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NEW PERMUTATION CODING AND EQUIDISTRIBUTION OF SET-VALUED STATISTICS

Dominique Foata and Guo-Niu Han

ABSTRACT. A new coding for permutations is explicitly constructed and its association with the classical Lehmer coding provides a bijection of the symmetric group onto itself serving to show that six bivariable set-valued statistics are equidistributed on that group. This extends a recent result due to Cori valid for integer-valued statistics.

1. Introduction

In a recent paper Cori [Cor08] updates a classical algorithm constructed by Ossona de Mendez and Rosenstiehl [OR04] that provides a one-to-one correspondence between rooted hypermaps and indecomposable permutations. He further constructs a bijection of the symmetric group \mathfrak{S}_n onto itself that maps each permutation having p cycles and q left-to-right maxima onto another one having q cycles and p left-to-right maxima. Moreover, by using an encoding of permutations by Dyck paths due to Roblet and Viennot [RV96] he also shows that three bivariable *integer-valued* statistics, introduced in the next paragraph, are equidistributed on \mathfrak{S}_n . The purpose of this paper is to show that all those results can be extended to *set-valued* statistics and that the construction of the underlying bijection is very simple; it involves two permutation codings called the *A-code* and the *B-code*.

The first one, classically referred to as the *Lehmer code* [Le60] or the *inversion table*, goes back, in fact, to more ancient authors (Rothe, Rodrigues, Netto), as knowledgeably stated by Knuth ([Kn98], Ex. 5.1.1-7, p. 14). The second one is a *new* coding that takes the cycle decomposition of permutations into account. Although the motivation of the paper was to prove the equidistribution of several set-valued statistics, its novelty is to fully describe that B-code and exploit its basic properties.

The set-valued statistics in question are introduced as follows. Let $w = x_1x_2 \cdots x_n$ be a word of length n , whose letters are positive integers. The **L**eft to right **m**aximum **p**lace set, $\text{Lmap } w$, of w is defined to be the set of all *places* i such that $x_j < x_i$ for all $j < i$, while the **R**ight to left **m**inimum **l**etter set, $\text{Rmil } w$, of w is the set of all *letters* x_i such that $x_j > x_i$ for all $j > i$.

When the word w is a permutation of $12 \cdots n$ that we shall preferably denote by $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ and the bijection $i \mapsto \sigma(i)$ ($1 \leq i \leq n$) has

r disjoint cycles, whose *minimum* elements are c_1, c_2, \dots, c_r , respectively, define $\text{Cyc } \sigma$ to be the set

$$\text{Cyc } \sigma := \{c_1, c_2, \dots, c_r\}.$$

When σ is a permutation, the *cardinalities* of $\text{Lmap } \sigma$, $\text{Rmil } \sigma$ and $\text{Cyc } \sigma$ are denoted by $\text{lmap } \sigma$, $\text{rmil } \sigma$ and $\text{cyc } \sigma$, respectively, and classically referred to as the *number of left-to-right maxima*, *number of right-to-left minima*, *number of cycles*.

In Fig. 1 the graphs of the permutation $\sigma = 5, 7, 1, 4, 9, 2, 6, 3, 8$ and its inverse $\sigma^{-1} = 3, 6, 8, 4, 1, 7, 2, 9, 5$ have been drawn. The set $\text{Lmap } \sigma$ (resp. $\text{Lmap } \sigma^{-1}$) is the set of the *abscissas* of the “bullets,” while $\text{Rmil } \sigma$ (resp. $\text{Rmil } \sigma^{-1}$) is the set of the *ordinates* of the “crosses.” The set-valued statistics “Leh,” “Rmil Leh” and “Max Leh” will be further introduced. Notice that $\text{lmap } \sigma = \text{rmil } \sigma^{-1} = 3$, $\text{rmil } \sigma = \text{lmap } \sigma^{-1} = 4$. As σ is the product of the disjoint cycles $(15983)(4)(276)$, we have $\text{Cyc } \sigma = \text{Cyc } \sigma^{-1} = \{1, 2, 4\}$ and $\text{cyc } \sigma = \text{cyc } \sigma^{-1} = 3$.

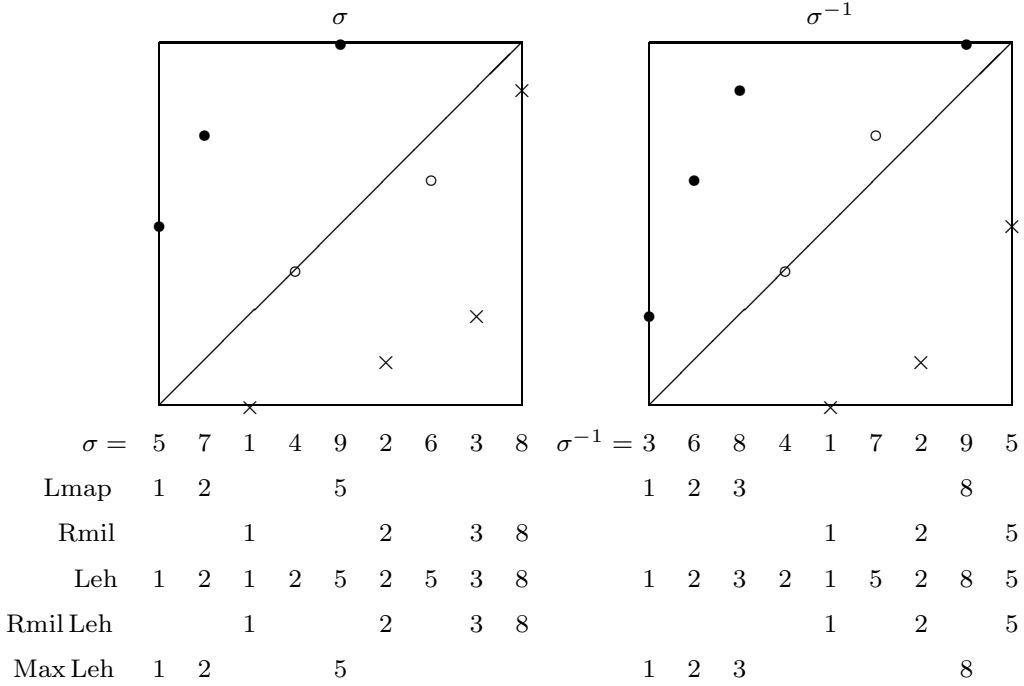


Fig. 1. Graphs of σ and of its inverse σ^{-1}

First, recall Cori’s result [Cor08].

The three pairs of integer-valued statistics $(\text{rmil}, \text{cyc})$, $(\text{cyc}, \text{rmil})$ and $(\text{lmap}, \text{rmil})$ are equidistributed on \mathfrak{S}_n .

The equidistribution of the first two pairs (resp. of the last two ones) is proved by updating the Ossona-de-Mendez-Rosenstiehl algorithm [OR04]

on hypermaps (resp. by using the Roblet-Viennot Dyck path encoding [RV96]). Second, the set-valued statistics “Cyc” and “Rmil” (or “Lmap”) are known to be equidistributed on \mathfrak{S}_n . This is one of the properties of the first fundamental transformation [Lo83, chap. 10]. Our main result is the following theorem.

Theorem 1. *The six bivariable set-valued statistics (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Rmil, Cyc), (Lmap, Rmil), (Lmap, Cyc) are all equidistributed on \mathfrak{S}_n .*

Based on two permutation codings, the A-code and B-code, introduced in Sections 2 and 3, respectively, we construct a bijection ϕ of \mathfrak{S}_n onto itself (see (4.1)) having the following property:

$$(1.1) \quad (\text{Lmap}, \text{Rmil}) \sigma = (\text{Lmap}, \text{Cyc}) \phi(\sigma) \quad (\sigma \in \mathfrak{S}_n).$$

Let $\mathbf{i} : \sigma \mapsto \sigma^{-1}$. As

$$(1.2) \quad \text{Cyc} \mathbf{i} \sigma = \text{Cyc} \sigma;$$

$$(1.3) \quad \text{Rmil} \mathbf{i} \sigma = \text{Lmap} \sigma;$$

(see Fig. 1 for the second relation), it follows from (1.1) that the chain

$$(1.4) \quad \begin{array}{ccccccccc} \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n & \xrightarrow{\phi^{-1}} & \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n & \xrightarrow{\phi} & \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n \\ \left(\begin{smallmatrix} \text{Cyc} \\ \text{Rmil} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Cyc} \\ \text{Lmap} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Rmil} \\ \text{Lmap} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Lmap} \\ \text{Rmil} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Lmap} \\ \text{Cyc} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Rmil} \\ \text{Cyc} \end{smallmatrix} \right) \end{array}$$

provides all the bijections needed to prove Theorem 1. Note that (1.1), on the one hand, and (1.2)–(1.3), on the other hand, are reproduced as

$$\begin{array}{ccc} \mathfrak{S}_n & \xrightarrow{\phi} & \mathfrak{S}_n \quad \text{and} \quad \mathfrak{S}_n & \xrightarrow{\mathbf{i}} & \mathfrak{S}_n \\ \left(\begin{smallmatrix} \text{Lmap} \\ \text{Rmil} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Lmap} \\ \text{Cyc} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Cyc} \\ \text{Rmil} \end{smallmatrix} \right) & & \left(\begin{smallmatrix} \text{Cyc} \\ \text{Lmap} \end{smallmatrix} \right) \end{array}$$

Let $A = (I_1, I_2, \dots, I_h)$ be an ordered partition of the set $[n] := \{1, 2, \dots, n\}$ into disjoint non-empty *intervals*, such that $\max I_j + 1 = \min I_{j+1}$ for $j = 1, 2, \dots, h-1$. A permutation σ from \mathfrak{S}_n is said to be *A-decomposable*, if each I_j is the *smallest interval* such that $\sigma(I_j) = I_j$ (see [Com74], p. 261, exercise 14). For instance, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$ is *A-decomposable*, with $A = (\{1, 2\}, \{3, 4, 5\})$. It is convenient to write $\text{Decomp} \sigma = A$, if σ is *A-decomposable*. A permutation is said to be *indecomposable*, if it is *A-decomposable*, with $A = ([n])$. The bijection ϕ defined in (4.1) has the further property

$$(1.5) \quad \text{Decomp} \phi(\sigma) = \text{Decomp} \sigma \quad (\sigma \in \mathfrak{S}_n).$$

As we evidently have

$$(1.6) \quad \text{Decomp} \mathbf{i} \sigma = \text{Decomp} \sigma,$$

the following result holds.

Theorem 2. *Let A be an ordered partition of the set $[n]$ into disjoint consecutive non-empty intervals. Then, $(\text{Cyc}, \text{Rmil})$, $(\text{Cyc}, \text{Lmap})$, $(\text{Rmil}, \text{Lmap})$, $(\text{Rmil}, \text{Cyc})$, $(\text{Lmap}, \text{Rmil})$, $(\text{Lmap}, \text{Cyc})$ are equidistributed on the set of all A -decomposable permutations from \mathfrak{S}_n .*

The next corollary is relevant to the study of hypermaps, as the set of rooted hypermaps with darts $1, 2, \dots, n$ is in one-to-one correspondence with the subset of indecomposable permutations from \mathfrak{S}_{n+1} (see [Cor08, CM92]).

Corollary 3. *The statistics $(\text{Cyc}, \text{Rmil})$, $(\text{Cyc}, \text{Lmap})$, $(\text{Rmil}, \text{Lmap})$, $(\text{Rmil}, \text{Cyc})$, $(\text{Lmap}, \text{Rmil})$, $(\text{Lmap}, \text{Cyc})$ are equidistributed on the set of all indecomposable permutations from \mathfrak{S}_n .*

The construction of the bijection ϕ together with the proofs of Theorem 2, and Corollary 3 are given in Section 4. It is followed by the algorithmic definitions of both A-code and B-code in Section 5. Tables and concluding remarks are reproduced in Section 6.

2. The A-code

The *Lehmer code* [Le60] of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ is defined to be the sequence $\text{Leh } w = (a_1, a_2, \dots, a_n)$, where for each $i = 1, 2, \dots, n$

$$a_i := \#\{j : 1 \leq j \leq i, \sigma(j) \leq \sigma(i)\}.$$

The sequence $\text{Leh } w$ belongs to SE_n of all sequences $a = (a_1, a_2, \dots, a_n)$, called *subexcedant*, such that $1 \leq a_i \leq i$ for each $i = 1, 2, \dots, n$. For such a sequence it makes sense to define the set, denoted by $\text{Max } a$, of all letters (or places!) a_i such that $a_i = i$.

Under the graphs drawn in Fig. 1 the Lehmer codes $\text{Leh } \sigma$ and $\text{Leh } \sigma^{-1}$ have been calculated, as well as the four sets $\text{Rmil } \text{Leh } \sigma$, $\text{Rmil } \text{Leh } \sigma^{-1}$, $\text{Max } \text{Leh } \sigma$ and $\text{Max } \text{Leh } \sigma^{-1}$. The next Proposition is geometrically evident and given without proof. It shows that the set-valued statistics “Lmap” and “Rmip” can be directly read from the Lehmer code.

Proposition 4. *For each permutation σ we have:*

$$(2.1) \quad \text{Rmil } \text{Leh } \sigma = \text{Rmil } \sigma;$$

$$(2.2) \quad \text{Max } \text{Leh } \sigma = \text{Lmap } \sigma.$$

We then define the A-code of a permutation σ to be

$$(2.3) \quad \text{A-code } \sigma := \text{Leh } \mathbf{i} \sigma.$$

Hence, $\text{Max } \text{A-code } \sigma = \text{Max } \text{Leh } \mathbf{i} \sigma = \text{Lmap } \mathbf{i} \sigma = \text{Rmil } \sigma$. Furthermore, $\text{Rmil } \text{A-code } \sigma = \text{Rmil } \text{Leh } \mathbf{i} \sigma = \text{Rmil } \mathbf{i} \sigma = \text{Lmap } \sigma$. As Leh is a bijection of the symmetric group \mathfrak{S}_n onto SE_n , we obtain the following result.

Theorem 5. *The A-code is a bijection of \mathfrak{S}_n onto SE_n having the property:*

$$(2.4) \quad (\text{Rmil}, \text{Lmap}) \sigma = (\text{Max}, \text{Rmil}) \text{ A-code } \sigma \quad (\sigma \in \mathfrak{S}_n).$$

An algorithmic definition of the A-code will be given in Section 5.

3. The B-code

The B-code is based on the decomposition of each permutation as product of disjoint cycles. For a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and each $i = 1, 2, \dots, n$ let $k := k(i)$ be the *smallest* integer $k \geq 1$ such that $\sigma^{-k}(i) \leq i$. Then, define

$$\text{B-code } \sigma = (b_1, b_2, \dots, b_n) \quad \text{with} \quad b_i := \sigma^{-k(i)}(i) \quad (1 \leq i \leq n).$$

For example, with the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix}$ we have:
 $\sigma^{-1}(1) = 6, \sigma^{-2}(1) = 5, \sigma^{-3}(1) = 3, \sigma^{-4}(1) = 1$, so that $b_1 = 1$;
 $\sigma^{-1}(2) = 4, \sigma^{-2}(2) = 2$, so that $b_2 = 2$; $\sigma^{-1}(3) = 1$, so that $b_3 = 1$;
 $\sigma^{-1}(4) = 2$, so that $b_4 = 2$; $\sigma^{-1}(5) = 3$, so that $b_5 = 3$;
 $\sigma^{-1}(6) = 5$, so that $b_6 = 5$. Thus, B-code $\sigma = (1, 2, 1, 2, 3, 5)$.

An alternate definition is the following. First, the B-code of the unique permutation from \mathfrak{S}_1 is defined to be the sequence $(1) \in \text{SE}_1$. Let $n \geq 2$. When writing each permutation $\sigma \in \mathfrak{S}_n$ of order $n \geq 2$ as a product of its disjoint cycles, the removal of n yields a permutation σ' of order $(n-1)$. Let $b' = (b'_1, b'_2, \dots, b'_{n-1})$ be the B-code of σ' . We define the B-code of σ to be $b := (b'_1, b'_2, \dots, b'_{n-1}, \sigma^{-1}(n))$. By induction on n , we immediately see that the B-code is a bijection of \mathfrak{S}_n onto SE_n .

The following Theorem shows that the set-valued statistics “Lmap” and “Cyc” can be directly read from the B-code.

Theorem 6. *The B-code is a bijection of \mathfrak{S}_n onto SE_n having the property:*

$$(3.1) \quad (\text{Cyc}, \text{Lmap}) \sigma = (\text{Max}, \text{Rmil}) \text{ B-code } \sigma \quad (\sigma \in \mathfrak{S}_n).$$

Proof. By induction, suppose that $\text{Lmap } \sigma' = \text{Rmil } b'$ and $\text{Cyc } \sigma' = \text{Max } b'$. If n is a fixed point of σ , so that $\sigma^{-1}(n) = n$ and $b = (b'_1, \dots, b'_{n-1}, n)$, then $\text{Lmap } \sigma = \text{Lmap } \sigma' \cup \{n\} = \text{Rmil } b' \cup \{n\} = \text{Rmil } \sigma$. Also, $\text{Cyc } \sigma = \text{Cyc } \sigma' \cup \{n\} = \text{Max } b' \cup \{n\} = \text{Max } b$.

When n is not a fixed point of σ , then σ is a product of the form

$$\sigma = \cdots (\cdots \sigma^{-1}(n) n \sigma(n) \cdots) \cdots$$

while σ' may be expressed as

$$\sigma' = \cdots (\cdots \sigma^{-1}(n) \sigma(n) \cdots) \cdots$$

In particular, $\sigma^{-1}(n) < n$, $\sigma(n) < n$ and $\sigma'(\sigma^{-1}(n)) = \sigma(n)$. We have $\text{Cyc } \sigma = \text{Cyc } \sigma' = \text{Max } b' = \text{Max } b$ since $\sigma^{-1}(n) < n$.

To prove $\text{Lmap } \sigma = \text{Rmil } b$, three cases are to be considered, (i) $\sigma(n) = n-1$; (ii) $\sigma(n) \neq n-1$ and $\sigma^{-1}(n-1) < \sigma^{-1}(n)$; (iii) $\sigma(n) \neq n-1$ and $\sigma^{-1}(n-1) > \sigma^{-1}(n)$, each of them materialized by the following three tableaux:

$$(i) \quad \begin{array}{ccccccc} \text{Id} = & 1 & \cdots & \sigma^{-1}(n) & \cdots & n-1 & n \\ \sigma = & \sigma(1) & \cdots & n & \cdots & \sigma(n-1) & \sigma(n) = n-1 \\ \sigma' = & \sigma(1) & \cdots & \sigma(n) = n-1 & \cdots & \sigma(n-1) & * \end{array}$$

$$(ii) \quad \begin{array}{ccccccc} \text{Id} = & 1 & \cdots & \sigma^{-1}(n-1) & \cdots & \sigma^{-1}(n) & \cdots & n-1 & n \\ \sigma = & \sigma(1) & \cdots & n-1 & \cdots & n & \cdots & \sigma(n-1) & \sigma(n) \\ \sigma' = & \sigma(1) & \cdots & n-1 & \cdots & \sigma(n) & \cdots & \sigma(n-1) & * \end{array}$$

$$(iii) \quad \begin{array}{ccccccc} \text{Id} = & 1 & \cdots & \sigma^{-1}(n) & \cdots & \sigma^{-1}(n-1) & \cdots & n-1 & n \\ \sigma = & \sigma(1) & \cdots & n & \cdots & n-1 & \cdots & \sigma(n-1) & \sigma(n) \\ \sigma' = & \sigma(1) & \cdots & \sigma(n) & \cdots & n-1 & \cdots & \sigma(n-1) & * \end{array}$$

In case (i) we get $\text{Lmap } \sigma = \text{Lmap } \sigma'$, $b' = (\dots, \sigma^{-1}(n))$ and $b = (\dots, \sigma^{-1}(n), \sigma^{-1}(n))$, then $\text{Rmil } b = \text{Rmil } b'$.

In case (ii) we clearly have: $\text{Lmap } \sigma = \text{Lmap } \sigma' \cup \{\sigma^{-1}(n)\}$. Also, $b' = (\dots, \sigma^{-1}(n-1))$ and $b = (\dots, \sigma^{-1}(n-1), \sigma^{-1}(n))$. Hence, $\text{Lmap } \sigma = \text{Lmap } \sigma' \cup \{\sigma^{-1}(n)\} = \text{Rmil } b' \cup \{\sigma^{-1}(n)\} = \text{Rmil } b$.

Finally, comes case (iii), which is the hardest one. We have $\text{Lmap } \sigma = (\text{Lmap } \sigma' \cap [1, \sigma^{-1}(n) - 1]) \cup \{\sigma^{-1}(n)\}$, also $b' = (\dots, b'_{n-2}, \sigma^{-1}(n-1))$, $b = (\dots, b'_{n-2}, \sigma^{-1}(n-1), \sigma^{-1}(n))$. But as $\sigma^{-1}(n) < \sigma^{-1}(n-1)$, we have $\text{Rmil } b = (\text{Rmil } b' \cap [1, \sigma^{-1}(n) - 1]) \cup \{\sigma^{-1}(n)\} = (\text{Lmap } \sigma' \cap [1, \sigma^{-1}(n) - 1]) \cup \{\sigma^{-1}(n)\} = \text{Lmap } \sigma$. \square

4. The bijection ϕ

The bijection ϕ , which is the main ingredient in the chain displayed in (1.4), is simply defined as

$$(4.1) \quad \phi := (\text{B-code})^{-1} \circ \text{A-code}.$$

It follows from Theorems 6 and 5 that

$$\begin{aligned} (\text{Cyc}, \text{Lmap}) \phi(\sigma) &= (\text{Max}, \text{Rmil}) \text{B-code } \phi(\sigma) \\ &= (\text{Max}, \text{Rmil}) \text{A-code } \sigma = (\text{Rmil}, \text{Lmap}) \sigma. \end{aligned}$$

This proves relation (1.1) and consequently Theorem 1. It also follows from Theorem 5 and/or 6 that the distribution of each pair of statistics stated in Theorem 1 is also equal to the distribution of $(\text{Max}, \text{Rmil})$ on SE_n .

It remains to prove identity (1.5) to achieve the proofs of Theorem 2 and its Corollary. Let $A = ([p_1, q_1], [p_2, q_2], \dots, [p_h, q_h])$ be an ordered partition of $[n]$ into disjoint non-empty intervals, such that $p_j + 1 = q_{j+1}$ for $j = 1, 2, \dots, h-1$ and $p_1 = 1, q_h = n$. Let $G(\sigma) = \{(i, \sigma(i)) : 1 \leq i \leq n\}$ be the graph of a permutation σ from \mathfrak{S}_n . Referring to Fig. 2, where the square $[p_j, q_j] \times [p_j, q_j]$ has been materialized by the four points B, B'', D'', D , we see that σ is A -indecomposable, if for every $j = 1, 2, \dots, h$

- (i) the square $[BB''D''D]$ contains the subgraph $\{(i, \sigma(i)) : p_j \leq i \leq q_j\}$;
- (ii) for every l such that $p_j + 1 \leq l \leq q_j$ the rectangle $[B'B''C''C']$ contains at least one element from $G(\sigma)$.

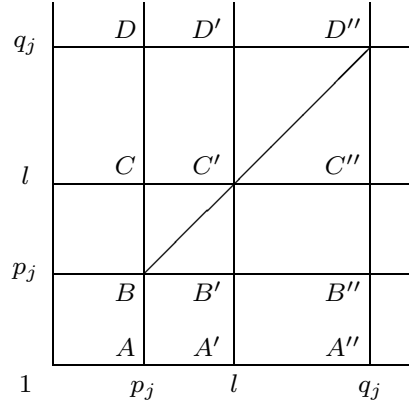


Fig. 2. Graphs of σ and c

We are then led to the following definition.

Definition. Each subexcedant sequence $c = (c_1, c_2, \dots, c_n)$ from SE_n is said to be A -decomposable, if for every $j = 1, 2, \dots, h$

- (i) the triangle $[BB''D'']$ contains the subgraph $\{(i, c_i) : p_j \leq i \leq q_j\}$;
- (ii) for every l such that $p_j + 1 \leq l \leq q_j$ the rectangle $[B'B''C''C']$ contains at least one element (i, c_i) ($l \leq i \leq q_j$).

Proposition 6. A permutation σ from \mathfrak{S}_n is A -decomposable, if and only if its A -code (resp. B -code) is A -decomposable.

Proof. Let $a = (a_1, a_2, \dots, a_n)$ be the A -code of a permutation σ . If σ is A -decomposable, then for every $j = 1, 2, \dots, h$ and $l = p_j, p_j + 1, \dots, q_j$ the point $(\sigma^{-1}(l), l)$ belongs to the square $[BB''D''D]$. As a_l is equal to 1 plus the number of points $(i, \sigma(i))$ such that $1 \leq i < \sigma^{-1}(l)$ and $\sigma(i) < l$, we have $a_l \geq p_j$, so that the point (l, a_l) belongs to the triangle $[BB''D'']$. Conversely, if $(l, a_l) \in [BB''D'']$, then $(\sigma^{-1}(l), l) \in [BB''D''D]$.

Now, the rectangle $[B'B''C''C']$ contains no element from $G(\sigma)$ if and only if all the points $(\sigma^{-1}(l), l), \dots, (\sigma^{-1}(q_j), q_j)$ are in the square $[C'C''D''D']$. This is equivalent to saying that all the quantities $\sigma^{-1}(l), l, \dots, \sigma^{-1}(q_j), q_j$ lie between l and q_j , which is also equivalent to the fact

that a_l, \dots, a_{q_j} lie between l and q_j , that is, the rectangle $[B'B''C''C']$ has no element (i, a_i) ($l \leq i \leq q_j$).

Next, let $b = (b_1, b_2, \dots, b_n)$ be the B-code of σ . If σ is A -decomposable, the restriction of σ to the interval $[p_j, q_j]$ is a product of cycles all elements of which lie between p_j and q_j . By definition of the B-code all the terms b_{p_j}, \dots, b_{q_j} also lie between p_j and q_j and conversely, if it is the case, all the points $(p_j, \sigma(p_j)), \dots, (q_j, \sigma(q_j))$ belong to the square $[BB''D''D]$. The same argument can be applied when all the points $(l, \sigma(l)), \dots, (q_j, \sigma(q_j))$ belong to the square $[C'C''D''D']$. All terms b_l, \dots, b_{q_j} are greater than or equal to l and the rectangle $[B'B''C''C']$ contains no element of the form (i, b_i) with $l \leq i \leq q_j$. \square

Thus, if σ is A -decomposable, so are A-code σ and the composition product $(\text{B-code})^{-1} \text{A-code}(\sigma) = \phi(\sigma)$. This proves identity (1.5) and then Theorem 2 and its corollary.

5. Algorithmic definitions and examples

Although the A-code has been greatly described in various forms (see, e.g., [Kn98], p. 14), we give a full algorithmic definition, which is to be compared with the analogous definition for the B-code.

Algorithmic definition of A-code. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation of $12\cdots n$. By definition the A-code of σ is the sequence $a = (a_1, a_2, \dots, a_n)$ where for each $i = 1, 2, \dots, n$

$$a_i := \#\{j : 1 \leq j \leq i, \sigma^{-1}(j) \leq \sigma^{-1}(i)\},$$

or still

$$(5.1) \quad a_i := \#\{\sigma(k) : 1 \leq \sigma(k) \leq i, k \leq \sigma^{-1}(i)\}.$$

Thus, a_i is equal to 1 *plus the number of letters less than i , to the left of i , in the word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$.*

For instance, with $\sigma = 4, 6, 1, 2, 3, 5$ the A-code of σ is equal to $a = (1, 2, 3, 1, 5, 2)$: $a_1 = 1$, $a_2 = 2$ as 1 is to the left of 2, $a_3 = 3$ as 1 and 2 are to the left of 3, $a_4 = 1$, as 4 is the leftmost letter of σ , etc. Thus,

$$(5.2) \quad \text{A-code}(4, 6, 1, 2, 3, 5) = (1, 2, 3, 1, 5, 2).$$

Algorithmic definition of A-code⁻¹. Given $a = (a_1, a_2, \dots, a_n) \in \text{SE}_n$ write a word with n empty places numbered 1 to n from left to right. First, move the letter n to the a_n -th leftmost place; let σ_n be the resulting word (having one non-empty letter!). Next, move $(n-1)$ to the place having $a_{n-1} - 1$ empty letters to its left. Let σ_{n-2} be the resulting word (having

two non-empty letters). Move $(n-2)$ to the place having $a_{n-2} - 1$ empty letters to its left, etc. Thus, $\text{A-code}^{-1}(a)$ is the final permutation σ_1 .

For instance, start with $a = (1, 2, 1, 2, 3, 5)$. We successively get:

$$\begin{array}{rcl}
 & * & * & * & * & * & * \\
 \sigma_6 = & * & * & * & * & 6 & * & a_6 = 5 \\
 \sigma_5 = & * & * & 5 & * & 6 & * & a_5 = 3 \\
 \sigma_4 = & * & 4 & 5 & * & 6 & * & a_4 = 2 \\
 \sigma_3 = & 3 & 4 & 5 & * & 6 & * & a_3 = 1 \\
 \sigma_2 = & 3 & 4 & 5 & * & 6 & 2 & a_2 = 2 \\
 \sigma_1 = & 3 & 4 & 5 & 1 & 6 & 2 & a_1 = 1
 \end{array}$$

Thus

$$(5.3) \quad \text{A-code}^{-1}(1, 2, 1, 2, 3, 5) = 3, 4, 5, 1, 6, 2.$$

Algorithmic definition of B-code. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$. Its B-code $b = (b_1, b_2, \dots, b_n)$ is calculated as follows. First, b_n is the place occupied by n in $\sigma_n := \sigma$. Permute the two letters n and $\sigma(n)$ in the word σ . Let σ_{n-1} be the resulting word. Then, b_{n-1} is the place occupied by $(n-1)$ in σ_{n-1} . Next, permute the two letters $(n-2)$ and $\sigma(n-2)$ in σ_{n-1} and let σ_{n-2} be the resulting word. Let b_{n-2} is the place occupied by $(n-2)$ in σ_{n-2} . Permute $(n-3)$ and $\sigma(n-3)$ in σ_{n-2} , etc. The B-code of σ is (b_1, b_2, \dots, b_n) .

Start with $\sigma = 3, 4, 5, 2, 6, 1$. We successively get:

$$\begin{array}{rcl}
 \text{Id} = & 1 & 2 & 3 & 4 & 5 & 6 \\
 \sigma_6 = & 3 & 4 & 5 & 2 & 6 & 1 & b_6 = 5 \\
 \sigma_5 = & 3 & 4 & 5 & 2 & 1 & \mathbf{6} & b_5 = 3 \\
 \sigma_4 = & 3 & 4 & 1 & 2 & \mathbf{5} & \mathbf{6} & b_4 = 2 \\
 \sigma_3 = & 3 & 2 & 1 & \mathbf{4} & \mathbf{5} & \mathbf{6} & b_3 = 1 \\
 \sigma_2 = & 1 & 2 & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & b_2 = 2 \\
 \sigma_1 = & 1 & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & b_1 = 1
 \end{array}$$

Thus

$$(5.4) \quad \text{B-code}(3, 4, 5, 2, 6, 1) = (1, 2, 1, 2, 3, 5).$$

Algorithmic definition of B-code⁻¹. Let $b = (b_1, b_2, \dots, b_n) \in \text{SE}_n$. Start with the identity permutation $\sigma_1 = 1, 2, \dots, n$. In σ_1 exchange 2 and the letter at the b_2 -th place. Let σ_2 be the resulting word. In σ_2 permute 3 and the letter at the b_3 -th place. Let σ_3 be the resulting word. In σ_3 permute 4 and the letter at the b_4 -th place, etc. The permutation $\sigma = \text{B-code}^{-1} b$ is the permutation σ_n .

For example, starting with $b = (1, 2, 3, 1, 5, 2)$. We successively form:

$$\begin{array}{ll} \sigma_1 = 1 \mathbf{2} \mathbf{3} \mathbf{4} \mathbf{5} \mathbf{6} & b_1 = 1 \\ \sigma_2 = 1 \mathbf{2} \mathbf{3} \mathbf{4} \mathbf{5} \mathbf{6} & b_2 = 2 \\ \sigma_3 = 1 \mathbf{2} \mathbf{3} \mathbf{4} \mathbf{5} \mathbf{6} & b_3 = 3 \\ \sigma_4 = 4 \mathbf{2} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{6} & b_4 = 1 \\ \sigma_5 = 4 \mathbf{2} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{6} & b_5 = 5 \\ \sigma_6 = 4 \mathbf{6} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{2} & b_6 = 2 \end{array}$$

Thus,

$$(5.5) \quad \text{B-code}^{-1}(1, 2, 3, 1, 5, 2) = 4, 6, 3, 1, 5, 2.$$

Let $\Phi := \mathbf{i}\phi\mathbf{i}\phi^{-1}\mathbf{i}$ be the product of the bijections occurring in (1.4). With $\sigma = 6, 4, 1, 2, 3, 5$ the computation of $\Phi(\sigma)$ can be made as follows.

$$\begin{array}{ll} \text{Id} = 1 \mathbf{2} \mathbf{3} \mathbf{4} \mathbf{5} \mathbf{6} & \\ \sigma = 6 \mathbf{4} \mathbf{1} \mathbf{2} \mathbf{3} \mathbf{5} & \\ \mathbf{i}\sigma = 3 \mathbf{4} \mathbf{5} \mathbf{2} \mathbf{6} \mathbf{1} & \\ \text{B-code } \mathbf{i}\sigma = 1 \mathbf{2} \mathbf{1} \mathbf{2} \mathbf{3} \mathbf{5} & \text{(by (5.4))} \\ \text{A-code}^{-1} \text{B-code } \mathbf{i}\sigma = \phi^{-1} \mathbf{i}\sigma = 3 \mathbf{4} \mathbf{5} \mathbf{1} \mathbf{6} \mathbf{2} & \text{(by (5.3))} \\ \mathbf{i}\phi^{-1} \mathbf{i}\sigma = 4 \mathbf{6} \mathbf{1} \mathbf{2} \mathbf{3} \mathbf{5} & \\ \text{A-code } \mathbf{i}\phi^{-1} \mathbf{i}\sigma = 1 \mathbf{2} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{2} & \text{(by (5.2))} \\ \text{B-code}^{-1} \text{A-code } \mathbf{i}\phi^{-1} \mathbf{i}\sigma = \phi \mathbf{i}\phi^{-1} \mathbf{i}\sigma = 4 \mathbf{6} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{2} & \text{(by (5.5))} \\ \Phi(\sigma) = \mathbf{i}\phi\mathbf{i}\phi^{-1}\mathbf{i}\sigma = 4 \mathbf{6} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{2}. & \end{array}$$

We verify that

$$(\text{Cyc}, \text{Rmil}) \sigma = (\text{Rmil}, \text{Cyc}) \Phi(\sigma) = (\{1, 2\}, \{1, 2, 3, 5\}).$$

6. Concluding remarks and Tables

The bijection constructed by Cori [Cor08] only preserves the *cardinalities* “cyc” and “lmap”, but not the sets “Cyc” and “Lmap.” With the example used in his paper, the permutation

$$\theta = 6, 5, 7, 4, 2, 10, 3, 8, 9 = (1, 6, 10)(2, 5)(3, 7)(4)(8)(9)$$

is mapped onto

$$\theta' = 4, 6, 5, 7, 3, 8, 1, 9, 10, 2 = (1, 4, 7)(2, 6, 8, 9, 10)(3, 5),$$

so that $(\text{Lmap}, \text{Cyc}) \theta' = (\{1, 2, 4, 6, 8, 9\}, \{1, 2, 3\}) \neq (\{1, 2, 3, 4, 8, 9\}, \{1, 3, 6\}) = (\text{Cyc}, \text{Lmap}) \theta$. However, $(\text{cyc}, \text{lmap}) \theta = (\text{lmap}, \text{cyc}) \theta' = (6, 3)$.

In our case, we take the bijection $\phi \mathbf{i} \phi^{-1}$ that satisfies (see (1.4))

$$(\text{Cyc}, \text{Lmap}) \theta = (\text{Lmap}, \text{Cyc}) \phi \mathbf{i} \phi^{-1}(\theta).$$

The calculation of $\phi \mathbf{i} \phi^{-1}(\theta)$ is made for the same θ , together with the relevant set-valued statistics. We successively get:

$$\begin{aligned}\theta &= 6, 5, 7, 4, 2, 10, 3, 8, 9 \\ &= (1, 6, 10)(2, 5)(3, 7)(4)(8)(9) \\ \text{B-code } \theta &= 1, 2, 3, 4, 2, 1, 3, 8, 9, 6 \\ \text{A-code}^{-1} \text{ B-code } \theta &= \phi^{-1}(\theta) = 6, 1, 7, 5, 2, 10, 3, 4, 8, 9 \\ \mathbf{i} \phi^{-1}(\theta) &= 2, 5, 7, 8, 4, 1, 3, 9, 10, 6 \\ \text{A-code } \mathbf{i} \phi^{-1}(\theta) &= 1, 1, 3, 2, 2, 6, 3, 4, 8, 9 \\ \phi \mathbf{i} \phi^{-1}(\theta) &= 2, 5, 7, 8, 4, 6, 3, 9, 10, 1 \\ &= (1, 2, 5, 4, 8, 9, 10)(3, 7)(6)\end{aligned}$$

Thus $(\text{Cyc}, \text{Lmap})\theta = (\text{Lmap}, \text{Cyc})\phi \mathbf{i} \phi^{-1}(\theta) = (\{1, 2, 3, 4, 8, 9\}, \{1, 3, 6\})$.

In Fig. 3 the common distribution over \mathfrak{S}_n of each bivariable statistic (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Rmil, Cyc), (Lmap, Rmil), (Lmap, Cyc) has been reproduced for $n = 1, 2, 3, 4$. On each cell (A, B) , where $A, B \subset [n]$, is written the number of permutations σ from \mathfrak{S}_n such that $(\text{Cyc}, \text{Rmil})\sigma = (A, B)$. In the table for $n = 4$ the total sums occurring at the bottom and on the right are the numbers $\#\{\sigma \in \mathfrak{S}_4 : \text{cyc}\sigma = k\}$ for $k = 4, 3, 2, 1$, which are the coefficients of the polynomial $x(x+1)(x+2)(x+3)$ ([Ri58], chap. 4, § 3). It will be noticed that all those tables are symmetric with respect to the main diagonal.

B=	1
A=1	1

$n = 1$

B=	1, 2	2
A=1,2	1	
	2	1

$n = 2$

B=	1, 2, 3	1, 3	2, 3	3
A=1,2,3	1			
	1,3		1	
	2,3			1
	3		1	1

$n = 3$

B=	1, 2, 3, 4	1, 2, 4	1, 3, 4	2, 3, 4	1, 4	2, 4	3, 4	4	Σ
A=1,2,3,4	1								1
1,2,4		1							6
1,3,4			1		1				
2,3,4				1			1	1	
1,4			1		1				11
2,4						1	1	1	
3,4				1		1	2	2	
4				1		1	2	2	6
Σ	1	6			11			6	

$$n = 4$$

Fig. 3. Distribution of (Cyc, Rmil) over \mathfrak{S}_n .

There exist other bijections $\sigma \mapsto a$ such that the sum $\sum_i (a_i - 1)$ is equal to a statistic different from the inversion number “inv,” but having interesting properties. Let us quote the *Tompkins-Paige method* ([To56, Le60, We61]) for generating permutations on a computer. That method was further used in [Ha92, Ha94] to show that the corresponding sum $\sum_i (a_i - 1)$ is equal to the *major index* “maj”. Let us also mention the *Denert coding* [FZ90, Ha94], whose sum $\sum_i (a_i - 1)$ is equal to the *Denert statistic* “den”. Those codings serve to prove that the statistics “inv,” “maj” and “den” are equidistributed on \mathfrak{S}_n , their common distribution being called *Mahonian*.

Let $b = (b_1, b_2, \dots, b_n)$ be the B-code of a permutation $\sigma \in \mathfrak{S}_n$. In its turn the sum $\text{env } \sigma := \sum_i (b_i - 1)$ becomes a new Mahonian statistic. Moreover, it follows from the properties of the bijection ϕ defined in (4.1) that the two three-variable statistics (env, Cyc, Lmap) and (inv, Rmil, Lmap) are equidistributed on \mathfrak{S}_n .

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