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# Predicate Transformers, (co)Monads and Resolutions

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**Abstract.** This short note contains random thoughts about a factorization theorem for closure/interior operators on a powerset which is reminiscent to the notion of *resolution* for a monad/comonad. The question originated from formal topology but is interesting in itself.

The result holds constructively (even if it classically has several variations); but usually not predicatively (in the sense that the interpolant will no be given by a set). For those not familiar with predicativity issues, we look at a “classical” version where we bound the size of the interpolant.

## Introduction

A very general theorem states that any monotonic operator  $F : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  can be factorized in the form  $\mathcal{P}(X) \rightarrow \mathcal{P}(Z) \rightarrow \mathcal{P}(Y)$ , where  $Z$  is an appropriate set; and the first predicate transformer commutes with arbitrary unions and the second commutes with arbitrary intersections.

We prove similar factorization for interior and closure operators on a powerset; the idea being to “resolve” the operator as is usually done for (co)monad in categories. We then look at the constructive version of those factorizations.

## 1 Relations and Predicate Transformers

We start by introducing the basic notions:

**Definition 1.** *If  $X$  and  $Y$  are sets, a (binary) relation between  $X$  and  $Y$  is a subset of the cartesian product  $X \times Y$ . The converse of a relation  $r \subseteq X \times Y$  is the relation  $r^\sim \subseteq Y \times X$  defined as  $(y, x) \in r^\sim \equiv (x, y) \in r$ .*

*A predicate transformer from  $X$  to  $Y$  is an operator from the powerset  $\mathcal{P}(X)$  to the powerset  $\mathcal{P}(Y)$ .*

Since most of our predicate transformers will be monotonic (with respect to inclusion), we drop the adjective when no confusion is possible.

**Definition 2.** Suppose  $r$  is a relation between  $X$  and  $Y$ ; we define two monotonic predicate transformers from  $Y$  to  $X$ :

$$\begin{aligned} \langle r \rangle : \mathcal{P}(Y) &\rightarrow \mathcal{P}(X) \\ V &\mapsto \{x \in X \mid (\exists y \in Y) (x, y) \in r \ \& \ y \in V\} \end{aligned}$$

and

$$\begin{aligned} [r] : \mathcal{P}(Y) &\rightarrow \mathcal{P}(X) \\ V &\mapsto \{x \in X \mid (\forall y \in Y) (x, y) \in r \Rightarrow y \in V\}; \end{aligned}$$

and an antitonic predicate transformer:

$$\begin{aligned} [r^\perp] : \mathcal{P}(Y) &\rightarrow \mathcal{P}(X) \\ V &\mapsto \{x \in X \mid (\forall y \in V) (x, y) \in r\}. \end{aligned}$$

Concerning notation:

- $\langle r \rangle$  and  $[r]$  are somewhat common in the refinement calculus, even though the main reference ([1]) uses  $\{r\}$  instead of  $\langle r \rangle$ . The problem is that this clashes with set theoretic notation.
- In [2], Birkhoff uses  $V^\leftarrow$  for  $[r]$ , but this supposes that  $r$  is clear from the context. (The notation  $V^\rightarrow$  would then be  $[r^\sim](U)$ .)
- The linear logic community would use  $V^\perp$  for the same thing, but this also supposes that the relation is called “ $\perp$ ”.

We will always be in a “typed” context; *i.e.* subsets will always be subset of some ambient set. We write  $\neg$  for complementation with respect to that ambient set.

**Lemma 1.** Suppose  $r$  is a relation between  $X$  and  $Y$ ; we have:

1.  $\langle r \rangle \cdot \neg = \neg \cdot [r]$ ;
2.  $\langle \neg r \rangle = \neg \cdot [r]$ . (where  $(x, y) \in \neg r \equiv (x, y) \notin r$ )

A very interesting property is the following Galois connections:

**Lemma 2.** Suppose  $r$  is a relation between  $X$  and  $Y$ ; then  $\langle r \rangle \vdash [r]$ , and  $[r]$  is Galois-connected to itself:

1.  $\langle r \rangle(V) \subseteq U \Leftrightarrow V \subseteq [r^\sim](U)$ ;
2.  $U \subseteq [r](V) \Leftrightarrow V \subseteq [r^\sim](U)$ .

Those predicate transformers satisfy:

**Lemma 3.** If  $r$  is a relation between  $X$  and  $Y$ , then

- $\langle r \rangle$  commutes with arbitrary unions; (*i.e.* it is a sup-lattice morphism<sup>3</sup>)
- $[r]$  commutes with arbitrary intersections; (*i.e.* it is an inf-lattice morphism)
- $[r]$  transforms arbitrary unions into intersections.

<sup>3</sup> All of our lattices are complete, so we do not bother writing “complete” all the time...

and moreover:

- any sup-lattice morphism from  $\mathcal{P}(Y)$  to  $\mathcal{P}(X)$  is of the form  $\langle r \rangle$  for some  $r \subseteq X \times Y$ ;
- any inf-lattice morphism from  $\mathcal{P}(Y)$  to  $\mathcal{P}(Y)$  is of the form  $[r]$  for some  $r \subseteq X \times Y$ ;
- any predicate transformer from  $Y$  to  $X$  taking arbitrary unions to intersections is of the form  $[r]$  for some  $r \subseteq X \times Y$ .

*Proof.* The sup-lattice part is easy; and the rest is an application of Lemma 1.  $\square$

Just like it is possible to factorize any relation as the composition of a total function and the inverse of a total function, it is possible to factorize an monotonic predicate transformer as the composition of a  $\langle r \rangle$  and a  $[s]$  (see [3] for a detailed categorical construction).

**Proposition 1.** *Suppose  $F$  is a monotonic predicate transformer from  $X$  to  $Y$ ; then there is an  $X'$  and there are relations  $s \subseteq X' \times X$  and  $r \subseteq Y \times X'$  such that  $F = \langle r \rangle \cdot [s]$ .*

*Proof.* Let  $F$  be a monotonic predicate transformer, and define  $X' = \mathcal{P}(X)$  together with  $(U, x) \in s \equiv x \in U$  and  $(y, U) \in r \equiv y \in F(U)$ . We have:

$$\begin{aligned}
& y \in \langle r \rangle \cdot [s](U) \\
& \Leftrightarrow \{ \text{definition of } \langle r \rangle \} \\
& (\exists V \in X') (y, V) \in r \ \& \ V \in [s](U) \\
& \Leftrightarrow \{ \text{definition of } X' \text{ and } r \} \\
& (\exists V \subseteq X) y \in F(V) \ \& \ V \in [s](U) \\
& \Leftrightarrow \{ \text{definition of } [s] \} \\
& (\exists V \subseteq X) y \in F(V) \ \& \ (\forall x) (V, x) \in s \Rightarrow x \in U \\
& \Leftrightarrow \{ \text{definition of } s \} \\
& (\exists V \subseteq X) y \in F(V) \ \& \ (\forall x) V \subseteq U \\
& \Leftrightarrow \{ \text{since } F \text{ is monotonic} \} \\
& y \in F(U)
\end{aligned}$$

The result thus holds, but the proof doesn't bring much information...  $\square$

And as a direct application of Lemma 1:

**Corollary 1.** *Any monotonic predicate transformer can be factorized as a  $[r] \cdot \langle s \rangle$  or as a  $[r] \cdot [s]$ .*

*Similarly, any antitonic predicate transformer can be written has one of  $[r] \cdot [s]$ ,  $[r] \cdot \langle s \rangle$ ,  $\langle r \rangle \cdot [s]$  or  $[r] \cdot [s]$ .*

## 2 Interior and Closure Operators

**Definition 3.** If  $(X, \leq)$  is a partial order, we say that  $F : X \rightarrow X$  is an interior operator if:

- $F$  is monotonic;
- $F$  is contractive:  $F(x) \leq x$ ;
- $F(x) \leq FF(x)$ .

We say that it is a closure operator if:

- $F$  is monotonic;
- $F$  is expansive:  $x \leq F(x)$ ;
- $FF(x) \leq F(x)$ .

It is well known that the composition of two Galois connected operators yield interior/closure operators, so that we have:

**Lemma 4.** If  $r$  is a relation between  $X$  and  $Y$ , then

- $\langle r \rangle \cdot [r^\sim]$  is an interior operator on  $\mathcal{P}(X)$ ;
- $[r] \cdot \langle r^\sim \rangle$  is a closure operator on  $\mathcal{P}(X)$ ;
- $[r] \cdot [r^\sim]$  is a closure operator on  $\mathcal{P}(X)$ .

Some other consequences of the Galois connection are listed below:

- $\langle r \rangle \cdot [r^\sim] \cdot \langle r \rangle = \langle r \rangle$ ;
- $[r] \cdot \langle r^\sim \rangle \cdot [r] = [r]$ ;
- $[r] \cdot [r^\sim] \cdot [r] = [r]$ .

The problem is now to mimic Proposition 1.

**Definition 4.** If  $F$  is an interior operator on  $\mathcal{P}(X)$ , a resolution for  $F$  is given by a set  $Y$  (called the interpolant) together with a relation  $r \subseteq Y \times X$  such that  $F = \langle r \rangle \cdot [r^\sim]$ .

**Proposition 2.** If  $F$  is an interior operator on  $\mathcal{P}(X)$ , then it has a resolution.

The proof relies on the following lemma:

**Lemma 5.** Let  $F$  be an interior operator on a complete sup-lattice  $(X, \leq, \bigvee)$ ; write  $\mathbf{Fix}(F)$  for the collection of fixed-point for  $F$ . We have that  $(\mathbf{Fix}(F), \leq, \bigvee)$  is a complete sup-lattice; and for any  $x \in X$

$$F(x) = \bigvee \{y \in \mathbf{Fix}(F) \mid y \leq x\} .$$

*Proof.* That  $(\mathbf{Fix}(F), \leq, \bigvee)$  is a complete sup-lattice is left as an easy exercise; for the second point, let  $x \in X$ ;

- we know that  $F(x)$  is a fixed point of  $F$ , and that  $F(x) \leq x$ . This implies that  $F(x) \in \{y \in \mathbf{Fix}(F) \mid y \leq x\}$ ; and so  $F(x) \leq \bigvee \{y \in \mathbf{Fix}(F) \mid y \leq x\}$ ;

- suppose  $y \in \mathbf{Fix}(F)$  and  $y \leq x$ ; this implies that  $F(y) \leq F(x)$ , i.e. that  $y \leq F(x)$ . We can conclude that  $\bigvee \{y \in \mathbf{Fix}(F) \mid y \leq x\} \leq F(x)$ .

□

*Proof (of proposition 2).* Suppose  $F$  is an interior operator on  $\mathcal{P}(X)$ ; define  $Y = \mathbf{Fix}(F)$  and  $(U, x) \in r \equiv x \in U$ . We have:

$$\begin{aligned}
& x \in \langle r \rangle \cdot [r^\sim](U) \\
& \Leftrightarrow \{ \text{definition of } r \} \\
& (\exists V \in \mathbf{Fix}(F)) \ x \in V \ \& \ (\forall y) \ y \in V \Rightarrow y \in U \\
& \Leftrightarrow \\
& (\exists V \in \mathbf{Fix}(F)) \ x \in V \ \& \ V \subseteq U \\
& \Leftrightarrow \\
& x \in \bigcup \{V \in \mathbf{Fix}(F) \mid V \subseteq U\} \\
& \Leftrightarrow \{ \text{Lemma 5} \} \\
& x \in F(U)
\end{aligned}$$

which concludes the proof. □

Just like for Proposition 1, the statement of the theorem is interesting, but the proof hardly tells us anything about the structure of  $F$ . To gain a little more information about  $F$ , we will try to “bound” the size of the interpolant set  $Y$ .

**Definition 5.** *If  $(X, \leq, \bigvee)$  is a complete sup-lattice, we say that a family  $(x_i)_{i \in I}$  of element of  $X$  is a basis if, for any  $y \in X$ , we have*

$$y = \bigvee \{x_i \mid x_i \leq y\} .$$

**Corollary 2.** *Suppose  $F$  is an interior operator on  $\mathcal{P}(X)$ ; if  $(\mathbf{Fix}(F), \subseteq, \bigcup)$  has a basis of cardinality  $\kappa$ , then we can find a resolution of  $F$  with an interpolant  $Y$  of cardinality  $\kappa$ .*

*Proof.* It is easy to see that in the above proof of Proposition 2, we can replace  $\mathbf{Fix}(F)$  by a basis of  $(\mathbf{Fix}(F), \subseteq, \bigcup)$ . □

In particular, if there is a basis of  $\mathbf{Fix}(F)$  which has cardinality less than the cardinality of  $X$ ; we can use  $X$  as the interpolant and use a relation  $r \subseteq X \times X$  to obtain a resolution of  $F$ .

We now show that this result is optimal:

**Lemma 6.** *Let  $F$  be an interior operator on  $\mathcal{P}(X)$ ; and suppose there are no basis of  $\mathbf{Fix}(F)$  of cardinality  $\kappa$ ; then there is no interpolant of cardinality less than  $\kappa$ .*

*Proof.* To show that, we will construct a basis of  $\mathbf{Fix}(F)$  indexed by any interpolant for  $F$ . Suppose  $Y$  and  $r$  form a resolution for  $F$ . For any  $y \in Y$ , define  $U_y \equiv \langle r \rangle \{y\}$ . We will show that  $(U_y)_{y \in Y}$  is a basis for  $\mathbf{Fix}(F)$ .

Each  $U_y$  is a fixed point for  $F$ :

$$\begin{aligned} U_y &= \langle r \rangle \{y\} && \{\text{definition}\} \\ &= \langle r \rangle \cdot [r^\sim] \cdot \langle r \rangle \{y\} && \{\text{second part of Lemma 4}\} \\ &= F \cdot \langle r \rangle \{y\} && \{r \text{ is a resolution of } F\} \\ &= F(U_y) && \{\text{definition}\} \end{aligned}$$

Let  $U$  be a fixed point of  $F$ :

$$\begin{aligned} U &= \langle r \rangle \cdot [r^\sim](U) && \{U \text{ is a fixed point of } F\} \\ &= \langle r \rangle (\bigcup \{ \{y\} \mid y \in [r^\sim](U) \}) && \{\text{basic logic}\} \\ &= \bigcup \{ \langle r \rangle \{y\} \mid \{y\} \subseteq [r^\sim](U) \} && \{\langle r \rangle \text{ commutes with unions}\} \\ &= \bigcup \{ \langle r \rangle \{y\} \mid \langle r \rangle \{y\} \subseteq U \} && \{\text{Galois connection between } \langle r \rangle \text{ and } [r^\sim]\} \\ &= \bigcup \{ U_y \mid U_y \subseteq U \} && \{\text{definition}\} \end{aligned}$$

which concludes the proof that  $(U_y)_{y \in Y}$  is a basis for  $\mathbf{Fix}(F)$ .  $\square$

Let's look at an example of interior operator on  $\mathcal{P}(X)$  which *cannot* be resolved using  $X$  as an interpolant. Let  $X$  be a countable infinite set (natural numbers for example); and define  $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  as follows:

$$F(U) = \begin{cases} \emptyset & \text{if } U \text{ is finite} \\ U & \text{if } U \text{ is infinite} \end{cases}$$

The sup-lattice  $\mathbf{Fix}(F)$  is given by the collection of infinite subsets of  $X$ ; and this lattice doesn't have a countable basis. To prove that, it is enough to do it for any particular countable infinite set. Take  $C$  to be the set of finite strings over  $\{0, 1\}$ . If  $\alpha$  is an infinite string of 0's and 1's, define  $U_\alpha \subseteq C$  to be the set of finite prefixes of  $\alpha$ . Each  $U_\alpha$  is an element of  $\mathbf{Fix}(F)$ ; but no countable family of infinite subsets can "generate" all the  $U_\alpha$ : since  $\alpha \neq \beta$  implies that  $U_\alpha \cap U_\beta$  is finite, if  $V_i \subseteq U_\alpha$  and  $V_j \subseteq U_\beta$  then  $i \neq j$ . In other words, a family which generates all the  $U_\alpha$ 's needs to have the cardinality of the collection of the  $U_\alpha$ 's, *i.e.* uncountable.

Using Lemma 1, we can now extend all what has been done for interior operators for closure operators: if  $F$  is a closure operator on  $\mathcal{P}(X)$ , then  $\neg \cdot F \cdot \neg$  is an interior operator on  $\mathcal{P}(X)$ .

**Corollary 3.** *If  $F$  is a closure operator on  $\mathcal{P}(X)$ , then it has a resolution as a composition  $[r] \cdot \langle r^\sim \rangle$  or as  $[r] \cdot [r^\sim]$ .*

*As for interior operators, the possible cardinalities of the interpolant are given by the cardinalities of the bases for the inf-lattice  $\mathbf{Fix}(F)$ .*

In particular, for linear logicians, it is not the case that any closure operator can be written as a biorthogonal...

### 3 Comonad and Monads

It seems that the traditional way to look at monads in a category is to see them as a kind of generalized monoid; at least in my part of the world. Another view<sup>4</sup>

<sup>4</sup> which has given me a much better understanding of what (co)monads are

is to view them as a generalization of closure operators. This is the view taken in the introduction of [4].

**Definition 6.** A monad on a category  $\mathcal{C}$  is a morphism  $F : \mathcal{C} \rightarrow \mathcal{C}$  together with two natural transformations  $\eta : \mathbf{Id}_{\mathcal{C}} \rightarrow F$  and  $\mu : FF \rightarrow F$  s.t. some diagram commute.

If one takes the partial order category  $\mathcal{P}(X)$ , then a monad is an operator on  $\mathcal{P}(X)$  s.t.:

- it acts on morphisms: if  $i : U \subseteq V$  then  $F_i : F(U) \subseteq F(V)$ ; i.e.  $F$  is monotonic;
- there is a natural transformation  $\eta_U : U \subseteq F(U)$ ;
- there is a natural transformation  $\mu_U : FF(U) \subseteq F(U)$ .

All the “coherence” conditions are trivially satisfied in a partial order category, since every diagram commute! What we’ve just shown is that a monad on  $\mathcal{P}(X)$  is nothing more than a closure operator on  $\mathcal{P}(X)$ ; and vice and versa. Similarly, a comonad corresponds to an interior operator.

In categories, a *resolution* for a monad corresponds to factorizing the functor  $F$  as the composition of two adjoint functors. Adjointness  $H \vdash G$  between the functors  $G : \mathcal{D} \rightarrow \mathcal{C}$  and  $H : \mathcal{C} \rightarrow \mathcal{D}$  in a locally small category means that  $\mathcal{C}[A, G(B)] \simeq \mathcal{D}[H(A), B]$  which, in the case of a partial order category simplifies to “ $U \subseteq G(V)$  iff  $H(U) \subseteq V$ ” i.e. is exactly the Galois connection  $H \vdash G$ .

Category theory tells us that there always is a resolution since we have two degenerate resolutions: something like  $F = \mathbf{Id} \cdot F$  and  $F = F \cdot \mathbf{Id}$ . The first one is given by the Eleinberg-Moore category. In the context of order theory, it amounts to taking: interpolant category to partial orders: if  $F$  is a closure operator on  $X$ , take  $\mathcal{E}(X, F)$  to be the the collection of fixed points for  $F$ , with the ordering inherited from  $X$ .<sup>5</sup> We have  $F : X \rightarrow \mathcal{E}(X, F)$  and  $\mathbf{Id} : \mathcal{E}(X, F) \rightarrow X$  which are adjoint:  $F \vdash \mathbf{Id}$ .

$$\begin{aligned} x &\leq \mathbf{Id}(f) \\ &\Rightarrow \{ F \text{ is monotonic} \} \\ F(x) &\leq F(f) \\ &\Leftrightarrow \{ f \in \mathcal{E}(X, F), \text{ i.e. } f \text{ is a fixed point for } F \} \\ F(x) &\leq f \end{aligned}$$

and  $F(x) \leq f \Rightarrow x \leq f = \mathbf{Id}(f)$  since  $x \leq F(x)$ .

The second resolution ( $F \cdot \mathbf{Id}$ ) is obtained via the Kleisli category. In the context of order theory, it amounts to taking the order  $(X, \sqsubseteq)$  where  $x \sqsubseteq y$  iff  $F(x) \leq F(y)$ .

What is important to us is that those resolutions are trivial and uninteresting. More abstract nonsense states that there is a category of resolutions for any given monad; and that those two trivial resolutions are respectively initial/terminal.

<sup>5</sup> In a partial order, an algebra for  $F$  is just a post fixed point:  $x \leq F(x)$ . If  $F$  is a closure, then it is also a fixed point for  $F$ .

What we have done with Proposition 2 and Corollary 3 corresponds to finding a non-trivial *resolution* for the monad corresponding to the interior/closure operator. It is even more than a resolution, since the interpolant is itself a complete and cocomplete category (and one of the functors preserves limits while the other one preserves colimits; but this is a general fact about adjoints). The feeling is that this resolution lies “exactly in the middle” between the initial and terminal resolutions. I don’t know if this kind of “strong” resolution has been considered in category theory.<sup>6</sup>

The resolution constructed here is quite different from the Eleinberg-Moore resolution (even though it uses the fixed-points as a basis). We do not construct functors from  $\mathcal{P}(X)$  to  $\mathcal{E}(X, F)$  and back; but from  $\mathcal{P}(X)$  to  $\mathcal{P}(\mathcal{E}(X, F))$  and back. In particular, as noted above, the interpolant is complete and cocomplete;<sup>7</sup> which is not the case for the Eleinberg-Moore category: fixed points for an interior are closed under unions but not under intersections; and conversely for closure operators.

## 4 Revisiting Section 2 in a Constructive Setting

### 4.1 Impredicative

In the previous section we used Lemma 1 to generalize results on interior to closures. It is thus natural to ask whether (1) we can make the original proof constructive; (2) we can avoid using this lemma and prove the result for closure constructively. I will not into the details but just provide some hints about that.

- Galois connections from Lemma 2 are constructive.
- Lemma 3: the first part is trivial. The second part is easy: if  $F$  commutes with unions, take  $(x, y) \in r$  iff  $y \in F\{x\}$ ; if  $F$  commutes with intersections, take  $(x, y) \in r$  iff  $(\forall U) y \in F(U) \Rightarrow x \in U$ ; and if  $F$  transforms unions into intersections, take  $(x, y) \in r$  iff  $y \in F\{x\}$ .
- Proposition 1: the proof is constructive.
- Lemma 3 is constructive.
- Lemma 5 and the corresponding lemma for closure operator are constructive.
- Proposition 2 is constructive.
- we can mimic the proof of Proposition 2 to obtain a resolution of a closure operator as  $F = [r] \cdot [r^\sim]$ , but *not* to obtain a resolution as  $F = [r] \cdot \langle r^\sim \rangle$ .
- I doubt we can constructively obtain a resolution of a closure operator as  $F = [r] \cdot \langle r^\sim \rangle$ .
- all of the lemmas about the size of interpolant are constructive at least if we read them as “if  $B$  is a basis for  $\mathbf{Fix}(F)$  then we can use  $B$  as an interpolant”.

As a proof of concept, all this (except the last point) has been proved in the proof assistant COQ.<sup>8</sup>

<sup>6</sup> *i.e.* if  $\mathcal{C}$  is a complete and cocomplete category and  $F$  is a monad on  $\mathcal{C}$ , a “strong resolution” is a resolution with a complete and cocomplete interpolant. Any reference to something similar in the literature would be most welcome.

<sup>7</sup> *i.e.* is a complete lattice since we deal with partial orders

<sup>8</sup> proof scripts available from <http://iml.univ-mrs.fr/~hyvernac/academics.html>

## 4.2 Predicative

In a predicative setting like Martin-Löf type theory (see [5]) or CZF (constructive ZF, see [6]) set theory, many of the results are not provable. The reason being that we do not allow quantification on a power-set. The main result about resolution would become something like:

**Proposition 3.** *Suppose  $\mathbf{Fix}(F)$  (proper type) has a set-indexed basis, then  $F$  as a resolution as  $\langle r \rangle \cdot [r^\sim]$  (if  $F$  is an interior) or as  $[r] \cdot [r^\sim]$  (if  $F$  is a closure). If  $F$  as a resolution, then  $\mathbf{Fix}(F)$  as a set-index basis.*

Note that having a set-indexed basis is equivalent to being “set presented” in the terminology of P. Aczel ([6,7]). Note that in the case of an interior operator, this implies that the predicate transformer is “set-based” (in the sense that it has a factorization as in Proposition 1, where the interpolant  $X'$  is a set). It doesn't seem that the existence of a resolution for a closure operator implies that the original predicate transformer is itself set-presented.

## Conclusion

Nothing revolutionary has really been done, but the statement of Propositions 1 and 2 is, in an abstract setting, quite neat. However, as the proofs show, this is mostly abstract nonsense. The best example is probably Proposition 1, where  $y \in F(U)$  is factorized as “there is a  $V$  such that  $s \in F(V)$  and  $V \subseteq U$ ”. The proof of Proposition 2 is slightly subtler, but is hardly interesting. In the end, the most interesting and informative thing is probably Lemma 6, in its “positive” version: if  $F$  has  $Y$  as an interpolant, then  $\mathbf{Fix}(F)$  has a basis indexed by  $Y$ , which is hardly a breakthrough in mathematics...

I do nevertheless hope that it might interest some people, since while Proposition 1 is known to many (especially the refinement calculus people), it seems that Proposition 2 isn't stated anywhere. I also hope the link between monad and closure operator (together with the Eleinberg-Moore category being the partial order of fixed points) will gain in popularity, as I see it as a much better way of seeing monads, at least as far as intuition is concerned.

A final word about the motivation for this: the starting point was the question about whether it is possible to represent any “basic topology” (see the forthcoming [8]) as the formal side of a basic pair, impredicatively speaking. A basic topology is a structure  $(X, \mathcal{A}, \mathcal{J})$  where  $X$  is a set, and  $\mathcal{A}$  and  $\mathcal{J}$  are closure and interior operators on  $\mathcal{P}(X)$  such that  $\mathcal{A}(U) \mathbin{\vDash} \mathcal{J}(V) \Rightarrow U \mathbin{\vDash} \mathcal{J}(V)$ .<sup>9</sup> The answer is obviously *no* since the  $\mathcal{A}$  and  $\mathcal{J}$  arising from a formal pair are classically dual (*i.e.*  $\mathcal{A} \cdot \neg = \neg \cdot \mathcal{J}$ ) and there are basic topologies which are provably not dual. The question then turned into: “can any interior operator be written as the formal interior of a basic pair?” and similar for closure operators. The answers are *yes* (Proposition 2) and *I don't know* (the constructive resolution of a closure is of the form  $[r] \cdot [r^\sim]$ , and not of the form  $[r] \cdot \langle r^\sim \rangle$ .)

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<sup>9</sup>  $U \mathbin{\vDash} V$  is the constructive version of  $U \cap V \neq \emptyset$ .

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