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# NEW PERMUTATION CODING AND EQUIDISTRIBUTION OF SET-VALUED STATISTICS 

Dominique Foata and Guo-Niu Han


#### Abstract

A new coding for permutations is explicitly constructed and its association with the classical Lehmer coding provides a bijection of the symmetric group onto itself serving to show that six bivariable set-valued statistics are equidistributed on that group. This extends a recent result due to Cori valid for integer-valued statistics.


## 1. Introduction

In a recent paper Cori [Cor08] updates a classical algorithm constructed by Ossona de Mendez and Rosenstiehl [OR04] that provides a one-to-one correspondence between rooted hypermaps and indecomposable permutations. He further constructs a bijection of the symmetric group $\mathfrak{S}_{n}$ onto itself that maps each permutation having $p$ cycles and $q$ left-to-right maxima onto another one having $q$ cycles and $p$ left-to-right maxima. Moreover, by using an encoding of permutations by Dyck paths due to Roblet and Viennot [RV96] he also shows that three bivariable integervalued statistics, introduced in the next paragraph, are equidistributed on $\mathfrak{S}_{n}$. The purpose of this paper is to show that all those results can be extended to set-valued statistics and that the construction of the underlying bijection is very simple; it involves two permutation codings called the A-code and the B-code.

The first one, classically referred to as the Lehmer code [Le60] or the inversion table, goes back, in fact, to more ancient authors (Rothe, Rodrigues, Netto), as knowledgeably stated by Knuth ([Kn98], Ex. 5.1.1-7, p. 14). The second one is a new coding that takes the cycle decomposition of permutations into account. Although the motivation of the paper was to prove the equidistribution of several set-valued statistics, its novelty is to fully describe that B-code and exploit its basic properties.

The set-valued statistics in question are introduced as follows. Let $w=x_{1} x_{2} \cdots x_{n}$ be a word of length $n$, whose letters are positive integers. The Left to right maximum place set, Lmap $w$, of $w$ is defined to be the set of all places $i$ such that $x_{j}<x_{i}$ for all $j<i$, while the Right to left minimum letter set, Rmil $w$, of $w$ is the set of all letters $x_{i}$ such that $x_{j}>x_{i}$ for all $j>i$.

When the word $w$ is a permutation of $12 \cdots n$ that we shall preferably denote by $\sigma=\sigma(1) \sigma(2) \cdots \sigma(n)$ and the bijection $i \mapsto \sigma(i)(1 \leq i \leq n)$ has

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$r$ disjoint cylces, whose minimum elements are $c_{1}, c_{2}, \ldots, c_{r}$, respectively, define Cyc $\sigma$ to be the set

$$
\text { Сус } \sigma:=\left\{c_{1}, c_{2}, \ldots, c_{r}\right\}
$$

When $\sigma$ is a permutation, the cardinalities of $\operatorname{Lmap} \sigma, \operatorname{Rmil} \sigma$ and $\operatorname{Cyc} \sigma$ are denoted by $\operatorname{lmap} \sigma, \operatorname{rmil} \sigma$ and $\operatorname{cyc} \sigma$, respectively, and classically referred to as the number of left-to-right maxima, number of right-to-left minima, number of cycles.

In Fig. 1 the graphs of the permutation $\sigma=5,7,1,4,9,2,6,3,8$ and its inverse $\sigma^{-1}=3,6,8,4,1,7,2,9,5$ have been drawn. The set Lmap $\sigma$ (resp. Lmap $\sigma^{-1}$ ) is the set of the abscissas of the "bullets," while Rmil $\sigma$ (resp. Rmil $\sigma^{-1}$ ) is the set of the ordinates of the "crosses." The set-valued statistics "Leh," "Rmil Leh" and "Max Leh" will be further introduced. Notice that $\operatorname{lmap} \sigma=\operatorname{rmil} \sigma^{-1}=3$, $\operatorname{rmil} \sigma=\operatorname{lmap} \sigma^{-1}=4$. As $\sigma$ is the product of the disjoint cycles $(15983)(4)(276)$, we have Cyc $\sigma=$ Cyc $\sigma^{-1}=\{1,2,4\}$ and $\operatorname{cyc} \sigma=\operatorname{cyc} \sigma^{-1}=3$.


Fig. 1. Graphs of $\sigma$ and of its inverse $\sigma^{-1}$
First, recall Cori's result [Cor08].
The three pairs of integer-valued statistics (rmil, cyc), (cyc, rmil) and (lmap, rmil) are equidistributed on $\mathfrak{S}_{n}$.

The equidistribution of the first two pairs (resp. of the last two ones) is proved by updating the Ossona-de-Mendez-Rosenstiehl algorithm [OR04]
on hypermaps (resp. by using the Roblet-Viennot Dyck path encoding [RV96]). Second, the set-valued statistics "Cyc" and "Rmil" (or "Lmap") are known to be equidistributed on $\mathfrak{S}_{n}$. This is one of the properties of the first fundamental transformation [Lo83, chap. 10]. Our main result is the following theorem.
Theorem 1. The six bivariable set-valued statistics (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Rmil, Cyc), (Lmap, Rmil), (Lmap, Cyc) are all equidistributed on $\mathfrak{S}_{n}$.

Based on two permutation codings, the A-code and B-code, introduced in Sections 2 and 3, respectively, we construct a bijection $\phi$ of $\mathfrak{S}_{n}$ onto itself (see (4.1)) having the following property:

$$
\begin{equation*}
\left(\text { Lmap, Rmil) } \sigma=(\text { Lmap, Cyc }) \phi(\sigma) \quad\left(\sigma \in \mathfrak{S}_{n}\right)\right. \tag{1.1}
\end{equation*}
$$

Let $\mathbf{i}: \sigma \mapsto \sigma^{-1}$. As

$$
\begin{align*}
\operatorname{Cyc} \mathbf{i} \sigma & =\operatorname{Cyc} \sigma ;  \tag{1.2}\\
\operatorname{Rmil} \mathbf{i} \sigma & =\operatorname{Lmap} \sigma ; \tag{1.3}
\end{align*}
$$

(see Fig. 1 for the second relation), it follows from (1.1) that the chain

provides all the bijections needed to prove Theorem 1. Note that (1.1), on the one hand, and (1.2)-(1.3), on the other hand, are reproduced as

Let $A=\left(I_{1}, I_{2}, \ldots, I_{h}\right)$ be an ordered partition of the set $[n]:=$ $\{1,2, \ldots, n\}$ into disjoint non-empty intervals, such that $\max I_{j}+1=$ $\min I_{j+1}$ for $j=1,2, \ldots, h-1$. A permutation $\sigma$ from $\mathfrak{S}_{n}$ is said to be A-decomposable, if each $I_{j}$ is the smallest interval such that $\sigma\left(I_{j}\right)=I_{j}$ (see [Com74], p. 261, exercise 14). For instance, $\sigma=\binom{12345}{21543}$ is $A$ decomposable, with $A=(\{1,2\},\{3,4,5\})$. It is convenient to write Decomp $\sigma=A$, if $\sigma$ is $A$-decomposable. A permutation is said to be indecomposable, if it is $A$-decomposable, with $A=([n])$. The bijection $\phi$ defined in (4.1) has the further property

$$
\begin{equation*}
\operatorname{Decomp} \phi(\sigma)=\operatorname{Decomp} \sigma \quad\left(\sigma \in \mathfrak{S}_{\mathfrak{n}}\right) \tag{1.5}
\end{equation*}
$$

As we evidently have

$$
\begin{equation*}
\text { Decomp } \mathbf{i} \sigma=\operatorname{Decomp} \sigma \tag{1.6}
\end{equation*}
$$

the following result holds.

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Theorem 2. Let $A$ be an ordered partition of the set $[n]$ into disjoint consecutive non-empty intervals. Then, (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Rmil, Cyc), (Lmap, Rmil), (Lmap, Cyc) are equidistributed on the set of all $A$-decomposable permutations from $\mathfrak{S}_{n}$.

The next corollary is relevant to the study of hypermaps, as the set of rooted hypermaps with darts $1,2, \ldots, n$ is in one-to-one correspondence with the subset of indecomposable permutations from $\mathfrak{S}_{n+1}$ (see [Cor08, CM92]).
Corollary 3. The statistics (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Rmil, Cyc), (Lmap, Rmil), (Lmap, Cyc) are equidistributed on the set of all indecomposable permutations from $\mathfrak{S}_{n}$.

The construction of the bijection $\phi$ together with the proofs of Theorem 2, and Corollary 3 are given in Section 4. It is followed by the algorithmic definitions of both A-code and B-code in Section 5. Tables and concluding remarks are reproduced in Section 6.

## 2. The A-code

The Lehmer code [Le60] of a permutation $\sigma=\sigma(1) \sigma(2) \cdots \sigma(n)$ of $12 \cdots n$ is defined to be the sequence Leh $w=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$, where for each $i=1,2, \ldots, n$

$$
a_{i}:=\#\{j: 1 \leq j \leq i, \sigma(j) \leq \sigma(i)\} .
$$

The sequence Leh $w$ belongs to $\mathrm{SE}_{n}$ of all sequences $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$, called subexcedant, such that $1 \leq a_{i} \leq i$ for each $i=1,2, \ldots, n$. For such a sequence it makes sense to define the set, denoted by $\operatorname{Max} a$, of all letters (or places!) $a_{i}$ such that $a_{i}=i$.

Under the graphs drawn in Fig. 1 the Lehmer codes Leh $\sigma$ and Leh $\sigma^{-1}$ have been calculated, as well as the four sets $\operatorname{Rmil} \operatorname{Leh} \sigma$, $\operatorname{Rmil} \operatorname{Leh} \sigma^{-1}$, Max Leh $\sigma$ and MaxLeh $\sigma^{-1}$. The next Proposition is geometrically evident and given without proof. It shows that the set-valued statistics "Lmap" and "Rmip" can be directly read from the Lehmer code.
Proposition 4. For each permutation $\sigma$ we have:

$$
\begin{align*}
\operatorname{Rmil} \operatorname{Leh} \sigma & =\operatorname{Rmil} \sigma  \tag{2.1}\\
\operatorname{Max} \operatorname{Leh} \sigma & =\operatorname{Lmap} \sigma . \tag{2.2}
\end{align*}
$$

We then define the A-code of a permutation $\sigma$ to be

$$
\begin{equation*}
\text { A-code } \sigma:=\operatorname{Leh} \mathbf{i} \sigma \tag{2.3}
\end{equation*}
$$

Hence, Max A-code $\sigma=\operatorname{Max} \operatorname{Leh} \mathbf{i} \sigma=\operatorname{Lmap} \mathbf{i} \sigma=\operatorname{Rmil} \sigma$. Furthermore, Rmil A-code $\sigma=\operatorname{Rmil} \operatorname{Leh} \mathbf{i} \sigma=\operatorname{Rmil} \mathbf{i} \sigma=\operatorname{Lmap} \sigma$. As Leh is a bijection of the symmetric group $\mathfrak{S}_{n}$ onto $\mathrm{SE}_{n}$, we obtain the following result.

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Theorem 5. The A-code is a bijection of $\mathfrak{S}_{n}$ onto $\mathrm{SE}_{n}$ having the property:
(Rmil, Lmap) $\sigma=\left(\right.$ Max, Rmil) A-code $\sigma \quad\left(\sigma \in \mathfrak{S}_{n}\right)$.
An algorithmic definition of the A-code will be given in Section 5.

## 3. The B-code

The B-code is based on the decomposition of each permutation as product of disjoint cycles. For a permutation $\sigma=\sigma(1) \sigma(2) \cdots \sigma(n)$ and each $i=1,2, \ldots, n$ let $k:=k(i)$ be the smallest integer $k \geq 1$ such that $\sigma^{-k}(i) \leq i$. Then, define

$$
\text { B-code } \sigma=\left(b_{1}, b_{2}, \ldots, b_{n}\right) \quad \text { with } \quad b_{i}:=\sigma^{-k(i)}(i) \quad(1 \leq i \leq n) .
$$

For example, with the permutation $\sigma=\binom{123456}{345261}$ we have:
$\sigma^{-1}(1)=6, \sigma^{-2}(1)=5, \sigma^{-3}(1)=3, \sigma^{-4}(1)=1$, so that $b_{1}=1$;
$\sigma^{-1}(2)=4, \sigma^{-2}(2)=2$, so that $b_{2}=2 ; \quad \sigma^{-1}(3)=1$, so that $b_{3}=1$;
$\sigma^{-1}(4)=2$, so that $b_{4}=2 ; \quad \sigma^{-1}(5)=3$, so that $b_{5}=3$;
$\sigma^{-1}(6)=5$, so that $b_{6}=5 . \quad$ Thus, B-code $\sigma=(1,2,1,2,3,5)$.
An alternate definition is the following. First, the B-code of the unique permutation from $\mathfrak{S}_{1}$ is defined to be the sequence (1) $\in \mathrm{SE}_{1}$. Let $n \geq 2$. When writing each permutation $\sigma \in \mathfrak{S}_{n}$ of order $n \geq 2$ as a product of its disjoint cycles, the removal of $n$ yields a permutation $\sigma^{\prime}$ of order $(n-1)$. Let $b^{\prime}=\left(b_{1}^{\prime}, b_{2}^{\prime}, \ldots, b_{n-1}^{\prime}\right)$ be the B-code of $\sigma^{\prime}$. We define the B-code of $\sigma$ to be $b:=\left(b_{1}^{\prime}, b_{2}^{\prime}, \ldots, b_{n-1}^{\prime}, \sigma^{-1}(n)\right)$. By induction on $n$, we immediately see that the B-code is a bijection of $\mathfrak{S}_{n}$ onto $\mathrm{SE}_{n}$.

The following Theorem shows that the set-valued statistics "Lmap" and "Cyc" can be directly read from the B-code.

Theorem 6. The B-code is a bijection of $\mathfrak{S}_{n}$ onto $\mathrm{SE}_{n}$ having the property:

$$
\begin{equation*}
(\mathrm{Cyc}, \text { Lmap }) \sigma=(\text { Max, Rmil }) \mathrm{B}-\operatorname{code} \sigma \quad\left(\sigma \in \mathfrak{S}_{n}\right) \tag{3.1}
\end{equation*}
$$

Proof. By induction, suppose that $\operatorname{Lmap} \sigma^{\prime}=\operatorname{Rmil} b^{\prime}$ and $\mathrm{Cyc} \sigma^{\prime}=$ $\operatorname{Max} b^{\prime}$. If $n$ is a fixed point of $\sigma$, so that $\sigma^{-1}(n)=n$ and $b=$ $\left(b_{1}^{\prime}, \cdots, b_{n-1}^{\prime}, n\right)$, then $\operatorname{Lmap} \sigma=\operatorname{Lmap} \sigma^{\prime} \cup\{n\}=\operatorname{Rmil} b^{\prime} \cup\{n\}=\operatorname{Rmil} \sigma$. Also, $\operatorname{Cyc} \sigma=\operatorname{Cyc} \sigma^{\prime} \cup\{n\}=\operatorname{Max} b^{\prime} \cup\{n\}=\operatorname{Max} b$.

When $n$ is not a fixed point of $\sigma$, then $\sigma$ is a product of the form

$$
\sigma=\cdots\left(\cdots \sigma^{-1}(n) n \sigma(n) \cdots\right) \cdots
$$

while $\sigma^{\prime}$ may be expressed as

$$
\sigma^{\prime}=\cdots\left(\cdots \sigma^{-1}(n) \sigma(n) \cdots\right) \cdots
$$

In particular, $\sigma^{-1}(n)<n, \sigma(n)<n$ and $\sigma^{\prime}\left(\sigma^{-1}(n)\right)=\sigma(n)$. We have $\operatorname{Cyc} \sigma=\operatorname{Cyc} \sigma^{\prime}=\operatorname{Max} b^{\prime}=\operatorname{Max} b$ since $\sigma^{-1}(n)<n$.

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To prove $\operatorname{Lmap} \sigma=$ Rmilb, three cases are to be considered, $(i)$ $\sigma(n)=n-1$; (ii) $\sigma(n) \neq n-1$ and $\sigma^{-1}(n-1)<\sigma^{-1}(n)$; (iii) $\sigma(n) \neq n-1$ and $\sigma^{-1}(n-1)>\sigma^{-1}(n)$, each of them materialized by the following three tableaux:
(i) $\quad \sigma=\begin{array}{lllllll}\sigma(1) & \cdots & n & \cdots & \sigma(n-1) & \sigma(n)=n-1\end{array}$
$\sigma^{\prime}=\begin{array}{llll}\sigma(1) & \cdots & \sigma(n)=n-1 & \cdots\end{array} \sigma(n-1) \quad *$
(ii)

$$
\text { (ii) } \begin{array}{cccccccccc}
\mathrm{Id}= & 1 & \cdots & \sigma^{-1}(n-1) & \cdots & \sigma^{-1}(n) & \cdots & n-1 & n \\
\sigma= & \sigma(1) & \cdots & n-1 & \cdots & n & \cdots & \sigma(n-1) & \sigma(n) \\
& \sigma^{\prime}= & \sigma(1) & \cdots & n-1 & \cdots & \sigma(n) & \cdots & \sigma(n-1) & *
\end{array}
$$

$$
\mathrm{Id}=\begin{array}{llllllll}
1 & \cdots & \sigma^{-1}(n) & \cdots & \sigma^{-1}(n-1) & \cdots & n-1 & n
\end{array}
$$

$$
\text { (iii) } \quad \sigma=\begin{array}{llllllll}
\sigma(1) & \cdots & n & \cdots & n-1 & \cdots & \sigma(n-1) & \sigma(n)
\end{array}
$$

$$
\sigma^{\prime}=\begin{array}{lllllll}
\sigma(1) & \cdots & \sigma(n) & \cdots & n-1 & \cdots & \sigma(n-1)
\end{array} *
$$

In case $(i)$ we get $\operatorname{Lmap} \sigma=\operatorname{Lmap} \sigma^{\prime}, b^{\prime}=\left(\ldots, \sigma^{-1}(n)\right)$ and $b=$ $\left(\ldots, \sigma^{-1}(n), \sigma^{-1}(n)\right)$, then Rmil $b=$ Rmil $b^{\prime}$.

In case (ii) we clearly have: $\operatorname{Lmap} \sigma=\operatorname{Lmap} \sigma^{\prime} \cup\left\{\sigma^{-1}(n)\right\}$. Also, $b^{\prime}=\left(\ldots, \sigma^{-1}(n-1)\right)$ and $b=\left(\ldots, \sigma^{-1}(n-1), \sigma^{-1}(n)\right)$. Hence, Lmap $\sigma=$ Lmap $\sigma^{\prime} \cup\left\{\sigma^{-1}(n)\right\}=$ Rmil $b^{\prime} \cup\left\{\sigma^{-1}(n)\right\}=$ Rmil $b$.

Finally, comes case (iii), which is the hardest one. We have Lmap $\sigma=$ $\left(\operatorname{Lmap} \sigma^{\prime} \cap\left[1, \sigma^{-1}(n)-1\right]\right) \cup\left\{\sigma^{-1}(n)\right\}$, also $b^{\prime}=\left(\ldots, b_{n-2}^{\prime}, \sigma^{-1}(n-1)\right)$, $b=\left(\ldots, b_{n-2}^{\prime}, \sigma^{-1}(n-1), \sigma^{-1}(n)\right)$. But as $\sigma^{-1}(n)<\sigma^{-1}(n-1)$, we have Rmil $b=\left(\right.$ Rmil $\left.b^{\prime} \cap\left[1, \sigma^{-1}(n)-1\right]\right) \cup\left\{\sigma^{-1}(n)\right\}=\left(\operatorname{Lmap} \sigma^{\prime} \cap\left[1, \sigma^{-1}(n)-\right.\right.$ 1]) $\cup\left\{\sigma^{-1}(n)\right\}=\operatorname{Lmap} \sigma$.

## 4. The bijection $\phi$

The bijection $\phi$, which is the main ingredient in the chain displayed in (1.4), is simply defined as

$$
\begin{equation*}
\phi:=(\mathrm{B}-\text { code })^{-1} \circ \mathrm{~A} \text {-code } . \tag{4.1}
\end{equation*}
$$

It follows from Theorems 6 and 5 that

$$
\begin{aligned}
(\text { Cyc, Lmap }) \phi(\sigma) & =(\text { Max, Rmil) B-code } \phi(\sigma) \\
& =(\text { Max, Rmil) A-code } \sigma=(\text { Rmil, Lmap }) \sigma .
\end{aligned}
$$

This proves relation (1.1) and consequently Theorem 1. It also follows from Theorem 5 and/or 6 that the distribution of each pair of statistics stated in Theorem 1 is also equal to the distribution of (Max, Rmil) on $\mathrm{SE}_{n}$.

It remains to prove identity (1.5) to achieve the proofs of Theorem 2 and its Corollary. Let $A=\left(\left[p_{1}, q_{1}\right],\left[p_{2}, q_{2}\right], \ldots,\left[p_{h}, q_{h}\right]\right)$ be an ordered partition of $[n]$ into disjoint non-empty intervals, such that $p_{j}+1=q_{j+1}$ for $j=1,2, \ldots, h-1$ and $p_{1}=1, q_{h}=n$. Let $G(\sigma)=\{(i, \sigma(i)): 1 \leq i \leq n\}$ be the graph of a permutation $\sigma$ from $\mathfrak{S}_{n}$. Referring to Fig. 2, where the square $\left[p_{j}, q_{j}\right] \times\left[p_{j}, q_{j}\right]$ has been materialized by the four points $B, B^{\prime \prime}$, $D^{\prime \prime}, D$, we see that $\sigma$ is $A$-indecomposable, if for every $j=1,2, \ldots, h$
(i) the square $\left[B B^{\prime \prime} D^{\prime \prime} D\right]$ contains the subgraph $\left\{(i, \sigma(i)): p_{j} \leq i \leq q_{j}\right\}$;
(ii) for every $l$ such that $p_{j}+1 \leq l \leq q_{j}$ the rectangle $\left[B^{\prime} B^{\prime \prime} C^{\prime \prime} C^{\prime}\right]$ contains at least one element from $G(\sigma)$.


Fig. 2. Graphs of $\sigma$ and $c$
We are then led to the following definition.
Definition. Each subexcedant sequence $c=\left(c_{1}, c_{2}, \ldots, c_{n}\right)$ from $\mathrm{SE}_{n}$ is said to be $A$-decomposable, if for every $j=1,2, \ldots, h$
(i) the triangle $\left[B B^{\prime \prime} D^{\prime \prime}\right]$ contains the subgraph $\left\{\left(i, c_{i}\right): p_{j} \leq i \leq q_{j}\right\}$;
(ii) for every $l$ such that $p_{j}+1 \leq l \leq q_{j}$ the rectangle $\left[B^{\prime} B^{\prime \prime} C^{\prime \prime} C^{\prime}\right]$ contains at least one element $\left(i, c_{i}\right)\left(l \leq i \leq q_{j}\right)$.

Proposition 6. A permutation $\sigma$ from $\mathfrak{S}_{n}$ is $A$-decomposable, if and only if its $A$-code (resp. B-code) is $A$-decomposable.

Proof. Let $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ be the A-code of a permutation $\sigma$. If $\sigma$ is $A$-decomposable, then for every $j=1,2, \ldots, h$ and $l=p_{j}, p_{j}+1, \ldots, q_{j}$ the point ( $\left.\sigma^{-1}(l), l\right)$ belongs to the square $\left[B B^{\prime \prime} D^{\prime \prime} D\right]$. As $a_{l}$ is equal to 1 plus the number of points $(i, \sigma(i))$ such that $1 \leq i<\sigma^{-1}(l)$ and $\sigma(i)<l$, we have $a_{l} \geq p_{j}$, so that the point $\left(l, a_{l}\right)$ belongs to the triangle $\left[B B^{\prime \prime} D^{\prime \prime}\right]$. Conversely, if $\left(l, a_{l}\right) \in\left[B B^{\prime \prime} D^{\prime \prime}\right]$, then $\left(\sigma^{-1}(l), l\right) \in\left[B B^{\prime \prime} D^{\prime \prime} D\right]$.

Now, the rectangle $\left[B^{\prime} B^{\prime \prime} C^{\prime \prime} C^{\prime}\right]$ contains no element from $G(\sigma)$ if and only if all the points $\left(\sigma^{-1}(l), l\right), \ldots,\left(\sigma^{-1}\left(q_{j}\right), q_{j}\right)$ are in the square $\left[C^{\prime} C^{\prime \prime} D^{\prime \prime} D^{\prime}\right]$. This is equivalent to saying that all the quantities $\sigma^{-1}(l), l$, $\ldots, \sigma^{-1}\left(q_{j}\right), q_{j}$ lie between $l$ and $q_{j}$, which is also equivalent to the fact
that $a_{l}, \ldots, a_{q_{j}}$ lie between $l$ and $q_{j}$, that is, the rectangle $\left[B^{\prime} B^{\prime \prime} C^{\prime \prime} C^{\prime}\right]$ has no element $\left(i, a_{i}\right)\left(l \leq i \leq q_{j}\right)$.

Next, let $b=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ be the B -code of $\sigma$. If $\sigma$ is $A$-decomposable, the restriction of $\sigma$ to the interval $\left[p_{j}, q_{j}\right]$ is a product of cycles all elements of which lie between $p_{j}$ and $q_{j}$. By definition of the B-code all the terms $b_{p_{j}}$, $\ldots, b_{q_{j}}$ also lie between $p_{j}$ and $q_{j}$ and conversely, if it is the case, all the points $\left(p_{j}, \sigma\left(p_{j}\right)\right), \ldots,\left(q_{j}, \sigma\left(q_{j}\right)\right)$ belong to the square $\left[B B^{\prime \prime} D^{\prime \prime} D\right]$. The same argument can be applied when all the points $(l, \sigma(l)), \ldots,\left(q_{j}, \sigma\left(q_{j}\right)\right)$ belong to the square $\left[C^{\prime} C^{\prime \prime} D^{\prime \prime} D^{\prime}\right]$. All terms $b_{l}, \ldots, b_{q_{j}}$ are greater than or equal to $l$ and the rectangle $\left[B^{\prime} B^{\prime \prime} C^{\prime \prime} C^{\prime}\right]$ contains no element of the form $\left(i, b_{i}\right)$ with $l \leq i \leq q_{j}$.

Thus, if $\sigma$ is $A$-decomposable, so are A-code $\sigma$ and the composition product $(\mathrm{B}-\text { code })^{-1} \mathrm{~A}-\operatorname{code}(\sigma)=\phi(\sigma)$. This proves identity (1.5) and then Theorem 2 and its corollary.

## 5. Algorithmic definitions and examples

Although the A-code has been greatly described in various forms (see, e.g., [Kn98], p. 14), we give a full algorithmic definition, which is to be compared with the analogous definition for the B-code.

Algorithmic definition of A-code. Let $\sigma=\sigma(1) \sigma(2) \cdots \sigma(n)$ be a permutation of $12 \cdots n$. By definition the A-code of $\sigma$ is the sequence $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ where for each $i=1,2, \ldots, n$

$$
a_{i}:=\#\left\{j: 1 \leq j \leq i, \sigma^{-1}(j) \leq \sigma^{-1}(i)\right\},
$$

or still

$$
\begin{equation*}
a_{i}:=\#\left\{\sigma(k): 1 \leq \sigma(k) \leq i, k \leq \sigma^{-1}(i)\right\} . \tag{5.1}
\end{equation*}
$$

Thus, $a_{i}$ is equal to 1 plus the number of letters less than $i$, to the left of $i$, in the word $\sigma=\sigma(1) \sigma(2) \cdots \sigma(n)$.

For instance, with $\sigma=4,6,1,2,3,5$ the A-code of $\sigma$ is equal to $a=(1,2,3,1,5,2): a_{1}=1, a_{2}=2$ as 1 is to the left of $2, a_{3}=3$ as 1 and 2 are to the left of $3, a_{4}=1$, as 4 is the leftmost letter of $\sigma$, etc. Thus,

$$
\begin{equation*}
\text { A-code }(4,6,1,2,3,5)=(1,2,3,1,5,2) . \tag{5.2}
\end{equation*}
$$

Algorithmic definition of A-code ${ }^{-1}$. Given $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in \mathrm{SE}_{n}$ write a word with $n$ empty places numbered 1 to $n$ from left to right. First, move the letter $n$ to the $a_{n}$-th leftmost place; let $\sigma_{n}$ be the resulting word (having one non-empty letter!). Next, move $(n-1)$ to the place having $a_{n-1}-1$ empty letters to its left. Let $\sigma_{n-2}$ be the resulting word (having
two non-empty letters). Move $(n-2)$ to the place having $a_{n-2}-1$ empty letters to its left, etc. Thus, A-code ${ }^{-1}(a)$ is the final permutation $\sigma_{1}$.

For instance, start with $a=(1,2,1,2,3,5)$. We successively get:

$$
\begin{array}{rlr}
* * * * * * & \\
\sigma_{6} & =* * * * 6 * & a_{6}=5 \\
\sigma_{5} & =* * 5 * 6 * & a_{5}=3 \\
\sigma_{4} & =* 45 * 6 * & a_{4}=2 \\
\sigma_{3}=345 * 6 * & a_{3}=1 \\
\sigma_{2}=345 * 62 & a_{2}=2 \\
\sigma_{1}=345162 & a_{1}=1
\end{array}
$$

Thus

$$
\begin{equation*}
\text { A-code }{ }^{-1}(1,2,1,2,3,5)=3,4,5,1,6,2 \tag{5.3}
\end{equation*}
$$

Algorithmic definition of B-code. Let $\sigma=\sigma(1) \sigma(2) \cdots \sigma(n) \in \mathfrak{S}_{n}$. Its B -code $b=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ is calculated as follows. First, $b_{n}$ is the place occupied by $n$ in $\sigma_{n}:=\sigma$. Permute the two letters $n$ and $\sigma(n)$ in the word $\sigma$. Let $\sigma_{n-1}$ be the resulting word. Then, $b_{n-1}$ is the place occupied by $(n-1)$ in $\sigma_{n-1}$. Next, permute the two letters $(n-2)$ and $\sigma(n-2)$ in $\sigma_{n-1}$ and let $\sigma_{n-2}$ be the resulting word. Let $b_{n-2}$ is the place occupied by $(n-2)$ in $\sigma_{n-2}$. Permute $(n-3)$ and $\sigma(n-3)$ in $\sigma_{n-2}$, etc. The B-code of $\sigma$ is $\left(b_{1}, b_{2}, \cdots, b_{n}\right)$.

Start with $\sigma=3,4,5,2,6,1$. We successively get:

$$
\begin{array}{lll}
\mathrm{Id}=123456 & \\
\sigma_{6}=345261 & b_{6}=5 \\
\sigma_{5}=3452166 & b_{5}=3 \\
\sigma_{4}=3412566 & b_{4}=2 \\
\sigma_{3}=321456 & b_{3}=1 \\
\sigma_{2}=12 \mathbf{3} 56 & b_{2}=2 \\
\sigma_{1}=123456 & b_{1}=1
\end{array}
$$

Thus

$$
\begin{equation*}
\text { B-code }(3,4,5,2,6,1)=(1,2,1,2,3,5) . \tag{5.4}
\end{equation*}
$$

Algorithmic definition of B-code ${ }^{-1}$. Let $b=\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathrm{SE}_{n}$. Start with the identity permutation $\sigma_{1}=1,2, \ldots, n$. In $\sigma_{1}$ exchange 2 and the letter at the $b_{2}$-th place. Let $\sigma_{2}$ be the resulting word. In $\sigma_{2}$ permute 3 and the letter at the $b_{3}$-th place. Let $\sigma_{3}$ be the resulting word. In $\sigma_{3}$ permute 4 and the letter at the $b_{4}$-th place, etc. The permutation $\sigma=\mathrm{B}$-code ${ }^{-1} b$ is the permutation $\sigma_{n}$.

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For example, starting with $b=(1,2,3,1,5,2)$. We successively form:

$$
\begin{array}{ll}
\sigma_{1}=123456 & b_{1}=1 \\
\sigma_{2}=123456 & b_{2}=2 \\
\sigma_{3}=123456 & b_{3}=3 \\
\sigma_{4}=423156 & b_{4}=1 \\
\sigma_{5}=423156 & b_{5}=5 \\
\sigma_{6}=463152 & b_{6}=2
\end{array}
$$

Thus,

$$
\begin{equation*}
\text { B-code }{ }^{-1}(1,2,3,1,5,2)=4,6,3,1,5,2 \tag{5.5}
\end{equation*}
$$

Let $\Phi:=\mathbf{i} \phi \mathbf{i} \phi^{-1} \mathbf{i}$ be the product of the bijections occurring in (1.4). With $\sigma=6,4,1,2,3,5$ the computation of $\Phi(\sigma)$ can be made as follows.

$$
\begin{array}{rlr}
\mathrm{Id} & =123456 & \\
\sigma=641235 & \\
\mathbf{i} \sigma=345261 & \\
\mathrm{~B}-\operatorname{code} \mathbf{i} \sigma & =121235 & (\text { by }(5.4)) \\
\mathrm{A}-\operatorname{code}^{-1} \mathrm{~B}-\operatorname{code} \mathbf{i} \sigma=\phi^{-1} \mathbf{i} \sigma & =345162 & (\text { by }(5.3)) \\
\mathbf{i} \phi^{-1} \mathbf{i} \sigma & =461235 & \\
\mathrm{~A}-\operatorname{code} \mathbf{i} \phi^{-1} \mathbf{i} \sigma=123152 & (\text { by }(5.2)) \\
\mathrm{B}^{2} \mathrm{code}^{-1} \mathrm{~A}-\operatorname{code} \mathbf{i} \phi^{-1} \mathbf{i} \sigma=\phi \mathbf{i} \phi^{-1} \mathbf{i} \sigma=463152 & (\text { by }(5.5)) \\
\Phi(\sigma)=\mathbf{i} \phi \mathbf{i} \phi^{-1} \mathbf{i} \sigma=463152 . &
\end{array}
$$

We verify that

$$
(\text { Cyc }, \text { Rmil }) \sigma=(\text { Rmil, Cyc }) \Phi(\sigma)=(\{1,2\},\{1,2,3,5\})
$$

## 6. Concluding remarks and Tables

The bijection constructed by Cori [Cor08] only preserves the cardinalities "cyc" and "lmap", but not the sets "Cyc" and "Lmap." With the example used in his paper, the permutation

$$
\theta=6,5,7,4,2,10,3,8,9=(1,6,10)(2,5)(3,7)(4)(8)(9)
$$

is mapped onto

$$
\theta^{\prime}=4,6,5,7,3,8,1,9,10,2=(1,4,7)(2,6,8,9,10)(3,5)
$$

so that $\left(\right.$ Lmap, Cyc) $\theta^{\prime}=(\{1,2,4,6,8,9\},\{1,2,3\}) \neq(\{1,2,3,4,8,9\},\{1$, $3,6\})=\left(\mathrm{Cyc}\right.$, Lmap) $\theta$. However, (cyc, lmap) $\theta=($ lmap, cyc $) \theta^{\prime}=(6,3)$.

In our case, we take the bijection $\phi \mathbf{i} \phi^{-1}$ that satisfies (see (1.4))

$$
(\mathrm{Cyc}, \mathrm{Lmap}) \theta=(\mathrm{Lmap}, \mathrm{Cyc}) \phi \mathbf{i} \phi^{-1}(\theta) .
$$

The calculation of $\phi \mathbf{i} \phi^{-1}(\theta)$ is made for the same $\theta$, together with the relevant set-valued statistics. We successively get:

$$
\begin{aligned}
\theta & =6,5,7,4,2,10,3,8,9 \\
& =(1,6,10)(2,5)(3,7)(4)(8)(9) \\
\text { B-code } \theta & =1,2,3,4,2,1,3,8,9,6 \\
\text { A-code }{ }^{-1} \mathrm{~B}-\operatorname{code} \theta=\phi^{-1}(\theta) & =6,1,7,5,2,10,3,4,8,9 \\
\mathbf{i} \phi^{-1}(\theta) & =2,5,7,8,4,1,3,9,10,6 \\
\text { A-code } \mathbf{i} \phi^{-1}(\theta) & =1,1,3,2,2,6,3,4,8,9 \\
\phi \mathbf{i} \phi^{-1}(\theta) & =2,5,7,8,4,6,3,9,10,1 \\
& =(1,2,5,4,8,9,10)(3,7)(6)
\end{aligned}
$$

Thus $($ Cyc, Lmap $) \theta=($ Lmap, Cyc $) \phi \mathbf{i} \phi^{-1}(\theta)=(\{1,2,3,4,8,9\},\{1,3,6\})$.
In Fig. 3 the common distribution over $\mathfrak{S}_{n}$ of each bivariable statistic (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Rmil, Cyc), (Lmap, Rmil), (Lmap, Cyc)has been reproduced for $n=1,2,3,4$. On each cell $(A, B)$, where $A, B \subset[n]$, is written the number of permutations $\sigma$ from $\mathfrak{S}_{n}$ such that $(\mathrm{Cyc}, \mathrm{Rmil}) \sigma=(A, B)$. In the table for $n=4$ the total sums occurring at the bottom and on the right are the numbers $\#\left\{\sigma \in \mathfrak{S}_{4}\right.$ : cyc $\sigma=k\}$ for $k=4,3,2,1$, which are the coefficients of the polynomial $x(x+1)(x+2)(x+3)([\operatorname{Ri} 58]$, chap. $4, \S 3)$. It will be noticed that all those tables are symmetric with respect to the main diagonal.

$n=1$

| $\mathrm{B}=$ | 1,2 | 2 |
| ---: | :---: | :---: |
| $\mathrm{~A}=1,2$ | 1 |  |
| 2 |  | 1 |

$n=2$

| $\mathrm{B}=$ | $1,2,3$ | 1,3 | 2,3 | 3 |
| ---: | :---: | :---: | :---: | :---: |
| $\mathrm{~A}=1,2,3$ | 1 |  |  |  |
| 1,3 |  | 1 |  |  |
| 2,3 |  |  | 1 | 1 |
| 3 |  |  | 1 | 1 |

$n=3$

| $\mathrm{B}=$ | 1,2,3,4 | 1,2,4 | 1,3,4 | 2, 3, 4 | 1,4 | 2, 4 | 3,4 | 4 | $\Sigma$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{A}=1,2,3,4$ | 1 |  |  |  |  |  |  |  | 1 |
| 1,2,4 |  | 1 |  |  |  |  |  |  |  |
| 1,3,4 |  |  | 1 |  | 1 |  |  |  | 6 |
| 2,3,4 |  |  |  | 1 |  |  | 1 | 1 |  |
| 1,4 |  |  | 1 |  | 1 |  |  |  |  |
| 2,4 |  |  |  |  |  | 1 | 1 | 1 | 11 |
| 3,4 |  |  |  | 1 |  | 1 | 2 | 2 |  |
| 4 |  |  |  | 1 |  | 1 | 2 | 2 | 6 |
| $\Sigma$ | 1 | 6 |  |  | 11 |  |  | 6 |  |

Fig. 3. Distribution of (Cyc, Rmil) over $\mathfrak{S}_{n}$.
There exist other bijections $\sigma \mapsto a$ such that the sum $\sum_{i}\left(a_{i}-1\right)$ is equal to a statistic different from the inversion number "inv," but having interesting properties. Let us quote the Tompkins-Paige method ([To56, Le60, We61]) for generating permutations on a computer. That method was further used in [Ha92, Ha94] to show that the corresponding sum $\sum_{i}\left(a_{i}-1\right)$ is equal to the major index "maj". Let us also mention the Denert coding [FZ90, Ha94], whose sum $\sum_{i}\left(a_{i}-1\right)$ is equal to the Denert statistic "den". Those codings serve to prove that the statistics "inv," "maj" and "den" are equidistributed on $\mathfrak{S}_{n}$, their common distribution being called Mahonian.

Let $b=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ be the B-code of a permutation $\sigma \in \mathfrak{S}_{n}$. In its turn the sum env $\sigma:=\sum_{i}\left(b_{i}-1\right)$ becomes a new Mahonian statistic. Moreover, it follows from the properties of the bijection $\phi$ defined in (4.1) that the two three-variable statistics (env, Cyc, Lmap) and (inv, Rmil, Lmap) are equidistributed on $\mathfrak{S}_{n}$.

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