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# Delta-Modulation Coding Redesign for Feedback Controlled Systems

Carlos Canudas-de-Wit, Fabio Gómez-Estern, and Francisco R. Rubio

**Abstract**—This paper investigates the closed-loop properties of the differential coding scheme known as Delta-Modulation ( $\Delta$ - $M$ ) when used in feedback loops within the context of feedback controlled systems. We propose a new modified scheme of the original form of the  $\Delta$ - $M$  algorithm which is better suited for applications where the sensed information is used in feedback. A state feedback controller is implemented with the state estimated by a predictor-based differential decoder. Stability of the resulting closed-loop systems (controller-coder-decoder) are studied. These properties (stability and performance) depend on the quantization parameter  $\Delta$ , which is assumed constant in the first part of our work. In a further step, parameter  $\Delta$  is made adaptive, by defining an adaptation law exclusively in terms of information available at both the transmitter and receiver side. With this approach both stability and performance is improved.

**Index Terms**—Differential coding, Delta modulation, Networked controlled systems, NCS, Wireless Sensor Networks.

## I. INTRODUCTION

**D**ELTA modulation ( $\Delta$ - $M$ ) is a well-known differential coding technique used for reducing the data rate required for voice communication, see [20]. The standard technique is based on synchronizing a state predictor on emitter and receiver and just sending a one-bit error signal corresponding to the innovation of the sampled data with respect to the predictor. The prediction is then updated by adding a positive or negative quantity (determined by the bit that has been transmitted) of absolute value  $\Delta$ , a known parameter shared between emitter and receiver.

This paper studies different aspects of the use of differential coding as a means for transmitting sensing signals in feedback controlled linear systems. In particular we focus on the stability issues that appear when the  $\Delta$ -modulation scheme is embedded in a control system between the device measuring the state and the control signal. The problem is of interest in the area of Networked Controlled Systems (NCS) calling for data-compression algorithms, but also in digital control applications where sensed information is converted using analog-to-digital converters (ADC) with few bits (i.e. 1-bit ADC).

### A. Delta-modulation as a coding mechanism in NCS

Networked Control Systems is a multidisciplinary field where Control and Communications technologies and Information theory meet. Whereas the advantages of using networks

as a means of connecting plants and actuators is easily understood, the set of problems arisen by the technological convergence is complex. In one sense, stability of controllers may be compromised by the inclusion of a network, and new closed-loop analysis must be done, with specific quantization-modulation considerations, (see, for instance, [7] and the references therein). Another approach is to monitor the network quality of service (QoS) parameters, to consequentially tune the controller gains for maximum stable performance, as is presented in [22]–[23]. Finally, co-design approach (see [18]) addresses simultaneously the modification of the control strategy and the message transmission schedule for best overall results.

Applications concerned with real-time wireless networked controlled systems, as shown in Figure 1, seek for data-compression algorithms aiming at reducing the amount of information that may be transmitted throughout the communication channel, and therefore permitting a better resource allocation and/or an improvement of the permissible closed-loop system bandwidth (data-rate). Moreover, recent advances in this field are strongly focused on energy consumption, which is probably the key to cheap distributed sensing. Vendors of technologies like ZigBee and Bluetooth now claim that devices may be powered on the same battery for years, leaving a way to non replaceable battery devices into the market. Energy consumption is thus a goal together with bandwidth when dealing with modern sensor networks, and new communications technologies. The search for minimal data transfer devices in control has also been motivated by underwater applications, where data rates are normally bounded below 100 b.p.s. due to the strongly dissipative medium.

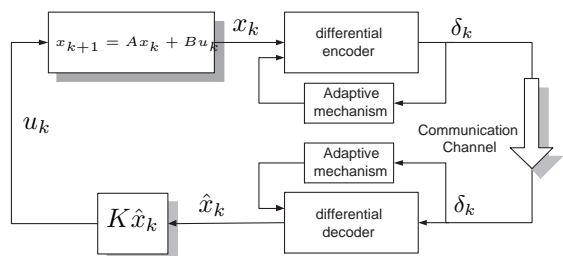


Fig. 1. Block diagram of the problem set up studied in this paper.

The Delta modulation ( $\Delta$ - $M$ ) algorithm can also be understood as the coarsest two-level (1-bit) quantizer. Optimization of the quantization levels is mandatory in large-scale systems, and may be of great interest while designing low cost transmitter/receiver components. Differential coding in feedback system is thus a simple alternative to reported control schemes

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concerning the use of quantizers in the context of NCS, i.e. [12], [7], [17], [14], [27], [16]. More precisely, the use of delta-modulators for networked control has been reported in [15], as a practical implementation without theoretical stability analysis.

The *modified form* of the ( $\Delta$ - $M$ ) algorithm proposed here, has the ability of enforce the *separation principle* between the control law and the estimation process (coding strategy). As a result, the (linear) feedback law is first designed disregarding the coding algorithm ( $x \equiv \hat{x}$ ). Then, the coding scheme is designed in order to preserve stability when embedded in the feedback loop. It is also worth to mention that this *modified form* of the  $\Delta$ - $M$  coding structure explicitly contains information about the model plant and the controller feedback gain.

Others works leading with some variants of the Delta modulation coding in feedback systems can be found in [4] for energy-aware coding, [3] for the combination of Delta modulation with entropy variable length coding, [11] for the use of more than one bit, and [5] for multivariable coding in the fixed quantization case.

### B. Delta-modulation as 1-bit converter

The other possible scenario where our results will be of interest, concerns embedded control systems where the sensors and digital controller are embedded in the same electronics (system on chip, SoC). With the aim of saving energy, cost, and space in the silicon layers, low-resolution analog-to-digital converters are preferable to high-resolution ones. Numerous examples of the use of Delta-modulators embedded in general microelectronic devices have been reported in the literature. For instance, [1] and [24] use FPGAs and ASICs for implementing Delta and Sigma-Delta-modulation A/D converters for current and voltage measurement. The need for oversampling the analog signals, is compensated by the technological advantage of reducing the number of ADC channels and comparators.

Interesting applications using Delta-modulation inside control loops are also found in gyroscopic and acceleration sensors in MEMS, which are configured as shown in Fig. 2. The principle, as described in [13]–[19], is based on feedback-sensing the capacitive mass displacement produced by an acceleration on the device. In [13] and [9], this analogous mass displacement is converted to a digital signal by using 1-bit Sigma-Delta modulation encoders. Then, the 1-bit signal is decoded and used in feedback to “cancel” the external acceleration that disturbs the small mass motion. If the control is successfully designed, then the control signal equals the external acceleration providing the desired measure. Note that this example is similar to the framework displayed in Figure 1 with the difference that the encoder signals are transmitted by wire.

### C. Main paper contribution

The paper contributions are presented in the following order.

First we study the case of the Delta-modulation algorithm under *fixed-gain*  $\Delta$ . To begin with, we address the continuous-time formulation case, which is relevant because it captures

the limiting case of the discrete-time formulation (finite data rate transmission), and provides a better understanding of the maximal stability properties that can be attained. It also shows the substantial simplification that can be reached in the stability analysis, as well as the separation properties between states and estimation obtained by using the proposed  $\Delta$ - $M$  modified form. In this part, we study the closed-loop stability properties of the original form of the  $\Delta$ - $M$  algorithm. Then, we propose a new coding law that improves over the original form of the  $\Delta$ - $M$  algorithm, and it allows for the *separation principle* mentioned previously.

Based in this modified form, we then study the discrete-time case, which permits a more clear assessment of the data rate constraints. It is shown that the system states converge to a ball enclosing the origin, and that the result is semi-global. The size of the attraction region and the system precision, depends on the coding gain  $\Delta$  linearly, and is also a function of the location of the unstable discrete-time open-loop poles. These results show that the stability properties improve as the sampling time is reduced, or equivalently as the transmission data rate is increased.

Finally, the last part of the paper illustrates an adaptation mechanism, consisting of making time variant the quantization interval  $\Delta$ . In our adaptive approach, closed-loop global asymptotic stability is proved for the noiseless case. The effect of random noise in the performance of the adaptive quantization scheme is also analyzed via numerical simulations.

### D. Comparison with other approaches

The problem of quantization with time-varying resolution in feedback loops has been addressed in [10] and [2]. The first of these works presents, in the case of fixed resolution, a scheme similar to [6], in the sense that the state estimation is computed through a filter built upon the closed-loop system matrix. However, the extension to variable-scale quantization is only defined in the *zooming-in* direction, and hence the initial states are upper bounded by the initial choice of the zoom factor. As a consequence, only semi-global stabilization is achieved. Here we propose a  $\Delta$ -update law that works well for both *in*-and-*out* zooming directions, thus providing a means to capture unbounded initial states in the *zooming-out* stage and guaranteeing global asymptotic stability. Moreover, by defining an explicit  $\Delta$ -update law in both directions, if the state is driven temporarily out of the domain of attraction at any time due to unattended disturbances, the system will recover stability, unlike the case of [10].

On the other hand, the work of [2] also guarantees global asymptotic stability with time-varying full-state quantization in two subsequent stages. Although our approach can be roughly understood as a particular case of the Theorems provided there, an important new feature introduced here is a state predictor that reduces the amount of data transmitted per sample by only sending the quantized prediction error. This reduces the data-rate to a minimum of 1 bit per sample, while the data-rate in Theorem 3 of [2] is  $\log_2(2M)$ , a value that cannot be taken arbitrarily low.  $M$  is actually defined as a function of the matrices  $A, B, K$  of the control system, and

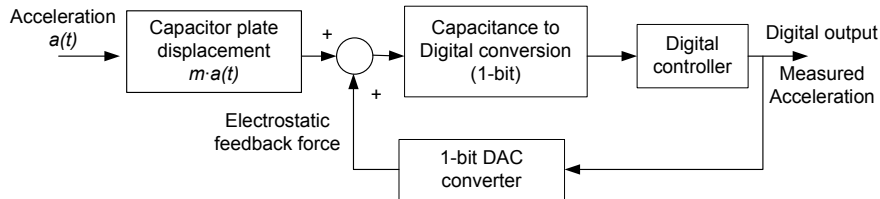


Fig. 2. Block diagram of MEMS accelerometer using  $\Delta$ - $M$  both in the inner analog capacitance-to-digital conversion and in the outer force-feedback loop.

it must be large enough to make the (thereby defined) scale factor small. In Theorem 4 of [2] a 1-bit per sample data-rate transmission scheme is also discussed, but here the separation principle is not present, as the feedback  $u = H(q(x))$  is no longer a linear feedback of the state estimated on the receiver side. But probably the most significant difference between that approach and the one presented here is the zooming factor. In [2], it is calculated in terms of the convergence of the state to an attractive ellipsoid, whose geometry depends in a complex way on the system matrices. In our approach the zoom factor is updated at all times with a simple law that only depends on the last two state estimations, irrespective of the system matrices (and hence not subject to the side effects of bad identification).

#### E. Definitions and notation

Let  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ . The following definitions and notation are in order:

- $\|x\|^2 = \sum_{i=1}^n x_i^2$
- $|x| = \sum_{i=1}^n |x_i|$ , where  $|x| \geq \|x\|$
- $\text{sgn}(x) = [\text{sgn}(x_1), \text{sgn}(x_2), \dots, \text{sgn}(x_n)]^T$ , with  $\text{sgn}(0) = \pm 1$  arbitrarily<sup>1</sup>, and
- $\|A\|$  is the induced Euclidean norm of  $A$ .

#### F. Control setup

The control setup under study is shown in Figure 3, which describes a closed-loop system under one-way communication channel. The control computations are assumed to be done at the plant side, whereas the sensor is remotely located. Information from sensor to controller is then transmitted through a data communication network, and coded by the differential encoder/decoder maps  $\Phi_E : x \mapsto \delta$ , and  $\Phi_D : \delta \mapsto \hat{x}$ , where  $\delta(t)$  is the encoded signal vector of dimension  $n$ , with only two-valued elements  $\delta^i(t) \in \{-1, 1\}$ ,  $\forall n = 1, 2, \dots, n$ , and  $\hat{x}(t) \in \mathbb{R}^n$  is the estimated value of  $x$  as obtained from the decoding map  $\Phi_D$ .

This architecture is commonly found in the NCS literature where a common actuator is used for controlling a system with distributed sensors (probably a large number of low cost wireless devices). Some practical applications where its use is clearly justified are:

- Energy-efficiency control in buildings. In such applications, it is quite frequent to distribute a set of low-power temperature and humidity sensors all over the premises, in order to gather distributed information and use it together

<sup>1</sup>When our encoding algorithm requires that value, we will allow it to issue a +1 or -1 arbitrarily.

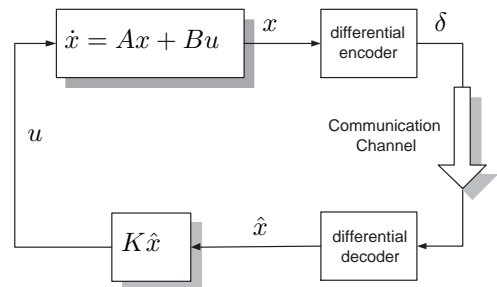


Fig. 3. Block diagram of the continuous-time version of the problem.

with heat flow or energy consumption models to design optimal control strategies. In those cases, the actuators are normally few and large in size (air conditioning units, window openers...) and they are wired to the controller (when not directly built-in).

- Another application where a great research effort is being done is in underwater vehicle control. In a scenario where a set of underwater explorer robots are remotely controlled and coordinated by a surface vessel, the limitations on data rate appear exclusively on the upward link, as the surface ship enjoys a significantly larger source of energy for transmitting.

#### G. Assumptions

The hypotheses used all along the paper, are the following:

- The coding strategy is assumed to be scalar, i.e. each component  $x_k^i$  of the vector signal  $x_k$  is coded independently of each other. Therefore, the encoding (respectively decoding  $\Phi_D$ ) process is defined as a  $n$ -dimensional map,  $\Phi_E = [\Phi_E^1, \Phi_E^2, \dots, \Phi_E^n]^T$ , with elements  $\Phi_E^i : x^i \mapsto \delta^i$ ,
- The encoded system outputs are binary, i.e.  $\delta_k^i \in \{-1, 1\}$
- The maximum bit rate per unit of time is  $R$  [b.p.s.],
- Transmission and ADC delays are neglected. Aspects related to transmission delays have been studied elsewhere, see for instance [26], and [25].

## II. CONTINUOUS-TIME FORMULATION

To simplify matters, we first consider in this section a fully continuous-time formulation. The more realistic discrete-time case, including data-rate constraints, will be studied in subsequent sections.

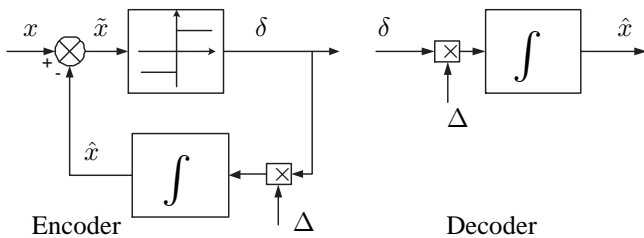


Fig. 4. Block diagram of standard continuous-time version of the  $\Delta$ - $M$  coding scheme. The encoder  $\Phi_E$  is depicted at the left, where the decoder  $\Phi_D$  is shown at the right. For  $n > 1$ ,  $\delta$  is defined as  $\delta = [\delta^1, \delta^2, \dots, \delta^n]^T$ , with  $\delta^i \in \{-1, 1\}$ .

We consider linear systems, with a linear feedback of the following form:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1a)$$

$$u(t) = -K\hat{x}(t) \quad (1b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  are controllable matrices,  $u(t)$  is the input vector.

#### A. Differential coding

There exists a large class of source coding algorithms aiming at compressing information for a more efficient data transmission.

Differential coding schemes belong to the temporal waveform coding class of algorithms. In differential coding, differences between successive samples are encoded rather than the samples themselves. Since the difference between samples is expected to be smaller than the actual sample amplitude, fewer bits are required to represent the differences. Delta modulation ( $\Delta$ - $M$ ), shown in Figure 4, is the simplest form of differential coding (see [20]), in which a two-level (1-bit) quantizer is used in conjunction with a first-order predictor.

The continuous-time version of the  $\Delta$ - $M$  coding algorithm is:

- Encoder map  $\Phi_E : x \mapsto \delta$

$$\dot{\hat{x}}(t) = \Delta(t) \cdot \delta(t), \quad \hat{x}(0) = 0 \quad (2a)$$

$$\delta(t) = \text{sgn}(x(t) - \hat{x}(t)). \quad (2b)$$

where  $\text{sgn}(\zeta) = [\text{sgn}(\zeta_1), \text{sgn}(\zeta_2), \dots, \text{sgn}(\zeta_n)]^T$ ,

- Transmitted information:  $\delta^i(t) \in \{-1, 1\}$ ,
- Decoder map  $\Phi_D : \delta \mapsto \hat{x}$

$$\hat{x}(t) = \int_0^t \Delta(\tau) \cdot \delta(\tau) d\tau, \quad \hat{x}(0) = 0. \quad (3)$$

$\Delta(t) \in \mathbb{R}^{n \times n}$  is the step size matrix, which in general is chosen to be constant (time-varying, or adaptive gains can also be considered, as shown later in this paper). For designing simplicity,  $\Delta(t)$  is frequently chosen to be diagonal. A mandatory choice is that the gain  $\Delta(t)$  must be strictly the same on both sides of the coding process, in order for the decoder to work properly.

To simplify notation, time-arguments will be dropped in this section in all variables when its explicit mention is not necessary.

#### B. Coding separation principle

The complete coding process can be seen as an estimation process defined by the coding map  $\Phi_C : x \mapsto \hat{x}$ , for the problem of state feedback stabilization. In the case of reliable noiseless channel transmission  $\Phi_C$  is simply defined as  $\Phi_C = \Phi_E \circ \Phi_D$ .

Behind the *separation principle* that will be advocated here, lies the idea of first designing the feedback gain  $K$  disregarding the coding algorithm; assuming that  $x \equiv \hat{x}$ , and then designing the coding map  $\Phi_C$  in such a way to preserve stability when  $\Phi_C$  is embedded in the feedback loop. In fact, the feedback law design is separated from the design of the coding algorithm, while the converse is not true, as it will be demonstrated later.

With this in mind, we first consider here that the linear feedback gain  $K$  is designed so that the matrix

$$A_c = A - BK$$

is strictly stable under the controllability assumption of the pair  $(A, B)$ . Then, we will see that some modifications on the standard  $\Delta$ - $M$  algorithm are in order, for this separation principle to be possible. Before introducing the modified form of the  $\Delta$ - $M$  algorithm, we first discuss the stability properties achievable with the standard form.

#### C. Standard $\Delta$ - $M$ coding algorithm in feedback

Consider the problem of finding a suitable value for  $\Delta$  so as to stabilize system (1), under the coding law (2)-(3). It is easy to see that the error equations read as:

$$\dot{\tilde{x}} = A_c x + BK \tilde{x} \quad (4a)$$

$$\dot{\tilde{x}} = \underbrace{A_c x + BK \tilde{x}}_{\dot{x}} - \Delta \text{sgn}(\tilde{x}) \quad (4b)$$

with  $\tilde{x} = x - \hat{x}$ , and  $\text{sgn}(\tilde{x})$  as defined in Section I-E. A first observation is that this system is fully coupled, and that the control and coding law can not be designed independently of each other. In particular we note that the gain  $\Delta$  should, somehow, be chosen large enough to locally dominate the rate of change of the system state,  $\dot{x}$ . Clearly, this can be done only under relatively “high-values” for  $\Delta$ .

Putting aside technicalities in the sense of existence of solutions (i.e. Filippov’s type, etc.) due to the discontinuous right-hand side of Equation (4), which are not central for the discussion here, the computation details given in Appendix VIII-A, show that if  $\Delta$  is constant, and of the form

$$\Delta = \Delta_0 \cdot I_{n \times n}, \quad \Delta_0 > 0$$

with  $\Delta_0$  being a scalar, one can select a sufficiently large  $\Delta_0$ , such as to ensure that the state and error trajectories  $\zeta = (x^T, \tilde{x}^T)^T$  tend to zero. That is,  $\Delta_0$  must satisfy a condition of the type:

$$\Delta_0 \geq c \cdot \|\zeta(0)\|$$

where  $c > 0$  is a constant depending on the system and controller parameters.

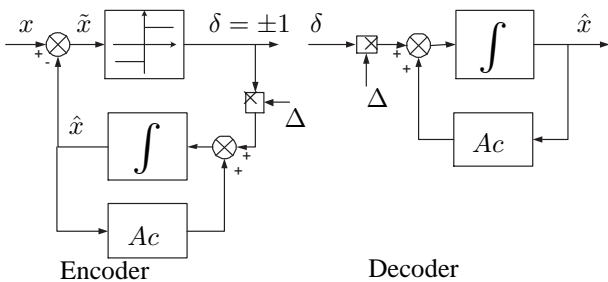


Fig. 5. Block diagram of standard continuous-time version of the  $\Delta$ - $M$  coder with the proposed modification.

#### D. A new structure for the $\Delta$ - $M$ coding

The previous discussion (summarized in Proposition 5) shows that relatively large gains for  $\Delta_0$  will be required to stabilize the system, as the estimation of the bound on  $\Delta_0$  would tend to be conservative, and the size of the region of attraction is suited to be large. Besides, in this continuous-time framework, high gains will result in an important “chattering” effect, thus increasing the estimation error variance when the algorithm is discretized, or adding noise to the measured signal. Although the analysis is conservative, and probably lower values  $\Delta_0$  may still be possible, it is interesting to study new structures of the  $\Delta$ - $M$  coding such as to reduce the necessary gain for stabilization. The algorithm presented next is a modification of the original form of the  $\Delta$ - $M$  algorithm, ad most importantly, it yields an estimation error equation decoupled from the system state  $x$ .

Consider now the following modified  $\Delta$ - $M$  algorithm

$$\dot{\hat{x}}(t) = A_c \hat{x} + \Delta(t) \cdot \delta(t) \quad (5)$$

together with system (1), and  $\delta(t)$  as defined before. Note that a new term  $A_c \hat{x}$  has been included in this case. As a consequence, the coding algorithm depends explicitly on the system model and the feedback gain used.

This equation describes both: the encoder and the decoder, as shown in the block diagram of Figure 5.

It can be shown that the new error equation resulting from this new structure is described as two systems in cascade interconnection, i.e.

$$\dot{x} = A_c x + BK \tilde{x} \quad (6a)$$

$$\dot{\tilde{x}} = A \tilde{x} - \Delta \cdot \text{sgn}(\tilde{x}) \quad (6b)$$

note that equation (6b) describes an autonomous system whose solution is the input of the stable linear system (6a).

The stability properties of this algorithm are simpler to analyze than the one presented previously. They are also more tractable, and certainly less conservative. They are given next, and also result in a system (6) which is semi-globally asymptotically stable.

*Proposition 1: MODIFIED  $\Delta$ - $M$  CODING.* Consider system (1) together with the modified  $\Delta$ - $M$  coding scheme (5). As before, assume that  $\Delta \in \mathbb{R}^{n \times n}$  is constant and has the following form:

$$\Delta = \Delta_0 \cdot I_{n \times n},$$

and that  $\Delta_0$  fulfills the following inequality,

$$\Delta_0 > a \cdot \|\tilde{x}(0)\|, \quad a = \frac{1}{2} \lambda_{sup} \{A + A^T\},$$

then,  $\zeta(t) = (x^T(t), \tilde{x}^T(t))^T$  is bounded and tends to zero as  $t \rightarrow \infty$ .

*Proof:* The proof is straightforward. Let  $V = \tilde{x}^T \tilde{x} / 2$ , and from equation (6b), we have that

$$\begin{aligned} \dot{V} &= \frac{1}{2} \tilde{x}^T (A \tilde{x} - \Delta \text{sgn}(\tilde{x})) + \frac{1}{2} (A \tilde{x} - \Delta \text{sgn}(\tilde{x}))^T \tilde{x} \\ &\leq -\Delta_0 |\tilde{x}| + \frac{1}{2} \tilde{x}^T (A + A^T) \tilde{x} \\ &\leq -\Delta_0 \|\tilde{x}\| + \frac{1}{2} \tilde{x}^T (A + A^T) \tilde{x} \\ &\leq -\Delta_0 \|\tilde{x}\| + a \|\tilde{x}\|^2 = -\|\tilde{x}\| (\Delta_0 - a \|\tilde{x}\|) \end{aligned}$$

where we have used the relation  $-|x| \leq -\|x\|$ . From here it can be seen that the condition  $\Delta_0 > a \cdot \|\tilde{x}(0)\|$ , with  $a$  given as in the last Proposition, makes  $V$  decrease, and hence ensures that  $\|\tilde{x}(t)\| < \|\tilde{x}(0)\|$ , and that  $\tilde{x}(t)$  remains bounded and tends to zero in finite time. Finite-time convergence is typical in switching systems of the form (6b), and will not be demonstrated here. From this analysis we can also conclude that  $\tilde{x}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ .

To complete the proof note that equation (6a) describes a strictly stable linear system with input<sup>2</sup>  $\tilde{x}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ , hence we can conclude that  $x(t)$  is also bounded and tends to zero, as the Proposition states. ■

*Remark 1:* Note that the proposed new coding form, in addition to simplify the stability conditions by making them only depend on the estimation initial error, the new structure introduces a side effect of a low pass filtering action that will improve the filtering properties of the decoding dynamic equations.

### III. DISCRETE-TIME ALGORITHM

In this section we present the extension of the continuous-time differential coding to the more realistic case of a discrete-time framework. We first introduce a simple case of a one-dimensional unstable system for which the optimal gain  $\Delta$  and the attraction domain are easily found. Then, we extend this result to systems of higher dimension.

#### A. One-dimensional system example

Consider the following one-dimensional discrete-time system, together with the control law, and the differential coding modified law:

- Open-loop system, and encoder:

$$x_{k+1} = ax_k + bu_k \quad (7)$$

$$\hat{x}_{k+1} = [a - bK] \hat{x}_k + \Delta \cdot \delta_k \quad (8)$$

$$\delta_k = \text{sgn}(\tilde{x}_k) \quad (9)$$

- Transmitted information:  $\delta_k \in \{-1, 1\}$ ,

<sup>2</sup>Because  $\tilde{x}(t)$  is bounded and tends to zero with Lyapunov function  $\dot{V} < \beta \|x\|^2$  for some  $\beta$ , locally.

- Decoder and control law:

$$\hat{x}_{k+1} = [a - bK]\hat{x}_k + \Delta \cdot \delta_k \quad (10)$$

$$u_k = -K\hat{x}_k \quad (11)$$

with,  $K \in \mathbb{R}$ ,  $a \geq 1$ ,  $a_c = (a - bK)$ ;  $|a_c| < 1$ , and  $\tilde{x}_k = x_k - \hat{x}_k$ .

The *modified* differential coder (8)-(10) differs from its *standard* form in that the term within the square brackets depends on the system model parameters  $a$ , and  $b$ , and on the control gain  $K$ . In the *standard* form this term is equal to one, i.e. the encoder is described by a delayed integral term. As mentioned before, the advantages of this modification is that the coding error equations become decoupled from the system state equations, making the design of the feedback gain  $K$  independent of the coding structure.

This algorithm gives the following error equations, in cascade form:

$$x_{k+1} = a_c x_k + bK \tilde{x}_k \quad (12)$$

$$\tilde{x}_{k+1} = a \tilde{x}_k - \Delta \text{sgn}(\tilde{x}_k) \quad (13)$$

stability of the whole system can thus be tackled by only studying the stability of the coding error equation (13).

Let  $V_k = \tilde{x}_k^2$ , and  $\nabla V_k = V_{k+1} - V_k$ , then

$$\nabla V_k = \tilde{x}_{k+1}^2 - \tilde{x}_k^2 = (a^2 - 1)\tilde{x}_k^2 - 2a\Delta|\tilde{x}_k| + \Delta^2 \quad (14)$$

The right hand side of the equality defines a second order polynomial of the form  $\alpha r^2 + \beta r + \Delta^2 = 0$ , with  $r = |\tilde{x}_k|$  and roots  $r_1, r_2$  given by:

$$r_1 = \frac{a-1}{a^2-1}\Delta, \quad r_2 = \frac{a+1}{a^2-1}\Delta$$

Note that, for unstable systems<sup>3</sup>, these roots are always real and positive since  $a > 1$ , and that these values define three zones, where  $\nabla V_k$  changes sign, i.e.

$$\nabla V_k = \begin{cases} \geq 0 & \text{if } |\tilde{x}_k| \leq r_1 \\ < 0 & \text{if } r_1 < |\tilde{x}_k| < r_2 \\ \geq 0 & \text{if } r_2 \leq |\tilde{x}_k| \end{cases}$$

For some constant  $\Delta$ , this means that if the initial condition  $|\tilde{x}_0|$  is taken within the region where  $\nabla V_k$  is negative, then the function  $V_k$  will decrease until  $|\tilde{x}_k|$  enters in the region  $|\tilde{x}_k| < r_1$ , where  $\nabla V_k$  changes sign. At the next sampling time, the system is pushed away from the region  $r_1 \leq |\tilde{x}_k|$ . Due to the discrete nature of the problem, special care must be taken in order to check that this repulsive jump (at  $k+1$ ) does not lead the system state out of  $|\tilde{x}_{k+1}| < r_2$ , and hence guarantee that the region  $|\tilde{x}| < r_2$  is indeed an invariant set. The condition ensuring such a property depends on  $a$  and  $\Delta$  as follows.

From (14) we have that the maximum possible jump of  $x_{k+1}$  obtained from any value in  $|\tilde{x}_k| < r_1$  happens at  $\tilde{x}_{k^*} = 0$ , and has a magnitude equal to  $\Delta$ . Therefore, it is straightforward to see that the state remains within the region  $|\tilde{x}_k| < r_2$  for all  $k > k^* + 1$ , as long as the following relation holds:

$$\Delta < r_2 = \frac{a+1}{a^2-1}\Delta$$

<sup>3</sup>The case of stable systems is simpler. We will only discuss here the more involved case of unstable ones.

This inequality is satisfied if and only if  $a < 2$ , which is the well known necessary and sufficient condition for stabilization (see [21]).

Summarizing, we have that if the initial error coding variable verifies  $|\tilde{x}_0| < r_2$ , then the error coding state  $|\tilde{x}_k|$  is locally attracted to a threshold delimited by the value of  $r_1$ . After that, the state  $|\tilde{x}_k|$  is ultimately bounded by  $\Delta$ , which is below  $r_2$  as long as  $a < 2$ .

It is important to remark that the value of  $\Delta$  plays an important role in both; the system stability and the system precision:

- $|\tilde{x}_k| < \frac{a+1}{a^2-1}\Delta$  defined the domain of attraction (DOA) of the trajectories delimiting the system stability domain. It is enlarged by increasing  $\Delta$ .
- $|\tilde{x}_k| < \Delta$  is an invariant set (included in the previous DOA) defining the system precision. Better precision is reach with small values for.

Therefore, larger values for  $\Delta$  will make the system more stable but less precise; conversely, the reduction of the gain  $\Delta$  will lead to small estimation errors, but at the same time it will reduce the domain where the system remains stable. This analysis is summarized below.

*Proposition 2:* DISCRETE ONE-DIMENSIONAL SYSTEM. Consider system (7)-(11), with constant  $\Delta$  and  $a < 2$ . Then if the initial conditions of the coding error are such that  $|\tilde{x}_0| < r_2$  then, the following holds:

- $|\tilde{x}_k| < r_2, \quad \forall k \geq 0$ ,
- $\exists k_0 : |\tilde{x}_k| \leq \Delta, \quad \forall k \geq k_0$ , and
- $\lim_{k \rightarrow \infty} d(x_k, \mathcal{B}_\gamma) = 0$ .

where  $r_2 = \Delta(a+1)/(a^2-1)$  and  $d(x_k, \mathcal{B}_\gamma)$  is the minimum distance from  $x_k$  to any point within the interval

$$\mathcal{B}_\gamma := \{x \in \mathbb{R} : |x| < \gamma\}, \quad \gamma = \frac{Kb}{1-a_c}\Delta$$

*Proof:* See Appendix VIII-B ■

*Remark 2:* This result displays an inherent trade-off between stability and precision for discrete-time differential coding when the gain  $\Delta$  is fixed. This suggests the search of other coding strategies with varying gains. Note also that, as the sampling time<sup>4</sup> is chosen small,  $a$  approaches 1 and the precision is increased. Indeed,  $\lim_{a \rightarrow 1} r_1(a) = 0$ , thus by making  $T_s$  infinitely small, the limit case of the continuous-time precision is approached.

### B. Noisy one-dimensional system

The above analysis will be extended to the case where the open-loop model is affected by bounded noise. In this case, the state equations turn into

$$x_{k+1} = ax_k + bu_k + w_k$$

$$u_k = -K\hat{x}_k$$

$$\hat{x}_{k+1} = a_c \hat{x}_k + \Delta \text{sgn}(\tilde{x}_k)$$

where  $w_k$  is a bounded noise ( $|w_k| < W$ ). With the proposed feedback, the dynamics of the error variable  $\tilde{x}_k$  turns into

$$\tilde{x}_{k+1} = a\tilde{x}_k - \Delta \text{sgn}(\tilde{x}_k) + w_k$$

<sup>4</sup>The pole of the discrete-time system  $a$  is related the open-loop continuous one as  $a = e^{\omega_{ol} T_s}$ .

Using again the Lyapunov function  $V_k = \tilde{x}_k^2$ , and  $\nabla V_k = V_{k+1} - V_k$ , then

$$\begin{aligned} \nabla V_k &= \\ (a^2 - 1)\tilde{x}_k^2 - 2(a\tilde{x}_k + w_k)\Delta \operatorname{sgn}(\tilde{x}_k) + 2a\tilde{x}_k w_k + \Delta^2 + w_k^2 \\ &\leq (a^2 - 1)\tilde{x}_k^2 - 2a|\tilde{x}_k|(\Delta - W) + (\Delta + W)^2 \end{aligned} \quad (15)$$

There is an interval of  $|\tilde{x}_k|$  such that  $\nabla V_k$  is negative as long as the last second order polynomial on  $|\tilde{x}_k|$  has real roots (otherwise it would be an everywhere positive parabola). This condition is implied by

$$4a^2(\Delta - W)^2 - 4(a^2 - 1)(\Delta + W)^2 > 0$$

for which it is necessary that

$$\frac{(\Delta - W)^2}{(\Delta + W)^2} > \frac{a^2 - 1}{a^2} = 1 - \frac{1}{a^2}. \quad (16)$$

It is clear that the right hand side of the inequality is less than one for all  $a \in \mathbb{R}$ . Now we will consider that  $\Delta$  is a tuning parameter and  $\lim_{\Delta \rightarrow \infty} \frac{(\Delta - W)^2}{(\Delta + W)^2} = 1$

Hence, the left hand side of (16) can be made arbitrarily close to 1, and greater than any value of the RHS. Rearranging terms, we have that the set of values of  $\Delta$  such that the polynomial (15) has real roots is

$$\Delta > W \frac{1 + \sqrt{1 + \frac{1}{a^2}}}{1 + \sqrt{1 - \frac{1}{a^2}}} \quad (17)$$

This inequality makes sense for unstable open-loop plants ( $a > 1$ ). Otherwise, (16) is trivially satisfied for all  $\Delta$ .

The roots of polynomial (15) determine the region of attraction in  $\tilde{x}$  of the proposed scheme, and its steady state error. These roots are  $r_{1,2} =$

$$\frac{a(\Delta - W) \pm \sqrt{a^2((\Delta - W)^2 - (\Delta + W)^2) + (\Delta + W)^2}}{2(a^2 - 1)}$$

From the analysis of this expression, the following observations are in order:

- i) considering that the variable of polynomial (15) is the *absolute value* of  $|\tilde{x}|$ , the existence of positive real roots guarantees the existence of an interval on  $\tilde{x}$  such that  $\Delta V < 0$ ;
- ii) a necessary condition for the polynomial to have positive real roots is (17);
- iii) if the latter holds, there are real values  $r_1, r_2$  such that  $r_1 < |\tilde{x}| < r_2$  implies  $\Delta V < 0$ , i.e. the domain of attraction is defined by  $|\tilde{x}| < r_2$ , and the estimation error is ultimately bounded by  $|\tilde{x}| < r_1$ .

Regarding the variable  $x_k$ , an analysis analogous to the previous subsection can be made by observing the following expression,  $x_k = \frac{1}{z - a_c}(Kb\tilde{x}_k + w_k) = \frac{1}{z - a_c}\tilde{w}_k$ . Now considering that the new input  $\tilde{w}_k$  is ultimately bounded as  $|\tilde{w}_k| < r_1 + W$  (from the upper bounds on  $|w_k|$  and  $|\tilde{x}_k|$ ), the same arguments of the previous subsection lead to the conclusion that the state  $x$  asymptotically approaches the interval  $|x_k| \leq \tilde{\gamma}$  with  $\tilde{\gamma} = \frac{Kbr_1 + W}{1 - a_c}$ .

### C. n-dimensional systems

The previous study can be generalized to systems with higher dimension of the form

$$x_{k+1} = Ax_k + Bu_k \quad (18)$$

$$u_k = -Kx_k \quad (19)$$

where  $x_k \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ .  $(A, B)$  are stabilizable pair.

For the sake of space, the noise considerations will be spared for simplicity. We consider systems whose matrix  $A$  has distinct and real eigenvalues, i.e. there exists a transform matrix  $T$  such that  $\Lambda = TAT^{-1} = \operatorname{diag}\{\lambda_i\}$ , with  $\lambda_i \neq 0, \forall i$ .

The differential encoding modified law is:

$$\hat{x}_{k+1} = A_c \hat{x}_k + \Delta \operatorname{sgn}(T\tilde{x}_k) \quad (20)$$

with,  $A_c = (A - BK)$ ;  $|\lambda_i\{A_c\}| < 1$ , and  $\tilde{x}_k = x_k - \hat{x}_k, \in \mathbb{R}^n$ ,  $\Delta \in \mathbb{R}^{n \times n}$ . Note that the sign function depends on the transform coordinates  $\tilde{z}_k = T\tilde{x}_k$ , and

$$\operatorname{sgn}(\tilde{z}_k) = \operatorname{sgn}(T\tilde{x}_k) = [\operatorname{sgn}(\tilde{z}_{1,k}), \operatorname{sgn}(\tilde{z}_{2,k}), \dots, \operatorname{sgn}(\tilde{z}_{n,k})]^T$$

The error equations in the  $x_k$ , and  $\tilde{z}_k$  coordinates are:

$$x_{k+1} = A_c x_k + BK T^{-1} \tilde{z}_k \quad (21)$$

$$\tilde{z}_{k+1} = \Lambda \tilde{z}_k - T \Delta \operatorname{sgn}(\tilde{z}_k) \quad (22)$$

**Proposition 3: DISCRETE N-DIMENSIONAL SYSTEM.** Let  $\lambda_m$  and  $\lambda_M$  be the smaller and the larger eigenvalues of  $\Lambda$ , respectively. Consider the differential encoding law (20), in feedback with system (18)-(19), with the gain  $\Delta$  given as

$$\begin{aligned} \Delta &= \Delta_0 \cdot T^{-1} \cdot \tilde{\Delta} \\ \tilde{\Delta} &= \operatorname{diag}\{\operatorname{sgn}(\lambda_1), \operatorname{sgn}(\lambda_2), \dots, \operatorname{sgn}(\lambda_n)\} \end{aligned}$$

where  $\Delta_0$  is a positive scalar constant. Then, if  $\lambda_m^2 > (\lambda_M^2 - 1)$ , and the initial condition of the coded state is such that  $\|\tilde{z}_0\| < r_2$  the following holds:

- $\|\tilde{z}_k\| \leq r_2, \forall k \geq 0$ ,
- $\exists k_0 : |\tilde{z}_k| \leq r_1, \forall k \geq k_0$ , and
- $\lim_{k \rightarrow \infty} d(x_k, \mathcal{B}_\beta) = 0$ .

where  $0 < r_1 < r_2$  are given as:

$$r_{1,2} = \frac{\Delta_0}{\lambda_M^2 - 1} \left[ \lambda_m \pm \sqrt{\lambda_m^2 - \lambda_M^2 + 1} \right]$$

and  $d(x_k, \mathcal{B}_\beta)$  is the minimum Euclidean distance from  $x_k$  to any point within the ball

$$\mathcal{B}_\beta := \{x \in \mathbb{R}^n : \|x\| < \beta\},$$

and  $\beta$  is a constant that can be computed as in the proof of Proposition 2.

*Proof:* The proof follows along similar steps to the previous proof. Introduce  $V_k = \tilde{z}_k^T \tilde{z}_k$ , and  $\nabla V_k = V_{k+1} - V_k$ , then

$$\begin{aligned} \nabla V_k &= \tilde{z}_{k+1}^T \tilde{z}_{k+1} - \tilde{z}_k^T \tilde{z}_k = \tilde{z}_k^T (\Lambda^2 - \mathcal{I}) \tilde{z}_k \\ &\quad - 2 \operatorname{sgn}(\tilde{z}_k) \tilde{\Delta} \Lambda \tilde{z}_k + \Delta_0^2 \operatorname{sgn}(\tilde{z}_k)^T \tilde{\Delta}^2 \operatorname{sgn}(\tilde{z}_k) \\ &= \tilde{z}_k^T (\Lambda^2 - \mathcal{I}) \tilde{z}_k - 2 \operatorname{sgn}(\tilde{z}_k) \tilde{\Delta} \Lambda \tilde{z}_k + \Delta_0^2 \end{aligned}$$



Note that

$$\begin{aligned} -\text{sgn}(\tilde{z}_k)\tilde{\Delta}\Lambda\tilde{z}_k &= -\sum_i \Delta_0|\tilde{z}_{i,k}| \cdot |\lambda_i| \\ &\leq -\Delta_0\lambda_m \sum_i |\tilde{z}_{i,k}| = -\Delta_0\lambda_m\|\tilde{z}_k\| \leq -\Delta_0\lambda_m\|\tilde{z}_k\| \end{aligned}$$

Then,

$$\nabla V_k \leq (\lambda_M^2 - 1)\|\tilde{z}_k\|^2 - \Delta_0\lambda_m\|\tilde{z}_k\| + \Delta_0^2 \equiv \varphi(\|\tilde{z}_k\|)$$

$\varphi(\|\tilde{z}_k\|) = 0$  defines a second order polynomial with roots,  $r_1$ , and  $r_2$ , as defined previously. A necessary condition for this polynomial to have real roots, or equivalently, for the minimum of  $\varphi(\|\tilde{z}_k\|)$  to be negative is that  $\lambda_m^2 > (\lambda_M^2 - 1)$ . However, this condition is only sufficient for making  $\nabla V_k$  negative in the domain  $r_1 \leq \|\tilde{z}_k\| \leq r_2$ . Other less conservative conditions may be found.

We have, as before, three regions:

$$\nabla V_k = \begin{cases} \geq 0 & \text{if } \|\tilde{z}_k\| \leq r_1 \\ < 0 & \text{if } r_1 < \|\tilde{z}_k\| < r_2 \\ \geq 0 & \text{if } r_2 \leq \|\tilde{z}_k\| \end{cases}$$

and hence, the first two properties invoked in Proposition 3 follow exactly the same arguments than the ones used in previous sections. They are in consequence omitted here. The last statement follows from the relation  $x_k = G(z)\tilde{z}_k$ , and assuming that the  $x$ -subsystem (21) may also be diagonalized, the diagonalization leads to a set of  $n$  perturbed systems of the form (12) in new coordinates, from which we conclude boundedness of the state  $\|x_k\|$  based on the same arguments as in Proposition 2. ■

#### D. Data-Rates

It is assumed here that samples of  $x(t)$ , are digitalized with large enough resolution such that  $x_k$  digitalization errors are neglected. As the transmission is done by only one bit, the data transmission rate (number of bits transmitted per unit of time) associated to this scheme, is  $R_\delta = \frac{\text{one bit}}{T_s} = f_s$ , in  $[Bits/sec]$  where  $f_s = 1/T_s$  is the sampling frequency, and the system precision is determined by the size of  $\Delta$ , the minimum quantization step in the decoding process.

Let  $R$ , in  $[Bits/sec]$ , be the network data-rate. Then if the maximum network capabilities are used, i.e.  $R_\delta = R = f_s$ , the system precision and the domain of stability can be rewritten as a function of  $R$ . For instance, in the one-dimensional case, the expression for  $r_1$  and  $r_2$  is rewritten as:

$$r_1 = \frac{e^{\alpha/R} - 1}{e^{2\alpha/R} - 1}\Delta, \quad r_2 = \frac{e^{\alpha/R} + 1}{e^{2\alpha/R} - 1}\Delta$$

where  $\alpha$  is the unstable pole of the continuous-time original system. As expected, increasing the transmission rate  $R$  improves precision ( $\lim_{R \rightarrow \infty} r_1(R) = 0$ ), and stability ( $\lim_{R \rightarrow \infty} r_2(R) = \infty$ ).

#### IV. ADAPTIVE CODING

The proposed scheme can be improved by making the quantization factor  $\Delta$  variable. We will show that proper choice of a  $\Delta$ -adaptation mechanism relating the quantization

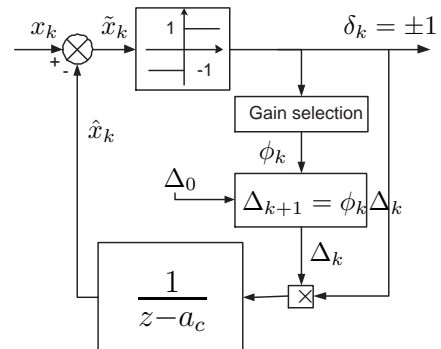


Fig. 6. Adaptive coding scheme (only the encoder structure is depicted). The figure shows the case of one-dimensional systems. The selection gain block toggles the value of  $\phi_k$  according to Equation (24).

step to the amplitude of the state results in global asymptotic convergence of the estimation error and the system states to zero. This is a significant achievement with respect to the fixed-gain scheme presented before, which was limited to finite domains of attraction and convergence to finite balls<sup>5</sup>. The main ideas of the adaptive scheme will be introduced here, while the details and the stability proof has been reported in [8].

One possibility is to make the adaptation law for  $\Delta$  be explicitly state depended. However, the state  $x$  is not available at the receiver side. Therefore, the adaptation law must be defined exclusively in terms of the information available *both at the receiver and transmitter*; which is not the state itself, but the coding signal  $\delta_k$ , and also the *estimation* of the state  $\hat{x}_k$ . This is an additional difficulty resulting for the hypotheses made in this work.

A reasonable approach is to enlarge  $\Delta_k$  for large values of the estimated error prediction, and decrease it for smaller values, in the same way as used in the standard adaptive delta modulation for open-loop scenarios [20], i.e.

$$\Delta_{k+1} = K^{(\delta_k \delta_{k-1})} \Delta_k, \quad K > 1.$$

With this adaptation law, the value of  $\Delta_k$  increases when two consecutive signals  $\delta_k, \delta_{k-1}$  have the same sign.  $K$  is then selected to minimize the total distortion. There exist also other variations to this law, like the continuously variable slope delta modulation (CVSD). However, it is important to stress out that in the context of feedback systems, instabilities may occur if rate of variation of  $\Delta_k$  does not accommodate to the maximum (or minimum) rate of variation of the dynamics of the system to be controlled. This is the main difficulty in designing adaptive Delta-modulation laws for control.

Under these considerations, a  $\Delta$ -adaptive mechanism, with minimal storage and computation power requirements, can be designed under the following assumptions:

- 1) If  $\delta_k = \delta_{k-1}$ , the error prediction is assumed to be growing, thus  $\Delta_k$  must be increased.
- 2) If  $\delta_k \neq \delta_{k-1}$ , the error prediction is assumed to converge (oscillating close to zero) and  $\Delta_k$  must be decreased.

<sup>5</sup>The attraction domain can be enlarged arbitrarily (by increasing  $\Delta$ ), at the price of increasing also the amplitude of the remaining oscillation (granularity).

which leads to the following update law:

$$\Delta_{k+1} = \phi_{k+1} \Delta_k, \quad \Delta_0 > 0, \quad (23)$$

$$\phi_{k+1} = \begin{cases} \lambda^+ & \text{if } \delta_k = \delta_{k-1} \\ \lambda^- & \text{if } \delta_k \neq \delta_{k-1} \end{cases} \quad (24)$$

where  $0 < \lambda^- < 1$  is the exponential decay rate of  $\Delta_k$ , and  $\lambda^+ > 1$  is the exponential growth rate. The block scheme is shown in Figure 6.

Heuristic and simple as it may seem, this adaptation law guarantees global asymptotic stability of the close-loop system, under the conditions stated in the following proposition.

*Proposition 4:* For any initial condition, the state  $x_k$  of system (7)-(11) with the adaptive  $\Delta$ -modulation coding scheme 24, asymptotically converges to zero as  $k \rightarrow \infty$  if there exist parameters  $\lambda^+ > 1$ ,  $\lambda^- \in (0, 1)$  satisfying the following inequalities:

$$\lambda^+ > a \quad (25)$$

$$\lambda^- < (\lambda^+)^{-\frac{\beta}{2}}, \quad (26)$$

where

$$\beta(a, \lambda^-, \lambda^+) \triangleq 1 + \log_\rho \left( 1 + \frac{a(a - \lambda^-)(\rho - 1)}{(\lambda^-)^2} \right)$$

$$\rho \triangleq \frac{\lambda^+}{a}$$

Moreover,  $\Delta_k$  also converges to zero regardless its initial value  $\Delta_0$ .

*Remark 3:* A practical implementation of the algorithm should avoid too low values of  $\Delta_k$ . If the zoom-in stage is allowed to run for a long time,  $\Delta_k$  would become very small. In that situation, a disturbance driving the state far from the origin would originate a zoom-out stage equally long, as  $\Delta_k$  would have to undo all the previous reduction steps, for catching up the state. This is properly addressed in the Simulations Section.

The details of the proof of this proposition can be found in [8], and are omitted here for the sake of clarity. However, a key consideration is that a choice of parameters  $\lambda^+$  and  $\lambda^-$  fulfilling the stability conditions (25)-(26) is not always feasible. Indeed, (26) is an implicit equation and its solvability depends on the particular value of  $a$ .

In Fig. 7 we have depicted the expression  $\lambda^- - (\lambda^+)^{-\frac{\beta}{2}}$  from (26). The left part of the figure shows that for  $a = 1.2$ , the surface is below zero over a small area, and hence there are solutions fulfilling Proposition 4. However, the right part of the figure indicates that for  $a = 1.4$  no valid pair  $\lambda^+, \lambda^-$  can be found, and hence the system cannot be properly tuned. Moreover, as the right hand side of (26) decreases with  $a$ , there will be no more solutions for larger  $a$ .

#### A. $n$ -dimensional adaptive coding.

The results of section III-C can be easily extrapolated to the adaptive coding results. Indeed, under the condition of diagonalizing the system matrix  $A$ , it is easy to check that 22 become a set of independent scalar equations, that can be rewritten as

$$\{\tilde{z}_i\}_{k+1} = \lambda_i \{\tilde{z}_i\}_k - \{\Delta_i\}_k \text{sgn}(\{\tilde{z}_i\}_k), \quad i = 1 \dots n$$

Now by substituting each  $\Delta_i$  by an adaptive parameter in the form (23),

$$\{\Delta_i\}_{k+1} = \{\phi_i\}_{k+1} \{\Delta_i\}_k, \quad \Delta_0 > 0, \quad (27)$$

$$\{\phi_i\}_{k+1} = \begin{cases} \lambda^+ & \text{if } \text{sgn}(\{\tilde{z}_i\}_k) = \text{sgn}(\{\tilde{z}_i\}_{k-1}) \\ \lambda^- & \text{if } \text{sgn}(\{\tilde{z}_i\}_k) \neq \text{sgn}(\{\tilde{z}_i\}_{k-1}) \end{cases} \quad (28)$$

we then conclude that each  $\{\tilde{z}_i\}$  will converge to zero by the same arguments presented in the previous scalar analysis; and from such convergence, the stability of the system is directly implied.

#### B. Relation with the $N$ & $S$ condition for stabilization under channel limitations.

A further issue that must be addressed with respect to the adaptive  $\Delta$ - $M$  scheme is the question whether the limitation on the open-loop poles (parameter  $a$ ) to be below a certain value is a structural property or a limitation due to the sufficient nature of the result.

In any case, our limitation should be consistent with the necessary and sufficient condition for stabilization presented in [21]. This condition indicates that the minimal data rate required for stabilizing a discrete-time system via a communication channel of maximum rate capacity  $R$  [b.p.u.]<sup>6</sup> is related to the unstable open-loop poles ( $\lambda_i^{un}$ ) as:  $R > \sum \log_2(\lambda_i^{un})$ , which in our case simplifies to:

$$R > \log_2 a \quad (29)$$

The implicit assumption made within the framework of our discrete-time formulation is that the channel can reliably transmit one bit-per-unit of time, that is that  $R = 1$ . This means that with regard to condition (29), the maximum admissible value for  $a$  is  $a < 2$ , which is consistent with our sufficient condition  $a < 1.313$ , probably due to the technicalities used for the stability analysis, or either due to the particular structure of the proposed adaptation law. This also indicates that there is some conservatism in the computation of the admissible set of the parameter  $\lambda^+$  and  $\lambda^-$ . Some alternative adaptation strategies could be devised in order to improve this bound. For example, a more sophisticated adaptive algorithm based on older samples of  $\delta_k$  could possibly be designed, at the cost of higher complexity.

## V. SIMULATIONS

System (7)-(11) has been simulated for the set of values  $R = 1$  [b.p.s.],  $a = 1.1$ ,  $b = 1$ ,  $K = 0.2$ . Initial conditions for the states are:  $x_0 = 100$ ,  $\hat{x}_0 = 0$ , and  $\Delta_0 = 5$  for the adaptive algorithm. Its adaptation gains are  $\lambda^- = 0.4$ ,  $\lambda^+ = 1.21$ , satisfying conditions (25) and (26).

For the sake of comparison with the non-adaptive scheme, a first simulation has been made with constant  $\Delta$ . Fig. 8 shows the behavior of the closed-loop system when  $\Delta$  is given fixed values, (it only switches at specific times, between values 20, 10, 5, and then 20). We have chosen large values of  $\Delta$  in order to highlight its connection with the chattering amplitude. The plot indicates clearly that both magnitudes have the same

<sup>6</sup> $R$  is given in dimensionless units, i.e. bits per unit of time [b.p.u.]

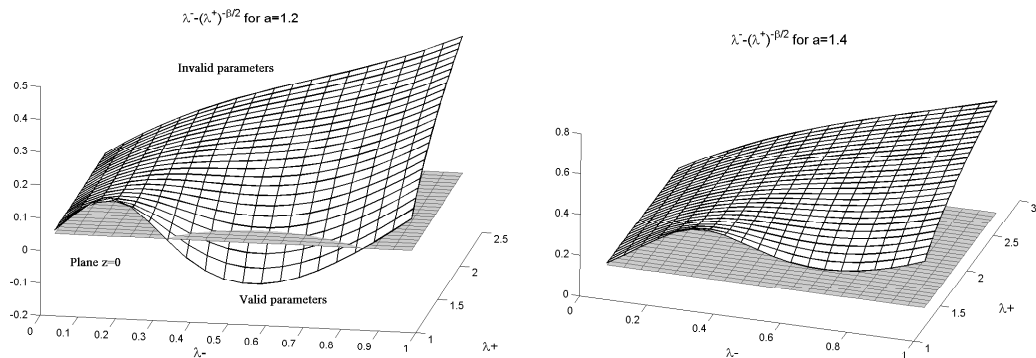


Fig. 7. (Left) Expression  $\lambda^- - (\lambda^+)^{-\frac{\beta}{2}}$  for  $a = 1.2$ ; (Right) The same expression for  $a = 1.4$ .

order. An important issue here is that  $\Delta$  cannot be fixed too low (thus reducing the granularity) without compromising the domain of attraction.

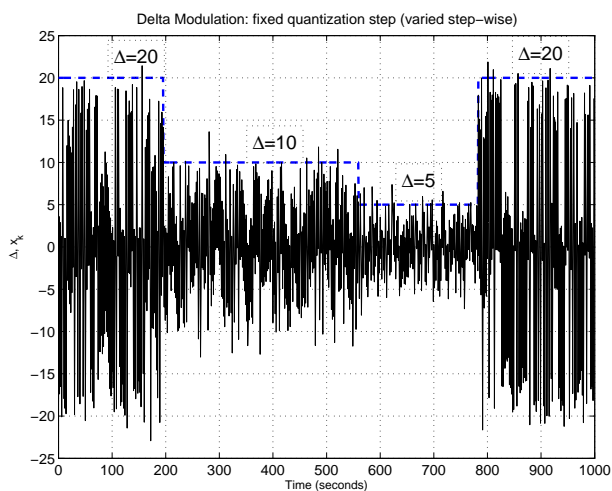


Fig. 8. Simulation results for non-adaptive  $\Delta$ -M scheme, with  $\Delta$  fixed along specific intervals.

Fortunately, this limitation has been successfully tackled with the new adaptive approach, as is illustrated in Fig. 9. In this plot, the state  $x_k$ ,  $\hat{x}$ , and the adaptive quantization parameter  $\Delta_k$  for the noiseless system are depicted. On the upper plot, the state  $x_k$ ,  $\hat{x}$  are plotted together along the whole simulation. As expected, convergence of both the estimation and the state to zero is obtained regardless the initial values. In the lower plot of that figure,  $\Delta_k$  can be compared to the state  $x_k$  at the initial stages of the simulation, where the zoom-out and zoom-in periods can be distinguished.

Although we do not have a conclusive theoretical analysis on the performance of the adaptive scheme in the presence of noise, we have observed via simulations, that the inclusion of white noise in the system dynamics, no matter the amplitude, does not cause instability of the system. Yet, naturally, the steady state presents variations around the origin whose amplitude is directly related to the amplitude of the added noise (Fig. 10).

In all simulations, some granularity has been allowed by constraining  $\Delta_k$  to remain always above  $\Delta_{min} = 2$ . This is a

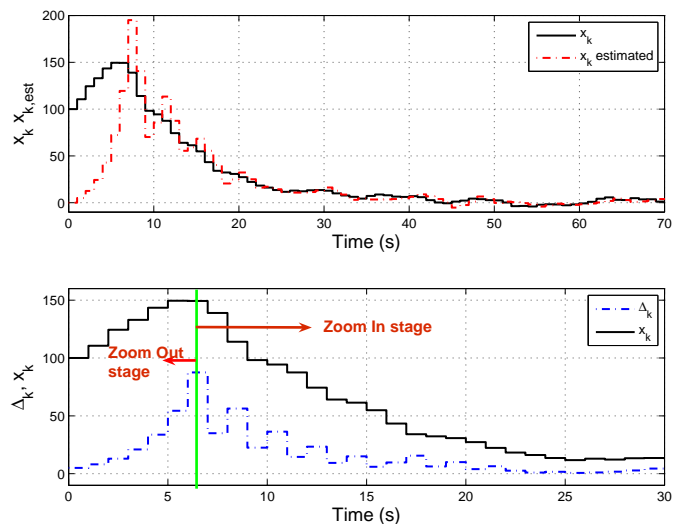


Fig. 9. Simulation results for adaptive  $\Delta$ -M scheme, noiseless system.

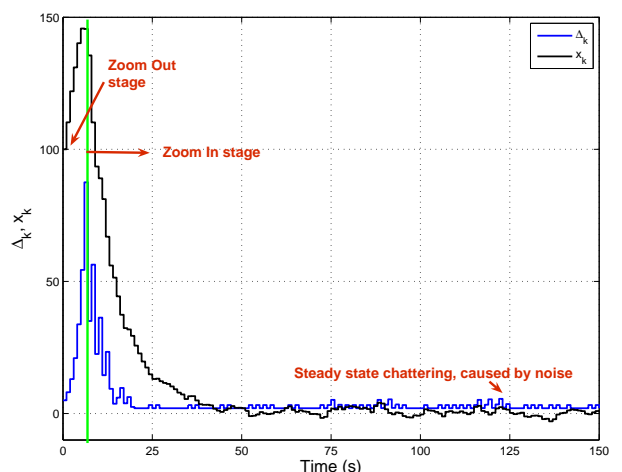


Fig. 10. Simulation results for adaptive  $\Delta$ -M scheme, noisy system.

practical adjustment done for improving the transient behavior (see Remark 3). Without this saturation,  $\Delta_k$  would keep tending to zero in steady state, and whenever a disturbance drives the state away from the origin, a large number of samples would be required for (23) to make  $\Delta_k$  large enough

to capture again such state (i.e. the zoom-out transient time would undesirably depend on the length of the previous zoom-in stage). In the noisy scenario, it is very important to properly tune the minimum allowable value of  $\Delta_k$  to minimize the steady-state chattering.

## VI. CONCLUSIONS

In this paper we have investigated the stability properties of the Delta-modulation coding rule, when used as an Analog to Digital Converter or a transmission means in networked controlled linear systems.

It was first shown that the standard form of the  $\Delta$ - $M$  algorithm can be modified, including information about the system and the controller, to enlarge the attraction domain of the closed-loop system equilibrium. Then, we have shown that in the discrete-time case, a trade-off between system precision and size of the stability domain can be assessed.

These results were extended to the case of adaptive  $\Delta$ - $M$ . An explicit adaptation rule has been discussed, with still significantly easy implementation. This scheme guarantees global asymptotic stability under the constrain imposed by a limit on the maximum unstable eigenvalues of the system that are compatible with the ones given in [21]. The effect of random noise has been analyzed for all schemes.

The practical application of the proposed technique is the growing field of low-cost wireless sensor networks, where minimum data transmission per sample results in a significant improvement of battery life and optimal bandwidth management.

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## VIII. APPENDIX

### A. Stability of the continuous-time standard $\Delta$ - $M$ algorithm.

*Proposition 5: STANDARD  $\Delta$ - $M$  CODING.* Consider system (1) where  $\hat{x} \in \mathbb{R}^n$  is the state estimate computed according to the  $\Delta$ - $M$  standard scheme (2)-(3). Let  $\tilde{x} = x - \hat{x}$ , and assume that  $\Delta \in \mathbb{R}^{n \times n}$  is constant, and has the form  $\Delta = \Delta_0 \cdot I_{n \times n}$ ,

Let  $\zeta = (x^T, \tilde{x}^T)^T$ , then there exists a constant  $c > 0$  such that for any possible value of initial conditions  $\zeta(0)$ , there exists a corresponding scalar constant value  $\Delta_0 \geq c \cdot \|\zeta(0)\|$  such that  $\zeta(t) \rightarrow 0$ , as  $t \rightarrow \infty$ .

The constant  $c > \max(c_5, c_5 c_6) > 0$ , and the  $c_i$  given by the following relations:  $c_1 = \|2PBK\|$ ,  $c_2 = \|A_c\|$ ,  $c_3 = \|BK\|$ ,  $c_4 = \max\{c_1, c_2\}$ ,  $c_5 = c_3 + \frac{c_4^2}{4q}$ ,  $c_6 = \sqrt{\frac{\lambda_M}{\lambda_m}}$ . Where  $\lambda_M = \lambda_{sup}\{\mathbf{P}\}$ ,  $\lambda_m = \lambda_{min}\{\mathbf{P}\}$ , with  $\mathbf{P} = diag\{P, I\}$ , and  $q = \lambda_{min}Q$ .  $P = P^T > 0$ , and  $Q > 0$ , are solutions of

$$PA_c + A_c^T P = -Q.$$

*Proof:* Consider the quadratic Lyapunov function  $V = x^T P x + \tilde{x}^T \tilde{x}$ . Evaluating  $\dot{V}$  along solutions of system (4), and using  $-|x| \leq -\|\tilde{x}\|$ , gives  $\dot{V} \leq$

$$\begin{aligned} &\leq -q\|x\|^2 + c_1\|x\|\|\tilde{x}\| + c_2\|x\|\|\tilde{x}\| + c_3\|\tilde{x}\|^2 - \Delta_0\|\tilde{x}\| \\ &\leq -q\|x\|^2 + c_1\|x\|\|\tilde{x}\| + c_2\|x\|\|\tilde{x}\| + c_3\|\tilde{x}\|^2 - \Delta_0\|\tilde{x}\| \\ &\leq -q\|x\|^2 + c_4\|x\|\|\tilde{x}\| + c_3\|\tilde{x}\|^2 - \Delta_0\|\tilde{x}\| \\ &\leq -(\|x\|, \|\tilde{x}\|) \begin{pmatrix} q & -\frac{c_4}{2} \\ -\frac{c_4}{2} & \frac{\Delta_0}{\|\tilde{x}\|} - c_3 \end{pmatrix} \begin{pmatrix} \|x\| \\ \|\tilde{x}\| \end{pmatrix} \end{aligned}$$

From here we can see that  $\dot{V}$  is negative as long as the matrix in the equality above is positive definite, i.e. for values of  $\Delta_0$ , such that  $q \left( \frac{\Delta_0}{\|\tilde{x}\|} - c_3 \right) > \frac{c_4^2}{4}$ , or equivalently if  $\Delta_0 > c_5 \|\zeta\| \geq c_5 \|\tilde{x}\|$ . Nevertheless, this condition by itself does not define the domain of attraction. For this we proceed further as follows.

Define  $\lambda_M = \lambda_{sup}\{\mathbf{P}\}$ , and  $\lambda_m = \lambda_{min}\{\mathbf{P}\}$ , with  $\mathbf{P} = diag\{P, I\}$ . With this definition we have the following bounds on  $V(\zeta)$ ,

$$\begin{aligned} \lambda_M \|\zeta\|^2 &\geq V(\zeta) \geq \lambda_m \|\zeta\|^2 \quad \text{and} \\ -\lambda_M \|\zeta\|^2 &\leq -V(\zeta) \leq -\lambda_m \|\zeta\|^2 \end{aligned}$$

using these bounds, and assuming that  $\Delta_0 > c_5 \|\zeta\|$ , then there exists a scalar function  $\epsilon(\|\zeta\|) > 0$ , such that

$$\dot{V} \leq -\epsilon(\|\zeta\|) \cdot \|\zeta\|^2 \leq -\epsilon(\|\zeta\|) \frac{V}{\lambda_M} \quad (30)$$

Integration on both sides of this equation along the time-interval  $[0, t]$ , gives,

$$V(t) \leq V(0) \exp(-\varphi(\epsilon(\|\zeta\|))),$$

where  $\varphi(t) = \int_0^t \frac{\epsilon(\|\zeta(\tau)\|)}{\lambda_M} d\tau$ . Note that  $\varphi(t) > 0$ , as long as  $\Delta_0 > c_5 \|\zeta\|$ . Using again the bound on  $V$  in the expression above, we obtain

$$\begin{aligned} \|\zeta(t)\|^2 &\leq \frac{\lambda_m}{\lambda_M} \|\zeta(0)\|^2 \exp(-\varphi(t)) \\ \|\zeta(t)\| &\leq \sqrt{\frac{\lambda_m}{\lambda_M}} \|\zeta(0)\| \exp(-\varphi(t)/2) \\ &= c_6 \|\zeta(0)\| \exp(-\varphi(t)/2), \end{aligned}$$

and using the relation given in Proposition 5, i.e.  $\Delta_0 \geq c \cdot \|\zeta(0)\| > c_5 c_6 \|\zeta(0)\|$  in the above inequality, we get  $\|\zeta(t)\| < \frac{\Delta_0}{c_5} \exp(-\varphi(t)/2)$ . Therefore, under the relation  $\Delta_0 > c_5 \|\zeta(0)\|$ , as proposed in Proposition 5, and using  $\exp(-\varphi(t)/2) < 1 \quad \forall t$ , we conclude that  $\|\zeta(t)\| < \Delta_0/c_5$  as required for Eq. (30) to hold. As a final consequence, the time derivative of the Lyapunov function is negative definite and the convergence  $\|\zeta(t)\| \rightarrow 0$  is guaranteed. ■

### B. Proof of Proposition 3

*Proof:* The first two items result from the previous development. The last item of the claim derives from the following arguments. First note that  $x_k = \frac{Kb}{z-a_c} \tilde{x}_k$ . Then, defining the discrete-time transfer function  $H(z) = \frac{Kbz}{z-a_c}$ ,

with impulse response  $h(n) = Kba_c^n, n \geq 0$ , yields the input–output relation  $x_k = H(z)z^{-1}\tilde{x}_k$  which in time domain corresponds to the convolution

$$x_k = \sum_{m=0}^{\infty} \tilde{x}_{m-1}h(k-m) = \sum_{m=0}^{\infty} \tilde{x}_m h(k-m-1)$$

because  $\tilde{x}_m = 0 \forall m < 0$  is assumed. Now using the fact that  $x_k$  is *absolutely* bounded ( $|\tilde{x}_k| < r_2, \forall k \geq 0$ ) and also *ultimately* bounded by  $\Delta$  (second item of the Proposition), we can split the convolution as follows

$$x_k = \sum_{m=0}^{k_0} \tilde{x}_m h(k-m-1) + \sum_{m=k_0+1}^{\infty} \tilde{x}_m h(k-m-1)$$

hence

$$|x_k| \leq r_2 \sum_{m=0}^{k_0} |h(k-m-1)| + \Delta \sum_{m=k_0+1}^{n-1} |h(k-m-1)|$$

where the causality of  $h(n)$  has been used. Substituting the impulse response into this expression yields

$$\begin{aligned} |x_k| &< Kbr_2 \left( \frac{a_c^k(1-a_c^{-k_0-1})}{a_c-1} \right) + Kb\Delta \left( \frac{a_c^{k-k_0-1}-1}{a_c-1} \right) \\ &= \frac{Kb\Delta}{1-a_c} + v(k) \end{aligned}$$

where  $v(k)$  stands for

$$v(k) = \frac{Kb(r_2(1-a_c^{-k_0-1}) + \Delta a_c^{-k_0-1})}{a_c-1} a_c^k,$$

and as  $a_c < 1$ , we conclude that  $\lim_{k \rightarrow \infty} v(k) = 0$  and the state  $x$  asymptotically approaches the interval  $|x_k| \leq \frac{Kb}{1-a_c} \Delta$ . ■

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