the base X^*/\mathbb{C} is not compact, then this bundle is holomorphically trivial, i. e. X^* is biholomorphic to $\mathbb{C} \times R$ where R is a non-compact Riemann surface. Since R is Stein, the same is true for X^* and for $X^*/\mathbb{Z} \cong \mathbb{C}^* \times R$. Since X/\mathbb{Z} is locally Stein, see [Mie08], in the Stein manifold X^*/\mathbb{Z} , the claim follows from [DG60]. \square

Therefore, the crucial step in the proof of our main result consists in showing that \mathbb{C} acts properly on X^* under the assumption $\dim X = 2$.

3. Local properness

Let X be a hyperbolic Stein \mathbb{R} -manifold. Suppose that X is taut or that it admits the Bergman metric and $H^1(X, \mathbb{R}) = \{0\}$. We show that then \mathbb{C} acts locally properly on X^* .

3.1. Locally proper actions. — Recall that the action of a Lie group G on a manifold M is called locally proper if every point in M admits a G -invariant open neighborhood on which the G acts properly.

Lemma 3.1. — *Let $G \times M \rightarrow M$ be locally proper.*

- (1) *For every $x \in M$ the isotropy group G_x is compact.*
- (2) *Every G -orbit admits a geometric slice.*
- (3) *The orbit space M/G is a smooth manifold which is in general not Hausdorff.*
- (4) *All G -orbits are closed in M .*
- (5) *The G -action on M is proper if and only if M/G is Hausdorff.*

Proof. — The first claim is elementary to check. The second claim is proven in [DK00]. The third one is a consequence of (2) since the slices yield charts on M/G which are smoothly compatible because the transitions are given by the smooth action of G on M . Assertion (4) follows from (3) because in locally Euclidian topological spaces points are closed. The last claim is proven in [Pal61]. \square

Remark. — Since \mathbb{R} acts properly on X , the \mathbb{R} -action on X^* is locally proper.

3.2. Local properness of the \mathbb{C} -action on X^* . — Recall that we assume that

$$(3.1) \quad X \text{ is taut}$$

or that

$$(3.2) \quad X \text{ admits the Bergman metric and } H^1(X, \mathbb{R}) = \{0\}.$$

We first show that assumption (3.1) implies that \mathbb{C} acts locally properly on X^* .

Since X^* is the universal globalization of the induced local \mathbb{C} -action on X , we know that X is orbit-connected in X^* . This means that for every $x \in X^*$ the set $\Sigma(x) = \{t \in \mathbb{C}; t \cdot x \in X\}$ is a strip in \mathbb{C} . In the following we will exploit the properties of the thickness of this strip.

Since $\Sigma(x)$ is \mathbb{R} -invariant, there are “numbers” $u(x) \in \mathbb{R} \cup \{-\infty\}$ and $o(x) \in \mathbb{R} \cup \{\infty\}$ for every $x \in X^*$ such that

$$\Sigma(x) = \{t \in \mathbb{C}; u(x) < \text{Im}(t) < o(x)\}.$$

The functions $u: X^* \rightarrow \mathbb{R} \cup \{-\infty\}$ and $o: X^* \rightarrow \mathbb{R} \cup \{\infty\}$ so obtained are upper and lower semicontinuous, respectively. Moreover, u and o are \mathbb{R} -invariant and $i\mathbb{R}$ -equivariant:

$$u(it \cdot x) = u(x) - t \quad \text{and} \quad o(it \cdot x) = o(x) - t.$$

Proposition 3.2. — *The functions $u, -o: X^* \rightarrow \mathbb{R} \cup \{-\infty\}$ are plurisubharmonic. Moreover, u and o are continuous on $X^* \setminus \{u = -\infty\}$ and $X^* \setminus \{o = \infty\}$, respectively.*

Proof. — It is proven in [For96] that u and $-o$ are plurisubharmonic on X . By equivariance, we obtain this result for X^* .

Now we prove that the function $u: X \setminus \{u = -\infty\} \rightarrow \mathbb{R}$ is continuous which was remarked without complete proof in [Ian03]. For this let (x_n) be a sequence in X which converges to $x_0 \in X \setminus \{u = -\infty\}$. Since u is upper semi-continuous, we have $\limsup_{n \rightarrow \infty} u(x_n) \leq u(x_0)$. Suppose that u is not continuous in x_0 . Then, after replacing (x_n) by a subsequence, we find $\varepsilon > 0$ such that $u(x_n) \leq u(x_0) - \varepsilon < u(x_0)$ holds for all $n \in \mathbb{N}$. Consequently, we have $\Sigma(x_0) = \{t \in \mathbb{C}; u(x_0) < \operatorname{Im}(t) < o(x_0)\} \subset \Sigma := \{t \in \mathbb{C}; u(x_0) - \varepsilon < \operatorname{Im}(t) < o(x_0)\} \subset \Sigma(x_n)$ for all n and hence obtain the sequence of holomorphic functions $f_n: \Sigma \rightarrow X$, $f_n(t) := t \cdot x_n$. Since X is taut and $f_n(0) = x_n \rightarrow x_0$, the sequence (f_n) has a subsequence which compactly converges to a holomorphic function $f_0: \Sigma \rightarrow X$. Because of $f_0(iu(x_0)) = \lim_{n \rightarrow \infty} f_n(iu(x_0)) = \lim_{n \rightarrow \infty} iu(x_0) \cdot x_n = iu(x_0) \cdot x_0 \notin X$ we arrive at a contradiction. Thus the function $u: X \setminus \{u = -\infty\} \rightarrow \mathbb{R}$ is continuous. By $(i\mathbb{R})$ -equivariance, u is also continuous on $X^* \setminus \{u = -\infty\}$. A similar argument shows continuity of $-o: X^* \setminus \{o = \infty\} \rightarrow \mathbb{R}$. \square

Let us consider the sets

$$\mathcal{N}(o) := \{x \in X^*; o(x) = 0\} \quad \text{and} \quad \mathcal{P}(o) := \{x \in X^*; o(x) = \infty\}.$$

The sets $\mathcal{N}(u)$ and $\mathcal{P}(u)$ are similarly defined. Since $X = \{x \in X^*; u(x) < 0 < o(x)\}$, we can recover X from X^* with the help of u and o .

Lemma 3.3. — *The action of \mathbb{R} on X^* is proper.*

Proof. — Let ∂^*X denote the boundary of X in X^* . Since the functions u and $-o$ are continuous on $X^* \setminus \mathcal{P}(u)$ and $X^* \setminus \mathcal{P}(o)$ one verifies directly that $\partial^*X = \mathcal{N}(u) \cup \mathcal{N}(o)$ holds. As a consequence, we note that if $x \in \partial^*X$, then for every $\varepsilon > 0$ the element $(i\varepsilon) \cdot x$ is not contained in ∂^*X .

Let (t_n) and (x_n) be sequences in \mathbb{R} and X^* such that $(t_n \cdot x_n, x_n)$ converges to (y_0, x_0) in $X^* \times X^*$. We may assume without loss of generality that x_0 and hence x_n are contained in X for all n . Consequently, we have $y_0 \in X \cup \partial^*X$. If $y_0 \in \partial^*X$ holds, we may choose an $\varepsilon > 0$ such that $(i\varepsilon) \cdot y_0$ and $(i\varepsilon) \cdot x_0$ lie in X . Since the \mathbb{R} -action on X is proper, we find a convergent subsequence of (t_n) which was to be shown. \square

Lemma 3.4. — *We have:*

- (1) $\mathcal{N}(u)$ and $\mathcal{N}(o)$ are \mathbb{R} -invariant.
- (2) We have $\mathcal{N}(u) \cap \mathcal{N}(o) = \emptyset$.
- (3) The sets $\mathcal{P}(u)$ and $\mathcal{P}(o)$ are closed, \mathbb{C} -invariant and pluripolar in X^* .
- (4) $\mathcal{P}(u) \cap \mathcal{P}(o) = \emptyset$.

Proof. — The first claim follows from the \mathbb{R} -invariance of u and o .

The second claim follows from $u(x) < o(x)$.

The third one is a consequence of the \mathbb{R} -invariance and $i\mathbb{R}$ -equivariance of u and o .

If there was a point $x \in \mathcal{P}(u) \cap \mathcal{P}(o)$, then $\mathbb{C} \cdot x$ would be a subset of X which is impossible since X is hyperbolic. \square

Lemma 3.5. — *If o is not identically ∞ , then the map*

$$\varphi: i\mathbb{R} \times \mathcal{N}(o) \rightarrow X^* \setminus \mathcal{P}(o), \quad \varphi(it, z) = it \cdot z,$$

is an $i\mathbb{R}$ -equivariant homeomorphism. Since \mathbb{R} acts properly on $\mathcal{N}(o)$, it follows that \mathbb{C} acts properly on $X^* \setminus \mathcal{P}(o)$. The same holds when o is replaced by u .

Proof. — The inverse map φ^{-1} is given by $x \mapsto (-io(x), io(x) \cdot x)$. \square

Corollary 3.6. — *The \mathbb{C} -action on X^* is locally proper. If $\mathcal{P}(o) = \emptyset$ or $\mathcal{P}(u) = \emptyset$ hold, then \mathbb{C} acts properly on X^* .*

From now on we suppose that X fulfills the assumption (3.2). Recall that the Bergman form ω is a Kähler form on X invariant under the action of $\text{Aut}(X)$. Let ξ denote the complete holomorphic vector field on X which corresponds to the \mathbb{R} -action, i. e. we have $\xi(x) = \frac{\partial}{\partial t} \Big|_0 \varphi_t(x)$. Hence, $\iota_\xi \omega = \omega(\cdot, \xi)$ is a 1-form on X and since $H^1(X, \mathbb{R}) = \{0\}$ there exists a function $\mu^\xi \in C^\infty(X)$ with $d\mu^\xi = \iota_\xi \omega$.

Remark. — This means that μ^ξ is a momentum map for the \mathbb{R} -action on X .

Lemma 3.7. — *The map $\mu^\xi: X \rightarrow \mathbb{R}$ is an \mathbb{R} -invariant submersion.*

Proof. — The claim follows from $d\mu^\xi(x)J\xi_x = \omega_x(J\xi_x, \xi_x) > 0$. \square

Proposition 3.8. — *The \mathbb{C} -action on X^* is locally proper.*

Proof. — Since μ^ξ is a submersion, the fibers $(\mu^\xi)^{-1}(c)$, $c \in \mathbb{R}$, are real hypersurfaces in X . Then

$$\frac{d}{dt} \Big|_0 \mu^\xi(it \cdot x) = \omega_x(J\xi_x, \xi_x) > 0$$

implies that every $i\mathbb{R}$ -orbit intersects $(\mu^\xi)^{-1}(c)$ transversally. Since X is orbit-connected in X^* , the map $i\mathbb{R} \times (\mu^\xi)^{-1}(c) \rightarrow X^*$ is injective and therefore a diffeomorphism onto its open image. Together with the fact that $(\mu^\xi)^{-1}(c)$ is \mathbb{R} -invariant this yields the existence of differentiable local slices for the \mathbb{C} -action. \square

3.3. A necessary condition for X/\mathbb{Z} to be Stein. — We have the following necessary condition for X/\mathbb{Z} to be a Stein manifold.

Proposition 3.9. — *If the quotient manifold X/\mathbb{Z} is Stein, then X^* is Stein and the \mathbb{C} -action on X^* is proper.*

Proof. — Suppose that X/\mathbb{Z} is a Stein manifold. By [CTIT00] this implies that X^* is Stein as well.

Next we will show that the \mathbb{C}^* -action on X^*/\mathbb{Z} is proper. For this we will use as above a moment map for the S^1 -action on X^*/\mathbb{Z} .

By compactness of S^1 we may apply the complexification theorem from [Hei91] which shows that X^*/\mathbb{Z} is also a Stein manifold and in particular Hausdorff. Hence, there exists a smooth strictly plurisubharmonic exhaustion function $\rho: X^*/\mathbb{Z} \rightarrow \mathbb{R}^{>0}$ invariant under S^1 . Consequently, $\omega := \frac{i}{2} \partial \bar{\partial} \rho \in \mathcal{A}^{1,1}(X^*)$ is an S^1 -invariant Kähler form. Associated to ω we have the S^1 -invariant moment map

$$\mu: X^*/\mathbb{Z} \rightarrow \mathbb{R}, \quad \mu^\xi(x) := \frac{d}{dt} \Big|_0 \rho(\exp(it\xi) \cdot x),$$

where ξ is the complete holomorphic vector field on X^*/\mathbb{Z} which corresponds to the S^1 -action. Now we can apply the same argument as above in order to deduce that \mathbb{C}^* acts locally properly on X^*/\mathbb{Z} .

We still must show that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. To see this, let $\mathbb{C}^* \cdot x_j$, $j = 0, 1$, be two different orbits in X^*/\mathbb{Z} . Since \mathbb{C}^* acts locally properly, these are closed and therefore there exists a function $f \in \mathcal{O}(X^*/\mathbb{Z})$ with $f|_{\mathbb{C}^* \cdot x_j} = j$ for $j = 0, 1$. Again we may assume that f is S^1 - and consequently \mathbb{C}^* -invariant. Hence, there is a continuous function on $(X^*/\mathbb{Z})/\mathbb{C}^*$ which separates the two orbits, which implies that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. This proves that \mathbb{C}^* acts properly on X^*/\mathbb{Z} .

Since we know already that the \mathbb{C} -action on X^* is locally proper, it is enough to show that X^*/\mathbb{C} is Hausdorff. But this follows from the properness of the \mathbb{C}^* -action on X^*/\mathbb{Z} since $X^*/\mathbb{C} \cong (X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. \square

4. Properness of the \mathbb{C} -action

Let X be a hyperbolic Stein \mathbb{R} -manifold. Suppose that X fulfills (3.1) or (3.2). We have seen that \mathbb{C} acts locally properly on X^* . In this section we prove that under the additional assumption $\dim X = 2$ the orbit space X^*/\mathbb{C} is Hausdorff. This implies that \mathbb{C} acts properly on X^* if $\dim X = 2$.

4.1. Stein surfaces with \mathbb{C} -actions. — For every function $f \in \mathcal{O}(\Delta)$ which vanishes only at the origin, we define

$$X_f := \{(x, y, z) \in \Delta \times \mathbb{C}^2; f(x)y - z^2 = 1\}.$$

Since the differential of the defining equation of X_f is given by $(f'(x)y - 2z)$, we see that 1 is a regular value of $(x, y, z) \mapsto f(x)y - z^2$. Hence, X_f is a smooth Stein surface in $\Delta \times \mathbb{C}^2$.

There is a holomorphic \mathbb{C} -action on X_f defined by

$$t \cdot (x, y, z) := (x, y + 2tz + t^2 f(x), z + tf(x)).$$

One can directly check that this defines an action.

Lemma 4.1. — *The \mathbb{C} -action on X_f is free, and all orbits are closed.*

Proof. — Let $t \in \mathbb{C}$ such that $(x, y + 2tz + t^2 f(x), z + tf(x)) = (x, y, z)$ for some $(x, y, z) \in X_f$. If $f(x) \neq 0$, then $z + tf(x) = z$ implies $t = 0$. If $f(x) = 0$, then $z \neq 0$ and $y + 2tz = y$ gives $t = 0$.

The map $\pi: X_f \rightarrow \Delta$, $(x, y, z) \mapsto x$, is \mathbb{C} -invariant. If $a \in \Delta^*$, then $f(a) \neq 0$ and we have

$$\frac{z}{f(a)} \cdot (a, f(a)^{-1}, 0) = (a, y, z) \in X_f,$$

which implies $\pi^{-1}(a) = \mathbb{C} \cdot (a, f(a)^{-1}, 0)$. A similar calculation gives $\pi^{-1}(0) = \mathbb{C} \cdot p_1 \cup \mathbb{C} \cdot p_2$ with $p_1 = (0, 0, i)$ and $p_2 = (0, 0, -i)$. Consequently, every \mathbb{C} -orbit is closed. \square

Remark. — The orbit space X_f/\mathbb{C} is the unit disc with a doubled origin and in particular not Hausdorff.

We calculate slices at the point p_j , $j = 1, 2$, as follows. Let $\varphi_j: \Delta \times \mathbb{C} \rightarrow X_f$ be given by $\varphi_1(z, t) := t \cdot (z, 0, i)$ and $\varphi_2(w, s) = s \cdot (w, 0, -i)$. Solving the equation $s \cdot (w, 0, -i) = t \cdot (z, 0, i)$ for (w, s) yields the transition function $\varphi_{12} = \varphi_2^{-1} \circ \varphi_1: \Delta^* \times \mathbb{C} \rightarrow \Delta^* \times \mathbb{C}$,

$$(z, t) \mapsto \left(z, t + \frac{2i}{f(z)} \right).$$

The function $\frac{1}{f}$ is a meromorphic function on Δ without zeros and with the unique pole 0.

Lemma 4.2. — *Let \mathbb{R} act on X_f via $\mathbb{R} \hookrightarrow \mathbb{C}$, $t \mapsto ta$, for some $a \in \mathbb{C}^*$. Then there is no \mathbb{R} -invariant domain $D \subset X_f$ with $D \cap \mathbb{C} \cdot p_j \neq \emptyset$ for $j = 1, 2$ on which \mathbb{R} acts properly.*

Proof. — Suppose that $D \subset X_f$ is an \mathbb{R} -invariant domain with $D \cap \mathbb{C} \cdot p_j \neq \emptyset$ for $j = 1, 2$. Without loss of generality we may assume that $p_1 \in D$ and $\zeta \cdot p_2 = (0, -2\zeta i, -i) \in D$ for some $\zeta \in \mathbb{C}$. We will show that the orbits $\mathbb{R} \cdot p_1$ and $\mathbb{R} \cdot (\zeta \cdot p_2)$ cannot be separated by \mathbb{R} -invariant open neighborhoods.

Let $U_1 \subset D$ be an \mathbb{R} -invariant open neighborhood of p_1 . Then there are $r, r' > 0$ such that $\Delta_r^* \times \Delta_{r'} \times \{i\} \subset U_1$ holds. Here, $\Delta_r = \{z \in \mathbb{C}; |z| < r\}$. For $(\varepsilon_1, \varepsilon_2) \in \Delta_r^* \times \Delta_{r'}$ and $t \in \mathbb{R}$ we have

$$t \cdot (\varepsilon_1, \varepsilon_2, i) = (\varepsilon_1, \varepsilon_2 + 2(ta)i + (ta)^2 f(\varepsilon_1), i + (ta)f(\varepsilon_1)) \in U_1.$$

We have to show that for all $r_2, r_3 > 0$ there exist $(\tilde{\varepsilon}_2, \tilde{\varepsilon}_3) \in \Delta_{r_2} \times \Delta_{r_3}$, $(\varepsilon_1, \varepsilon_2) \in \Delta_r^* \times \Delta_{r'}$ and $t \in \mathbb{R}$ such that

$$(4.1) \quad (\varepsilon_1, \varepsilon_2 + 2(ta)i + (ta)^2 f(\varepsilon_1), i + (ta)f(\varepsilon_1)) = (\varepsilon_1, -2\zeta i + \tilde{\varepsilon}_2, -i + \tilde{\varepsilon}_3)$$

holds.

Let $r_2, r_3 > 0$ be given. From (4.1) we obtain $\tilde{\varepsilon}_3 = taf(\varepsilon_1) + 2i$ or, equivalently, $ta = \frac{\tilde{\varepsilon}_3 - 2i}{f(\varepsilon_1)}$. Setting $\tilde{\varepsilon}_2 = \varepsilon_2$ we obtain from $2(ta)i + (ta)^2 f(\varepsilon_1) = -2\zeta i$ the equivalent expression

$$(4.2) \quad f(\varepsilon_1) = -2i \frac{\zeta + ta}{(ta)^2}.$$

for $t \neq 0$. Choosing a real number $t \gg 1$, we find an $\varepsilon_1 \in \Delta_r^*$ such that (4.2) is fulfilled. After possibly enlarging t we have $\tilde{\varepsilon}_3 := taf(\varepsilon_1) + 2i = -2i \frac{\zeta}{ta} \in \Delta_{r_3}$. Together with $\varepsilon_2 = \tilde{\varepsilon}_2$ equation (4.1) is fulfilled and the proof is finished. \square

Thus, the Stein surface X_f cannot be obtained as globalization of the local \mathbb{C} -action on any \mathbb{R} -invariant domain $D \subset X_f$ on which \mathbb{R} acts properly.

4.2. The quotient X^*/\mathbb{C} is Hausdorff. — Suppose that X^*/\mathbb{C} is not Hausdorff and let $x_1, x_2 \in X$ be such that the corresponding \mathbb{C} -orbits cannot be separated in X^*/\mathbb{C} . Since we already know that \mathbb{C} acts locally proper on X^* we find local holomorphic slices $\varphi_j: \Delta \times \mathbb{C} \rightarrow U_j \subset X$, $\varphi_j(z, t) = t \cdot s_j(z)$ at each $\mathbb{C} \cdot x_j$ where $s_j: \Delta \rightarrow X$ is holomorphic with $s_j(0) = x_j$. Consequently, we obtain the transition function $\varphi_{12}: (\Delta \setminus A) \times \mathbb{C} \rightarrow (\Delta \setminus A) \times \mathbb{C}$ for some closed subset $A \subset \Delta$ which must be of the form $(z, t) \mapsto (z, t + f(z))$ for some $f \in \mathcal{O}(\Delta \setminus A)$. The following lemma applies to show that A is discrete and

that f is meromorphic on Δ . Hence, we are in one of the model cases discussed in the previous subsection.

Lemma 4.3. — *Let Δ_1 and Δ_2 denote two copies of the unit disk $\{z \in \mathbb{C}; |z| < 1\}$. Let $U \subset \Delta_j$, $j = 1, 2$, be a connected open subset and $f: U \subset \Delta_1 \rightarrow \mathbb{C}$ a non-constant holomorphic function on U . Define the complex manifold*

$$M := (\Delta_1 \times \mathbb{C}) \cup (\Delta_2 \times \mathbb{C}) / \sim,$$

where \sim is the relation $(z_1, t_1) \sim (z_2, t_2) :\Leftrightarrow z_1 = z_2 =: z \in U$ and $t_2 = t_1 + f(z)$.

Suppose that M is Hausdorff. Then the complement A of U is discrete and f extends to a meromorphic function on Δ_1 .

Proof. — We first prove that for every sequence (x_n) , $x_n \in U$, with $\lim_{n \rightarrow \infty} x_n = p \in \partial U$, one has $\lim_{n \rightarrow \infty} |f(x_n)| = \infty \in \mathbb{P}_1(\mathbb{C})$. Assume the contrary, i.e. there is a sequence (x_n) , $x_n \in U$, with $\lim_{n \rightarrow \infty} x_n = p \in \partial U$ such that $\lim_{n \rightarrow \infty} f(x_n) = a \in \mathbb{C}$. Choose now $t_1 \in \mathbb{C}$, consider the two points $(p, t_1) \in \Delta_1 \times \mathbb{C}$ and $(p, t_1 + a) \in \Delta_2 \times \mathbb{C}$ and note their corresponding points in M as q_1 and q_2 . Then $q_1 \neq q_2$. The sequences $(x_n, t_1) \in \Delta_1 \times \mathbb{C}$ and $(x_n, t_1 + f(x_n)) \in \Delta_2 \times \mathbb{C}$ define the same sequence in M having q_1 and q_2 as accumulation points. So M is not Hausdorff, a contradiction.

In particular we have proved that the zeros of f do not accumulate to ∂U in Δ_1 . So there is an open neighborhood V of ∂U in Δ_1 such that the restriction of f to $W := U \cap V$ does not vanish. Let $g := 1/f$ on W . Then g extends to a continuous function on V taking the value zero outside of U . The theorem of Rado implies that this function is holomorphic on V . It follows that the boundary ∂U is discrete in Δ_1 and that f has a pole in each of the points of this set, so f is a meromorphic function on Δ_1 . \square

Theorem 4.4. — *The orbit space X^*/\mathbb{C} is Hausdorff. Consequently, \mathbb{C} acts properly on X^* .*

Proof. — By virtue of the above lemma, in a neighborhood of two non-separable \mathbb{C} -orbits X is isomorphic to a domain in one of the model Stein surfaces discussed in the previous subsection. Since we have seen there that these surfaces are never globalizations, we arrive at a contradiction. Hence, all \mathbb{C} -orbits are separable. \square

5. Examples

In this section we discuss several examples which illustrate our results.

5.1. Hyperbolic Stein surfaces with proper \mathbb{R} -actions. — Let R be a compact Riemann surface of genus $g \geq 2$. It follows that the universal covering of R is given by the unit disc $\Delta \subset \mathbb{C}$ and hence that R is hyperbolic. The fundamental group $\pi_1(R)$ of R contains a normal subgroup N such that $\pi_1(R)/N \cong \mathbb{Z}$. Let $\tilde{R} \rightarrow R$ denote the corresponding normal covering. Then \tilde{R} is a hyperbolic Riemann surface with a holomorphic \mathbb{Z} -action such that $\tilde{R}/\mathbb{Z} = R$. Note that \mathbb{Z} is not contained in a one parameter group of automorphisms of \tilde{R} .

We have two mappings

$$\begin{array}{ccc} X := \mathbb{H} \times_{\mathbb{Z}} \tilde{R} & \xrightarrow{q} & \tilde{R}/\mathbb{Z} = R \\ p \downarrow & & \\ \mathbb{H}/\mathbb{Z} \cong \Delta \setminus \{0\}. & & \end{array}$$

The map $p: X \rightarrow \Delta \setminus \{0\}$ is a holomorphic fiber bundle with fiber \tilde{R} . Since the Serre problem has a positive answer if the fiber is a non-compact Riemann surface ([Mok82]), the suspension $X = \mathbb{H} \times_{\mathbb{Z}} \tilde{R}$ is a hyperbolic Stein surface. The group \mathbb{R} acts on $\mathbb{H} \times \tilde{R}$ by $t \cdot (z, x) = (z + t, x)$ and this action commutes with the diagonal action of \mathbb{Z} . Consequently, we obtain an action of \mathbb{R} on X .

Lemma 5.1. — *The universal globalization of the local \mathbb{C} -action on X is given by $X^* = \mathbb{C} \times_{\mathbb{Z}} \tilde{R}$. Moreover, \mathbb{C} acts properly on X^* .*

Proof. — One checks directly that $t \cdot [z, x] := [z + t, x]$ defines a holomorphic \mathbb{C} -action on $X^* = \mathbb{C} \times_{\mathbb{Z}} \tilde{R}$ which extends the \mathbb{R} -action on X . We will show that X is orbit-connected in X^* : Since $[z + t, x]$ lies in X if and only if there exist elements $(z', x') \in \mathbb{H} \times \tilde{R}$ and $m \in \mathbb{Z}$ such that $(z + t, x) = (z' + m, m \cdot x')$, we conclude $\mathbb{C}[z, x] = \{t \in \mathbb{C}; \operatorname{Im}(t) > -\operatorname{Im}(z)\}$ which is connected.

In order to show that \mathbb{C} acts properly on X^* it is sufficient to show that $\mathbb{C} \times \mathbb{Z}$ acts properly on $\mathbb{C} \times \tilde{R}$. Hence, we choose sequences $\{t_n\}$ in \mathbb{C} , $\{m_n\}$ in \mathbb{Z} and $\{(z_n, x_n)\}$ in $\mathbb{C} \times \tilde{R}$ such that

$$((t_n, m_n) \cdot (z_n, x_n), (z_n, x_n)) = ((z_n + t_n + m_n, m_n \cdot x_n), (z_n, x_n)) \rightarrow ((z_1, x_1), (z_0, x_0))$$

holds. Since \mathbb{Z} acts properly on \tilde{R} , it follows that $\{m_n\}$ has a convergent subsequence, which in turn implies that $\{t_n\}$ has a convergent subsequence. Hence, the lemma is proven. \square

Proposition 5.2. — *The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein. The quotient X^*/\mathbb{C} is biholomorphically equivalent to $\tilde{R}/\mathbb{Z} = R$.*

Proof. — It is sufficient to note that the map $\Phi: X = \mathbb{H} \times_{\mathbb{Z}} \tilde{R} \rightarrow \Delta^* \times R$, $\Phi[z, x] := (e^{2\pi iz}, [x])$, induces a biholomorphic map $X/\mathbb{Z} \rightarrow \Delta^* \times R$. \square

Proposition 5.3. — *The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein.*

Thus we have found an example for a hyperbolic Stein surface X endowed with a proper \mathbb{R} -action such that the associated \mathbb{Z} -quotient is not holomorphically separable. Moreover, the \mathbb{R} -action on X extends to a proper \mathbb{C} -action on a Stein manifold X^* containing X as an orbit-connected domain such that X^*/\mathbb{C} is any given compact Riemann surface of genus $g \geq 2$.

5.2. Counterexamples with domains in \mathbb{C}^n . — There is a bounded Reinhardt domain D in \mathbb{C}^2 endowed with a holomorphic action of \mathbb{Z} such that D/\mathbb{Z} is not Stein. However, this \mathbb{Z} -action does not extend to an \mathbb{R} -action. We give quickly the construction.

Let $\lambda := \frac{1}{2}(3 + \sqrt{5})$ and

$$D := \{(x, y) \in \mathbb{C}^2 \mid |x| > |y|^\lambda, |y| > |x|^\lambda\}.$$

It is obvious that D is a bounded Reinhardt domain in \mathbb{C}^2 avoiding the coordinate hyperplanes. The holomorphic automorphism group of D is a semidirect product $\Gamma \ltimes (S^1)^2$, where the group $\Gamma \simeq \mathbb{Z}$ is generated by the automorphism $(x, y) \mapsto (x^3 y^{-1}, x)$ and $(S^1)^2$ is the rotation group. Therefore the group Γ is not contained in a one-parameter group. Furthermore the quotient D/Γ is the (non-Stein) complement of the singular point in a 2-dimensional normal complex Stein space, a so-called "cusp singularity". These singularities are intensively studied in connection with Hilbert modular surfaces and Inoue-Hirzebruch surfaces, see e.g. [vdG88] and [Zaf01].

In the rest of this subsection we give an example of a hyperbolic domain of holomorphy in a 3-dimensional Stein solvmanifold endowed with a proper \mathbb{R} -action such that the \mathbb{Z} -quotient is not Stein. While this domain is not simply-connected, its fundamental group is much simpler than the fundamental groups of our two-dimensional examples.

Let $G := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}; a, b, c \in \mathbb{C} \right\}$ be the complex Heisenberg group and let us consider its discrete subgroup

$$\Gamma := \left\{ \begin{pmatrix} 1 & m & \frac{m^2}{2} + 2\pi i k \\ 0 & 1 & m + 2\pi i l \\ 0 & 0 & 1 \end{pmatrix}; m, k, l \in \mathbb{Z} \right\}.$$

Note that Γ is isomorphic to $\mathbb{Z}_m \ltimes \mathbb{Z}_{(k,l)}^2$. We let Γ act on \mathbb{C}^2 by

$$(z, w) \mapsto \left(z + mw - \frac{m^2}{2} - 2\pi i k, w - m - 2\pi i l \right).$$

Proposition 5.4. — *The group Γ acts properly and freely on \mathbb{C}^2 , and the quotient manifold \mathbb{C}^2/Γ is holomorphically separable but not Stein.*

Proof. — Since $\Gamma' \cong \mathbb{Z}^2$ is a normal subgroup of Γ , we obtain $\mathbb{C}^2/\Gamma \cong (\mathbb{C}^2/\Gamma')/(\Gamma/\Gamma')$. The map $\mathbb{C}^2 \rightarrow \mathbb{C}^* \times \mathbb{C}^*$, $(z, w) \mapsto (\exp(z), \exp(w))$, identifies \mathbb{C}^2/Γ' with $\mathbb{C}^* \times \mathbb{C}^*$. The induced action of $\Gamma/\Gamma' \cong \mathbb{Z}$ on $\mathbb{C}^* \times \mathbb{C}^*$ is given by

$$(z, w) \mapsto \left(e^{-m^2/2} z w^m, e^{-m} w \right)$$

which shows that Γ acts properly and freely on \mathbb{C}^2 . Moreover, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^* & \longrightarrow & Y := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z} \\ \downarrow (z,w) \mapsto w & & \downarrow \\ \mathbb{C}^* & \longrightarrow & T := \mathbb{C}^*/\mathbb{Z}. \end{array}$$

The group \mathbb{C}^* acts by multiplication in the first factor on $\mathbb{C}^* \times \mathbb{C}^*$ and this action commutes with the \mathbb{Z} -action. One checks directly that the joint $(\mathbb{C}^* \times \mathbb{Z})$ -action on $\mathbb{C}^* \times \mathbb{C}^*$ is proper which implies that the map $Y \rightarrow T$ is a \mathbb{C}^* -principal bundle. Consequently, Y is not Stein.

In order to show that Y is holomorphically separable, note that by [Oel92] this \mathbb{C}^* -principal bundle $Y \rightarrow T$ extends to a line bundle $p: L \rightarrow T$ with first Chern class $c_1(L) = -1$. Therefore the zero section of $p: L \rightarrow T$ can be blown down and we obtain a singular normal Stein space $\bar{Y} = Y \cup \{y_0\}$ where $y_0 = \text{Sing}(\bar{Y})$ is the blown down zero section. Thus Y is holomorphically separable. \square

Let us now choose a neighborhood of the singularity $y_0 \in \bar{Y}$ biholomorphic to the unit ball and let U be its inverse image in \mathbb{C}^2 . It follows that U is a hyperbolic domain with smooth strictly Levi-convex boundary in \mathbb{C}^2 and in particular Stein. In order to obtain a proper action of \mathbb{R} we form the suspension $D = \mathbb{H} \times_{\Gamma} U$ where Γ acts on $\mathbb{H} \times U$ by $(t, z, w) \mapsto (t + m, z + mw - \frac{m^2}{2} - 2\pi ik, w - m - 2\pi il)$.

Proposition 5.5. — *The suspension $D = \mathbb{H} \times_{\Gamma} U$ is isomorphic to a Stein domain in the Stein manifold G/Γ .*

Proof. — We identify $\mathbb{H} \times U$ with the $\mathbb{R} \times \Gamma$ -invariant domain

$$\Omega := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}; \text{Im}(a) > 0, (c, b) \in U \right\}$$

in G .

Since $\mathbb{H} \times U$ is Stein, it follows that $\mathbb{H} \times_{\Gamma} U$ is locally Stein in G/Γ . Hence, by virtue of [DG60] we only have to show that G/Γ is Stein.

For this we note first that G is a closed subgroup of $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^2$ which implies that G/Γ is a closed complex submanifold of $X := (\text{SL}(2, \mathbb{C}) \times \mathbb{C}^2)/\Gamma$. By [Oel92] the manifold X is holomorphically separable, hence G/Γ is holomorphically separable. Since G is solvable, a result of Huckleberry and Oeljeklaus ([HO86]) yields the Steinness of G/Γ .

One checks directly that the action of $\mathbb{R} \times \Gamma$ on $\mathbb{H} \times U$ is proper which implies that \mathbb{R} acts properly on $\mathbb{H} \times_{\Gamma} U$. \square

Because of $(\mathbb{H} \times_{\Gamma} U)/\mathbb{Z} \cong \Delta^* \times (U/\Gamma)$ this quotient manifold is not Stein but holomorphically separable.

6. Bounded domains with proper \mathbb{R} -actions

In this section we give the proof of our main result.

6.1. Proper \mathbb{R} -actions on D . — Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\text{Aut}(D)^0$ be the connected component of the identity in $\text{Aut}(D)$.

Lemma 6.1. — *A proper \mathbb{R} -action by holomorphic transformations on D exists if and only if the group $\text{Aut}(D)^0$ is non-compact.*

The proof follows from the existence of a diffeomorphism $K \times V \rightarrow \text{Aut}(D)^0$ where K is a maximal compact subgroup of $\text{Aut}(D)^0$ and V is a linear subspace of the Lie algebra of $\text{Aut}(D)^0$.

6.2. Steinness of D/\mathbb{Z} . — Now we give the proof of our main result.

Theorem 6.2. — *Let D be a simply-connected bounded domain of holomorphy in \mathbb{C}^2 . Suppose that the group \mathbb{R} acts properly by holomorphic transformations on D . Then the complex manifold D/\mathbb{Z} is biholomorphically equivalent to a domain of holomorphy in \mathbb{C}^2 .*

Proof. — Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy. Since the Serre problem is solvable if the fiber is D , see [Siu76], the universal globalization D^* is a simply-connected Stein surface, [CTIT00]. Moreover, we have shown in Theorem 4.4, that \mathbb{C} acts properly on D^* . Since the Riemann surface D^*/\mathbb{C} is also simply-connected, it must be Δ , \mathbb{C} or $\mathbb{P}_1(\mathbb{C})$. In all three cases the bundle $D^* \rightarrow D^*/\mathbb{C}$ is holomorphically trivial. So we can exclude the case that D^*/\mathbb{C} is compact and it follows that $D/\mathbb{Z} \cong \mathbb{C}^* \times (D^*/\mathbb{C})$ is a Stein domain in \mathbb{C}^2 . \square

6.3. A normal form for domains with non-compact $\text{Aut}(D)^0$. — Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy such that the identity component of its automorphism group is non-compact. As we have seen, this yields a proper \mathbb{R} -action on D by holomorphic transformations and the universal globalization of the induced local \mathbb{C} -action on D is isomorphic to $\mathbb{C} \times S$ where S is either Δ or \mathbb{C} and where \mathbb{C} acts by translation in the first factor.

Moreover, there are plurisubharmonic functions $u, -o: \mathbb{C} \times S \rightarrow \mathbb{R} \cup \{-\infty\}$ which fulfill

$$u(t \cdot (z_1, z_2)) = u(z_1, z_2) - \text{Im}(t) \quad \text{and} \quad o(t \cdot (z_1, z_2)) = o(z_1, z_2) - \text{Im}(t)$$

such that $D = \{(z_1, z_2) \in \mathbb{C} \times S; u(z_1, z_2) < 0 < o(z_1, z_2)\}$. From this we conclude $u(z_1, z_2) = u(0, z_2) - \text{Im}(z_1)$, $o(z_1, z_2) = o(0, z_2) - \text{Im}(z_1)$ and define $u'(z_2) := u(0, z_2)$, $o'(z_2) := o(0, z_2)$.

We summarize our remarks in the following

Theorem 6.3. — *Let D be a simply-connected bounded domain of holomorphy in \mathbb{C}^2 admitting a non-compact connected identity component of its automorphism group. Then D is biholomorphic to a domain of the form*

$$\tilde{D} = \{(z_1, z_2) \in \mathbb{C} \times S; u'(z_2) < \text{Im}(z_1) < o'(z_2)\},$$

where the functions $u', -o'$ are subharmonic in S .

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