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ON THE FREQUENCY OF PARTIAL QUOTIENTS OF REGULAR CONTINUED FRACTIONS

AI-HUA FAN, LING-MIN LIAO, AND JI-HUA MA

Abstract. We consider sets of real numbers in $[0, 1)$ with prescribed frequencies of partial quotients in their regular continued fraction expansions. It is shown that the Hausdorff dimensions of these sets, always bounded from below by $1/2$, are given by a modified variational principle.

1. Introduction

Let $\mathbb{Q}^c$ denote the set of irrational number. It is well-known that each $x \in [0, 1) \cap \mathbb{Q}^c$ possesses a unique continued fraction expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}}.$$  \hfill (1.1)

where $a_k(x) \in \mathbb{N} := \{1, 2, 3, \cdots\}$ is the $k$-th partial quotient of $x$. This expansion is usually denoted by $x = [a_1(x), a_2(x), \cdots]$. For each $j \in \mathbb{N}$, define the frequency of the digit $j$ in the continued fraction expansion of $x$ by

$$\tau_j(x) := \lim_{n \to \infty} \frac{\tau_j(x, n)}{n},$$

whenever the limit exists, where

$$\tau_j(x, n) := \text{Card}\{1 \leq k \leq n : a_k(x) = j\}.$$

This paper is concerned with sets of real numbers with prescribed digit frequencies in their continued fraction expansions. To be precise, let $\vec{p} = (p_1, p_2, \ldots)$ be a probability vector with $p_j \geq 0$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} p_j = 1$, which will be called a frequency vector in the sequel. Our purpose is to determine the Hausdorff dimension of the set

$$\mathcal{E}_{\vec{p}} := \{x \in [0, 1) \cap \mathbb{Q}^c : \tau_j(x) = p_j \ \forall j \geq 1\}.$$

Let us first recall some notation. For any $a_1, a_2, \cdots, a_n \in \mathbb{N}$, we call

$$I(a_1, a_2, \cdots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \cdots, a_n(x) = a_n\}$$

a rank-$n$ basic interval. Let $T : [0, 1) \to [0, 1)$ be the Gauss transformation defined by

$$T(0) = 0, \quad T(x) = 1/x \quad \text{(mod 1)} \quad \text{for } x \in (0, 1).$$

For a given frequency vector $\vec{p} = (p_1, p_2, \ldots)$, we denote by $\mathcal{N}(\vec{p})$ the set of $T$-invariant ergodic probability measures $\mu$ such that

$$\int |\log x| d\mu < \infty \quad \text{and} \quad \mu(I(j)) = p_j \quad \text{for all } j \geq 1.$$  \hfill (1.2)
Let \( h_\mu \) stand for the measure-theoretical entropy of \( \mu \), and \( \dim_H A \) for the Hausdorff dimension of a set \( A \). The main result of this paper can be stated as follows.

**Theorem 1.1.** For any frequency vector \( \vec{p} \), one has

\[
\dim_H(\mathcal{E}_{\vec{p}}) = \max \left\{ \frac{1}{2}, \sup_{\mu \in \mathcal{N}(\vec{p})} \left( \frac{h_\mu}{2 \int |\log x| d\mu} \right) \right\},
\]

where the “sup” is set to be zero if \( \mathcal{N}(\vec{p}) = \emptyset \).

By virtue of \( \log |T'(x)| = 2|\log x| \), we see that \( 2 \int |\log x| d\mu \) is the Liapunov exponent of the measure \( \mu \). Therefore, the “sup” in the above is a variational formula which relates the Hausdorff dimension to the entropy and Liapunov exponent of measures.

Theorem 1.1 provides a complete solution to the long standing problem that requests an exact formula for \( \dim_H(\mathcal{E}_{\vec{p}}) \). Let us recall some partial results in the literature. In 1966, under the condition that \( \sum_{j=1}^{\infty} p_j \log j < \infty \), Kinney and Pitcher [9] showed that

\[
\dim_H(\mathcal{E}_{\vec{p}}) \geq -\sum_{j=1}^{\infty} p_j \log p_j -\frac{1}{2} \sum_{j \geq 1} \log j \int |\log x| d\mu_{\vec{p}}.
\]

where \( \mu_{\vec{p}} \) is the Bernoulli measure on \([0,1]\) defined by

\[
\mu_{\vec{p}}(I(a_1, a_2, \ldots, a_n)) = \prod_{j=1}^{n} p_{a_j}.
\]

The above lower bound is just the Hausdorff dimension of the Bernoulli measure \( \mu_{\vec{p}} \). However, by a result of Kifer, Peres and Weiss in 2001, this is not an optimal lower bound. Indeed, it is shown in [8] that, for any Bernoulli measure \( \mu_{\vec{p}} \),

\[
\dim_H \mu_{\vec{p}} \leq 1 - 10^{-7}
\]

This surprising fact indicates that the collection of Bernoulli measures are insufficient for providing the correct lower bound for \( \dim_H(\mathcal{E}_{\vec{p}}) \).

In 1975, under the same condition \( \sum_{j=1}^{\infty} p_j \log j < \infty \), Billingsley and Henningsen [2] obtained an improved lower bound

\[
\dim_H(\mathcal{E}_{\vec{p}}) \geq \sup_{\mu \in \mathcal{N}(\vec{p})} \left( \frac{h_\mu}{2 \int |\log x| d\mu} \right).
\]

Moreover, they proved that, for any fixed \( N \in \mathbb{N} \), this lower bound is the exact Hausdorff dimension of the set

\[
\{ x \in \mathcal{E}_{\vec{p}} : a_n(x) \leq N \text{ for all } n \geq 1 \}
\]

provided that \( p_j = 0 \) for all \( j > N \). It is therefore quite natural to guess that this lower bound is the right value for \( \dim_H \mathcal{E}_{\vec{p}} \) in general. However, as will be shown in Theorem 1.3, this is not the case.

Actually, the lower bound due to Billingsley and Henningsen is only a half of the correct lower bound. The other half of the lower bound, namely, \( \dim_H(\mathcal{E}_{\vec{p}}) \geq 1/2 \), can be proved basing on Lemma 2.4 in [11]. However, we will give a direct proof in this paper.

The upper bound estimate is more difficult. In its proof, we will use techniques from [11] and [3] to estimate the lengths of basic intervals. Not incidentally, an entropy-involved combinatorial lemma borrowed from [3] (see Lemma 2.7) will play an important role.
ON THE FREQUENCY OF PARTIAL QUOTIENTS OF REGULAR CONTINUED FRACTIONS

The paper is organized as follows. In Section 2, we give some preliminaries on the basic intervals and on the entropy of finite words. In Section 3, we establish the upper bound in Theorem 1.1. In Section 4, we prove that \( \dim_H(\mathcal{C}_p) \geq 1/2 \) and show that we can drop the condition \( \sum_{j=1}^{\infty} p_j \log j < \infty \) in Billingsley and Henningsen’s theorem and then obtain the lower bound in Theorem 1.1. The last section serves as a remark.

2. Preliminary

Let \( x = [a_1(x), a_2(x), \cdots] \in [0, 1) \cap \mathbb{Q}^c \). The \( n \)-th convergent in the continued fraction expansion of \( x \) is defined by

\[
\frac{p_n}{q_n} := \frac{p_n(a_1(x), \cdots, a_n(x))}{q_n(a_1(x), \cdots, a_n(x))} = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots + \frac{1}{a_n(x)}}}.
\]

For ease of notation, we shall drop the argument \( x \) in what follows. It is known (see [6] p.9) that \( p_n, q_n \) can be obtained by the recursive relations:

\[
\begin{align*}
    p_{n-1} &= 1, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 2), \\
    q_{n-1} &= 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 2).
\end{align*}
\]

By the above recursion relations, we have the following results.

Lemma 2.1 ([6]). Let \( q_n = q_n(a_1, \cdots, a_n) \) and \( p_n = p_n(a_1, \cdots, a_n) \), we have

(i) \( p_{n-1}q_n - p_nq_{n-1} = (-1)^n \);

(ii) \( q_n \geq 2^{n-1}, \quad \prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n} (a_k + 1) \).

Lemma 2.2 ([14]). For any \( a_1, a_2, \cdots, a_n, b \in \mathbb{N} \),

\[
\frac{b+1}{2} \leq \frac{q_{n+1}(a_1, \cdots, a_j, b, a_{j+1}, \cdots, a_n)}{q_n(a_1, \cdots, a_j, a_{j+1}, \cdots, a_n)} \leq b + 1 \quad (\forall 1 \leq j < n).
\]

Recall that for any \( a_1, a_2, \cdots, a_n \in \mathbb{N} \), the set

\[ I(a_1, a_2, \cdots, a_n) = \{ x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \cdots, a_n(x) = a_n \} \]

is a rank-\( n \) basic interval. We write \( |I| \) for the length of an interval \( I \).

Lemma 2.3 ([10] p.18). The basic interval \( I(a_1, a_2, \cdots, a_n) \) is an interval with endpoints \( p_n/q_n \) and \( (p_n + p_{n-1})/(q_n + q_{n-1}) \). Consequently, one has

\[
\left| I(a_1, \cdots, a_n) \right| = \frac{1}{q_n(q_n + q_{n-1})}, \quad (2.1)
\]

and

\[
\frac{1}{2q_n^2} \leq \left| I(a_1, \cdots, a_n) \right| \leq \frac{1}{q_n}. \quad (2.2)
\]

Lemma 2.4. We have

\[
|I(x_1, \cdots, j, \cdots, x_n)| \leq \frac{8}{(j + 1)^2} \left| I(x_1, \cdots, j, \cdots, x_n) \right|,
\]

where the notation \( \hat{j} \) means “deleting the digit \( j \).”
We will simply denote by \( I_n(x) \) the rank \( n \) basic interval containing \( x \). Suppose that \( a_n := a_n(x) \geq 2 \) and consider

\[ I'_n(x) = I(a_1, \cdots, a_{n-1}, a_n - 1) \quad \text{and} \quad I''_n(x) = I(a_1, \cdots, a_{n-1}, a_n + 1) \]

which are two rank \( n \) basic intervals adjacent to \( I_n(x) \). By the recursive equation of \( q_n \) and (2.2), one has the following lemma.

**Lemma 2.5.** Suppose that \( a_n := a_n(x) \geq 2 \). Then the lengths of the adjacent intervals \( I'_n(x) \) and \( I''_n(x) \) are bounded by \( \frac{4}{3} |I_n(x)| \) from below and by \( 3 |I_n(x)| \) from above.

For any \( x \in [0, 1] \setminus \mathbb{Q} \) and any word \( i_1 \cdots i_k \in \mathbb{N}^k, (k \geq 1) \), denote by \( \tau_{i_1 \cdots i_k} (x, n) \) the number of \( j, 1 \leq j \leq n \), for which

\[ a_j(x) \cdots a_{j+k-1}(x) = i_1 \cdots i_k. \]

For \( N \in \mathbb{N} \), define \( \Sigma_N := \{1, \ldots, N\} \). We shall use the following estimate in (3).

**Lemma 2.6 (3).** Let \( N \geq 1 \) and \( n \geq 1 \). For any \( x = [x_1, x_2, \cdots] \in [0, 1] \cap \mathbb{Q}^c \) with \( x_j \in \Sigma_N \) for \( 1 \leq j \leq n \). Then for any \( k \geq 1 \), we have

\[
\log |I_n(x)| \leq 2 \sum_{i_1, \cdots, i_k \in \Sigma_N} \tau_{i_1 \cdots i_k} (x, n) \log \frac{p_k(i_1, \cdots, i_k)}{q_k(i_1, \cdots, i_k)} + 8 + \frac{8n}{2k}. \tag{2.3}
\]

Now we turn to the key combinatorial lemma which will be used in the upper bound estimation. Let \( \phi : [0, 1] \to \mathbb{R} \) denote the function

\[ \phi(0) = 0, \quad \text{and} \quad \phi(t) = -t \log t \quad \text{for} \quad 0 < t \leq 1. \]

For every word \( \omega \in \Sigma_N^n \) of length \( n \) and every word \( u \in \Sigma_N^k \) of length \( k \), denote by \( p(u|\omega) \) the frequency of appearances of \( u \) in \( \omega \), i.e.,

\[ p(u|\omega) = \frac{\tau_u(\omega)}{n - k + 1}, \]

where \( \tau_u(\omega) \) denote the number of \( j, 1 \leq j \leq n - k + 1 \), for which

\[ \omega_j \cdots \omega_{j+k-1} = u. \]

Define

\[ H_k(\omega) := \sum_{u \in \Sigma_N^k} \phi(p(u|\omega)). \]

We have the following counting lemma.

**Lemma 2.7 (3).** For any \( h > 0, \epsilon > 0, k \in \mathbb{N} \), and for any \( n \in \mathbb{N} \) large enough, we have

\[ \text{Card}\{\omega \in \Sigma_N^n : H_k(\omega) \leq kh\} \leq \exp(n(h + \epsilon)). \]
3. Upper Bound

3.1. Some Lemmas. Let \((p(i_1, \ldots, i_k))_{i_1, \ldots, i_k} \in \mathbb{N}^k\) be a probability vector indexed by \(\mathbb{N}^k\). As usual, we denote by \(q_k(a_1, \ldots, a_k)\) the denominator of the \(k\)-th convergent of a real number with leading continued fraction digits \(a_1, \ldots, a_k\).

**Lemma 3.1.** For each \(k \in \mathbb{N}\) and each probability vector \((p(i_1, \ldots, i_k))_{i_1, \ldots, i_k} \in \mathbb{N}^k\),
\[
\sum_{i_1, \ldots, i_k} -p(i_1, \ldots, i_k) \log p(i_1, \ldots, i_k) \leq \sum_{i_1, \ldots, i_k} -p(i_1, \ldots, i_k) \log |I(i_1, \ldots, i_k)|.
\]

**Proof.** Applying Jesen’s inequality to the concave function \(\log\), we have
\[
\sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) \log \frac{|I(i_1, \ldots, i_k)|}{p(i_1, \ldots, i_k)} \leq \log \sum_{i_1, \ldots, i_k} |I(i_1, \ldots, i_k)| = 0.
\]

**Lemma 3.2.** Let \(\vec{p} = (p_1, p_2, \ldots)\) be a probability vector and \(\vec{q} = (q_1, q_2, \ldots)\) a positive vector. Suppose \(-\sum_{j=1}^{\infty} p_j \log q_j = \infty\) and \(\sum_{j=1}^{\infty} q_j^s < \infty\) for some positive number \(s\). Then
\[
\limsup_{n \to \infty} \frac{-\sum_{j=1}^{n} p_j \log p_j}{-\sum_{j=1}^{n} p_j \log q_j} \leq s.
\]

**Proof.** This is a consequence of the following inequality (see [13], p.217): for non-negative numbers \(s_j (1 \leq j \leq m)\) such that \(\sum_{j=1}^{m} s_j = 1\) and any real numbers \(t_j (1 \leq j \leq m)\), we have
\[
\sum_{j=1}^{m} s_j (t_j - \log s_j) \leq \log(\sum_{j=1}^{m} e^{t_j}). \quad (3.1)
\]

Fix \(n \geq 1\). Let \(s_j = p_j\) for \(1 \leq j \leq n\) and \(s_{n+1} = \sum_{j=n+1}^{\infty} p_j\). Let \(t_j = s \log q_j\) for \(1 \leq j \leq n\) and \(t_{n+1} = 0\). Applying the above inequality (3.1) with \(m = n + 1\), we get
\[
s \sum_{j=1}^{n} p_j \log q_j - \sum_{j=1}^{n} p_j \log p_j - (\sum_{j=n+1}^{\infty} p_j) \log(\sum_{j=n+1}^{\infty} p_j) \leq \log(1 + \sum_{j=1}^{n} q_j^s).
\]

Consequently,
\[
-\sum_{j=1}^{n} p_j \log p_j \leq s + (\sum_{j=n+1}^{\infty} p_j) \log(\sum_{j=n+1}^{\infty} p_j) - \sum_{j=1}^{n} p_j \log q_j - (\sum_{j=n+1}^{\infty} p_j) \log(\sum_{j=n+1}^{\infty} p_j) - \sum_{j=1}^{n} p_j \log q_j.
\]

Using the facts \(-\sum_{j=1}^{\infty} p_j \log q_j = \infty\) and \(\sum_{j=1}^{\infty} q_j^s < \infty\), we finish the proof by letting \(n \to \infty\).

**Lemma 3.2** implies the following lemma. Recall that \(\Sigma_N^k = \{1, \ldots, N\}^k\).

**Lemma 3.3.** Let \(k \geq 1\). If \((p(i_1, \ldots, i_k))_{i_1, \ldots, i_k} \in \mathbb{N}^k\) is a probability vector such that
\[
\sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) q_k(i_1, \ldots, i_k) = \infty,
\]
then we have
\[
\limsup_{N \to \infty} \frac{-\sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) \log p(i_1, \ldots, i_k)}{2 \sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) \log q_k(i_1, \ldots, i_k)} \leq \frac{1}{2}.
\]
Proof. By Lemma 3.1, for any \( k \in \mathbb{N} \) and any \( s > 1/2 \), we have
\[
\sum_{i_1, \ldots, i_k} q_k(i_1, \ldots, i_k)^{-2s} \leq \sum_{i_1, \ldots, i_k} (i_1 \cdots i_k)^{-2s} = (\sum_{j=1}^{\infty} j^{-2s})^k < \infty.
\]
Thus we get the result by Lemma 3.3.

3.2. Proof of the upper bound. To prove the upper bound, we shall make use of multi-step Markov measures. Let \( k \geq 1 \), by a \((k-1)\)-step Markov measure, we mean a \( T \)-invariant probability measure \( P \) on \([0, 1]\) satisfying the Markov property
\[
P(I(a_1, \ldots, a_n)) = \frac{P(I(a_{n-k}, \ldots, a_n))}{P(I(a_{n-k}, \ldots, a_{n-1}))}
\]
for all \( n \geq 1 \) and \( a_1, \ldots, a_n \in \mathbb{N} \) (see [3], p.9). We may regard a Bernoulli measure as a \( 0 \)-step Markov measure.

For each \( N \geq 2 \), we denote by \( \mathbb{P}_N^k = \mathbb{P}_N^k(\mathbb{P}) \) the collection of \((k-1)\)-step Markov measures satisfying the condition
\[
P(I(j)) = p_j \text{ for } 1 \leq j \leq N - 1 \text{ and } P(I(N)) = 1 - \sum_{j=1}^{N-1} p_j.
\]
These Markov measures are supported by the set of continued fractions for which the partial quotients are bounded from above by \( N \).

For each \( i_1 \cdots i_k \in \{1, 2, \ldots, N\}^k \), write \( p(i_1, \ldots, i_k) = P(I(i_1, \ldots, i_k)) \). Put
\[
\alpha_{N,k} := \sup_{P \in \mathbb{P}_N^k} -\frac{1}{k} \sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) \log p(i_1, \ldots, i_k)
\]
(3.4)
The argument in [3] (pp.171-172) shows that the following limit
\[
\alpha_N := \lim_{k \to \infty} \alpha_{N,k}
\]
essists and coincides with each of the following three limits:
\[
\alpha'_{N} := \lim_{k \to \infty} \sup_{P \in \mathbb{P}_N^k} -\frac{1}{2k} \sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) \log p(i_1, \ldots, i_k)
\]
(3.5)
and
\[
\alpha''_{N} := \lim_{k \to \infty} \sup_{P \in \mathbb{P}_N^k} -\frac{1}{k} \sum_{i_1, \ldots, i_k} p(i_1, \ldots, i_k) \log \left| I(i_1, \ldots, i_k) \right|
\]
(3.6)
and
\[
\alpha'''_{N} := \lim_{k \to \infty} \sup_{P \in \mathbb{P}_N^k} \frac{h_P}{2 \int \log x \, dP}.
\]
Let
\[
\alpha := \lim_{N \to \infty} \sup_{N \to \infty} \alpha_N = \lim_{N \to \infty} \sup_{N \to \infty} \alpha'_N = \lim_{N \to \infty} \sup_{N \to \infty} \alpha''_N = \lim_{N \to \infty} \sup_{N \to \infty} \alpha'''_N.
\]
To prove the upper bound, we need only to prove the following two propositions.

Proposition 3.4. For any \( N \in \mathbb{N} \) large enough, we have
\[
\dim_H(E_P) \leq \max \left\{ \frac{1}{2}, \alpha_N \right\}.
\]
Proposition 3.5. We have
\[ \alpha \leq \max \left\{ \frac{1}{2}, \sup_{\mu \in N(\vec{p})} \frac{h_\mu}{2 \int |\log x| d\mu} \right\}. \]

Remark that we will finally establish the formula in Theorem 1.1, thus by Proposition 3.4 and Proposition 3.5 we have
\[ \max \left\{ \frac{1}{2}, \liminf_{N \to \infty} \alpha_N \right\} = \max \left\{ \frac{1}{2}, \limsup_{N \to \infty} \alpha_N \right\} = \max \left\{ \frac{1}{2}, \sup_{\mu \in N(\vec{p})} h_\mu \right\}, \]
and if \( \liminf_{N \to \infty} \alpha_N \geq \frac{1}{2} \), then the limit of \( \alpha_N \) exists and equals to
\[ \sup_{\mu \in N(\vec{p})} h_\mu. \]

Proof of Proposition 3.5. By virtue of (3.5), we set
\[ \alpha = \limsup_{N \to \infty} \limsup_{k \to \infty} \sup_{P \in P_N^k} \frac{\sum p(i_1, \ldots, i_k) \log p(i_1, \ldots, i_k)}{2 \sum p(i_1, \ldots, i_k) \log q_k(i_1, \ldots, i_k)}. \]

Denote
\[ D := \limsup_{N \to \infty} \limsup_{k \to \infty} \sup_{P \in P_N^k} 2 \int |\log x| dP. \]

If \( D = \infty \), then by Lemma 3.3, we have \( \alpha \leq \frac{1}{2} \).

Now suppose that \( D < \infty \), which is equivalent to
\[ \limsup_{N \to \infty} \limsup_{k \to \infty} \sup_{P \in P_N^k} \int |\log x| dP < \infty. \]

By (3.5),
\[ \alpha = \limsup_{N \to \infty} \limsup_{k \to \infty} \sup_{P \in P_N^k} \frac{h_P}{2 \int |\log x| dP}. \]

Without loss of generality, we may suppose that there is a sequence of Markov measures \( P_{N,k} \in P_N^k \) converging to a measure \( \mu \in N(\vec{p}) \) in the weak*-topology such that
\[ \limsup_{N \to \infty} \limsup_{k \to \infty} \frac{h_{P_{N,k}}}{2 \int |\log x| dP_{N,k}} = \alpha. \]

Then by the upper semi-continuity of the entropy function and the weak convergence, we have
\[ \alpha \leq \sup_{\mu \in N(\vec{p})} \frac{h_\mu}{2 \int |\log x| d\mu}. \]

\[ \square \]

Proof of Proposition 3.4. For any fixed integer \( N \) which is large enough, and any \( \epsilon > 0 \), we have
\[ E_{\vec{p}} \subset \bigcup_{\ell = 1}^{\infty} \bigcap_{n=\ell}^{\infty} H_n(\epsilon, N), \]

where
\[ H_n(\epsilon, N) := \left\{ x \in [0,1) \setminus \mathbb{Q} : \left| \frac{\tau_j(x,n)}{n} - p_j \right| < \epsilon, 1 \leq j \leq N \right\}. \]
For any $\gamma > \max \{1/2, \alpha_N\}$ and for any integer $k \in \mathbb{N}$, we have the $\gamma$-Hausdorff measure (see [3], for the definition of $\mathcal{H}^\gamma$)

$$
\mathcal{H}^\gamma \left( \bigcap_{n=\ell}^{\infty} H_n(\epsilon, N) \right) 
\leq \sum_{\lceil \frac{\tau_j(x, n)}{n} \rceil \epsilon_1 \leq j \leq N} |I_n(x)|^\gamma 
(\forall n \geq \ell)
= \sum_{n(p_j - \epsilon) < m_j < n(p_j + \epsilon), 1 \leq j \leq N} \sum_{x_1 \cdots x_n \in A} |I(x_1, \cdots, x_n)|^\gamma,
$$
where

$$A := \{x_1 \cdots x_n \in \Sigma_N^\infty : \tau_j(x_1 \cdots x_n) = m_j, 1 \leq j \leq N\}.$$ 

(We recall that $\tau_j(x_1 \cdots x_n)$ denotes the times of appearances of $j$ in $x_1 \cdots x_n$.)

Let $\tilde{n} := \sum_{j=1}^{N} m_j$. By Lemma 2.6, we have the following estimate by deleting the digits $j > N$ in the first $n$ partial quotients $x_1, \ldots, x_n$ of $x \in I_n(x)$:

$$
\sum_{x_1 \cdots x_n \in A} |I(x_1, \cdots, x_n)|^\gamma \leq \left( \sum_{j=N+1}^{\infty} \frac{8}{(j+1)^{2\gamma}} \right)^{n-\tilde{n}} \sum_{x_1 \cdots x_n \in \tilde{A}} |I(x_1, \cdots, x_n)|^\gamma,
$$
where

$$\tilde{A} := \{x_1 \cdots x_{\tilde{n}} \in \Sigma_N^\infty : \tau_j(x_1 \cdots x_{\tilde{n}}) = m_j, 1 \leq j \leq N\}.$$ 

Since $\gamma > 1/2$, for $N$ large enough

$$\sum_{j=N+1}^{\infty} \frac{8}{(j+1)^{2\gamma}} < 1. \quad (3.6)$$

By applying Lemma 2.7 and noticing that $\tau_{i_1 \cdots i_k}(x, \tilde{n}) \leq \tau_{i_1 \cdots i_k}(x_1 \cdots x_{\tilde{n}}) + k$, we have

$$|I(x_1, \cdots, x_{\tilde{n}})| = \exp \{\log |I(x_1, \cdots, x_{\tilde{n}})|\}$$

$$\leq \exp \left\{ 2 \sum_{i_1 \cdots i_k \in \Sigma_N^k} \left( \tau_{i_1 \cdots i_k}(x_1 \cdots x_{\tilde{n}}) + k \right) \log \frac{p_k(i_1, \cdots, i_k)}{q_k(i_1, \cdots, i_k)} + 8 + \frac{8\tilde{n}}{2^k} \right\}.$$ 

Thus

$$\sum_{x_1 \cdots x_n \in \tilde{A}} |I(x_1, \cdots, x_n)|^\gamma$$

$$\leq \sum_{m_{i_1 \cdots i_k} x_1 \cdots x_n \in B} \exp \left\{ 2\gamma \sum_{i_1 \cdots i_k \in \Sigma_N^k} (m_{i_1 \cdots i_k} + k) \log \frac{p_k}{q_k} + 8 + \frac{8\tilde{n}\gamma}{2^k} \right\},$$
where

$$B := \{x_1 \cdots x_{\tilde{n}} \in \tilde{A} : \tau_{i_1 \cdots i_k}(x_1 \cdots x_{\tilde{n}}) = m_{i_1 \cdots i_k} \forall i_1 \cdots i_k \in \Sigma_N^k\}.$$ 

Take

$$h = \frac{1}{k} \sum_{i_1 \cdots i_k \in \Sigma_N^k} \phi \left( \frac{m_{i_1 \cdots i_k}}{\tilde{n} - k + 1} \right) \quad (3.7)$$
in Lemma 2.7. We have for any $\delta > 0$ and for $\tilde{n}$ large enough

$$\sum_{x_{1} \cdots x_{n} \in B} \exp \left\{ 2\gamma \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} (m_{i_{1} \cdots i_{k}} + k) \log \frac{p_{k}}{q_{k}} + 8\gamma + \frac{8\tilde{n}\gamma}{2^{k}} \right\}$$

$$\leq \exp \left\{ \tilde{n} (h + \delta) + 2\gamma \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} (m_{i_{1} \cdots i_{k}} + k) \log \frac{p_{k}}{q_{k}} + 8\gamma + \frac{8\tilde{n}\gamma}{2^{k}} \right\}.$$

Rewrite the right side of the above inequality as

$$\exp \left\{ \tilde{n} (L(\gamma, k, m_{i_{1} \cdots i_{k}})) \right\},$$

where

$$L(\gamma, k, m_{i_{1} \cdots i_{k}}) := h + 2\gamma \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} \frac{m_{i_{1} \cdots i_{k}} + k}{\tilde{n}} \log \frac{p_{k}}{q_{k}} + \frac{8\gamma}{\tilde{n}} + \frac{8\gamma}{2^{k}} + \delta.$$

Since there are at most $(\tilde{n} - k + 1)^{N_{k}}$ possible words of $i_{1} \cdots i_{k}$ in $\Sigma_{N}^{k}$, we have

$$\sum_{x_{1} \cdots x_{n} \in \tilde{A}} |I(x_{1}, \cdots, x_{n})|^\gamma$$

$$\leq (\tilde{n} - k + 1)^{N_{k}} \exp \left\{ \tilde{n} \left( \sup_{m_{i_{1} \cdots i_{k}}} L(\gamma, k, m_{i_{1} \cdots i_{k}}) \right) \right\}.$$

Notice that by the definition of $\tilde{A}$ and $B$, the possible values of $m_{i_{1} \cdots i_{k}}$ are restricted to satisfy the condition that the frequency of digit $j$ in $x_{1} \cdots x_{n}$ is about $p_{j}$. Then when $\tilde{n} \to \infty$, for $i_{1}, \cdots, i_{k} \in \Sigma_{N}^{k}$

$$\frac{m_{i_{1} \cdots i_{k}}}{\tilde{n} - k + 1} \to p(i_{1}, \cdots, i_{k}),$$

and $\{p(i_{1}, \cdots, i_{k}) : i_{1}, \cdots, i_{k} \in \Sigma_{N}^{k}\}$ defines a probability measure $P$ in $\mathbb{P}_{N}^{k}$.

Now take $\delta > 0$ small enough and $k$ large enough such that

$$\frac{8\gamma}{2^{k}} < \delta, \quad (3.9)$$

and

$$\gamma > -\frac{1}{2} \sum p(i_{1}, \cdots, i_{k}) \log p(i_{1}, \cdots, i_{k}) + \frac{5\delta}{2 - 2 \sum p(i_{1}, \cdots, i_{k}) \log(p_{k}(i_{1}, \cdots, i_{k})/q_{k}(i_{1}, \cdots, i_{k}))}. \quad (3.10)$$

The last inequality comes from the definition of $\alpha_{N}$, (3.4) and the assumption $\gamma > \alpha_{N}$.

By (3.8), for sufficiently large $\tilde{n}$, we have $\frac{\tilde{n}}{\tilde{n}} < \delta$ and

$$\left| \frac{1}{k} \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} \phi \left( \frac{m_{i_{1} \cdots i_{k}}}{\tilde{n} - k + 1} \right) - \frac{1}{k} \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} \phi(p(i_{1} \cdots i_{k})) \right| < \delta,$$

$$\left| \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} \frac{m_{i_{1} \cdots i_{k}} + k}{\tilde{n}} \log \frac{p_{k}}{q_{k}} - \sum_{i_{1} \cdots i_{k} \in \Sigma_{N}^{k}} p(i_{1} \cdots i_{k}) \log \frac{p_{k}}{q_{k}} \right| < \delta. \quad (3.11)$$
Kolmogorov’s strong law of large numbers (see [12] p.388), we have for each digit
that

\[ P \]

Consider a product Bernoulli probability

\[ \text{Proof of} \]

For any \( z \), \( \infty \), there exists a point \( \in [0,1] \), the digit \( \in \Sigma_N^k \).

Thus \( \in \Sigma_N^k \).

This implies that \( \dim_H(\mathcal{E}_p) \leq \max \{1/2, \alpha_N\} \) as desired.

\( \square \)

4. Lower bound

In this section, we first prove \( \dim_H(\mathcal{E}_p) \geq 1/2 \). Then we examine what happens if the condition \( \sum_{j=1}^{\infty} p_j \log j < \infty \) in Billingsley and Henningsen’s theorem is violated. We will see that if the condition is not satisfied, then \( \dim_H(\mathcal{E}_p) = 1/2 \) and \( N_p = 0 \).

The following is the key lemma for proving that \( \dim_H(\mathcal{E}_p) \geq 1/2 \).

**Lemma 4.1.** For any given sequence of positive integers \( \{c_n\}_{n \geq 1} \) tending to the infinity, there exists a point \( z = (z_1, z_2, \ldots) \in \mathcal{E}_p \) such that \( z_n \leq c_n \) for all \( n \geq 1 \).

**Proof.** For any \( n \geq 1 \), we construct a probability vector \( (p_1^{(n)}, p_2^{(n)}, \ldots, p_k^{(n)}, \ldots) \) such that \( p_k^{(n)} > 0 \) for all \( 1 \leq k \leq c_n \) and \( \sum_{k=1}^{c_n} p_k^{(n)} = 1 \), and that for any \( k \geq 1 \),

\[
\lim_{n \to \infty} p_k^{(n)} = p_k. \tag{4.1}
\]

Consider a product Bernoulli probability \( P \) supported by \( \prod_{n=1}^{\infty} \{1, \ldots, c_n\} \). For each digit \( k \geq 1 \), consider the random variables of \( x \in \mathbb{N}^N \), \( X_k(x) = 1_{(k)}(x_i) \), \( (i \geq 1) \). By Kolmogorov’s strong law of large numbers (see [12] p.388), we have for each digit \( k \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_{(k)}(x_i) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(1_{(k)}(x_i)) = 0 \quad P - a.s.,
\]

which implies

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_{(k)}(x_i) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} p_k = p_k \quad P - a.s.. \tag{4.2}
\]

That is to say, for \( P \) almost every point in the space \( \prod_{n=1}^{\infty} \{1, \ldots, c_n\} \), the digit \( k \) has the frequency \( p_k \). Considering each point in \( \mathbb{N}^N \) as a continued fraction expansion of a number in \([0,1]\), we complete the proof.

**Proof of** \( \dim_H(\mathcal{E}_p) \geq 1/2 \). Take \( c_n = n \) in Lemma 4.1, we find a point \( z \in \mathcal{E}_p \), such that

\[
z_n = a_n(z) \leq n \quad (\forall n \geq 1). \tag{4.3}
\]

For a positive number \( b > 1 \), set

\[
\mathcal{F}_b(z) := \{x \in [0,1] : a_{kz}(x) \in (b^{k^2}, 2b^{k^2}) \land a_k(x) = a_k(z) \text{ if } k \text{ is nonsquare}\}.
\]
It is clear that $F_z(b) \subset \mathcal{E}_b$ for all $b > 1$. We define a measure $\mu$ on $F_z(b)$. For $n^2 \leq m < (n+1)^2$, set

$$\mu(I_m(x)) = \prod_{k=1}^n \frac{1}{b^{k^2}}.$$  \hfill (4.4)

Denote by $B(x, r)$ the ball centered at $x$ with radius $r$. We will show that for any $\theta > 0$, there exists $b > 1$, such that for all $x \in F_z(b)$,

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq \frac{1}{2} - \theta.$$  \hfill (4.5)

In fact, for any positive number $r$, there exist integers $m$ and $n$ such that

$$|I_{m+1}(x)| < 3r \leq |I_m(x)| \quad \text{and} \quad n^2 \leq m < (n+1)^2.$$  \hfill (4.6)

By the construction of $F_z(b)$, $a_{n^2}(x) > b^{n^2} > 1$. Let $x = [x_1, x_2, \cdots]$. By Lemma 2.5, $B(x, r)$ is covered by the union of three adjacent rank $n^2$ basic intervals, i.e.,

$$B(x, r) \subset I(x_1, x_2, \cdots, x_{n^2} - 1) \cup I(x_1, x_2, \cdots, x_{n^2}) \cup I(x_1, x_2, \cdots, x_{n^2} + 1).$$

By the definition of $\mu$, the above three intervals admit the same measure. Hence by (4.6), we have

$$- \log \mu(I_1(x_1, x_2, \cdots, x_{n^2})) = - \log \prod_{k=1}^n \frac{1}{b^{k^2}} = \sum_{k=1}^n k^2 \log b.$$  \hfill (4.7)

On the other hand, by (2.2) and Lemma 2.1, we have

$$- \log |I(x_1, x_2, \cdots, x_{m+1})| \leq 2 + \sum_{k=1}^{m+1} 2 \log(x_k + 1).$$

Let us estimate the second term of the sum. First we have

$$\sum_{k=1}^{m+1} 2 \log(x_k + 1) \leq 2 \sum_{k=1}^{n^2} \log(x_k^2 + 1) + 2 \sum_{k=1}^{n^2} \log(z_k + 1).$$

Since $x_k^2 \leq 2b^{k^2}$ for all $k \geq 1$, we deduce

$$\sum_{k=1}^{m+1} \log(x_k^2 + 1) \leq \sum_{k=1}^{n^2} \log(2b^{k^2} + 1) \leq \sum_{k=1}^{n^2} \log(3b^{k^2})$$

$$= (n + 1) \log 3 + \sum_{k=1}^{n^2} k^2 \log b.$$  \hfill (4.8)

By (4.3), since $z_n \leq n$, for all $n \geq 1$, we know

$$\sum_{k=1}^{m+1} \log(z_k + 1) \leq \sum_{k=1}^{(n+1)^2} \log(k + 1).$$
Thus
\[- \log |I(x_1, x_2, \ldots, x_{m+1})| \]
\[
\leq \log 2 + 2(n + 1) \log 3 + 2 \sum_{k=1}^{n+1} k^2 \log b + 2 \sum_{k=1}^{(n+1)^2} \log(k + 1). \quad (4.9)
\]
Combining (4.8) and (4.9), for any \( \theta > 0 \), take \( b > 1 \) to be large enough, we have
\[
\lim \inf_{n \to \infty} \frac{\log \mu(I(x_1, x_2, \ldots, x_{m+1}))}{\log |I(x_1, x_2, \ldots, x_{m+1})|} \geq 1 - \theta, \quad \forall x \in F_z(b).
\]
Hence by (4.7), we obtain (4.5).

Since \( \theta \) can be arbitrary small, by Billingsley Theorem (\cite{1}), we have
\[
\dim H(E \vec{p}) \geq \frac{1}{2}.
\]
This implies \( D = \infty \) (see the proof of Proposition 3.5 for the definition of \( D \)). Then by Proposition 3.4, we have \( \dim_H(E \vec{p}) \leq \frac{1}{2} \). Since we have already proved \( \dim_H(E \vec{p}) \geq \frac{1}{2} \), we get
\[
\dim_H(E \vec{p}) = \frac{1}{2} \text{ if } \sum_{j=1}^{\infty} p_j \log j = \infty.
\]
This is in accordance with the formula of Theorem 1.1 under the convention that \( \sup \emptyset = 0 \), because (4.10) implies \( \mathcal{N}(\vec{p}) = \emptyset \).

Finally, we remark that \( \mathcal{N}(\vec{p}) = \emptyset \) if and only if \( \sum_{j=1}^{\infty} p_j \log j = \infty \). We have seen the “if” part. For the other part, assume \( \sum_{j=1}^{\infty} p_j \log j < \infty \). Then the Bernoulli measure \( \mu \) such that \( \mu(I(j)) = p_j \) satisfies
\[
\int |\log x| \, d\mu(x) \leq \sum_{j=1}^{\infty} \mu(I(j)) \log(j + 1) = \sum_{j=1}^{\infty} p_j \log(j + 1) < \infty.
\]
which implies that \( \mathcal{N}(\vec{p}) \neq \emptyset \).

5. A Remark

As suggested by the referee, we add a remark on a problematic argument appearing in the literature. To obtain an upper bound of the Hausdorff dimension of a set, one usually applies the Billingsley’s theorem by constructing a finite measure \( P \) on the set such that
\[
|U|^a \leq P(U)
\]
(see \cite{4}, p.67). For the set \( E_{\vec{p}} \), where \( p_i = 0 \) for some \( j \), the Markov measure \( P \) satisfying (3.3) does not match because the cylinders starting with \( j \) do not
charge the measure and the above inequality is obviously not true. This appeared unnoticed for long (see the remarks of Kifer [7], p. 2012).

This problem did exist in the proof of Theorem 2 in [2]. Let us briefly indicate how to get around the problem in the proof of Theorem 2 of [2] when \( p_j = 0 \) for some \( j \)'s. The basic idea is similar to that of Cajar (see [3], p. 67) and that of Kifer [7]. Recall that \( P \) is a \((k - 1)\)-step Markov measure supported on the set 

\[
\{ x \in [0, 1) : a_n(x) \leq N \text{ for all } n \geq 1 \}.
\]

It is uniquely determined by its values on the \( k \)-cylinders, namely,

\[
p(i_1, \ldots, i_k) = P([i_1, \ldots, i_k]), \quad (i_1 \cdots i_k) \in \{1, 2, \ldots, N\}^k.
\]

Let \( 0 < \epsilon < 1 \) and \( P_\epsilon \) be the perturbed \((k - 1)\)-Markov measure determined by

\[
p_\epsilon(i_1, \ldots, i_k) = (1 - \epsilon)P([i_1, \ldots, i_k]) + \frac{\epsilon}{N^k}, \quad (i_1 \cdots i_k) \in \{1, 2, \ldots, N\}^k.
\]

Now, we can apply the Billingsley’s theorem with \( P_\epsilon \) to find an upper bound, and then get the desired result by letting \( \epsilon \to 0 \).

In the present paper, we have intentionally avoided using the Billingsley’s theorem. The proof of upper bound consists of Propositions 3.4 and 3.5. Proposition 3.5 concerns some calculations for which the zero frequency of some digits will not cause any trouble. In the proof of Proposition 3.4, we have used a “covering argument” depending on the estimate (2.3) instead of using the Billingsley’s theorem. This enables us to get the upper bound of the Hausdorff dimension.

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References
