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# The Busy Beaver Competition: a historical survey

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## Abstract

Tibor Rado defined the Busy Beaver Competition in 1962. He used Turing machines to give explicit definitions for some functions that are not computable and grow faster than any computable function. He put forward the problem of computing the values of these functions on numbers 1, 2, 3, . . . . More and more powerful computers have made possible the computation of lower bounds for these values. In 1988, Brady extended the definitions to functions on two variables.

We give a historical survey of these works. The successive record holders in the Busy Beaver Competition are displayed, with their discoverers, the date they were found, and, for some of them, an analysis of their behavior.

*Keywords:* Turing machine, busy beaver.

Mathematics Subject Classification (2000, 2010): 03D10, 68Q05.

## 1 Introduction

### 1.1 Noncomputable functions

In 1936, Turing succeeded in making formal the intuitive notion of a function computable by a finite, mechanical, procedure. He defined what is now called a *Turing machine* and stated that a function on integers is intuitively computable if and only if it is computable by a Turing machine. Other authors, such as Church, Kleene, Post, and, later, Markov, defined other models of computation that turn out to compute the same functions as Turing machines do. See Soare (1996, 2007, 2009) for more details about the history of the *Church-Turing Thesis*, as is now named the capture of the intuitive notion of computability by the formal notion of Turing machine.

Given a model of computation, a *noncomputable* function can easily be defined by *diagonalization*. The list of all computable functions is written, and then a function is defined

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such that it is distinct from each function in the list. Then this function is noncomputable. Such a definition by diagonalization leaves too much room in the choice of the list and in the choice of the values of the final function. What is needed is a function whose definition is simple, natural and without ambiguousness.

In 1962, Rado succeeded in providing a natural definition for noncomputable functions on the integers. He defined a *Busy Beaver* game, leading to two functions  $\Sigma$  and  $S$  which are still the best examples of noncomputable functions that one can give nowadays. The values  $\Sigma(n)$  and  $S(n)$  are defined by considering the finite set of carefully defined Turing machines with two symbols and  $n$  states, and picking among these machines those with some maximal behavior. It makes sense to compute the values  $\Sigma(n)$ ,  $S(n)$  of these functions on small integers  $n = 1, 2, \dots$ . We have  $\Sigma(1) = S(1) = 1$ , trivially. Lin and Rado (1965) gave proofs for the values  $\Sigma(2)$ ,  $S(2)$ ,  $\Sigma(3)$  and  $S(3)$ , and Brady (1983) did for  $\Sigma(4)$  and  $S(4)$ . Only lower bounds had been provided for  $\Sigma(5)$  and  $S(5)$ , by the works of Green, Lynn, Schult, Uhing and eventually Marxen and Buntrock. The lower bounds for  $\Sigma(6)$  and  $S(6)$  are still an ongoing quest.

The initial Busy Beaver game, as defined by Rado, used Turing machines with two symbols. Brady (1988) generalized the problem to Turing machines with  $k$  symbols,  $k \geq 3$ . He defined a function  $S(n, k)$  such that  $S(n, 2)$  is Rado's  $S(n)$ , and gave some lower bounds. Michel (2004) resumed the computation of lower bounds for  $S(n, k)$  and another function  $\Sigma(n, k)$ , and the search is going on, with the works of Brady, Souris, Lafitte and Papazian, T. and S. Ligocki.

Since 2004, results are sent by email to Marxen and to Michel, who record them on their websites. This paper aims to give a published version of these records.

## 1.2 Big numbers

Consider Rado's functions  $S$  and  $\Sigma$ . Not only they are not computable, but they grow faster than any computable function. That is, for any computable function  $f$ , there exists an integer  $N$  such that, for all  $n > N$ , we have  $S(n) > f(n)$ . This property can be used to write big numbers. For example, if  $S^k(n)$  denotes  $S(S(\dots S(n)\dots))$ , iterated  $k$  times, then  $S^{9^9}(9)$  is a very big number, bigger than any number that was written with six symbols before the definition of the  $S$  function. See the paper on big numbers on Scott Aaronson's website.

## 1.3 Plan of the paper

The paper is structured as follows.

1. Introduction.
2. Preliminaries.
3. Historical overview.
4. Historical survey (lower bounds for  $S(n, k)$  and  $\Sigma(n, k)$ , and tables of the Turing machines that achieve these lower bounds).
5. Behaviors of busy beavers. We also display the relations between these behaviors and open problems in mathematics called Collatz-like problems and we resume some

machines with non-Collatz-like behaviors. We also present pairs of machines that have the same behaviors, but not the same numbers of states and symbols.

6. Relations between the busy beaver functions  $S(n)$  and  $\Sigma(n)$ .
7. Variants of busy beavers:
  - Busy beavers defined by 4-tuples.
  - Busy beavers whose head can stand still.
  - Two-dimensional busy beavers.
8. The methods.

## 2 Preliminaries

There are many possible definitions for a Turing machine. We will follow the conventions chosen by Rado (1962) in his definition of functions  $\Sigma$  and  $S$ . A Turing machine has a tape, made of cells, infinite to the left and to the right. On each cell a *symbol* is written. There is a finite set  $S = \{0, 1, \dots\}$  of symbols. The symbol 0 is the *blank symbol*. A Turing machine has a tape head, which reads and writes symbols on the tape, and can move in both *directions* left or right, denoted by  $L$  and  $R$ . A Turing machine has a finite set of *states*  $Q = \{A, B, \dots\}$ , plus a special state  $H$ , the *halting state*. A Turing machine has a *next move function*

$$\delta : Q \times S \longrightarrow (S \times \{L, R\} \times Q) \cup \{(1, R, H)\}.$$

If we have  $\delta(q, a) = (b, d, p)$ , then it means that, when the Turing machine is in state  $q$  and reads symbol  $a$  on the tape, then it writes symbol  $b$  instead of  $a$  on the cell currently read, it moves one cell in the direction  $d \in \{L, R\}$ , and it changes the state from  $q$  to  $p$ . Each application of next move function  $\delta$  is a *step* of the computation. If  $\delta(q, a) = (1, R, H)$ , then, when the machine is in state  $q$  reading symbol  $a$ , it writes a 1, moves right, enters state  $H$ , and stops. We follow Rado (1962) in not allowing the center direction, that is in compelling the tape head to move left or right at each step. Like Rado, we keep the halting state  $H$  out of the set of states. We differ from Rado in not allowing transitions  $\delta(q, a) = (b, d, H)$  with  $b \neq 1, d \neq R$ .

Note that such a machine is a universal model of computation. That is, any computable function on integers can be computed by a Turing machine as defined above. Initially, a finite string of symbols is written on the tape. It is called the input, and can be a code for an integer. All other cells contain the blank symbol. The tape head reads the leftmost symbol of the input and the state is the initial state  $A$ . Then the computation is launched according to the next move function. If it stops, by entering the halting state  $H$ , then the string of symbols written on the tape is the output, which can be a code for an integer. So a Turing machine defines a partial function on integers. Reciprocally, any computable partial function on integers can be computed by a Turing machine as defined above.

In order to define functions  $\Sigma$  and  $S$ , Rado (1962) considers Turing machines with  $n$  states and two symbols 0 and 1. His definitions can be easily extended to Turing machines with  $n$  states and  $k$  symbols,  $k \geq 3$ , as Brady (1988) does. We consider the set  $TM(n, k)$  of Turing machines with  $n$  states and  $k$  symbols. With our definitions, it is a finite set with

$(2kn + 1)^{kn}$  members. We launch each of these  $(2kn + 1)^{kn}$  Turing machines on a blank tape, that is a tape with the blank symbol 0 in each cell. Some of these machines never stop. The other ones, that eventually stop, are called *busy beavers*, and they are competing in two competitions, for the maximum number of steps and for the maximum number of non-blank symbols left on the tape. Let  $s(M)$  be the number of computation steps taken by the busy beaver  $M$  to stop. Let  $\sigma(M)$  be the number of non-blank symbols left on the tape by the busy beaver  $M$  when it stops. Then the busy beaver functions are

$$S(n, k) = \max\{s(M) : M \text{ is a busy beaver with } n \text{ states and } k \text{ symbols}\},$$

$$\Sigma(n, k) = \max\{\sigma(M) : M \text{ is a busy beaver with } n \text{ states and } k \text{ symbols}\}.$$

For  $k = 2$ , we find Rado's functions  $S(n) = S(n, 2)$  and  $\Sigma(n) = \Sigma(n, 2)$ .

Note that a permutation of the states, symbols or directions does not change the behavior of a Turing machine. The choice between machines that differ only by such permutations is settled by the following normalizing rule : when a Turing machine is launched on a blank tape, it enters states in the order  $A, B, C, \dots$ , it writes symbols in the order 1, 2, ..., and it first moves right. So, normally, the first transition is  $\delta(A, 0) = (1, R, B)$ .

### 3 Historical overview

The search for champions in the busy beaver competition can be roughly divided into the following stages. Note that, from the beginnings, computers have been tools to find good competitors, so better results follow more powerful computers.

**First stage: Following the definitions.** The definitions of the busy beaver functions  $\Sigma(n)$  and  $S(n)$  by Rado (1962) were quickly followed by conjectures and proofs for  $n = 2, 3$ , by Rado and Lin. Brady (1964) gave a conjecture for  $n = 4$ , and Green (1964) gave lower bounds for many values of  $n$ . Lynn (1972) improved these lower bounds for  $n = 5, 6$ . Brady proved his conjecture for  $n = 4$  in 1974, and published the result in 1983. Details on this first stage can be found in the articles of Lynn (1972) and Brady (1983, 1988).

**Second stage: Following the Dortmund contest.** More results for  $n = 5, 6$  followed the contest that was organized at Dortmund in 1983, and was won by Schult. Uhing improved twice the result in 1984 and in 1986. Marxen and Buntrock began a search for competitors for  $n = 5, 6$  in 1989. They quickly found a conjectural winner for  $n = 5$ , and went on founding many good machines for  $n = 6$ , up to 2001. Michel (1993) studied the behaviors of many competitors for  $n = 5$ , proving that they depend on well known open problems in number theory. Details on this second stage can be found in the articles of Dewdney (1984ab, 1985ab), Brady (1988), and Marxen and Buntrock (1990). From 1997, results began to be put on the web, either on Google groups, or on personal websites.

**Third stage: Machines with more than two symbols.** As soon as 1988, Brady extended the busy beaver competition to machines with more than two symbols and gave some lower bounds. Michel (2004) resumed the search, and his lower bounds were quickly overtaken by those from Brady. Between 2005 and 2008, more than forty new machines, each breaking a

1963	Rado, Lin	$S(2, 2) = 6, \Sigma(2, 2) = 4$ $S(3, 2) = 21, \Sigma(3, 2) = 6$
1964	Brady	(4,2)-TM: $s = 107, \sigma = 13$
1964	Green	(5,2)-TM: $\sigma = 17$ (6,2)-TM: $\sigma = 35$
1972	Lynn	(5,2)-TM: $s = 435, \sigma = 22$ (6,2)-TM: $s = 522, \sigma = 42$
1974	Lynn	(5,2)-TM: $s = 7,707, \sigma = 112$
1974	Brady	$S(4, 2) = 107, \Sigma(4, 2) = 13$
1983	Brady	(6,2)-TM: $s = 13,488, \sigma = 117$
January 1983	Schult	(5,2)-TM: $s = 134,467, \sigma = 501$ (6,2)-TM: $\sigma = 2,075$
December 1984	Uhing	(5,2)-TM: $s = 2,133,492, \sigma = 1,915$
February 1986	Uhing	(5,2)-TM: $s = 2,358,064$
1988	Brady	(2,3)-TM: $s = 38, \sigma = 9$ (2,4)-TM: $s = 7,195, \sigma = 90$
February 1990	Marxen, Buntrock	<b>(5,2)-TM</b> : $s = 47,176,870, \sigma = 4,098$ (6,2)-TM: $s = 13,122,572,797, \sigma = 136,612$
September 1997	Marxen, Buntrock	(6,2)-TM: $s = 8,690,333,381,690,951, \sigma = 95,524,079$
August 2000	Marxen, Buntrock	(6,2)-TM: $s > 5.3 \times 10^{42}, \sigma > 2.5 \times 10^{21}$
October 2000	Marxen, Buntrock	(6,2)-TM: $s > 6.1 \times 10^{925}, \sigma > 6.4 \times 10^{462}$
March 2001	Marxen, Buntrock	(6,2)-TM: $s > 3.0 \times 10^{1730}, \sigma > 1.2 \times 10^{865}$

Table 1: Busy Beaver Competition from 1963 to 2001

October 2004	Michel	(3,3)-TM: $s = 40,737$ , $\sigma = 208$
November 2004	Brady	(3,3)-TM: $s = 29,403,894$ , $\sigma = 5,600$
December 2004	Brady	(3,3)-TM: $s = 92,649,163$ , $\sigma = 13,949$
February 2005	T. and S. Ligocki	<b>(2,4)-TM</b> : $s = 3,932,964$ , $\sigma = 2,050$ (2,5)-TM: $s = 16,268,767$ , $\sigma = 4,099$ (2,6)-TM: $s = 98,364,599$ , $\sigma = 10,574$
April 2005	T. and S. Ligocki	(4,3)-TM: $s = 250,096,776$ , $\sigma = 15,008$ (3,4)-TM: $s = 262,759,288$ , $\sigma = 17,323$ (2,5)-TM: $s = 148,304,214$ , $\sigma = 11,120$ (2,6)-TM: $s = 493,600,387$ , $\sigma = 15,828$
July 2005	Souris	(3,3)-TM: $s = 544,884,219$ , $\sigma = 36,089$
August 2005	Lafitte, Papazian	(3,3)-TM: $s = 4,939,345,068$ , $\sigma = 107,900$ (2,5)-TM: $s = 8,619,024,596$ , $\sigma = 90,604$
September 2005	Lafitte, Papazian	(3,3)-TM: $s = 987,522,842,126$ , $\sigma = 1,525,688$ (2,5)-TM: $\sigma = 97,104$
October 2005	Lafitte, Papazian	(2,5)-TM: $s = 233,431,192,481$ , $\sigma = 458,357$ (2,5)-TM: $s = 912,594,733,606$ , $\sigma = 1,957,771$
December 2005	Lafitte, Papazian	(2,5)-TM: $s = 924,180,005,181$
April 2006	Lafitte, Papazian	(3,3)-TM: $s = 4,144,465,135,614$ , $\sigma = 2,950,149$
May 2006	Lafitte, Papazian	(2,5)-TM: $s = 3,793,261,759,791$ , $\sigma = 2,576,467$
June 2006	Lafitte, Papazian	(2,5)-TM: $s = 14,103,258,269,249$ , $\sigma = 4,848,239$
July 2006	Lafitte, Papazian	(2,5)-TM: $s = 26,375,397,569,930$
August 2006	T. and S. Ligocki	(3,3)-TM: $s = 4,345,166,620,336,565$ , $\sigma = 95,524,079$ (2,5)-TM: $s > 7.0 \times 10^{21}$ , $\sigma = 172,312,766,455$

Table 2: Busy Beaver Competition from 2004 to 2006

June 2007	Lafitte, Papazian	$S(2, 3) = 38, \Sigma(2, 3) = 9$
September 2007	T. and S. Ligocki	(3,4)-TM: $s > 5.7 \times 10^{52}, \sigma > 2.4 \times 10^{26}$ (2,6)-TM: $s > 2.3 \times 10^{54}, \sigma > 1.9 \times 10^{27}$
October 2007	T. and S. Ligocki	(4,3)-TM: $s > 1.5 \times 10^{1426}, \sigma > 1.1 \times 10^{713}$ (3,4)-TM: $s > 4.3 \times 10^{281}, \sigma > 6.0 \times 10^{140}$ (3,4)-TM: $s > 7.6 \times 10^{868}, \sigma > 4.6 \times 10^{434}$ (3,4)-TM: $s > 3.1 \times 10^{1256}, \sigma > 2.1 \times 10^{628}$ (2,5)-TM: $s > 5.2 \times 10^{61}, \sigma > 9.3 \times 10^{30}$ (2,5)-TM: $s > 1.6 \times 10^{211}, \sigma > 5.2 \times 10^{105}$
November 2007	T. and S. Ligocki	(6,2)-TM: $s > 8.9 \times 10^{1762}, \sigma > 2.5 \times 10^{881}$ <b>(3,3)-TM:</b> $s = 119,112,334,170,342,540, \sigma = 374,676,383$ (4,3)-TM: $s > 7.7 \times 10^{1618}, \sigma > 1.6 \times 10^{809}$ (4,3)-TM: $s > 3.7 \times 10^{1973}, \sigma > 8.0 \times 10^{986}$ (4,3)-TM: $s > 3.9 \times 10^{7721}, \sigma > 4.0 \times 10^{3860}$ (4,3)-TM: $s > 3.9 \times 10^{9122}, \sigma > 2.5 \times 10^{4561}$ (3,4)-TM: $s > 8.4 \times 10^{2601}, \sigma > 1.7 \times 10^{1301}$ (3,4)-TM: $s > 3.4 \times 10^{4710}, \sigma > 1.4 \times 10^{2355}$ (3,4)-TM: $s > 5.9 \times 10^{4744}, \sigma > 2.2 \times 10^{2372}$ <b>(2,5)-TM:</b> $s > 1.9 \times 10^{704}, \sigma > 1.7 \times 10^{352}$ (2,6)-TM: $s > 4.9 \times 10^{1643}, \sigma > 8.6 \times 10^{821}$ (2,6)-TM: $s > 2.5 \times 10^{9863}, \sigma > 6.9 \times 10^{4931}$
December 2007	T. and S. Ligocki	(6,2)-TM: $s > 2.5 \times 10^{2879}, \sigma > 4.6 \times 10^{1439}$ (4,3)-TM: $s > 7.9 \times 10^{9863}, \sigma > 8.9 \times 10^{4931}$ (4,3)-TM: $s > 5.3 \times 10^{12068}, \sigma > 4.2 \times 10^{6034}$ <b>(3,4)-TM:</b> $s > 5.2 \times 10^{13036}, \sigma > 3.7 \times 10^{6518}$
January 2008	T. and S. Ligocki	<b>(4,3)-TM:</b> $s > 1.0 \times 10^{14072}, \sigma > 1.3 \times 10^{7036}$ <b>(2,6)-TM:</b> $s > 2.4 \times 10^{9866}, \sigma > 1.9 \times 10^{4933}$
May 2010	Kropitz	<b>(6,2)-TM:</b> $s > 3.8 \times 10^{21132}, \sigma > 3.1 \times 10^{10566}$

Table 3: Busy Beaver Competition since 2007



record, were found by two teams: the French one made of Grégory Lafitte and Christophe Papazian, and the father-and-son collaboration of Terry and Shawn Ligocki. Three new machines for the classical busy beaver competition of machines with 6 states and 2 symbols were also found.

With the coming of the web age, researchers have faced two problems: how to announce results, and how to store them. In 1997, Heiner Marxen chose to post them on Google groups, but it seems that the oldest reports are no longer available. From 2004, most results have been announced by sending them by email to several people (for example, the new machines with 6 states and 2 symbols found by Terry and Shawn Ligocki in November and December 2007 were sent by email to six persons: Allen H. Brady, Grégory Lafitte, Heiner Marxen, Pascal Michel, Christophe Papazian and Myron P. Souris). Storing results have been made on web pages (see websites list after the references). Brady has stored results on machines with 3 states and 3 symbols on his own website. Both Marxen and Michel have kept account of all results on their websites. Moreover, Marxen has held simulations, with four variants, of each discovered machine. Michel has held theoretical analyses of many machines.

## 4 Historical survey

### 4.1 Turing machines with 2 states and 2 symbols

- Rado (1963) claimed that  $\Sigma(2, 2) = 4$ , but that  $S(2, 2)$  was yet unknown.
- The value  $S(2, 2) = 6$  was probably set by Lin in 1963. See [http://www.drb.insel.de/~heiner/BB/simTM22\\_bb.html](http://www.drb.insel.de/~heiner/BB/simTM22_bb.html) for a study of the winner by H. Marxen.

1963	Rado, Lin	$S(2, 2) = 6$	$\Sigma(2, 2) = 4$
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The winner and some other good machines:

A0	A1	B0	B1	$s(M)$	$\sigma(M)$
1RB	1LB	1LA	1RH	6	4
1RB	1RH	1LB	1LA	6	3
1RB	0LB	1LA	1RH	6	3

### 4.2 Turing machines with 3 states and 2 symbols

- Soon after the definition of the functions  $S$  and  $\Sigma$ , by Rado (1962), it was conjectured that  $S(3, 2) = 21$ , and  $\Sigma(3, 2) = 6$ .
- Lin (1963) proved this conjecture and this proof was eventually published by Lin and Rado (1965). See studies by Heiner Marxen of the winners for the  $S$  function in [http://www.drb.insel.de/~heiner/BB/simTM32\\_bbS.html](http://www.drb.insel.de/~heiner/BB/simTM32_bbS.html), and for the  $\Sigma$  function in [http://www.drb.insel.de/~heiner/BB/simTM32\\_bb0.html](http://www.drb.insel.de/~heiner/BB/simTM32_bb0.html).

1963	Rado, Lin	$S(3, 2) = 21$	$\Sigma(3, 2) = 6$
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The winners and some other good machines:

A0	A1	B0	B1	C0	C1	$s(M)$	$\sigma(M)$
1RB	1RH	1LB	0RC	1LC	1LA	21	5
1RB	1RH	0LC	0RC	1LC	1LA	20	5
1RB	1LA	0RC	1RH	1LC	0LA	20	5
1RB	1RH	0RC	1RB	1LC	1LA	14	6
1RB	1RC	1LC	1RH	1RA	0LB	13	6
1RB	1LC	1LA	1RB	1LB	1RH	13	6
1RB	1RA	1LC	1RH	1RA	1LB	12	6
1RB	1LC	1RC	1RH	1LA	0LB	11	6

### 4.3 Turing machines with 4 states and 2 symbols

- Brady (1964,1965,1966) found a machine  $M$  such that  $s(M) = 107$  and  $\sigma(M) = 13$  (see study by H. Marxen in [http://www.dr.b.insel.de/~heiner/BB/simTM42\\_bb.html](http://www.dr.b.insel.de/~heiner/BB/simTM42_bb.html)) and conjectured that  $S(4, 2) = 107$  and  $\Sigma(4, 2) = 13$ .
- Brady (1974,1975) proved this conjecture, and the proof was eventually published in Brady (1983).
- Independently, Machlin and Stout (1990) published another proof of the same result, first reported by Kopp (1981) (Kopp is the maiden name of Machlin).

1964	Brady	$s = 107$	$\sigma = 13$
1974	Brady	$S(4, 2) = 107$	$\Sigma(4, 2) = 13$

The winner and some other good machines:

A0	A1	B0	B1	C0	C1	D0	D1	$s(M)$	$\sigma(M)$
1RB	1LB	1LA	0LC	1RH	1LD	1RD	0RA	107	13
1RB	1LD	1LC	0RB	1RA	1LA	1RH	0LC	97	9
1RB	0RC	1LA	1RA	1RH	1RD	1LD	0LB	96	13
1RB	1LB	0LC	0RD	1RH	1LA	1RA	0LA	96	6
1RB	1LD	0LC	0RC	1LC	1LA	1RH	0LA	84	11
1RB	1RH	1LC	0RD	1LA	1LB	0LC	1RD	83	8
1RB	0RD	1LC	0LA	1RA	1LB	1RH	0RC	78	12

### 4.4 Turing machines with 5 states and 2 symbols

- Green (1964) found a machine  $M$  with  $\sigma(M) = 17$ .
- Lynn (1972) found machines  $M$  and  $N$  with  $s(M) = 435$  and  $\sigma(N) = 22$ .

- Lynn, cited by Brady (1983), found in 1974 machines  $M$  and  $N$  with  $s(M) = 7,707$  and  $\sigma(N) = 112$ .
- Uwe Schult, cited by Dewdney (1984a), found, in January 1983, a machine  $M$  with  $s(M) = 134,467$  and  $\sigma(M) = 501$ . This machine was analyzed by Robinson, cited by Dewdney (1984b), and independently by Michel (1993).
- George Uhing, cited by Dewdney (1985a,b), found, in December 1984, a machine  $M$  with  $s(M) = 2,133,492$  and  $\sigma(M) = 1,915$ . This machine was analyzed by Michel (1993).
- George Uhing, cited by Brady (1988), found, in February 1986, a machine  $M$  with  $s(M) = 2,358,064$  (and  $\sigma(M) = 1,471$ ). This machine was analyzed by Michel (1993). Machine 7 in Marxen bb-list, in  
<http://www.drb.insel.de/~heiner/BB/bb-list>  
 can be obtained from Uhing's one, as given by Brady (1988), by the permutation of states (A D B E) (see study by H. Marxen in  
[http://www.drb.insel.de/~heiner/BB/simmbL5\\_7.html](http://www.drb.insel.de/~heiner/BB/simmbL5_7.html)).
- Heiner Marxen and Jürgen Buntrock found, in August 1989, a machine  $M$  with  $s(M) = 11,798,826$  and  $\sigma(M) = 4,098$ . This machine was cited by Marxen and Buntrock (1990), and by Machlin and Stout (1990), and was analyzed by Michel (1993). See study by H. Marxen in  
[http://www.drb.insel.de/~heiner/BB/simmbL5\\_2.html](http://www.drb.insel.de/~heiner/BB/simmbL5_2.html).
- Heiner Marxen and Jürgen Buntrock found, in September 1989, a machine  $M$  with  $s(M) = 23,554,764$  (and  $\sigma(M) = 4,097$ ). This machine was cited by Machlin and Stout (1990), and was analyzed by Michel (1993). See study by H. Marxen in  
[http://www.drb.insel.de/~heiner/BB/simmbL5\\_3.html](http://www.drb.insel.de/~heiner/BB/simmbL5_3.html)  
 and analysis by P. Michel in Section 5.2.2.
- Heiner Marxen and Jürgen Buntrock found, in September 1989, a machine  $M$  with  $s(M) = 47,176,870$  and  $\sigma(M) = 4,098$ . This machine was cited by Marxen and Buntrock (1990), and was analyzed by Michel (1993). See study by H. Marxen in  
[http://www.drb.insel.de/~heiner/BB/simmbL5\\_1.html](http://www.drb.insel.de/~heiner/BB/simmbL5_1.html)  
 and analysis by P. Michel in Section 5.2.1. It is the current record holder.
- Marxen gives a list of machines  $M$  with high values of  $s(M)$  and  $\sigma(M)$  in  
<http://www.drb.insel.de/~heiner/BB/bb-list>.
- The study of Turing machines with 5 states and 2 symbols is still going on. Marxen and Buntrock (1990), Skelet, and Hertel (2009) created programs to detect never halting machines, and manually proved that some machines, undetected by their programs, never halt. In each case, about a hundred holdouts were resisting computer and manual analyses. See Skelet's study in  
<http://skelet.ludost.net/bb/index.html>

- Norbert B3tfaei, allowing transitions where the head can stand still, found, in August 2009, a machine  $M$  with  $s(M) = 70,740,810$  and  $\sigma(M) = 4098$ . Note that this machine does not follow the current rules of the busy beaver competition. See B3tfaei's study in <http://arxiv.org/abs/0908.4013>

1964	Green		$\sigma = 17$
1972	Lynn	$s = 435$	$\sigma = 22$
1974	Lynn	$s = 7,707$	$\sigma = 112$
January 1983	Schult	$s = 134,467$	$\sigma = 501$
December 1984	Uhing	$s = 2,133,492$	$\sigma = 1,915$
February 1986	Uhing	$s = 2,358,064$	
February 1990	Marxen, Buntrock	$s = 47,176,870$	$\sigma = 4,098$

The record holder and some other good machines:

A0	A1	B0	B1	C0	C1	D0	D1	E0	E1	$s(M)$	$\sigma(M)$
1RB	1LC	1RC	1RB	1RD	0LE	1LA	1LD	1RH	0LA	47,176,870	4098
1RB	0LD	1LC	1RD	1LA	1LC	1RH	1RE	1RA	0RB	23,554,764	4097
1RB	1RA	0LC	0RC	1RH	1RD	1LE	0LA	1LA	1LE	11,821,190	4096*
1RB	1RA	1LC	0RD	1LA	1LC	1RH	1RE	1LC	0LA	11,815,076	4096*
1RB	1RA	0LC	0RC	1RH	1RD	1LE	1RB	1LA	1LE	11,811,010	4096*
1RB	1RA	1LC	0RD	1LA	1LC	1RH	1RE	0LE	1RB	11,804,910	4096
1RB	1RA	1LC	0RD	1LA	1LC	1RH	1RE	1LC	1RB	11,804,896	4096
1RB	1RA	1LC	1LB	1RA	1LD	1RA	1LE	1RH	0LC	11,798,826	4098
1RB	1RA	1LC	1RD	1LA	1LC	1RH	0RE	1LC	1RB	11,798,796	4097
1RB	1RA	1LC	1RD	1LA	1LC	1RH	1RE	0LE	0RB	11,792,724	4097*
1RB	1RA	1LC	1RD	1LA	1LC	1RH	1RE	1RA	0RB	11,792,682	4097*
1RB	1RH	1LC	1RC	0RE	0LD	1LC	0LB	1RD	1RA	2,358,064	1471
1RB	1LC	0LA	0LD	1LA	1RH	1LB	1RE	0RD	0RB	2,133,492	1915
1RB	0LC	1RC	1RD	1LA	0RB	0RE	1RH	1LC	1RA	134,467	501

(Among the first eleven machines, the five starred ones are from B3tfaei. The six machines without star were discovered by Marxen and Buntrock, the next two ones were by Uhing, and the last one was by Schult. Heiner Marxen says there are no other  $\sigma$  values within the  $\sigma$  range above).

#### 4.5 Turing machines with 6 states and 2 symbols

- Green (1964) found a machine  $M$  with  $\sigma(M) = 35$ .
- Lynn (1972) found a machine  $M$  with  $s(M) = 522$  and  $\sigma(M) = 42$ .
- Brady (1983) found machines  $M$  and  $N$  with  $s(M) = 13,488$  and  $\sigma(N) = 117$ .
- Uwe Schult, cited by Dewdney (1984a), found a machine  $M$  with  $\sigma(M) = 2,075$ .
- Heiner Marxen and J3rger Buntrock found, in January 1990, a machine  $M$  with  $s(M) = 13,122,572,797$  and  $\sigma(M) = 136,612$ . This machine was cited by Marxen and Buntrock (1990). See study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simmbL6\\_1.html](http://www.drb.insel.de/~heiner/BB/simmbL6_1.html).

- Heiner Marxen and Jürgen Buntrock found, in January 1990, a machine  $M$  with  $s(M) = 8,690,333,381,690,951$  and  $\sigma(M) = 95,524,079$ . This machine was posted on the web (Google groups) on September 3, 1997. See machine 2 in Marxen's bb-list in  
<http://www.drb.insel.de/~heiner/BB/bb-list>.  
 See study by H. Marxen in  
[http://www.drb.insel.de/~heiner/BB/simmbL6\\_2.html](http://www.drb.insel.de/~heiner/BB/simmbL6_2.html).  
 See analysis by R. Munafo in his website:  
["http://mrob.com/pub/math/ln-notes1-4.html#mb-bb-1,](http://mrob.com/pub/math/ln-notes1-4.html#mb-bb-1)  
 and in Section 5.3.7.
- Heiner Marxen and Jürgen Buntrock found, in July 2000, a machine  $M$  with  $s(M) > 5.3 \times 10^{42}$  and  $\sigma(M) > 2.5 \times 10^{21}$ . This machine was posted on the web (Google groups) on August 5, 2000. See machine 3 in Marxen's bb-list:  
<http://www.drb.insel.de/~heiner/BB/bb-list>,  
 and machine k in Marxen's bb-6list:  
<http://www.drb.insel.de/~heiner/BB/bb-6list>.  
 See study by H. Marxen in:  
[http://www.drb.insel.de/~heiner/BB/simmbL6\\_3.html](http://www.drb.insel.de/~heiner/BB/simmbL6_3.html).
- Heiner Marxen and Jürgen Buntrock found, in August 2000, a machine  $M$  with  $s(M) > 6.1 \times 10^{119}$  and  $\sigma(M) > 1.4 \times 10^{60}$ . This machine was posted on the web (Google groups) on October 23, 2000. See machine o in Marxen's bb-6list in:  
<http://www.drb.insel.de/~heiner/BB/bb-6list>.  
 See study by H. Marxen in:  
[http://www.drb.insel.de/~heiner/BB/simmbL6\\_o.html](http://www.drb.insel.de/~heiner/BB/simmbL6_o.html).  
 See analysis by P. Michel in Section 5.3.6.
- Heiner Marxen and Jürgen Buntrock found, in August 2000, a machine  $M$  with  $s(M) > 6.1 \times 10^{925}$  and  $\sigma(M) > 6.4 \times 10^{462}$ . This machine was posted on the web (Google groups) on October 23, 2000. See machine q in Marxen's bb-6list in  
<http://www.drb.insel.de/~heiner/BB/bb-6list>.  
 See study by Marxen in  
[http://www.drb.insel.de/~heiner/BB/simmbL6\\_q.html](http://www.drb.insel.de/~heiner/BB/simmbL6_q.html).  
 See analyses by R. Munafo, the short one in  
[http://mrob.com/pub/math/ln-notes1-4.html#mb6q,](http://mrob.com/pub/math/ln-notes1-4.html#mb6q)  
 or the long one in  
[http://mrob.com/pub/math/ln-mb6q.html,](http://mrob.com/pub/math/ln-mb6q.html)  
 and see analysis by P. Michel in Section 5.3.5.

- Heiner Marxen and Jürgen Buntrock found, in February 2001, a machine  $M$  with  $s(M) > 3.0 \times 10^{1730}$  and  $\sigma(M) > 1.2 \times 10^{865}$ . This machine was posted on the web (Google groups) on March 5, 2001. See machine r in Marxen's bb-6list in <http://www.drb.insel.de/~heiner/BB/bb-6list>.  
See study by Marxen in [http://www.drb.insel.de/~heiner/BB/simmbL6\\_r.html](http://www.drb.insel.de/~heiner/BB/simmbL6_r.html).  
See analysis by P. Michel in Section 5.3.4.
- Terry and Shawn Ligocki found, in November 2007, a machine  $M$  with  $s(M) > 8.9 \times 10^{1762}$  and  $\sigma(M) > 2.5 \times 10^{881}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig62\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig62_a.html)).  
See analysis by P. Michel in Section 5.3.3.
- Terry and Shawn Ligocki found, in December 2007, a machine  $M$  with  $s(M) > 2.5 \times 10^{2879}$  and  $\sigma(M) > 4.6 \times 10^{1439}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig62\\_b.html](http://www.drb.insel.de/~heiner/BB/simLig62_b.html)).  
See analysis by P. Michel in Section 5.3.2.
- Pavel Kropitz found, in May 2010, a machine  $M$  with  $s(M) > 3.8 \times 10^{21132}$  and  $\sigma(M) > 3.1 \times 10^{10566}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simKro62\\_a.html](http://www.drb.insel.de/~heiner/BB/simKro62_a.html)).  
See analysis in Section 5.3.1. It is the current record holder.
- Marxen gives a list of machines  $M$  with high values of  $s(M)$  and  $\sigma(M)$  in <http://www.drb.insel.de/~heiner/BB/bb-6list>.

1964	Green		$\sigma = 35$
1972	Lynn	$s = 522$	$\sigma = 42$
1983	Brady	$s = 13,488$	$\sigma = 117$
January 1983	Schult		$\sigma = 2,075$
February 1990	Marxen, Buntrock	$s = 13,122,572,797$	$\sigma = 136,612$
September 1997	Marxen, Buntrock	$s = 8,690,333,381,690,951$	$\sigma = 95,524,079$
August 2000	Marxen, Buntrock	$s > 5.3 \times 10^{42}$	$\sigma > 2.5 \times 10^{21}$
October 2000	Marxen, Buntrock	$s > 6.1 \times 10^{925}$	$\sigma > 6.4 \times 10^{462}$
March 2001	Marxen, Buntrock	$s > 3.0 \times 10^{1730}$	$\sigma > 1.2 \times 10^{865}$
November 2007	T. and S. Ligocki	$s > 8.9 \times 10^{1762}$	$\sigma > 2.5 \times 10^{881}$
December 2007	T. and S. Ligocki	$s > 2.5 \times 10^{2879}$	$\sigma > 4.6 \times 10^{1439}$
May 2010	Kropitz	$s > 3.8 \times 10^{21132}$	$\sigma > 3.1 \times 10^{10566}$

The record holder and some other good machines:

A0	A1	B0	B1	C0	C1	D0	D1	E0	E1	F0	F1	$s(M) >$	$\sigma(M) >$
1RB	0LD	1RC	0RF	1LC	1LA	0LE	1RH	1LA	0RB	0RC	0RE	$3.8 \times 10^{21132}$	$3.1 \times 10^{10566}$
1RB	0LE	1LC	0RA	1LD	0RC	1LE	0LF	1LA	1LC	1LE	1RH	$2.5 \times 10^{2879}$	$4.6 \times 10^{1439}$
1RB	0RF	0LB	1LC	1LD	0RC	1LE	1RH	1LF	0LD	1RA	0LE	$8.9 \times 10^{1762}$	$2.5 \times 10^{881}$
1RB	0LF	0RC	0RD	1LD	1RE	0LE	0LD	0RA	1RC	1LA	1RH	$3.0 \times 10^{1730}$	$1.2 \times 10^{865}$
1RB	0LB	0RC	1LB	1RD	0LA	1LE	1LF	1LA	0LD	1RH	1LE	$6.1 \times 10^{925}$	$6.4 \times 10^{462}$
1RB	0LC	1LA	1RC	1RA	0LD	1LE	1LC	1RF	1RH	1RA	1RE	$6.1 \times 10^{119}$	$1.4 \times 10^{60}$
1RB	0LB	1LC	0RE	1RE	0LD	1LA	1LA	0RA	0RF	1RE	1RH	$5.5 \times 10^{99}$	$6.9 \times 10^{49}$
1RB	0LC	1LA	1LD	1RD	0RC	0LB	0RE	1RC	1LF	1LE	1RH	$3.2 \times 10^{98}$	$1.1 \times 10^{49}$
1RB	0LC	1LA	1RD	1RA	0LE	1RA	0RB	1LF	1LC	1RD	1RH	$2.0 \times 10^{95}$	$6.7 \times 10^{47}$
1RB	0LC	1LA	1RD	0LB	0LE	1RA	0RB	1LF	1LC	1RD	1RH	$2.0 \times 10^{95}$	$6.7 \times 10^{47}$
1RB	0RC	0LA	0RD	1RD	1RH	1LE	0LD	1RF	1LB	1RA	1RE	$5.3 \times 10^{42}$	$2.5 \times 10^{21}$

## 4.6 Turing machines with 2 states and 3 symbols

- Brady (1988) found a machine  $M$  with  $s(M) = 38$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simTM23\\_cb.html](http://www.drb.insel.de/~heiner/BB/simTM23_cb.html)).
- This machine was found independently by Michel (2004), who gave  $\sigma(M) = 9$  and conjectured that  $S(2, 3) = 38$  and  $\Sigma(2, 3) = 9$ .
- Lafitte and Papazian (2007) proved this conjecture. T. and S. Ligocki (unpublished) proved this conjecture, independently.

1988	Brady	$s = 38$	$\sigma = 9$
2007	Lafitte, Papazian	$S(2, 3) = 38$	$\Sigma(2, 3) = 9$

The record holder and some other good machines:

A0	A1	A2	B0	B1	B2	$s(M)$	$\sigma(M)$
1RB	2LB	1RH	2LA	2RB	1LB	38	9
1RB	0LB	1RH	2LA	1RB	1RA	29	8
1RB	2LA	1RH	1LB	1LA	0RA	26	6
1RB	1LA	1LB	0LA	2RA	1RH	26	6
1RB	1LB	1RH	2LA	2RB	1LB	24	7

## 4.7 Turing machines with 3 states and 3 symbols

- Michel (2004) found machines  $M$  and  $N$  with  $s(M) = 40,737$  and  $\sigma(N) = 208$ .
  - Brady found, in November 2004, a machine  $M$  with  $s(M) = 29,403,894$  and  $\sigma(M) = 5600$  (see <http://www.cs.unr.edu/~al/BusyBeaver.html>) (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simAB3Y\\_b.html](http://www.drb.insel.de/~heiner/BB/simAB3Y_b.html)).
  - Brady found, in December 2004, a machine  $M$  with  $s(M) = 92,649,163$  and  $\sigma(M) = 13,949$  (see <http://www.cs.unr.edu/~al/BusyBeaver.html>) (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simAB3Y\\_c.html](http://www.drb.insel.de/~heiner/BB/simAB3Y_c.html)).
- See analysis by P. Michel in Section 5.4.8.

- Myron P. Souris found, in July 2005 (M.P. Souris said: actually in 1995, but then no one seemed to care), machines  $M$  and  $N$  with  $s(M) = 544,884,219$  and  $\sigma(N) = 36,089$  (see study of  $M$  by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simMS33\\_b.html](http://www.drb.insel.de/~heiner/BB/simMS33_b.html) and study of  $N$  by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simMS33\\_a.html](http://www.drb.insel.de/~heiner/BB/simMS33_a.html)). See analysis of  $M$  by P. Michel in Section 5.4.6, and analysis of  $N$  by P. Michel in Section 5.4.7.
- Grégory Lafitte and Christophe Papazian found, in August 2005, a machine  $M$  with  $s(M) = 4,939,345,068$  and  $\sigma(M) = 107,900$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf33\\_b.html](http://www.drb.insel.de/~heiner/BB/simLaf33_b.html)). See analysis by P. Michel in Section 5.4.5.
- Grégory Lafitte and Christophe Papazian found, in September 2005, a machine  $M$  with  $s(M) = 987,522,842,126$  and  $\sigma(M) = 1,525,688$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf33\\_d.html](http://www.drb.insel.de/~heiner/BB/simLaf33_d.html)). See analysis by P. Michel in Section 5.4.4.
- Grégory Lafitte and Christophe Papazian found, in April 2006, a machine  $M$  with  $s(M) = 4,144,465,135,614$  and  $\sigma(M) = 2,950,149$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf33\\_e.html](http://www.drb.insel.de/~heiner/BB/simLaf33_e.html)). See analysis by P. Michel in Section 5.4.3.
- Terry and Shawn Ligoeki found, in August 2006, a machine  $M$  with  $s(M) = 4,345,166,620,336,565$  and  $\sigma(M) = 95,524,079$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig33\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig33_a.html)). See analysis in Section 5.4.2.
- Terry and Shawn Ligoeki found, in November 2007, a machine  $M$  with  $s(M) = 119,112,334,170,342,540$  and  $\sigma(M) = 374,676,383$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig33\\_b.html](http://www.drb.insel.de/~heiner/BB/simLig33_b.html)). See analysis by P. Michel in Section 5.4.1. It is the current record holder.
- Brady gives a list of machines with high values of  $s(M)$  in <http://www.cs.unr.edu/~al/BusyBeaver.html>.



October 2004	Michel	$s = 40,737$	$\sigma = 208$
November 2004	Brady	$s = 29,403,894$	$\sigma = 5,600$
December 2004	Brady	$s = 92,649,163$	$\sigma = 13,949$
July 2005	Souris	$s = 544,884,219$	$\sigma = 36,089$
August 2005	Lafitte, Papazian	$s = 4,939,345,068$	$\sigma = 107,900$
September 2005	Lafitte, Papazian	$s = 987,522,842,126$	$\sigma = 1,525,688$
April 2006	Lafitte, Papazian	$s = 4,144,465,135,614$	$\sigma = 2,950,149$
August 2006	T. and S. Ligocki	$s = 4,345,166,620,336,565$	$\sigma = 95,524,079$
November 2007	T. and S. Ligocki	$s = 119,112,334,170,342,540$	$\sigma = 374,676,383$

The record holder and some other good machines:

A0	A1	A2	B0	B1	B2	C0	C1	C2	$s(M)$	$\sigma(M)$
1RB	2LA	1LC	0LA	2RB	1LB	1RH	1RA	1RC	119,112,334,170,342,540	374,676,383
1RB	2RC	1LA	2LA	1RB	1RH	2RB	2RA	1LC	4,345,166,620,336,565	95,524,079
1RB	1RH	2LC	1LC	2RB	1LB	1LA	2RC	2LA	4,144,465,135,614	2,950,149
1RB	2LA	1RA	1RC	2RB	0RC	1LA	1RH	1LA	987,522,842,126	1,525,688
1RB	1RH	2RB	1LC	0LB	1RA	1RA	2LC	1RC	4,939,345,068	107,900
1RB	2LA	1RA	1LB	1LA	2RC	1RH	1LC	2RB	1,808,669,066	43,925
1RB	2LA	1RA	1LC	1LA	2RC	1RH	1LA	2RB	1,808,669,046	43,925
1RB	1LB	2LA	1LA	1RC	1RH	0LA	2RC	1LC	544,884,219	32,213
1RB	0LA	1LA	2RC	1RC	1RH	2LC	1RA	0RC	408,114,977	20,240
1RB	2RA	2RC	1LC	1RH	1LA	1RA	2LB	1LC	310,341,163	36,089
1RB	1RH	2LC	1LC	2RB	1LB	1LA	0RB	2LA	92,649,163	13,949
1RB	2LA	1LA	2LA	1RC	2RB	1RH	0LC	0RA	51,525,774	7,205
1RB	2RA	1LA	2LA	2LB	2RC	1RH	2RB	1RB	47,287,015	12,290
1RB	2RA	1LA	2LC	0RC	1RB	1RH	2LA	1RB	29,403,894	5,600

(The first two machines were discovered by Terry and Shawn Ligocki, the next five ones were by Lafitte and Papazian, the next three ones were by Souris, and the last four ones were by Brady).

#### 4.8 Turing machines with 4 states and 3 symbols

- Terry and Shawn Ligocki found, in April 2005, a machine  $M$  with  $s(M) = 250,096,776$  and  $\sigma(M) = 15,008$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig43_a.html)).
- This machine was superseded by the machines with 3 states and 3 symbols found in July 2005 by Myron P. Souris.
- Terry and Shawn Ligocki found, in October 2007, a machine  $M$  with  $s(M) > 1.5 \times 10^{1426}$  and  $\sigma(M) > 1.1 \times 10^{713}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_b.html](http://www.drb.insel.de/~heiner/BB/simLig43_b.html)).
- Terry and Shawn Ligocki found successively, in November 2007, machines  $M$  with
  - $s(M) > 7.7 \times 10^{1618}$  and  $\sigma(M) > 1.6 \times 10^{809}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_c.html](http://www.drb.insel.de/~heiner/BB/simLig43_c.html)),
  - $s(M) > 3.7 \times 10^{1973}$  and  $\sigma(M) > 8.0 \times 10^{986}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_d.html](http://www.drb.insel.de/~heiner/BB/simLig43_d.html)),

- $s(M) > 3.9 \times 10^{7721}$  and  $\sigma(M) > 4.0 \times 10^{3860}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_e.html](http://www.drb.insel.de/~heiner/BB/simLig43_e.html)),
- $s(M) > 3.9 \times 10^{9122}$  and  $\sigma(M) > 2.5 \times 10^{4561}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_f.html](http://www.drb.insel.de/~heiner/BB/simLig43_f.html)).
- Terry and Shawn Ligocki found successively, in December 2007, machines  $M$  with
  - $s(M) > 7.9 \times 10^{9863}$  and  $\sigma(M) > 8.9 \times 10^{4931}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_g.html](http://www.drb.insel.de/~heiner/BB/simLig43_g.html)),
  - $s(M) > 5.3 \times 10^{12068}$  and  $\sigma(M) > 4.2 \times 10^{6034}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_h.html](http://www.drb.insel.de/~heiner/BB/simLig43_h.html)).
- Terry and Shawn Ligocki found, in January 2008, a machine  $M$  with  $s(M) > 1.0 \times 10^{14072}$  and  $\sigma(M) > 1.3 \times 10^{7036}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig43\\_i.html](http://www.drb.insel.de/~heiner/BB/simLig43_i.html)).

It is the current record holder.

April 2005	T. and S. Ligocki	$s = 250,096,776$	$\sigma = 15,008$
July 2005	Souris	superseded by a (3,3)-TM	
October 2007	T. and S. Ligocki	$s > 1.5 \times 10^{1426}$	$\sigma > 1.1 \times 10^{713}$
November 2007	T. and S. Ligocki	$s > 7.7 \times 10^{1618}$	$\sigma > 1.6 \times 10^{809}$
		$s > 3.7 \times 10^{1973}$	$\sigma > 8.0 \times 10^{986}$
		$s > 3.9 \times 10^{7721}$	$\sigma > 4.0 \times 10^{3860}$
		$s > 3.9 \times 10^{9122}$	$\sigma > 2.5 \times 10^{4561}$
December 2007	T. and S. Ligocki	$s > 7.9 \times 10^{9863}$	$\sigma > 8.9 \times 10^{4931}$
		$s > 5.3 \times 10^{12068}$	$\sigma > 4.2 \times 10^{6034}$
January 2008	T. and S. Ligocki	$s > 1.0 \times 10^{14072}$	$\sigma > 1.3 \times 10^{7036}$

The record holder and the past record holders:

A0	A1	A2	B0	B1	B2	C0	C1	C2	D0	D1	D2	$s(M)$	$\sigma(M)$
1RB	1RH	2RC	2LC	2RD	0LC	1RA	2RB	0LB	1LB	0LD	2RC	$> 1.0 \times 10^{14072}$	$> 1.3 \times 10^{7036}$
1RB	0LB	1RD	2RC	2LA	0LA	1LB	0LA	0LA	1RA	0RA	1RH	$> 5.3 \times 10^{12068}$	$> 4.2 \times 10^{6034}$
1RB	1LD	1RH	1RC	2LB	2LD	1LC	2RA	0RD	1RC	1LA	0LA	$> 7.9 \times 10^{9863}$	$> 8.9 \times 10^{4931}$
1RB	2LD	1RH	2LC	2RC	2RB	1LD	0RC	1RC	2LA	2LD	0LB	$> 3.9 \times 10^{9122}$	$> 2.5 \times 10^{4561}$
1RB	1LA	1RD	2LC	0RA	1LB	2LA	0LB	0RD	2RC	1RH	0LC	$> 3.9 \times 10^{7721}$	$> 4.0 \times 10^{3860}$
1RB	1RA	0LB	2LC	1LB	1RC	0RD	2LC	1RA	2RA	1RH	1RC	$> 3.7 \times 10^{1973}$	$> 8.0 \times 10^{986}$
1RB	2RC	1RA	2LC	1LA	1LB	2LD	0LB	0RC	0RD	1RH	0RA	$> 7.7 \times 10^{1618}$	$> 1.6 \times 10^{809}$
1RB	0LC	1RH	2LC	1RD	0LB	2LA	1LC	1LA	1RB	2LD	2RA	$> 1.5 \times 10^{1426}$	$> 1.1 \times 10^{713}$
0RB	1LD	1RH	1LA	1RC	1RD	1RB	2LC	1RC	1LA	1LC	2RB	250,096,776	15,008

## 4.9 Turing machines with 2 states and 4 symbols

- Brady (1988) found a machine  $M$  with  $s(M) = 7,195$ .
- This machine was found independently and analyzed by Michel (2004), who gave  $\sigma(M) = 90$ . See study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simTM24\\_b.html](http://www.drb.insel.de/~heiner/BB/simTM24_b.html).  
See analysis by P. Michel in Section 5.5.2.

- Terry and Shawn Ligocki found, in February 2005, a machine  $M$  with  $s(M) = 3,932,964$  and  $\sigma(M) = 2,050$ . See study by H. Marxen in

[http://www.drb.insel.de/~heiner/BB/simLig24\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig24_a.html).

See analysis by P. Michel in Section 5.5.1.

It is the current record holder. There is no machine between the first two ones (Ligocki, Brady). There is no machine such that  $3,932,964 < s(M) < 200,000,000$  (Ligocki, September 2005).

1988	Brady	$s = 7,195$	$\sigma = 90$
February 2005	T. and S. Ligocki	$s = 3,932,964$	$\sigma = 2,050$

The record holder and some other good machines:

A0	A1	A2	A3	B0	B1	B2	B3	$s(M)$	$\sigma(M)$
1RB	2LA	1RA	1RA	1LB	1LA	3RB	1RH	3,932,964	2,050
1RB	3LA	1LA	1RA	2LA	1RH	3RA	3RB	7,195	90
1RB	3LA	1LA	1RA	2LA	1RH	3LA	3RB	6,445	84
1RB	3LA	1LA	1RA	2LA	1RH	2RA	3RB	6,445	84
1RB	2RB	3LA	2RA	1LA	3RB	1RH	1LB	2,351	60

#### 4.10 Turing machines with 3 states and 4 symbols

- Terry and Shawn Ligocki found, in April 2005, a machine  $M$  with  $s(M) = 262,759,288$  and  $\sigma(M) = 17,323$  (see study by H. Marxen in

[http://www.drb.insel.de/~heiner/BB/simLig34\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig34_a.html)).

- This machine was superseded by the machines with 3 states and 3 symbols found in July 2005 by Myron P. Souris.

- Terry and Shawn Ligocki found, in September 2007, a machine  $M$  with  $s(M) > 5.7 \times 10^{52}$  and  $\sigma(M) > 2.4 \times 10^{26}$  (see study by H. Marxen in

[http://www.drb.insel.de/~heiner/BB/simLig34\\_b.html](http://www.drb.insel.de/~heiner/BB/simLig34_b.html)),

- Terry and Shawn Ligocki found successively, in October 2007, machines  $M$  with

- $s(M) > 4.3 \times 10^{281}$  and  $\sigma(M) > 6.0 \times 10^{140}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_c.html](http://www.drb.insel.de/~heiner/BB/simLig34_c.html)),

- $s(M) > 7.6 \times 10^{868}$  and  $\sigma(M) > 4.6 \times 10^{434}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_d.html](http://www.drb.insel.de/~heiner/BB/simLig34_d.html)),

- $s(M) > 3.1 \times 10^{1256}$  and  $\sigma(M) > 2.1 \times 10^{628}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_e.html](http://www.drb.insel.de/~heiner/BB/simLig34_e.html)).

- Terry and Shawn Ligocki found successively, in November 2007, machines  $M$  with

- $s(M) > 8.4 \times 10^{2601}$  and  $\sigma(M) > 1.7 \times 10^{1301}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_f.html](http://www.drb.insel.de/~heiner/BB/simLig34_f.html)),

- $s(M) > 3.4 \times 10^{4710}$  and  $\sigma(M) > 1.4 \times 10^{2355}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_g.html](http://www.drb.insel.de/~heiner/BB/simLig34_g.html)),
- $s(M) > 5.9 \times 10^{4744}$  and  $\sigma(M) > 2.2 \times 10^{2372}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_h.html](http://www.drb.insel.de/~heiner/BB/simLig34_h.html)).

- Terry and Shawn Ligocki found, in December 2007, a machine  $M$  with  $s(M) > 5.2 \times 10^{13036}$  and  $\sigma(M) > 3.7 \times 10^{6518}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig34\\_i.html](http://www.drb.insel.de/~heiner/BB/simLig34_i.html)).

It is the current record holder.

April 2005	T. and S. Ligocki	$s = 262,759,288$	$\sigma = 17,323$
July 2005	Souris	superseded by a (3,3)-TM	
September 2007	T. and S. Ligocki	$s > 5.7 \times 10^{52}$	$\sigma > 2.4 \times 10^{26}$
October 2007	T. and S. Ligocki	$s > 4.3 \times 10^{281}$	$\sigma > 6.0 \times 10^{140}$
		$s > 7.6 \times 10^{868}$	$\sigma > 4.6 \times 10^{434}$
		$s > 3.1 \times 10^{1256}$	$\sigma > 2.1 \times 10^{628}$
November 2007	T. and S. Ligocki	$s > 8.4 \times 10^{2601}$	$\sigma > 1.7 \times 10^{1301}$
		$s > 3.4 \times 10^{4710}$	$\sigma > 1.4 \times 10^{2355}$
		$s > 5.9 \times 10^{4744}$	$\sigma > 2.2 \times 10^{2372}$
December 2007	T. and S. Ligocki	$s > 5.2 \times 10^{13036}$	$\sigma > 3.7 \times 10^{6518}$

The record holder and the past record holders:

A0	A1	A2	A3	B0	B1	B2	B3	C0	C1	C2	C3	$s(M)$	$\sigma(M)$
1RB	1RA	2LB	3LA	2LA	0LB	1LC	1LB	3RB	3RC	1RH	1LC	$> 5.2 \times 10^{13036}$	$> 3.7 \times 10^{6518}$
1RB	1RA	1LB	1RC	2LA	0LB	3LC	1RH	1LB	0RC	2RA	2RC	$> 5.9 \times 10^{4744}$	$> 2.2 \times 10^{2372}$
1RB	2LB	2RA	1LA	2LA	1RC	0LB	2RA	1RB	3LC	1LA	1RH	$> 3.4 \times 10^{4710}$	$> 1.4 \times 10^{2355}$
1RB	1LA	3LA	3RC	2LC	2LB	1RB	1RA	2LA	3LC	1RH	1LB	$> 8.4 \times 10^{2601}$	$> 1.7 \times 10^{1301}$
1RB	3LA	3RC	1RA	2RC	1LA	1RH	2RB	1LC	1RB	1LB	2RA	$> 3.1 \times 10^{1256}$	$> 2.1 \times 10^{628}$
1RB	0RB	3LC	1RC	0RC	1RH	2RC	3RC	1LB	2LA	3LA	2RB	$> 7.6 \times 10^{868}$	$> 4.6 \times 10^{434}$
1RB	3RB	2LC	3LA	0RC	1RH	2RC	1LB	1LB	2LA	3RC	2LC	$> 4.3 \times 10^{281}$	$> 6.0 \times 10^{140}$
1RB	1LA	1LB	1RA	0LA	2RB	2LC	1RH	3RB	2LB	1RC	0RC	$> 5.7 \times 10^{52}$	$> 2.4 \times 10^{26}$
1RB	3LC	0RA	0LC	2LC	3RC	0RC	1LB	1RA	0LB	0RB	1RH	262,759,288	17,323

#### 4.11 Turing machines with 2 states and 5 symbols

- Terry and Shawn Ligocki found, in February 2005, machines  $M$  and  $N$  with  $s(M) = 16,268,767$  and  $\sigma(N) = 4,099$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig25\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig25_a.html)).
- Terry and Shawn Ligocki found, in April 2005, a machine  $M$  with  $s(M) = 148,304,214$  and  $\sigma(M) = 11,120$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig25\\_c.html](http://www.drb.insel.de/~heiner/BB/simLig25_c.html)).
- Grégory Lafitte and Christophe Papazian found, in August 2005, a machine  $M$  with  $s(M) = 8,619,024,596$  and  $\sigma(M) = 90,604$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_a.html](http://www.drb.insel.de/~heiner/BB/simLaf25_a.html)).

- Grégory Lafitte and Christophe Papazian found, in September 2005, a machine  $M$  with  $\sigma(M) = 97, 104$  (and  $s(M) = 7, 543, 673, 517$ ) (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_c.html](http://www.drb.insel.de/~heiner/BB/simLaf25_c.html)).
- Grégory Lafitte and Christophe Papazian found, in October 2005, a machine  $M$  with  $s(M) = 233, 431, 192, 481$  and  $\sigma(M) = 458, 357$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_d.html](http://www.drb.insel.de/~heiner/BB/simLaf25_d.html)).
- Grégory Lafitte and Christophe Papazian found, in October 2005, a machine  $M$  with  $s(M) = 912, 594, 733, 606$  and  $\sigma(M) = 1, 957, 771$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_f.html](http://www.drb.insel.de/~heiner/BB/simLaf25_f.html)).  
See analysis by P. Michel in Section 5.6.6.
- Grégory Lafitte and Christophe Papazian found, in December 2005, a machine  $M$  with  $s(M) = 924, 180, 005, 181$  (and  $\sigma(M) = 1, 137, 477$ ) (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_g.html](http://www.drb.insel.de/~heiner/BB/simLaf25_g.html)).  
See analysis by P. Michel in Section 5.6.5.
- Grégory Lafitte and Christophe Papazian found, in May 2006, a machine  $M$  with  $s(M) = 3, 793, 261, 759, 791$  and  $\sigma(M) = 2, 576, 467$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_h.html](http://www.drb.insel.de/~heiner/BB/simLaf25_h.html)).  
See analysis by P. Michel in Section 5.6.4.
- Grégory Lafitte and Christophe Papazian found, in June 2006, a machine  $M$  with  $s(M) = 14, 103, 258, 269, 249$  and  $\sigma(M) = 4, 848, 239$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_i.html](http://www.drb.insel.de/~heiner/BB/simLaf25_i.html)).  
See analysis by P. Michel in Section 5.6.3.
- Grégory Lafitte and Christophe Papazian found, in July 2006, a machine  $M$  with  $s(M) = 26, 375, 397, 569, 930$  (and  $\sigma(M) = 143$ ) (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLaf25\\_j.html](http://www.drb.insel.de/~heiner/BB/simLaf25_j.html)).  
See comments in Section 5.8.
- Terry and Shawn Ligocki found, in August 2006, a machine  $M$  with
 
$$s(M) = 7, 069, 449, 877, 176, 007, 352, 687$$
 and  $\sigma(M) = 172, 312, 766, 455$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig25\\_j.html](http://www.drb.insel.de/~heiner/BB/simLig25_j.html)).  
See analysis in Section 5.6.2.
- Terry and Shawn Ligocki found, in October 2007, a machine  $M$  with  $s(M) > 5.2 \times 10^{61}$  and  $\sigma(M) > 9.3 \times 10^{30}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig25\\_k.html](http://www.drb.insel.de/~heiner/BB/simLig25_k.html)).

- Terry and Shawn Ligocki found, in October 2007, two machines  $M$  and  $N$  with  $s(M) = s(N) > 1.6 \times 10^{211}$  and  $\sigma(M) = \sigma(N) > 5.2 \times 10^{105}$  (see study by H. Marxen of  $M$  in [http://www.drb.insel.de/~heiner/BB/simLig25\\_1.html](http://www.drb.insel.de/~heiner/BB/simLig25_1.html), and study by H. Marxen of  $N$  in [http://www.drb.insel.de/~heiner/BB/simLig25\\_m.html](http://www.drb.insel.de/~heiner/BB/simLig25_m.html)).
- Terry and Shawn Ligocki found, in November 2007, a machine  $M$  with  $s(M) > 1.9 \times 10^{704}$  and  $\sigma(M) > 1.7 \times 10^{352}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig25\\_n.html](http://www.drb.insel.de/~heiner/BB/simLig25_n.html)). See analysis by P. Michel in Section 5.6.1. It is the current record holder.

February 2005	T. and S. Ligocki	$s = 16,268,767$	$\sigma = 4,099$
April 2005	T. and S. Ligocki	$s = 148,304,214$	$\sigma = 11,120$
August 2005	Lafitte, Papazian	$s = 8,619,024,596$	$\sigma = 90,604$
September 2005	Lafitte, Papazian		$\sigma = 97,104$
October 2005	Lafitte, Papazian	$s = 233,431,192,481$ $s = 912,594,733,606$	$\sigma = 458,357$ $\sigma = 1,957,771$
December 2005	Lafitte, Papazian	$s = 924,180,005,181$	
May 2006	Lafitte, Papazian	$s = 3,793,261,759,791$	$\sigma = 2,576,467$
June 2006	Lafitte, Papazian	$s = 14,103,258,269,249$	$\sigma = 4,848,239$
July 2006	Lafitte, Papazian	$s = 26,375,397,569,930$	
August 2006	T. and S. Ligocki	$s > 7.0 \times 10^{21}$	$\sigma = 172,312,766,455$
October 2007	T. and S. Ligocki	$s > 5.2 \times 10^{61}$ $s > 1.6 \times 10^{211}$	$\sigma > 9.3 \times 10^{30}$ $\sigma > 5.2 \times 10^{105}$
November 2007	T. and S. Ligocki	$s > 1.9 \times 10^{704}$	$\sigma > 1.7 \times 10^{352}$

Note: Two machines were discovered by T. and S. Ligocki in February 2005 with  $s(M) = 16,268,767$ , and two were in October 2007 with  $s(M) > 1.6 \times 10^{211}$ .

The record holder and some other good machines:

A0	A1	A2	A3	A4	B0	B1	B2	B3	B4	$s(M)$	$\sigma(M)$
1RB	2LA	1RA	2LB	2LA	0LA	2RB	3RB	4RA	1RH	$> 1.9 \times 10^{704}$	$> 1.7 \times 10^{352}$
1RB	2LA	4RA	2LB	2LA	0LA	2RB	3RB	4RA	1RH	$> 1.6 \times 10^{211}$	$> 5.2 \times 10^{105}$
1RB	2LA	4RA	2LB	2LA	0LA	2RB	3RB	1RA	1RH	$> 1.6 \times 10^{211}$	$> 5.2 \times 10^{105}$
1RB	2LA	4RA	1LB	2LA	0LA	2RB	3RB	2RA	1RH	$> 5.2 \times 10^{61}$	$> 9.3 \times 10^{30}$
1RB	0RB	4RA	2LB	2LA	2LA	1LB	3RB	4RA	1RH	$> 7.0 \times 10^{21}$	172,312,766,455
1RB	3LA	3LB	0LB	1RA	2LA	4LB	4LA	1RA	1RH	339,466,124,499,007,251	1,194,050,967
1RB	3RB	3RA	1RH	2LB	2LA	4RA	4RB	2LB	0RA	339,466,124,499,007,214	1,194,050,967
1RB	1RH	4LA	4LB	2RA	2LB	2RB	3RB	2RA	0RB	91,791,666,497,368,316	620,906,587
1RB	3LA	1LA	0LB	1RA	2LA	4LB	4LA	1RA	1RH	37,716,251,406,088,468	398,005,342
1RB	2RA	1LA	3LA	2RA	2LA	3RB	4LA	1LB	1RH	9,392,084,729,807,219	114,668,733
1RB	2RA	1LA	1LB	3LB	2LA	3RB	1RH	4RA	1LA	417,310,842,648,366	36,543,045

(These machines were discovered by Terry and Shawn Ligocki).

Previous record holders and some other good machines:

A0	A1	A2	A3	A4	B0	B1	B2	B3	B4	$s(M)$	$\sigma(M)$
1RB	3LA	1LA	4LA	1RA	2LB	2RA	1RH	0RA	0RB	26,375,397,569,930	143
1RB	3LB	4LB	4LA	2RA	2LA	1RH	3RB	4RA	3RB	14,103,258,269,249	4,848,239
1RB	3RA	4LB	2RA	3LA	2LA	1RH	4RB	4RB	2LB	3,793,261,759,791	2,576,467
1RB	3RA	1LA	1LB	3LB	2LA	4LB	3RA	2RB	1RH	924,180,005,181	1,137,477
1RB	3LB	1RH	1LA	1LA	2LA	3RB	4LB	4LB	3RA	912,594,733,606	1,957,771
1RB	2RB	3LA	2RA	3RA	2LB	2LA	3LA	4RB	1RH	469,121,946,086	668,420
1RB	3RB	3RB	1LA	3LB	2LA	3RA	4LB	2RA	1RH	233,431,192,481	458,357
1RB	3LA	1LB	1RA	3RA	2LB	3LA	3RA	4RB	1RH	8,619,024,596	90,604
1RB	2RB	3RB	4LA	3RA	0LA	4RB	1RH	0RB	1LB	7,543,673,517	97,104
1RB	4LA	1LA	1RH	2RB	2LB	3LA	1LB	2RA	0RB	7,021,292,621	37
1RB	2RB	3LA	2RA	3RA	2LB	2LA	1LA	4RB	1RH	4,561,535,055	64,665
1RB	3LA	4LA	1RA	1LA	2LA	1RH	4RA	3RB	1RA	148,304,214	11,120
1RB	3LA	4LA	1RA	1LA	2LA	1RH	1LA	3RB	1RA	16,268,767	3,685
1RB	3RB	2LA	0RB	1RH	2LA	4RB	3LB	2RB	3RB	15,754,273	4,099

(The first eleven machines were discovered by Lafitte and Papazian, and the last three ones were by T. and S. Ligocki).

## 4.12 Turing machines with 2 states and 6 symbols

- Terry and Shawn Ligocki found, in February 2005, machines  $M$  and  $N$  with  $s(M) = 98,364,599$  and  $\sigma(N) = 10,574$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig26\\_a.html](http://www.drb.insel.de/~heiner/BB/simLig26_a.html)).
  - Terry and Shawn Ligocki found, in April 2005, a machine  $M$  with  $s(M) = 493,600,387$  and  $\sigma(M) = 15,828$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig26\\_c.html](http://www.drb.insel.de/~heiner/BB/simLig26_c.html)).
  - This machine was superseded by the machine with 2 states and 5 symbols found in August 2005 by Grégory Lafitte and Christophe Papazian.
  - Terry and Shawn Ligocki found, in September 2007, a machine  $M$  with  $s(M) > 2.3 \times 10^{54}$  and  $\sigma(M) > 1.9 \times 10^{27}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig26\\_d.html](http://www.drb.insel.de/~heiner/BB/simLig26_d.html)).
  - This machine was superseded by the machine with 2 states and 5 symbols found in October 2007 by Terry and Shawn Ligocki.
  - Terry and Shawn Ligocki found successively, in November 2007, machines  $M$  with
    - $s(M) > 4.9 \times 10^{1643}$  and  $\sigma(M) > 8.6 \times 10^{821}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig26\\_e.html](http://www.drb.insel.de/~heiner/BB/simLig26_e.html)),
    - $s(M) > 2.5 \times 10^{9863}$  and  $\sigma(M) > 6.9 \times 10^{4931}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig26\\_f.html](http://www.drb.insel.de/~heiner/BB/simLig26_f.html)).
  - Terry and Shawn Ligocki found, in January 2008, a machine  $M$  with  $s(M) > 2.4 \times 10^{9866}$  and  $\sigma(M) > 1.9 \times 10^{4933}$  (see study by H. Marxen in [http://www.drb.insel.de/~heiner/BB/simLig26\\_g.html](http://www.drb.insel.de/~heiner/BB/simLig26_g.html)).
- It is the current record holder.

February 2005	T. and S. Ligocki	$s = 98,364,599$	$\sigma = 10,574$
April 2005	T. and S. Ligocki	$s = 493,600,387$	$\sigma = 15,828$
August 2005	Lafitte, Papazian	superseded by a (2,5)-TM	
September 2007	T. and S. Ligocki	$s > 2.3 \times 10^{54}$	$\sigma > 1.9 \times 10^{27}$
October 2007	T. and S. Ligocki	superseded by a (2,5)-TM	
November 2007	T. and S. Ligocki	$s > 4.9 \times 10^{1643}$	$\sigma > 8.6 \times 10^{821}$
		$s > 2.5 \times 10^{9863}$	$\sigma > 6.9 \times 10^{4931}$
January 2008	T. and S. Ligocki	$s > 2.4 \times 10^{9866}$	$\sigma > 1.9 \times 10^{4933}$

The record holder and the past record holders:

A0	A1	A2	A3	A4	A5	B0	B1	B2	B3	B4	B5	$s(M)$	$\sigma(M)$
1RB	2LA	1RH	5LB	5LA	4LB	1LA	4RB	3RB	5LB	1LB	4RA	$> 2.4 \times 10^{9866}$	$> 1.9 \times 10^{4933}$
1RB	1LB	3RA	4LA	2LA	4LB	2LA	2RB	3LB	1LA	5RA	1RH	$> 2.5 \times 10^{9863}$	$> 6.9 \times 10^{4931}$
1RB	2LB	4RB	1LA	1RB	1RH	1LA	3RA	5RA	4LB	0RA	4LA	$> 4.9 \times 10^{1643}$	$> 8.6 \times 10^{821}$
1RB	0RB	3LA	5LA	1RH	4LB	1LA	2RB	3LA	4LB	3RB	3RA	$> 2.3 \times 10^{54}$	$> 1.9 \times 10^{27}$
1RB	2LA	1RA	1RA	5LB	4LB	1LB	1LA	3RB	4LA	1RH	3LA	493,600,387	15,828
1RB	3LA	3LA	1RA	1RA	3LB	1LB	2LA	2RA	4RB	5LB	1RH	98,364,599	10,249
1RB	3LA	4LA	1RA	3RB	1RH	2LB	1LA	1LB	3RB	5RA	1RH	94,842,383	10,574

## 5 Behaviors of busy beavers

### 5.1 Introduction

How do good machines behave? We give below the tricks that allow them to reach high scores.

A *configuration* of the Turing machine  $M$  is a description of the tape. The position of the tape head and the state are indicated by writing together between parentheses the state and the symbol currently read by the tape head.

For example, the initial configuration on a blank tape is:

$$\dots 0(A0)0 \dots$$

We denote by  $a^k$  the string  $a \dots a$ ,  $k$  times. We write  $C \vdash (t) D$  if the next move function leads from configuration  $C$  to configuration  $D$  in  $t$  computation steps.

### 5.2 Turing machines with 5 states and 2 symbols

#### 5.2.1 Marxen and Buntrock's champion

This machine is the record holder in the Busy Beaver Competition for machines with 5 states and 2 symbols, since 1990.

		0	1
Marxen and Buntrock (1990)	A	1RB	1LC
$s(M) = 47, 176, 870 =? S(5, 2)$	B	1RC	1RB
$\sigma(M) = 4098 =? \Sigma(5, 2)$	C	1RD	0LE
	D	1LA	1LD
	E	1RH	0LA



Let  $C(n) = \dots 0(A0)1^n 0 \dots$   
Then we have, for all  $k \geq 0$ ,

$$\begin{array}{rcl} C(3k) & \vdash (5k^2 + 19k + 15) & C(5k + 6) \\ C(3k + 1) & \vdash (5k^2 + 25k + 27) & C(5k + 9) \\ C(3k + 2) & \vdash (6k + 12) & \dots 01(H0)1(001)^{k+1}10 \dots \end{array}$$

So we have:

$$\begin{array}{l} \dots 0(A0)0 \dots = \\ C(0) \vdash (15) \\ C(6) \vdash (73) \\ C(16) \vdash (277) \\ C(34) \vdash (907) \\ C(64) \vdash (2,757) \\ C(114) \vdash (7,957) \\ C(196) \vdash (22,777) \\ C(334) \vdash (64,407) \\ C(564) \vdash (180,307) \\ C(946) \vdash (504,027) \\ C(1,584) \vdash (1,403,967) \\ C(2,646) \vdash (3,906,393) \\ C(4,416) \vdash (10,861,903) \\ C(7,366) \vdash (30,196,527) \\ C(12,284) \vdash (24,576) \\ \dots 01(H0)1(001)^{4095}10 \dots \end{array}$$

### 5.2.2 Marxen and Buntrock's runner-up

	0	1
Marxen and Buntrock (1990) $s(M) = 23,554,764$ $\sigma(M) = 4097$	A	1RB 0LD
	B	1LC 1RD
	C	1LA 1LC
	D	1RH 1RE
	E	1RA 0RB

Let  $C(n) = \dots 0(A0)1^n 0 \dots$   
Then we have, for all  $k \geq 0$ ,

$$\begin{array}{rcl} C(3k) & \vdash (10k^2 + 10k + 4) & C(5k + 3) \\ C(3k + 1) & \vdash (3k + 3) & \dots 01(110)^k 11(H0)0 \dots \\ C(3k + 2) & \vdash (10k^2 + 26k + 12) & C(5k + 7) \end{array}$$

So we have:

$$\begin{aligned}
& \dots 0(A0)0\dots = \\
& C(0) \vdash (4) \\
& C(3) \vdash (24) \\
& C(8) \vdash (104) \\
& C(17) \vdash (392) \\
& C(32) \vdash (1,272) \\
& C(57) \vdash (3,804) \\
& C(98) \vdash (11,084) \\
& C(167) \vdash (31,692) \\
& C(282) \vdash (89,304) \\
& C(473) \vdash (250,584) \\
& C(792) \vdash (699,604) \\
& C(1,323) \vdash (1,949,224) \\
& C(2,208) \vdash (5,424,324) \\
& C(3,683) \vdash (15,087,204) \\
& C(6,142) \vdash (6,144) \\
& \dots 01(110)^{2047}11(H0)0\dots
\end{aligned}$$

### 5.3 Turing machines with 6 states and 2 symbols

#### 5.3.1 Kropitz's machine found in May 2010

This machine is the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, since May 2010.

	0	1
Kropitz (2010)	A	1RB 0LD
$s(M)$ and $S(6, 2) > 3.8 \times 10^{21132}$	B	1RC 0RF
$\sigma(M)$ and $\Sigma(6, 2) > 3.1 \times 10^{10566}$	C	1LC 1LA
	D	0LE 1RH
	E	1LA 0RB
	F	0RC 0RE

**Analysis adapted from Shawn Ligocki:**

Let  $C(n, k) = \dots 010^n 1(C1)1^{3k} 0 \dots$

Then we have, for all  $k \geq 0$ , all  $n \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (47) & C(5, 2) \\
C(0, k) & \vdash (3) & \dots 01(H0)1^{3k+10} \dots \\
C(1, k) & \vdash (3k + 37) & C(3k + 2, 2) \\
C(2, k) & \vdash (12k + 44) & C(4, k + 2) \\
C(3, k) & \vdash (3k + 57) & C(3k + 8, 2) \\
C(n + 4, k) & \vdash (27k^2 + 105k + 112) & C(n, 3k + 5)
\end{array}$$

So we have (the final configuration is reached in 22158 transitions):

$\dots 0(A0)0\dots \vdash (47)$   
 $C(5, 2) \vdash (430)$   
 $C(1, 11) \vdash (70)$   
 $C(35, 2) \vdash (430)$   
 $C(31, 11) \vdash (4, 534)$   
 $C(27, 38) \vdash (43, 090)$   
 $C(23, 119) \vdash (394, 954)$   
 $C(19, 362) \vdash (3, 576, 310)$   
 $C(15, 1091) \vdash (32, 252, 254)$   
 $C(11, 3278) \vdash (290, 466, 970)$   
 $C(7, 9839) \vdash (2, 614, 793, 074)$   
 $C(3, 29522) \vdash (88, 623)$   
 $C(88574, 2) \vdash (430)$   
 $C(88570, 11) \vdash (4, 534)$   
 $C(88566, 38) \vdash (43, 090)$   
 $\dots$

Note that  $C(4n+r, 2) \vdash (t_n) C(r, u_n)$ , with  $u_n = (3^{n+2} - 5)/2$ , and  $t_n = (3 \times 9^{n+3} - 80 \times 3^{n+3} + 584n - 27)/32$ .

Some configurations take a long time to halt. For example,  $C(1, 9) \vdash (t)$  END with  $t > 10^{10^{10^{10^{3520}}}}$ .

### 5.3.2 Ligockis' machine found in December 2007

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from December 2007 to May 2010.

		0	1
Terry and Shawn Ligocki (2007)	A	1RB	0LE
$s(M) > 2.5 \times 10^{2879}$	B	1LC	0RA
$\sigma(M) > 4.6 \times 10^{1439}$	C	1LD	0RC
	D	1LE	0LF
	E	1LA	1LC
	F	1LE	1RH

Let  $C(n, p) = \dots 0(A0)(10)^n R(\text{bin}(p))0\dots$ , where  $R(\text{bin}(p))$  is the number  $p$  written in binary in reverse order, so that  $C(n, 4m+1) = C(n+1, m)$ . The number of transitions between configurations  $C(n, p)$  is infinite, but only 18 transitions are used in the computation

on a blank tape. For all  $m \geq 0$ , all  $k \geq 0$ ,

$C(k, 4m + 3)$	$\vdash (4k + 6)$	$C(k + 2, m)$
$C(2k + 1, 4m)$	$\vdash (6k^2 + 52k + 98)$	$C(3k + 8, m)$
$C(4k, 4m)$	$\vdash (24k^2 + 36k + 13)$	$C(6k + 2, 2m + 1)$
$C(4k + 2, 4m)$	$\vdash (24k^2 + 60k + 27)$	$C(6k + 2, 128m + 86)$
$C(k, 8m + 2)$	$\vdash (4k + 14)$	$C(k + 2, 2m + 1)$
$C(2k + 1, 32m + 22)$	$\vdash (6k^2 + 64k + 160)$	$C(3k + 10, 2m + 1)$
$C(4k, 32m + 22)$	$\vdash (24k^2 + 36k + 29)$	$C(6k + 4, m)$
$C(4k + 2, 32m + 22)$	$\vdash (24k^2 + 60k + 43)$	$C(6k + 2, 1024m + 342)$
$C(k, 64m + 46)$	$\vdash (4k + 30)$	$C(k + 4, m)$
$C(k + 1, 128m + 6)$	$\vdash (8k + 66)$	$C(k + 6, 2m + 1)$
$C(2k, 256m + 14)$	$\vdash (6k^2 + 64k + 172)$	$C(3k + 11, m)$
$C(4k + 1, 256m + 14)$	$\vdash (24k^2 + 84k + 89)$	$C(6k + 8, 2m + 1)$
$C(4k + 3, 256m + 14)$	$\vdash (24k^2 + 108k + 127)$	$C(6k + 8, 128m + 86)$
$C(4k, 512m + 30)$	$\vdash (24k^2 + 156k + 173)$	$C(6k + 11, m)$
$C(4k + 2, 512m + 30)$	$\vdash (24k^2 + 60k + 57)$	$C(6k + 2, 16384m + 11134)$
$C(4k + 2, 131072m + 11134)$	$\vdash (24k^2 + 60k + 89)$	$C(6k + 2, 4194304m + 2848638)$
$C(4k, 131072m + 96126)$	$\vdash (24k^2 + 36k + 109)$	$C(6k + 10, m)$
$C(k + 1, 512m + 94)$	$\vdash (2k + 61)$	$\dots 0(10)^k 1(H0)1110110101R(\text{bin}(m))0\dots$

So we have (the final configuration is reached in 11026 transitions):

$\dots 0(A0)0\dots =$
$C(0, 0) \vdash (13)$
$C(3, 0) \vdash (156)$
$C(11, 0) \vdash (508)$
$C(23, 0) \vdash (1396)$
$C(41, 0) \vdash (3538)$
$C(68, 0) \vdash (7, 561)$
$C(105, 0) \vdash (19, 026)$
$C(164, 0) \vdash (41, 833)$
$C(249, 0) \vdash (98, 802)$
$C(380, 0) \vdash (220, 033)$
$C(573, 0) \vdash (505, 746)$
$C(866, 0) \vdash (1, 132, 731)$
$C(1298, 86) \vdash (2, 538, 907)$
$C(1946, 2390) \vdash (5, 697, 907)$
$C(2918, 76118) \vdash (12, 798, 367)$
$C(4376, 2435414) \vdash (1, 034, 066, 333)$
$C(6568, 76106) \vdash (26, 286)$
$C(6570, 19027) \vdash (26, 286)$
$C(6572, 4756) \vdash (64, 845, 937)$
$C(9860, 2379) \vdash (39, 446)$
$C(9862, 594) \vdash (39, 462)$
$C(9867, 2) \vdash (39, 482)$
$\dots$

### 5.3.3 Ligockis' machine found in November 2007

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from November to December 2007.

		0	1
Terry and Shawn Ligocki (2007)	A	1RB	0RF
$s(M) > 8.9 \times 10^{1762}$	B	0LB	1LC
$\sigma(M) > 2.5 \times 10^{881}$	C	1LD	0RC
	D	1LE	1RH
	E	1LF	0LD
	F	1RA	0LE

Let  $C(n, p) = \dots 0(F0)(10)^n R(\text{bin}(p))0\dots$ , where  $R(\text{bin}(p))$  is the number  $p$  written in binary in reverse order, so that  $C(n, 4m + 1) = C(n + 1, m)$ . The number of transitions between configurations  $C(n, p)$  is infinite, but only 12 transitions are used in the computation on a blank tape. For all  $m \geq 0$ , all  $k \geq 0$ ,

$\dots 0(A0)0\dots$	$\vdash (6)$	$C(0, 15)$
$C(k, 4m + 3)$	$\vdash (4k + 6)$	$C(k + 2, m)$
$C(2k, 4m)$	$\vdash (30k^2 + 20k + 15)$	$C(5k + 2, 2m + 1)$
$C(2k + 1, 4m)$	$\vdash (30k^2 + 40k + 25)$	$C(5k + 2, 32m + 20)$
$C(k, 8m + 2)$	$\vdash (8k + 20)$	$C(k + 3, 2m + 1)$
$C(2k, 16m + 6)$	$\vdash (30k^2 + 40k + 23)$	$C(5k + 2, 32m + 20)$
$C(2k + 1, 16m + 6)$	$\vdash (30k^2 + 80k + 63)$	$C(5k + 7, 2m + 1)$
$C(k, 32m + 14)$	$\vdash (4k + 18)$	$C(k + 3, 2m + 1)$
$C(2k, 128m + 94)$	$\vdash (30k^2 + 40k + 39)$	$C(5k + 2, 256m + 84)$
$C(2k + 1, 128m + 94)$	$\vdash (30k^2 + 80k + 79)$	$C(5k + 9, m)$
$C(k, 256m + 190)$	$\vdash (4k + 34)$	$C(k + 5, m)$
$C(k, 512m + 30)$	$\vdash (2k + 43)$	$\dots 0(10)^k 1(H0)10100101R(\text{bin}(m))0\dots$

So we have (the final configuration is reached in 3346 transitions):

$\dots 0(A0)0\dots \vdash (6)$   
 $C(0, 15) \vdash (6)$   
 $C(2, 3) \vdash (14)$   
 $C(4, 0) \vdash (175)$   
 $C(13, 0) \vdash (1, 345)$   
 $C(32, 20) \vdash (8, 015)$   
 $C(82, 11) \vdash (334)$   
 $C(84, 2) \vdash (692)$   
 $C(88, 0) \vdash (58, 975)$   
 $C(223, 0) \vdash (374, 095)$   
 $C(557, 20) \vdash (2, 329, 665)$   
 $C(1392, 180) \vdash (14, 546, 415)$   
 $C(3482, 91) \vdash (13, 934)$   
 $C(3484, 22) \vdash (91, 106, 623)$   
 $C(8712, 52) \vdash (569, 329, 215)$   
 $C(21782, 27) \vdash (87, 134)$   
 $C(21784, 6) \vdash (3, 559, 505, 623)$   
 $C(54462, 20) \vdash (22, 246, 365, 465)$   
 $C(136157, 11) \vdash (544, 634)$   
 $C(136159, 2) \vdash (1, 089, 292)$   
 $C(136163, 0) \vdash (139, 053, 400, 095)$   
 $C(340407, 20) \vdash (869, 078, 644, 415)$   
 $\dots$

#### 5.3.4 Marxen and Buntrock's machine found in March 2001

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from March 2001 to November 2007.

		0	1
Marxen and Buntrock (2001)	A	1RB	0LF
$s(M) > 3.0 \times 10^{1730}$	B	0RC	0RD
$\sigma(M) > 1.2 \times 10^{865}$	C	1LD	1RE
	D	0LE	0LD
	E	0RA	1RC
	F	1LA	1RH

Let  $C(n, p) = \dots 0(A0)(01)^n R(\text{bin}(p))0\dots$ , where  $R(\text{bin}(p))$  is the number  $p$  written in binary in reverse order, so that  $C(n, 4m + 2) = C(n + 1, m)$ . The number of transitions between configurations  $C(n, p)$  is infinite, but only 20 transitions are used in the computation

on a blank tape. For all  $m \geq 0$ , all  $k \geq 0$ ,

$$\begin{array}{lll}
C(2k, 4m) & \vdash (9k^2 + 25k + 9) & C(3k + 1, 2m + 1) \\
C(2k, 16m + 1) & \vdash (9k^2 + 25k + 17) & C(3k + 2, 2m + 1) \\
C(2k, 4m + 3) & \vdash (9k^2 + 25k + 9) & C(3k + 1, 2m) \\
C(2k, 64m + 53) & \vdash (9k^2 + 25k + 25) & C(3k + 3, 2m) \\
C(2k, 256m + 9) & \vdash (9k^2 + 25k + 29) & C(3k + 4, 2m + 1) \\
C(2k, 1024m + 57) & \vdash (9k^2 + 25k + 33) & C(3k + 2, 128m + 104) \\
C(2k, 1024m + 85) & \vdash (9k^2 + 25k + 41) & C(3k + 5, 2m + 1) \\
C(2k + 1, 16m) & \vdash (9k^2 + 25k + 21) & C(3k + 3, 2m + 1) \\
C(2k + 1, 4m + 1) & \vdash (9k^2 + 25k + 13) & C(3k + 1, 8m + 4) \\
C(2k + 1, 64m + 4) & \vdash (9k^2 + 25k + 29) & C(3k + 4, 2m + 1) \\
C(2k + 1, 64m + 3) & \vdash (9k^2 + 25k + 25) & C(3k + 1, 128m + 104) \\
C(2k + 1, 1024m + 104) & \vdash (9k^2 + 43k + 75) & C(3k + 7, 2m + 1) \\
C(2k + 1, 16m + 12) & \vdash (9k^2 + 25k + 21) & C(3k + 3, 2m) \\
C(2k + 1, 16m + 7) & \vdash (9k^2 + 25k + 17) & C(3k + 1, 32m + 16) \\
C(2k + 1, 256m + 15) & \vdash (9k^2 + 25k + 29) & C(3k + 1, 512m + 416) \\
C(2k + 1, 64m + 52) & \vdash (9k^2 + 25k + 29) & C(3k + 4, 2m) \\
C(2k + 1, 256m + 20) & \vdash (9k^2 + 25k + 37) & C(3k + 5, 2m + 1) \\
C(2k + 1, 4096m + 420) & \vdash (9k^2 + 43k + 89) & C(3k + 8, 2m + 1) \\
C(2k + 1, 256m + 211) & \vdash (9k^2 + 25k + 33) & C(3k + 1, 512m + 168) \\
C(2k + 1, 16m + 11) & \vdash (9k^2 + 13k + 10) & \dots 0(10)^{3k+1}11(H0)10R(\text{bin}(m))0\dots
\end{array}$$

So we have (the final configuration is reached in 4911 transitions):

$$\begin{aligned}
 & \dots 0(A0)0\dots = \\
 & C(0, 0) \vdash (9) \\
 & C(1, 1) \vdash (13) \\
 & C(1, 4) \vdash (29) \\
 & C(4, 1) \vdash (103) \\
 & C(8, 1) \vdash (261) \\
 & C(14, 1) \vdash (633) \\
 & C(23, 1) \vdash (1, 377) \\
 & C(34, 4) \vdash (3, 035) \\
 & C(52, 3) \vdash (6, 743) \\
 & C(79, 0) \vdash (14, 685) \\
 & C(120, 1) \vdash (33, 917) \\
 & C(182, 1) \vdash (76, 821) \\
 & C(275, 1) \vdash (172, 359) \\
 & C(412, 4) \vdash (387, 083) \\
 & C(619, 3) \vdash (867, 079) \\
 & C(928, 104) \vdash (1, 949, 273) \\
 & C(1393, 53) \vdash (4, 377, 157) \\
 & C(2089, 108) \vdash (9, 835, 545) \\
 & C(3135, 12) \vdash (22, 138, 597) \\
 & C(4704, 0) \vdash (49, 845, 945) \\
 & C(7057, 1) \vdash (112, 109, 269) \\
 & C(10585, 4) \vdash (252, 179, 705) \\
 & \dots
 \end{aligned}$$

**Note:** Clive Tooth posted an analysis of this machine on Google Groups (sci.math>The Turing machine known as #r), on June 28, 2002. He used the configurations  $S(n, x) = \dots 0101(B1)010(01)^n x 0 \dots$ . His analysis can be easily connected to the present one, by noting that

$$C(n, p) \vdash (15) S(n - 2, R(\text{bin}(p))).$$

### 5.3.5 Marxen and Buntrock's second machine

This machine was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols, from October 2000 to March 2001.

		0	1
Marxen and Buntrock (2000)	A	1RB	0LB
$s(M) > 6.1 \times 10^{925}$	B	0RC	1LB
$\sigma(M) > 6.4 \times 10^{462}$	C	1RD	0LA
	D	1LE	1LF
	E	1LA	0LD
	F	1RH	1LE



Let  $C(n) = \dots 01^n(B0)0\dots$   
Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (1) & C(1) \\
C(3k) & \vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86) & C(9 \times 2^{k+1} - 8) \\
C(3k+1) & \vdash (2048 \times (4^k - 1)/3 - 3 \times 2^{k+7} + 26k + 792) & C(2^{k+5} - 8) \\
C(3k+2) & \vdash (3k + 8) & \dots 01(H1)(011)^k(0101)0\dots
\end{array}$$

So we have:

$$\begin{array}{ll}
\dots 0(A0)0\dots & \vdash (1) \\
C(1) & \vdash (408) \\
C(24) & \vdash (14, 100, 774) \\
C(4600) & \vdash (2048 \times (4^{1533} - 1)/3 - 3 \times 2^{1540} + 40650) \\
C(2^{1538} - 8) & \vdash (2^{1538} - 2) \\
\dots 01(H1)(011)^p(0101)0\dots &
\end{array}$$

with  $p = (2^{1538} - 10)/3$ .

So the total time is  $T = 2048 \times (4^{1533} - 1)/3 - 11 \times 2^{1538} + 14141831$ , and the final number of 1 is  $2 \times (2^{1538} - 10)/3 + 4$ .

Note that

$$C(6k+1) \vdash ( ) \quad C(3m) \vdash ( ) \quad C(6p+4) \vdash ( ) \quad C(3q+2) \vdash ( ) \quad \text{END,}$$

with  $m = (2^{2k+5} - 8)/3$ ,  $p = 3 \times 2^m - 2$ ,  $q = (2^{2p+6} - 10)/3$ .

So all configurations  $C(n)$  lead to a halting configuration. Those taking the most time are  $C(6k+1)$ . For example:

$$C(7) \vdash (t) \text{ END} \quad \text{with} \quad t > 10^{3.9 \times 10^{12}}.$$

More generally:

$$C(6k+1) \vdash (t(k)) \text{ END} \quad \text{with} \quad t(k) > 10^{10^{10(3k+2)/5}}.$$

See also the analyses by Robert Munafo: the short one in <http://mrob.com/pub/math/ln-notes1-4.html#mb6q>, and the detailed one in <http://mrob.com/pub/math/ln-mb6q.html>.

### 5.3.6 Marxen and Buntrock's third machine

		0	1
Marxen and Buntrock (2000) $s(M) > 6.1 \times 10^{119}$ $\sigma(M) > 1.4 \times 10^{60}$	A	1RB	0LC
	B	1LA	1RC
	C	1RA	0LD
	D	1LE	1LC
	E	1RF	1RH
	F	1RA	1RE

Let  $C(n, x) = \dots 0(E0)1000(10)^n x 0 \dots$ , so that  $C(n, 10y) = C(n+1, y)$ . The number of transitions between configurations  $C(n, x)$  is infinite, but only 9 transitions are used in the computation on a blank tape. For all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (18) & C(1, 01) \\
C(2k, 01^n) & \vdash (6k^2 + 22k + 15) & C(3k+1, 01^{n+1}) \\
C(2k, 11) & \vdash (6k^2 + 34k + 41) & C(3k+4, 01) \\
C(2k, 111) & \vdash (6k^2 + 34k + 45) & C(3k+5, 01) \\
C(2k, 1111) & \vdash (6k^2 + 28k + 25) & \dots 01^{6k+11}(H0)0\dots \\
C(2k+1, 0) & \vdash (6k^2 + 34k + 43) & C(3k+4, 0) \\
C(2k+1, 01) & \vdash (6k^2 + 22k + 27) & C(3k+4, 01) \\
C(2k+1, 01^{n+2}) & \vdash (6k^2 + 22k + 23) & C(3k+4, 1^n) \\
C(2k+1, 1^{n+2}) & \vdash (6k^2 + 34k + 41) & C(3k+4, 01^n)
\end{array}$$

So we have (the final configuration is reached in 337 transitions):

$$\begin{array}{ll}
\dots 0(A0)0\dots & \vdash (18) \\
C(1, 01) & \vdash (27) \\
C(4, 01) & \vdash (83) \\
C(7, 011) & \vdash (143) \\
C(13, 0) & \vdash (463) \\
C(22, 0) & \vdash (983) \\
C(34, 01) & \vdash (2, 123) \\
C(52, 011) & \vdash (4, 643) \\
C(79, 0111) & \vdash (10, 007) \\
C(122, 0) & \vdash (23, 683) \\
C(184, 01) & \vdash (52, 823) \\
C(277, 011) & \vdash (117, 323) \\
C(418, 0) & \vdash (266, 699) \\
C(628, 01) & \vdash (598, 499) \\
C(943, 011) & \vdash (1, 341, 431) \\
C(1417, 0) & \vdash (3, 031, 699) \\
C(2128, 0) & \vdash (6, 815, 999) \\
C(3193, 01) & \vdash (15, 318, 435) \\
C(4792, 01) & \vdash (34, 497, 623) \\
C(7189, 011) & \vdash (77, 580, 107) \\
C(10786, 0) & \vdash (174, 625, 355) \\
& \dots
\end{array}$$

Note that, if  $C(n, m) = \dots 0(E0)1000(10)^n R(\text{bin}(m))0 \dots$ , where  $R(\text{bin}(m))$  is the number  $m$  written in binary in reverse order, so that  $C(n, 4m+1) = C(n+1, m)$ , then we have

also, for all  $k, m \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (18) & C(1, 2) \\
C(2k, 2m) & \vdash (6k^2 + 22k + 15) & C(3k + 1, 4m + 2) \\
C(2k, 32m + 3) & \vdash (6k^2 + 34k + 41) & C(3k + 4, 4m + 2) \\
C(2k, 128m + 7) & \vdash (6k^2 + 34k + 45) & C(3k + 5, 4m + 2) \\
C(2k, 32m + 15) & \vdash (6k^2 + 28k + 25) & \dots 01^{6k+11}(H0)R(\text{bin}(m))0\dots \\
C(2k + 1, 4m) & \vdash (6k^2 + 34k + 43) & C(3k + 4, 2m) \\
C(2k + 1, 32m + 2) & \vdash (6k^2 + 22k + 27) & C(3k + 4, 4m + 2) \\
C(2k + 1, 8m + 6) & \vdash (6k^2 + 22k + 23) & C(3k + 4, m) \\
C(2k + 1, 4m + 3) & \vdash (6k^2 + 34k + 41) & C(3k + 4, 2m)
\end{array}$$

### 5.3.7 Another Marxen and Buntrock's machine

This machine was discovered in January 1990, and was published on the web (Google groups) on September 3, 1997. It was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols up to July 2000.

		0	1
	A	1RB	1RA
	B	1LC	1LB
Marxen and Buntrock (1997)	C	0RF	1LD
$s(M) = 8, 690, 333, 381, 690, 951$	D	1RA	0LE
$\sigma(M) = 95, 524, 079$	E	1RH	1LF
	F	0LA	0LC

Note the likeness to the machine  $N$  with 3 states and 3 symbols discovered, in August 2006, by Terry and Shawn Ligocki, and studied in Section 5.4.2. For this machine  $N$ , we have  $s(N) = 4, 345, 166, 620, 336, 565$  and  $\sigma(N) = 95, 524, 079$ , that is, same value of  $\sigma$ , and almost half the value of  $s$ . See analysis of this similarity in Section 5.9.

#### Analysis by Robert Munafo:

Let  $C(n) = \dots 0(D0)1^n 0 \dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (3) & C(2) \\
C(4k) & \vdash (8k + 6) & \dots 01(H0)(10)^{2k}110\dots \\
C(4k + 1) & \vdash (20k^2 + 56k + 30) & C(10k + 9) \\
C(4k + 2) & \vdash (20k^2 + 56k + 33) & C(10k + 9) \\
C(4k + 3) & \vdash (20k^2 + 68k + 51) & C(10k + 12)
\end{array}$$

So we have:

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (3) \\
C(2) &\vdash (33) \\
C(9) &\vdash (222) \\
C(29) &\vdash (1, 402) \\
C(79) &\vdash (8, 563) \\
C(202) &\vdash (52, 833) \\
C(509) &\vdash (329, 722) \\
C(1, 279) &\vdash (2, 056, 963) \\
C(3, 202) &\vdash (12, 844, 833) \\
C(8, 009) &\vdash (80, 272, 222) \\
C(20, 029) &\vdash (501, 681, 402) \\
C(50, 079) &\vdash (3, 135, 358, 563) \\
C(125, 202) &\vdash (19, 595, 552, 833) \\
C(313, 009) &\vdash (122, 471, 892, 222) \\
C(782, 529) &\vdash (765, 448, 543, 902) \\
C(1, 956, 329) &\vdash (4, 784, 051, 443, 102) \\
C(4, 890, 829) &\vdash (29, 900, 316, 628, 602) \\
C(12, 227, 079) &\vdash (186, 876, 942, 247, 563) \\
C(30, 567, 702) &\vdash (1, 167, 980, 782, 060, 333) \\
C(76, 419, 259) &\vdash (7, 299, 879, 658, 619, 323) \\
C(191, 048, 152) &\vdash (382, 096, 310) \\
\dots 01(H0)(10)^{95524076}110\dots
\end{aligned}$$

## 5.4 Turing machines with 3 states and 3 symbols

### 5.4.1 Ligoekis' champion

This machine is the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, since November 2007.

$$\begin{aligned}
&\text{Terry and Shawn Ligoeki (2007)} \\
s(M) = 119, 112, 334, 170, 342, 540 =? S(3, 3) \\
\sigma(M) = 374, 676, 383 =? \Sigma(3, 3)
\end{aligned}$$

	0	1	2
A	1RB	2LA	1LC
B	0LA	2RB	1LB
C	1RH	1RA	1RC

Let  $C(n) = \dots 0(A0)2^n 0 \dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (3) & C(1) \\
C(8k+1) & \vdash (112k^2 + 116k + 13) & C(14k+3) \\
C(8k+2) & \vdash (112k^2 + 144k + 38) & C(14k+7) \\
C(8k+3) & \vdash (112k^2 + 172k + 54) & C(14k+8) \\
C(8k+4) & \vdash (112k^2 + 200k + 74) & C(14k+9) \\
C(8k+5) & \vdash (112k^2 + 228k + 97) & \dots 01(H1)2^{14k+9}0\dots \\
C(8k+6) & \vdash (112k^2 + 256k + 139) & C(14k+14) \\
C(8k+7) & \vdash (112k^2 + 284k + 169) & C(14k+15) \\
C(8k+8) & \vdash (112k^2 + 312k + 203) & C(14k+16)
\end{array}$$

So we have (in 34 transitions):

$$\begin{aligned}
& \dots 0(A0)0\dots \vdash (3) \\
& \quad C(1) \vdash (13) \\
& \quad C(3) \vdash (54) \\
& \quad C(8) \vdash (203) \\
& \quad C(16) \vdash (627) \\
& \quad C(30) \vdash (1915) \\
& \quad \dots \\
& C(122, 343, 306) \vdash (26, 193, 799, 261, 043, 238) \\
& C(214, 100, 789) \vdash (80, 218, 511, 093, 348, 089) \\
& \quad \dots 01(H1)2^{374676381}0\dots
\end{aligned}$$

#### 5.4.2 Ligockis' machine found in August 2006

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from August 2006 to November 2007.

Terry and Shawn Ligocki (2006) $s(M) = 4, 345, 166, 620, 336, 565$ $\sigma(M) = 95, 524, 079$		0	1	2
	A	1RB	2RC	1LA
	B	2LA	1RB	1RH
	C	2RB	2RA	1LC

Note the likeness to the machine  $N$  with 6 states and 2 symbols discovered, in January 1990, by Heiner Marxen and Jürgen Buntrock, and studied in Section 5.3.7. For this machine  $N$ , we have  $s(N) = 8, 690, 333, 381, 690, 951$  and  $\sigma(N) = 95, 524, 079$ , that is, same value of  $\sigma$ , and almost twice the value of  $s$ . See analysis of this similarity in Section 5.9.

#### Analysis by Shawn Ligocki:

Let  $C(n, 0) = \dots 0(A0)12^n 0\dots$ ,  
and  $C(n, 1) = \dots 0(C0)12^n 0\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{aligned}
C(2k, 0) & \vdash (40k^2 + 32k + 5) & C(5k + 1, 1) \\
C(2k + 1, 0) & \vdash (40k^2 + 82k + 42) & \dots 01^{10k+9}(H0)0\dots \\
C(2k + 1, 1) & \vdash (40k^2 + 52k + 19) & C(5k + 3, 1) \\
C(2k + 2, 1) & \vdash (40k^2 + 92k + 53) & C(5k + 5, 0)
\end{aligned}$$

So we have:

$$\begin{aligned}
& \dots 0(A0)0\dots = \\
& C(0,0) \vdash (5) \\
& C(1,1) \vdash (19) \\
& C(3,1) \vdash (111) \\
& C(8,1) \vdash (689) \\
& C(20,0) \vdash (4,325) \\
& C(51,1) \vdash (26,319) \\
& C(128,1) \vdash (164,609) \\
& C(320,0) \vdash (1,029,125) \\
& C(801,1) \vdash (6,420,819) \\
& C(2003,1) \vdash (40,132,111) \\
& C(5008,1) \vdash (250,830,689) \\
& C(12520,0) \vdash (1,567,704,325) \\
& C(31301,1) \vdash (9,797,713,819) \\
& C(78253,1) \vdash (61,235,789,611) \\
& C(195633,1) \vdash (382,723,880,691) \\
& C(489083,1) \vdash (2,392,024,743,391) \\
& C(1222708,1) \vdash (14,950,155,868,889) \\
& C(3056770,0) \vdash (93,438,477,237,325) \\
& C(7641926,1) \vdash (582,990,375,746,317) \\
& C(19104815,0) \vdash (3,649,939,963,043,376) \\
& \dots 01^{95524079}(H0)0\dots
\end{aligned}$$

### 5.4.3 Lafitte and Papazian's machine found in April 2006

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from April to August 2006.

Lafitte and Papazian (2006)		0	1	2
$s(M) = 4,144,465,135,614$	A	1RB	1RH	2LC
$\sigma(M) = 2,950,149$	B	1LC	2RB	1LB
	C	1LA	2RC	2LA

Let  $C(n,0) = \dots 0(A0)1^n 0\dots$ ,  
and  $C(n,1) = \dots 0(A0)1^n 210\dots$

Then we have, for all  $k \geq 0$  (note the likeness to Brady's machine of Section 5.4.8),

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (16) & C(6,0) \\
C(2k+1,0) & \vdash (4k+5) & \dots 01(H2)2^{2k}10\dots \\
C(2k+2,0) & \vdash (10k^2 + 27k + 23) & C(5k+6,1) \\
C(2k,1) & \vdash (10k^2 + 27k + 18) & C(5k+5,1) \\
C(2k+1,1) & \vdash (10k^2 + 51k + 60) & C(5k+12,0)
\end{array}$$

So we have:

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (16) \\
C(6, 0) &\vdash (117) \\
C(16, 1) &\vdash (874) \\
C(45, 1) &\vdash (6, 022) \\
C(122, 0) &\vdash (37, 643) \\
C(306, 1) &\vdash (238, 239) \\
C(770, 1) &\vdash (1, 492, 663) \\
C(1930, 1) &\vdash (9, 338, 323) \\
C(4830, 1) &\vdash (58, 387, 473) \\
C(12080, 1) &\vdash (364, 979, 098) \\
C(30205, 1) &\vdash (2, 281, 474, 302) \\
C(75522, 0) &\vdash (14, 259, 195, 543) \\
C(188806, 1) &\vdash (89, 121, 812, 989) \\
C(472020, 1) &\vdash (557, 013, 573, 288) \\
C(1180055, 1) &\vdash (3, 481, 348, 698, 727) \\
C(2950147, 0) &\vdash (5, 900, 297) \\
\dots 01(H2)2^{2950146}10\dots
\end{aligned}$$

Note that we have also, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (133) & C(16, 1) \\
C(2k, 1) & \vdash (10k^2 + 27k + 18) & C(5k + 5, 1) \\
C(4k + 1, 1) & \vdash (290k^2 + 737k + 468) & C(25k + 31, 1) \\
C(4k + 3, 1) & \vdash (40k^2 + 162k + 158) & \dots 01(H2)2^{10k+16}10\dots
\end{array}$$

#### 5.4.4 Lafitte and Papazian's machine found in September 2005

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from September 2005 to April 2006.

Lafitte and Papazian (2005) $s(M) = 987, 522, 842, 126$ $\sigma(M) = 1, 525, 688$		0	1	2
	A	1RB	2LA	1RA
	B	1RC	2RB	0RC
	C	1LA	1RH	1LA

Let  $C(n, 0) = \dots 0(A0)2^n 0 \dots$ ,  
and  $C(n, 1) = \dots 0(A0)2^n 10 \dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
C(4k, 0) & \vdash (14k^2 + 16k + 5) & C(7k + 2, 1) \\
C(4k + 1, 0) & \vdash (14k^2 + 30k + 15) & C(7k + 5, 0) \\
C(4k + 2, 0) & \vdash (14k^2 + 30k + 15) & C(7k + 5, 0) \\
C(4k + 3, 0) & \vdash (14k^2 + 44k + 35) & C(7k + 9, 1) \\
C(2k + 1, 1) & \vdash (4k + 3) & \dots 01(12)^k 01(H0)0\dots \\
C(4k, 1) & \vdash (14k^2 + 26k + 11) & C(7k + 4, 0) \\
C(4k + 2, 1) & \vdash (14k^2 + 40k + 29) & C(7k + 8, 1)
\end{array}$$

So we have:

$$\begin{aligned}
 & \dots 0(A0)0\dots = \\
 & C(0,0) \vdash (5) \\
 & C(2,1) \vdash (29) \\
 & C(8,1) \vdash (119) \\
 & C(18,0) \vdash (359) \\
 & C(33,0) \vdash (1,151) \\
 & C(61,0) \vdash (3,615) \\
 & C(110,0) \vdash (11,031) \\
 & C(194,0) \vdash (33,711) \\
 & C(341,0) \vdash (103,715) \\
 & C(600,0) \vdash (317,405) \\
 & C(1052,1) \vdash (975,215) \\
 & C(1845,0) \vdash (2,989,139) \\
 & C(3232,0) \vdash (9,153,029) \\
 & C(5658,1) \vdash (28,048,133) \\
 & C(9906,1) \vdash (85,927,133) \\
 & C(17340,1) \vdash (263,203,871) \\
 & C(30349,0) \vdash (806,103,591) \\
 & C(53114,0) \vdash (2,468,672,331) \\
 & C(92951,0) \vdash (7,560,436,829) \\
 & C(162668,1) \vdash (23,154,325,799) \\
 & C(284673,0) \vdash (70,910,514,191) \\
 & C(498181,0) \vdash (217,164,134,715) \\
 & C(871820,0) \vdash (665,064,835,635) \\
 & C(1525687,1) \vdash (3,051,375) \\
 & \dots 01(12)^{762843}01(H0)0\dots
 \end{aligned}$$

#### 5.4.5 Lafitte and Papazian's machine found in August 2005

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from August to September 2005.

Lafitte and Papazian (2005)		0	1	2
$s(M) = 4,939,345,068$	A	1RB	1RH	2RB
$\sigma(M) = 107,900$	B	1LC	0LB	1RA
	C	1RA	2LC	1RC

Let  $C(n,0) = \dots 0(C0)2^n 0\dots$ ,  
and  $C(n,1) = \dots 0(C0)2^n 10\dots$



Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (3) & C(1,1) \\
C(4k,0) & \vdash (14k^2 + 16k + 5) & C(7k + 2,1) \\
C(4k + 1,0) & \vdash (14k^2 + 22k + 7) & C(7k + 3,0) \\
C(4k + 2,0) & \vdash (14k^2 + 30k + 15) & C(7k + 5,0) \\
C(4k + 3,0) & \vdash (14k^2 + 36k + 23) & C(7k + 7,1) \\
C(2k,1) & \vdash (2k + 2) & \dots 01(21)^k 1(H0)0\dots \\
C(4k + 1,1) & \vdash (14k^2 + 20k + 9) & C(7k + 3,1) \\
C(4k + 3,1) & \vdash (14k^2 + 34k + 21) & C(7k + 6,0)
\end{array}$$

So we have:

$$\begin{array}{ll}
\dots 0(A0)0\dots & \vdash (3) \\
C(1,1) & \vdash (9) \\
C(3,1) & \vdash (21) \\
C(6,0) & \vdash (59) \\
C(12,0) & \vdash (179) \\
C(23,1) & \vdash (541) \\
C(41,0) & \vdash (1,627) \\
C(73,0) & \vdash (4,939) \\
C(129,0) & \vdash (15,047) \\
C(227,0) & \vdash (45,943) \\
C(399,1) & \vdash (140,601) \\
C(699,0) & \vdash (430,151) \\
C(1225,1) & \vdash (1,317,033) \\
C(2145,1) & \vdash (4,032,873) \\
C(3755,1) & \vdash (12,349,729) \\
C(6572,0) & \vdash (37,818,579) \\
C(11503,1) & \vdash (115,816,521) \\
C(20131,0) & \vdash (354,675,511) \\
C(35231,1) & \vdash (1,086,184,945) \\
C(61655,0) & \vdash (3,326,402,857) \\
C(107898,1) & \vdash (107,900) \\
\dots 01(21)^{53949} 1(H0)0\dots &
\end{array}$$

#### 5.4.6 Souris's machine for $S(3,3)$

This machine was the record holder in the Busy Beaver Competition for  $S(3,3)$ , from July to August 2005.

Souris (2005)		0	1	2
$s(M) = 544,884,219$	A	1RB	1LB	2LA
$\sigma(M) = 32,213$	B	1LA	1RC	1RH
	C	0LA	2RC	1LC

Let  $C(n,0) = \dots 0(A0)1^n 0\dots$ ,  
and  $C(n,1) = \dots 0(A0)1^n 20\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (4) & C(3,0) \\
C(3k+2,0) & \vdash (21k^2+43k+19) & \dots 011(H2)2^{7k+1}0\dots \\
C(3k+3,0) & \vdash (21k^2+43k+24) & C(7k+7,0) \\
C(3k+4,0) & \vdash (21k^2+43k+26) & C(7k+7,1) \\
C(3k+1,1) & \vdash (21k^2+61k+35) & \dots 011(H2)2^{7k+3}0\dots \\
C(3k+2,1) & \vdash (21k^2+61k+42) & C(7k+9,0) \\
C(3k+3,1) & \vdash (21k^2+61k+46) & C(7k+9,1)
\end{array}$$

So we have:

$$\begin{array}{ll}
\dots 0(A0)0\dots & \vdash (4) \\
C(3,0) & \vdash (24) \\
C(7,0) & \vdash (90) \\
C(14,1) & \vdash (622) \\
C(37,0) & \vdash (3,040) \\
C(84,1) & \vdash (17,002) \\
C(198,1) & \vdash (92,736) \\
C(464,1) & \vdash (507,472) \\
C(1087,0) & \vdash (2,752,290) \\
C(2534,1) & \vdash (15,010,582) \\
C(5917,0) & \vdash (81,666,440) \\
C(13804,1) & \vdash (444,833,917) \\
\dots 011(H2)2^{32210}0\dots &
\end{array}$$

#### 5.4.7 Souris's machine for $\Sigma(3,3)$

This machine was the record holder in the Busy Beaver Competition for  $\Sigma(3,3)$ , from July to August 2005.

Souris (2005)	0	1	2
$s(M) = 310,341,163$	A 1RB 2RA 2RC	B 1LC 1RH 1LA	C 1RA 2LB 1LC
$\sigma(M) = 36,089$			

Let  $C(n,0) = \dots 0(C0)1^n 0\dots$ ,  
and  $C(n,1) = \dots 0(C0)1^n 210\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (4) & C(1,1) \\
C(2k+2,0) & \vdash (5k^2+32k+17) & C(5k+5,0) \\
C(2k+3,0) & \vdash (5k^2+32k+21) & C(5k+4,1) \\
C(2k+1,1) & \vdash (5k^2+32k+15) & C(5k+4,0) \\
C(2k+2,1) & \vdash (5k^2+37k+30) & \dots 012^{5k+5}1(H2)10\dots
\end{array}$$

So we have:

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (4) \\
C(1, 1) &\vdash (15) \\
C(4, 0) &\vdash (54) \\
C(10, 0) &\vdash (225) \\
C(25, 0) &\vdash (978) \\
C(59, 1) &\vdash (5, 148) \\
C(149, 0) &\vdash (29, 002) \\
C(369, 1) &\vdash (175, 183) \\
C(924, 0) &\vdash (1, 077, 374) \\
C(2310, 0) &\vdash (6, 695, 525) \\
C(5775, 0) &\vdash (41, 737, 353) \\
C(14434, 1) &\vdash (260, 620, 302) \\
\dots 012^{36085}1(H2)10\dots
\end{aligned}$$

Note that we have also, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (19) & C(4, 0) \\
C(2k + 2, 0) & \vdash (5k^2 + 32k + 17) & C(5k + 5, 0) \\
C(4k + 3, 0) & \vdash (145k^2 + 299k + 93) & \dots 012^{25k+10}1(H2)10\dots \\
C(4k + 5, 0) & \vdash (145k^2 + 444k + 281) & C(25k + 24, 0)
\end{array}$$

#### 5.4.8 Brady's machine

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from December 2004 to July 2005.

Brady (2004)		0	1	2
$s(M) = 92, 649, 163$	A	1RB	1RH	2LC
$\sigma(M) = 13, 949$	B	1LC	2RB	1LB
	C	1LA	0RB	2LA

Let  $C(n, 0) = \dots 0(A0)1^n 0 \dots$ ,  
and  $C(n, 1) = \dots 0(A0)1^n 210 \dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (6) & C(0, 1) \\
C(2k + 1, 0) & \vdash (4k + 5) & \dots 01(H2)2^{2k}10\dots \\
C(2k + 2, 0) & \vdash (10k^2 + 15k + 10) & C(5k + 3, 1) \\
C(2k, 1) & \vdash (10k^2 + 27k + 18) & C(5k + 5, 1) \\
C(2k + 1, 1) & \vdash (10k^2 + 51k + 60) & C(5k + 12, 0)
\end{array}$$

So we have:

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (6) \\
C(0,1) &\vdash (18) \\
C(5,1) &\vdash (202) \\
C(22,0) &\vdash (1,160) \\
C(53,1) &\vdash (8,146) \\
C(142,0) &\vdash (50,060) \\
C(353,1) &\vdash (318,796) \\
C(892,0) &\vdash (1,986,935) \\
C(2228,1) &\vdash (12,440,056) \\
C(5575,1) &\vdash (77,815,887) \\
C(13947,0) &\vdash (27,897) \\
\dots 01(H2)2^{13946}10\dots
\end{aligned}$$

Note that we have also, for all  $k \geq 0$ ,

$$\begin{array}{rcl}
\dots 0(A0)0\dots & \vdash (6) & C(0,1) \\
C(2k,1) & \vdash (10k^2 + 27k + 18) & C(5k + 5,1) \\
C(4k + 1,1) & \vdash (290k^2 + 677k + 395) & C(25k + 28,1) \\
C(4k + 3,1) & \vdash (40k^2 + 162k + 158) & \dots 01(H2)2^{10k+16}10\dots
\end{array}$$

## 5.5 Turing machines with 2 states and 4 symbols

### 5.5.1 Ligockis' champion

This machine is the record holder in the Busy Beaver Competition for machines with 2 states and 4 symbols, since February 2005.

Terry and Shawn Ligocki (2005)		0	1	2	3
$s(M) = 3,932,964 =? S(2,4)$	A	1RB	2LA	1RA	1RA
$\sigma(M) = 2,050 =? \Sigma(2,4)$	B	1LB	1LA	3RB	1RH

Let  $C(n,1) = \dots 0(A0)2^n 10\dots$ ,  
and  $C(n,2) = \dots 0(A0)2^n 110\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{rcl}
\dots 0(A0)0\dots & \vdash (6) & C(1,2) \\
C(3k,1) & \vdash (15k^2 + 9k + 3) & C(5k + 1,1) \\
C(3k + 1,1) & \vdash (15k^2 + 24k + 13) & \dots 013^{5k+2}1(H1)0\dots \\
C(3k + 2,1) & \vdash (15k^2 + 29k + 17) & C(5k + 4,2) \\
C(3k,2) & \vdash (15k^2 + 11k + 3) & C(5k + 1,2) \\
C(3k + 1,2) & \vdash (15k^2 + 21k + 7) & C(5k + 3,1) \\
C(3k + 2,2) & \vdash (15k^2 + 36k + 23) & \dots 013^{5k+4}1(H1)0\dots
\end{array}$$

So we have:

$$\begin{aligned}
&\dots 0(A0)0\dots \vdash (6) \\
&\quad C(1, 2) \vdash (7) \\
&\quad C(3, 1) \vdash (27) \\
&\quad C(6, 1) \vdash (81) \\
&\quad C(11, 1) \vdash (239) \\
&\quad C(19, 2) \vdash (673) \\
&\quad C(33, 1) \vdash (1, 917) \\
&\quad C(56, 1) \vdash (5, 399) \\
&\quad C(94, 2) \vdash (15, 073) \\
&\quad C(158, 1) \vdash (42, 085) \\
&\quad C(264, 2) \vdash (117, 131) \\
&\quad C(441, 2) \vdash (325, 755) \\
&\quad C(736, 2) \vdash (905, 527) \\
&\quad C(1228, 1) \vdash (2, 519, 044) \\
&\quad \dots 013^{2047}1(H1)0\dots
\end{aligned}$$

### 5.5.2 Brady's runner-up

This machine was the record holder in the Busy Beaver Competition for machines with 2 states and 4 symbols, from 1988 to February 2005.

Brady (1988)		0	1	2	3
$s(M) = 7, 195$	A	1RB	3LA	1LA	1RA
$\sigma(M) = 90$	B	2LA	1RH	3RA	3RB

Let  $C(n, 0) = \dots 0(A0)3^n 0\dots$ ,  
and  $C(n, 1) = \dots 0(A0)3^n 20\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{aligned}
C(3k, 0) &\vdash (15k^2 + 7k + 3) && C(5k + 1, 1) \\
C(3k + 1, 0) &\vdash (15k^2 + 22k + 11) && \dots 013^{5k+1}1(H0)0\dots \\
C(3k + 2, 0) &\vdash (15k^2 + 27k + 13) && C(5k + 4, 0) \\
C(3k, 1) &\vdash (15k^2 + 28k + 16) && \dots 013^{5k+3}1(H0)0\dots \\
C(3k + 1, 1) &\vdash (15k^2 + 33k + 19) && C(5k + 5, 0) \\
C(3k + 2, 1) &\vdash (15k^2 + 43k + 33) && C(5k + 7, 1)
\end{aligned}$$

So we have:

$$\begin{aligned}
&\dots 0(A0)0\dots = \\
&\quad C(0, 0) \vdash (3) \\
&\quad C(1, 1) \vdash (19) \\
&\quad C(5, 0) \vdash (55) \\
&\quad C(9, 0) \vdash (159) \\
&\quad C(16, 1) \vdash (559) \\
&\quad C(30, 0) \vdash (1, 573) \\
&\quad C(51, 1) \vdash (4, 827) \\
&\quad \dots 013^{88}1(H0)0\dots
\end{aligned}$$

## 5.6 Turing machines with 2 states and 5 symbols

### 5.6.1 Ligockis' champion

This machine is the record holder in the Busy Beaver Competition for machines with 2 states and 5 symbols, since November 2007.

Terry and Shawn Ligocki (2007)		0	1	2	3	4
$s(M)$ and $S(2, 5) > 1.9 \times 10^{704}$	A	1RB	2LA	1RA	2LB	2LA
$\sigma(M)$ and $\Sigma(2, 5) > 1.7 \times 10^{352}$	B	0LA	2RB	3RB	4RA	1RH

Let  $C(n, 1) = \dots 013^n(B0)0\dots$ ,  
and  $C(n, 2) = \dots 023^n(B0)0\dots$ ,  
and  $C(n, 3) = \dots 03^n(B0)0\dots$ ,  
and  $C(n, 4) = \dots 04113^n(B0)0\dots$ ,  
and  $C(n, 5) = \dots 04123^n(B0)0\dots$ ,  
and  $C(n, 6) = \dots 0413^n(B0)0\dots$ ,  
and  $C(n, 7) = \dots 0423^n(B0)0\dots$ ,  
and  $C(n, 8) = \dots 043^n(B0)0\dots$

Then we have, for all  $k \geq 0$ ,

$C(2k, 1)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 1, 1)$
$C(2k + 1, 1)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 1, 2)$
$C(2k, 2)$	$\vdash (3k^2 + 14k + 9)$	$C(3k + 2, 1)$
$C(2k + 1, 2)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 2, 3)$
$C(2k, 3)$	$\vdash (3k^2 + 8k + 2)$	$C(3k, 1)$
$C(2k + 1, 3)$	$\vdash (3k^2 + 8k + 22)$	$C(3k + 1, 4)$
$C(2k, 4)$	$\vdash (3k^2 + 8k + 8)$	$C(3k + 3, 1)$
$C(2k + 1, 4)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 1, 5)$
$C(2k, 5)$	$\vdash (3k^2 + 14k + 13)$	$C(3k + 4, 1)$
$C(2k + 1, 5)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 2, 6)$
$C(2k, 6)$	$\vdash (3k^2 + 8k + 6)$	$C(3k + 2, 1)$
$C(2k + 1, 6)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 1, 7)$
$C(2k, 7)$	$\vdash (3k^2 + 14k + 11)$	$C(3k + 3, 1)$
$C(2k + 1, 7)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 2, 8)$
$C(2k, 8)$	$\vdash (3k^2 + 8k + 4)$	$C(3k + 1, 1)$
$C(2k + 1, 8)$	$\vdash (3k^2 + 5k + 3)$	$\dots 01(H2)2^{3k}0\dots$

So we have:

$\dots 0(A0)0\dots$	$\vdash (1)$
$C(0, 1)$	$\vdash (4)$
$C(1, 1)$	$\vdash (4)$
$C(1, 2)$	$\vdash (4)$
$C(2, 3)$	$\vdash (13)$
$C(3, 1)$	$\vdash (15)$
$C(4, 2)$	$\vdash (49)$
$C(8, 1)$	$\vdash (84)$
$C(13, 1)$	$\vdash (160)$
$C(19, 2)$	$\vdash (319)$
$C(29, 3)$	$\vdash (722)$
$C(43, 4)$	$\vdash (1495)$
$C(64, 5)$	$\vdash (3533)$
$C(100, 1)$	$\vdash (7904)$
$\dots$	

### 5.6.2 Ligockis' machine found in August 2006

This machine was the record holder in the Busy Beaver Competition for machines with 2 states and 5 symbols, from August 2006 to October 2007.

Terry and Shawn Ligocki (2006)		0	1	2	3	4
$s(M) = 7, 069, 449, 877, 176, 007, 352, 687$	A	1RB	0RB	4RA	2LB	2LA
$\sigma(M) = 172, 312, 766, 455$	B	2LA	1LB	3RB	4RA	1RH

#### Analysis by Shawn Ligocki:

Let  $C(n, 1) = \dots 03^n(B0)0\dots$ ,  
and  $C(n, 2) = \dots 013^n(B0)0\dots$ ,  
and  $C(n, 3) = \dots 01403^n(B0)0\dots$ ,  
and  $C(n, 4) = \dots 01413^n(B0)0\dots$

Then we have, for all  $k \geq 0$ ,

$\dots 0(A0)0\dots$	$\vdash (1)$	$C(0, 2)$
$C(2k, 1)$	$\vdash (5k^2 + 14k + 3)$	$C(5k + 1, 2)$
$C(2k + 1, 1)$	$\vdash (5k^2 + 14k + 7)$	$C(5k + 3, 2)$
$C(2k, 2)$	$\vdash (5k^2 + 14k + 3)$	$C(5k + 1, 1)$
$C(2k + 1, 2)$	$\vdash (5k^2 + 14k + 11)$	$C(5k + 2, 3)$
$C(2k, 3)$	$\vdash (5k^2 + 14k + 3)$	$C(5k + 1, 4)$
$C(2k + 1, 3)$	$\vdash (5k^2 + 14k + 9)$	$C(5k + 4, 1)$
$C(2k, 4)$	$\vdash (5k^2 + 14k + 3)$	$C(5k + 1, 3)$
$C(2k + 1, 4)$	$\vdash (5k^2 + 9k + 4)$	$\dots 011(H1)2^{5k+2}0\dots$

So we have (in 30 transitions):

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (1) \\
C(0, 2) &\vdash (3) \\
C(1, 1) &\vdash (7) \\
C(3, 2) &\vdash (30) \\
C(7, 3) &\vdash (96) \\
C(19, 1) &\vdash (538) \\
&\dots \\
C(4411206821, 1) &\vdash (24, 323, 432, 041, 896, 588, 247) \\
C(11028017053, 2) &\vdash (152, 021, 450, 201, 199, 582, 755) \\
C(27570042632, 3) &\vdash (950, 134, 063, 605, 862, 157, 707) \\
C(68925106581, 4) &\vdash (5, 938, 337, 896, 640, 612, 100, 114) \\
&\dots 011(H1)2^{172312766452}0\dots
\end{aligned}$$

### 5.6.3 Lafitte and Papazian's machine found in June 2006

This machine was the record holder in the Busy Beaver Competition for  $\Sigma(2, 5)$ , from June to August 2006.

G. Lafitte and C. Papazian (2006)

$$s(M) = 14, 103, 258, 269, 249$$

$$\sigma(M) = 4, 848, 239$$

	0	1	2	3	4
A	1RB	3LB	4LB	4LA	2RA
B	2LA	1RH	3RB	4RA	3RB

Let  $C(n, 1) = \dots 0132^n 33(B0)0\dots$ ,

and  $C(n, 2) = \dots 01342^n 33(B0)0\dots$ ,

and  $C(n, 3) = \dots 0142^n 33(B0)0\dots$ ,

and  $C(n, 4) = \dots 012^n 33(B0)0\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (10) & C(0, 1) \\
C(2k, 1) & \vdash (3k^2 + 12k + 15) & C(3k + 2, 2) \\
C(2k + 1, 1) & \vdash (3k^2 + 12k + 11) & C(3k + 2, 3) \\
C(2k, 2) & \vdash (3k^2 + 12k + 9) & C(3k + 2, 1) \\
C(2k + 1, 2) & \vdash (3k^2 + 18k + 30) & C(3k + 5, 2) \\
C(2k, 3) & \vdash (3k^2 + 12k + 9) & C(3k + 2, 4) \\
C(2k + 1, 3) & \vdash (3k^2 + 18k + 28) & C(3k + 4, 2) \\
C(2k, 4) & \vdash (3k^2 + 12k + 13) & C(3k + 1, 2) \\
C(2k + 1, 4) & \vdash (3k^2 + 9k + 5) & \dots 01(H4)4^{3k+2}20\dots
\end{array}$$



So we have (in 36 transitions):

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (10) \\
C(0, 1) &\vdash (15) \\
C(2, 2) &\vdash (24) \\
C(5, 1) &\vdash (47) \\
C(8, 3) &\vdash (105) \\
C(14, 4) &\vdash (244) \\
&\dots \\
C(957674, 2) &\vdash (687, 860, 363, 760) \\
C(1436513, 1) &\vdash (1, 547, 683, 663, 691) \\
C(2154770, 3) &\vdash (3, 482, 288, 243, 304) \\
C(3232157, 4) &\vdash (7, 835, 138, 850, 959) \\
\dots 01(H4)4^{4848236}20\dots &
\end{aligned}$$

#### 5.6.4 Lafitte and Papazian's machine found in May 2006

This machine was the record holder in the Busy Beaver Competition for machines with 2 states and 5 symbols, from May to June 2006.

G. Lafitte and C. Papazian (2006)		0	1	2	3	4
$s(M) = 3, 793, 261, 759, 791$	A	1RB	3RA	4LB	2RA	3LA
$\sigma(M) = 2, 576, 467$	B	2LA	1RH	4RB	4RB	2LB

Let  $C(n, 1) = \dots 014^n(B0)0\dots$ ,  
and  $C(n, 2) = \dots 034^n(B0)0\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (1) & C(0, 1) \\
C(3k, 1) & \vdash (4k^2 + 17k + 11) & C(4k + 3, 1) \\
C(3k + 1, 1) & \vdash (4k^2 + 25k + 20) & C(4k + 4, 1) \\
C(3k + 2, 1) & \vdash (4k^2 + 17k + 13) & C(4k + 3, 2) \\
C(3k, 2) & \vdash (4k^2 + 17k + 11) & C(4k + 3, 1) \\
C(3k + 1, 2) & \vdash (4k^2 + 25k + 20) & C(4k + 4, 1) \\
C(3k + 2, 2) & \vdash (4k^2 + 21k + 24) & \dots 01(H2)23^{4k+3}20\dots
\end{array}$$

So we have (in 45 transitions):

$$\begin{array}{l}
\dots 0(A0)0\dots \vdash (1) \\
C(0, 1) \vdash (11) \\
C(3, 1) \vdash (32) \\
C(7, 1) \vdash (86) \\
C(12, 1) \vdash (143) \\
C(19, 1) \vdash (314) \\
\dots \\
C(815207, 1) \vdash (295, 364, 260, 408) \\
C(1086943, 2) \vdash (525, 094, 796, 254) \\
C(1449260, 1) \vdash (933, 496, 546, 059) \\
C(1932347, 2) \vdash (1, 659, 550, 059, 339) \\
\dots 01(H2)23^{2576463}20\dots
\end{array}$$

Note that we have also, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (1) & C(0, 1) \\
C(3k, 1) & \vdash (4k^2 + 17k + 11) & C(4k + 3, 1) \\
C(3k + 1, 1) & \vdash (4k^2 + 25k + 20) & C(4k + 4, 1) \\
C(9k + 2, 1) & \vdash (100k^2 + 151k + 45) & C(16k + 7, 1) \\
C(9k + 5, 1) & \vdash (100k^2 + 239k + 120) & C(16k + 12, 1) \\
C(9k + 8, 1) & \vdash (100k^2 + 279k + 186) & \dots 01(H2)23^{16k+15}20\dots
\end{array}$$

Note: The machine obtained by replacing  $B4 \rightarrow 2LB$  by  $B4 \rightarrow 3LB$  has the same behavior but final configuration  $\dots 01(H3)3^{2576464}20\dots$

### 5.6.5 Lafitte and Papazian's machine found in December 2005

This machine was the record holder in the Busy Beaver Competition for  $S(2, 5)$ , from December 2005 to May 2006.

G. Lafitte and C. Papazian (2005)		0	1	2	3	4
$s(M) = 924, 180, 005, 181$	A	1RB	3RA	1LA	1LB	3LB
$\sigma(M) = 1, 137, 477$	B	2LA	4LB	3RA	2RB	1RH

Let  $C(n, 1) = \dots 012^n(B0)0\dots$ ,  
and  $C(n, 2) = \dots 032^n(B0)0\dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (69) & C(8, 1) \\
C(2k + 1, 1) & \vdash (15k^2 + 37k + 31) & \dots 01221(H1)1^{5k+1}20\dots \\
C(2k + 2, 1) & \vdash (15k^2 + 32k + 19) & C(5k + 3, 2) \\
C(2k, 2) & \vdash (15k^2 + 32k + 19) & C(5k + 3, 1) \\
C(2k + 1, 2) & \vdash (15k^2 + 62k + 70) & C(5k + 9, 1)
\end{array}$$

So we have:

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (69) \\
C(8, 1) &\vdash (250) \\
C(18, 2) &\vdash (1, 522) \\
C(48, 1) &\vdash (8, 690) \\
C(118, 2) &\vdash (54, 122) \\
C(298, 1) &\vdash (333, 315) \\
C(743, 2) &\vdash (2, 087, 687) \\
C(1864, 1) &\vdash (13, 031, 226) \\
C(4658, 2) &\vdash (81, 438, 162) \\
C(11648, 1) &\vdash (508, 796, 290) \\
C(29118, 2) &\vdash (3, 179, 933, 122) \\
C(72798, 1) &\vdash (19, 873, 380, 815) \\
C(181993, 2) &\vdash (124, 209, 722, 062) \\
C(454989, 1) &\vdash (776, 311, 217, 849) \\
\dots 01221(H1)1^{1137471}20\dots
\end{aligned}$$

Note that we have also, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (69) & C(8, 1) \\
C(2k + 1, 1) & \vdash (15k^2 + 37k + 31) & \dots 01221(H1)1^{5k+1}20\dots \\
C(4k + 2, 1) & \vdash (435k^2 + 524k + 166) & C(25k + 14, 1) \\
C(4k + 4, 1) & \vdash (435k^2 + 884k + 453) & C(25k + 23, 1)
\end{array}$$

### 5.6.6 Lafitte and Papazian's machine found in October 2005

This machine was the record holder in the Busy Beaver Competition for  $\Sigma(2, 5)$ , from October 2005 to May 2006.

G. Lafitte and C. Papazian (2005)	0	1	2	3	4
$s(M) = 912, 594, 733, 606$	A	1RB	3LB	1RH	1LA
$\sigma(M) = 1, 957, 771$	B	2LA	3RB	4LB	3RA

Let  $C(n, 1) = \dots 0(A0)1^n 20 \dots$ ,  
and  $C(n, 2) = \dots 0(A0)1^n 40 \dots$ ,  
and  $C(n, 3) = \dots 0(A0)1^n 320 \dots$

Then we have, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (11) & C(3, 1) \\
C(2k + 1, 1) & \vdash (5k^2 + 28k + 26) & C(5k + 6, 1) \\
C(2k + 2, 1) & \vdash (5k^2 + 18k + 11) & C(5k + 3, 2) \\
C(2k, 2) & \vdash (5k^2 + 18k + 11) & C(5k + 3, 1) \\
C(2k + 1, 2) & \vdash (5k^2 + 18k + 13) & C(5k + 3, 3) \\
C(2k + 1, 3) & \vdash (5k^2 + 18k + 9) & C(5k + 3, 1) \\
C(2k + 2, 3) & \vdash (5k^2 + 23k + 17) & \dots 013^{5k+4}1(H0)0\dots
\end{array}$$

So we have:

$$\begin{aligned}
\dots 0(A0)0\dots &\vdash (11) \\
C(3,1) &\vdash (59) \\
C(11,1) &\vdash (291) \\
C(31,1) &\vdash (1,571) \\
C(81,1) &\vdash (9,146) \\
C(206,1) &\vdash (53,867) \\
C(513,2) &\vdash (332,301) \\
C(1283,3) &\vdash (2,065,952) \\
C(3208,1) &\vdash (12,876,910) \\
C(8018,2) &\vdash (80,432,578) \\
C(20048,1) &\vdash (502,483,070) \\
C(50118,2) &\vdash (3,140,218,478) \\
C(125298,1) &\vdash (19,624,987,195) \\
C(313243,2) &\vdash (122,653,507,396) \\
C(783108,3) &\vdash (766,577,764,781) \\
\dots 013^{1957769}1(H0)0\dots &
\end{aligned}$$

Note that we have also, for all  $k \geq 0$ ,

$$\begin{array}{lll}
\dots 0(A0)0\dots & \vdash (11) & C(3,1) \\
C(2k+1,1) & \vdash (5k^2+28k+26) & C(5k+6,1) \\
C(2k+2,1) & \vdash (5k^2+18k+11) & C(5k+3,2) \\
C(2k,2) & \vdash (5k^2+18k+11) & C(5k+3,1) \\
C(4k+1,2) & \vdash (145k^2+176k+45) & C(25k+8,1) \\
C(4k+3,2) & \vdash (145k^2+321k+167) & \dots 013^{25k+19}1(H0)0\dots
\end{array}$$

## 5.7 Collatz-like problems

Sameness of behaviors of the Turing machines above is striking. Their behaviors depend on transitions in the following form:

$$C(ak+b) \vdash ( ) C(ck+d),$$

where  $a, c$  are fixed, and  $b = 0, \dots, a-1$ . Sometimes, another parameter is added:  $C(ak+b, p)$ .

These transitions can be compared to the following problem. Let  $T$  be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

This can also be written

$$\begin{aligned}
T(2m) &= m \\
T(2m+1) &= 3m+2
\end{aligned}$$

When  $T$  is iterated over positive integers, do we always reach the loop:  $T(2) = 1, T(1) = 2$ ? This question is a famous open problem in mathematics, called  $3x+1$  problem, or *Collatz problem*.

A similar question can be asked about iterating transitions of configurations  $C(ak+b, p)$  on positive integers. Do the iterated transitions always reach a halting configuration? For

all the machines above (except for the machine with 6 states and 2 symbols in Section 5.3.5), this question is presently an open problem in mathematics. Because of likeness to Collatz problem, these problems are called *Collatz-like problems*. Thus, for each machine above (except for the machine with 6 states and 2 symbols in Section 5.3.5), the halting problem (that is, on what inputs does this machine stop?) depends on an open Collatz-like problem.

## 5.8 Non-Collatz-like behaviors

Some Turing machines run a large number of steps on a small piece of tape. Such machines do not seem to be Collatz-like. We list below some interesting machines with this sort of behavior.

### 5.8.1 Turing machines with 3 states and 3 symbols

A. H. Brady (November 2004) $s(M) = 2,315,619$ $\sigma(M) = 31$		0	1	2
	A	1RB	2LB	1LC
	B	1LA	2RB	1RB
	C	1RH	2LA	0LC

Brady called this machine “Surprise-in-a-Box”.

See also the simulation by Heiner Marxen:

[http://www.drb.insel.de/~heiner/BB/simAB3Y\\_SB.html](http://www.drb.insel.de/~heiner/BB/simAB3Y_SB.html)

### 5.8.2 Turing machines with 2 states and 5 symbols

#### (a) First machine

G. Lafitte and C. Papazian (July 2006) $s(M) = 26,375,397,569,930$ $\sigma(M) = 143$		0	1	2	3	4
	A	1RB	3LA	1LA	4LA	1RA
	B	2LB	2RA	1RH	0RA	0RB

This machine was the record holder for  $S(2,5)$ , from July to August 2006.

See also the simulation by Heiner Marxen:

[http://www.drb.insel.de/~heiner/BB/simLaf25\\_j.html](http://www.drb.insel.de/~heiner/BB/simLaf25_j.html)

#### (b) Second machine

G. Lafitte and C. Papazian (July 2006) $s(M) = 7,021,292,621$ $\sigma(M) = 37$		0	1	2	3	4
	A	1RB	4LA	1LA	1RH	2RB
	B	2LB	3LA	1LB	2RA	0RB

## 5.9 Turing machines in distinct classes with similar behaviors

In this section, we give examples of machines that have similar behaviors, but not the same numbers of states and symbols.

### 5.9.1 (2,4)-TM and (3,3)-TM

Terry and Shawn Ligoeki (2005)		0	1	2	3
$s(M) = 3,932,964 =? S(2,4)$	A	1RB	2LA	1RA	1RA
$\sigma(M) = 2,050 =? \Sigma(2,4)$	B	1LB	1LA	3RB	1RH

This machine is the record holder in the Busy Beaver Competition for machines with 2 states and 4 symbols, since February 2005.

A. H. Brady (2004)		0	1	2
$s(M) = 3,932,964$	A	1RB	1LC	1RH
$\sigma(M) = 2,050$	B	1LA	1LC	2RB
	C	1RB	2LC	1RC

There is a step-by-step correspondence between the configurations of these machines.

### 5.9.2 (6,2)-TM and (3,3)-TM

		0	1
	A	1RB	1RA
Marxen and Buntrock (1997)	B	1LC	1LB
$s(M) = 8,690,333,381,690,951$	C	0RF	1LD
$\sigma(M) = 95,524,079$	D	1RA	0LE
	E	1RH	1LF
	F	0LA	0LC

This machine was discovered in January 1990, and was published on the web (Google groups) on September 3, 1997. It was the record holder in the Busy Beaver Competition for machines with 6 states and 2 symbols up to July 2000.

		0	1	2
Terry and Shawn Ligoeki (2006)	A	1RB	2RC	1LA
$s(M) = 4,345,166,620,336,565$	B	2LA	1RB	1RH
$\sigma(M) = 95,524,079$	C	2RB	2RA	1LC

This machine was the record holder in the Busy Beaver Competition for machines with 3 states and 3 symbols, from August 2006 to November 2007.

Note that these machines have same  $\sigma$  value, and the  $s$  value of the first one is almost twice the  $s$  value of the second one.

The behaviors of these machines can be related as follows.

Giving the analyses of the (6,2)-TM in Section 5.3.7 and the (3,3)-TM in Section 5.4.2, the following functions  $f$  and  $g$  can be defined:

$$\left\{ \begin{array}{l} f(4k) \\ f(4k+1) \\ f(4k+2) \\ f(4k+3) \end{array} \right. = \left\{ \begin{array}{l} \text{undefined,} \\ 10k+9, \\ 10k+9, \\ 10k+12. \end{array} \right. \quad \left\{ \begin{array}{l} g(2k,0) \\ g(2k+1,0) \\ g(2k,1) \\ g(2k+1,1) \end{array} \right. = \left\{ \begin{array}{l} (5k+1,1), \\ \text{undefined,} \\ (5k,0), \\ (5k+3,1). \end{array} \right.$$

Now, let  $h$  be defined by

$$\begin{aligned} h(n,0) &= 10n+2, \\ h(n,1) &= 10n-1. \end{aligned}$$

Then:  $h \circ g = f \circ h$ .

There is no step-by-step correspondence between these machines, but there is a phase correspondence, according to functions  $f$  and  $g$ .

## 6 Relations between the busy beaver functions

Recall that  $S(n) = S(n,2)$ ,  $\Sigma(n) = \Sigma(n,2)$ , are the functions defined by Rado (1962). It is immediate that  $\Sigma(n) \leq S(n)$ . Can  $S(n)$  be bounded by a simple expression involving function  $\Sigma$ ? The following inequalities were founded.

- Rado (1962) proved that

$$S(n) < (n+1)\Sigma(5n)2^{\Sigma(5n)}.$$

- Julstrom (1992) proved that

$$S(n) < \Sigma(20n).$$

- Wang and Xu (1995) proved that

$$S(n) < \Sigma(10n).$$

- Yang, Ding and Xu (1997) proved that

$$S(n) < \Sigma(8n),$$

and that there is a constant  $c$  such that

$$S(n) < \Sigma(3n+c).$$

- Ben-Amram, Julstrom and Zwick (1996) proved that

$$S(n) < \Sigma(3n+6),$$

and

$$S(n) < (2n-1)\Sigma(3n+3).$$

- Ben-Amram and Petersen (2002) proved that there is a constant  $c$  such that

$$S(n) < \Sigma(n + 8n/\log_2 n + c).$$

## 7 Variants of busy beavers

### 7.1 Busy beavers defined by 4-tuples

The Turing machines used for regular busy beavers are based on 5-tuples. For example, the initial transition is

$$(A,0) \longrightarrow (1,R,B)$$

and generally a transition is

$$(\text{state, scanned symbol}) \longrightarrow (\text{new written symbol, move of the head, new state})$$

Instead of both writing a symbol and moving the head in one transition, these actions can be split up into two transitions, in the form of a 4-tuple:

$$(\text{state, scanned symbol}) \longrightarrow (\text{new written symbol or move of the head, new state})$$

This alternative definition was introduced by Post in 1947 (Recursive unsolvability of a problem of Thue, *The Journal of Symbolic Logic*, Vol. 12, 1-11). So Turing machines defined by 4-tuples are also called *Post machines*, or *Post-Turing machines*.

A busy beaver competition for such machines was studied by Oberschelp, Schmidt-Göttsch and Todt (1988), who defined two busy beaver functions, for the number of non-blank symbols, and for the number of steps, and gave some values and lower bounds for these functions.

The busy beaver competition for such machines are also studied by P. Machado and F. Pereira (See

<http://eden.dei.uc.pt/~machado/research/BB.html>),

and B. van Heuveln and his team (See

<http://www.cs.rpi.edu/~kelleo/busybeaver/index.html>).

### 7.2 Busy beavers whose head can stand still

In the definition of the Turing machines used for regular busy beavers, the tape head has to move one cell right or left at each step, and cannot stand still. If we allow the tape head to stand still, new machines come into the competition, and they can beat the current champions.

So Norbert B{á}tfai found, in August 2009, a Turing machine  $M$  with 5 states and 2 symbols with  $s(M) = 70,740,810$  and  $\sigma(M) = 4098$ . This machine beats the current champion for the number of steps ( $s = 47,176,870$ ). It seems that relaxing this condition on moves does not allow us to obtain machines with behaviors different from those of regular busy beavers. But the study is still to be done.



### 7.3 Two-dimensional busy beavers

The Turing machines used for regular busy beavers have a one-dimensional tape. Turing machines with two-dimensional or higher-dimensional tapes were first defined by Hartmanis and Stearns in 1965 (On the computational complexity of algorithms, *Transactions of the AMS*, Vol. 117, 285-306).

Brady (1988) launched the busy beaver competition for two-dimensional Turing machines. He also defined, first, “TurNing machines”, where the head reorients itself at each step, and, second, machines that work on a triangular grid.

Tim Hutton resumed the search for two-dimensional busy beavers (See <http://code.google.com/p/ruletablerepository/wiki/TwoDimensionalTuringMachines>). He gave the following results:

For  $S_2(k, n)$ : ( $k$  states,  $n$  symbols)

3 symbols	38	?	
2 symbols	6	32	4632 ?
	2 states	3 states	4 states

For  $\Sigma_2(k, n)$ : ( $k$  states,  $n$  symbols)

3 symbols	10	?	
2 symbols	4	11	244 ?
	2 states	3 states	4 states

Note that

$$S_2(3, 2) = 32 > S(3, 2) = 21,$$

and

$$\Sigma_2(3, 2) = 11 > \Sigma(3, 2) = 6.$$

Tim Hutton also studied higher-dimensional machines and found that, for all  $n > 0$ ,  $S_n(2, 2) = 6$  and  $\Sigma_n(2, 2) = 4$ .

He also studied one-dimensional and higher-dimensional Turing machines with *relative movements*, that is, where the head has an orientation and reorients itself at each step.

## 8 The methods

The machines presented in this paper were discovered by means of computer programs. These programs contain procedures that achieve the following tasks:

1. To enumerate Turing machines without repetition.
2. To simulate Turing machines efficiently.
3. To recognize non-halting Turing machines.

Note that these procedures are often mixed together in real programs.

If the purpose is to prove a value for the busy beaver functions, then all Turing machines in a class have to be studied. The machines that pass through the three procedures above are either halting machines, from which the better one is selected, or holdouts waiting for better programs or for hand analyses.

If the purpose is to find lower bounds, a systematic enumeration of machines is not necessary. Terry and Shawn Ligocki said they used simulated annealing to find some of their machines.

The following references can be consulted for more information:

- Brady (1983) for (4,2)-TM,
- Marxen and Buntrock (1990), Machlin and Stout (1990), and Hertel (2009) for (5,2)-TM,
- Lafitte and Papazian (2007) for (2,3)-TM,
- Page about Macro Machines on Marxen's website (See <http://www.drb.insel.de/~heiner/BB/macro.html>).

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