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Decomposition of the deformations of a thin shell.
Asymptotic behavior of the Green-St Venant's strain tensor.

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Abstract. We investigate the behavior of the deformations of a thin shell, whose thickness δ tends to zero, through a decomposition technique of these deformations. The terms of the decomposition of a deformation v are estimated in terms of the L^2 -norm of the distance from ∇v to $SO(3)$. This permits in particular to derive accurate nonlinear Korn's inequalities for shells (or plates). Then we use this decomposition technique and estimates to give the asymptotic behavior of the Green-St Venant's strain tensor when the "strain energy" is of order less than $\delta^{3/2}$.

1. Introduction

The concern of this paper is twofold. We first give a decomposition technique for the deformation of a shell which allows to established a nonlinear Korn's type inequality for shells. In a second part of the paper, we use such a decomposition to derive the asymptotic behavior of the Green-St Venant's strain tensor.

In the first part, we introduce two decompositions of an admissible deformation of a shell (i.e. which is H^1 with respect to the variables and is fixed on a part of the lateral boundary) which take into account the fact that the thickness 2δ of such a domain is small. This decomposition technique has been developed in the framework of linearized elasticity for thin structures in [14], [15], [16] and for thin curved rods in nonlinear elasticity in [4]. As far as large deformations are concerned these decompositions are obtained through using the "Rigidity Theorem" proved in [11] together with the geometrical precision of this result given in [4]. Let us consider a shell with mid-surface S and thickness 2δ . The two decompositions of a deformation v defined on this shell are of the type

$$v = \mathcal{V} + s_3 \mathbf{Rn} + \bar{v},$$

where s_3 is the variable in the direction \mathbf{n} which is a unit vector field normal to S . In the above expression, the fields \mathcal{V} and \mathbf{R} are defined on S while \bar{v} is a field still defined on the 3D shell. Let us emphasize that the terms of the decompositions \mathcal{V} , \mathbf{R} and \bar{v} have at least the same regularity than v and satisfy the corresponding boundary conditions. Loosely speaking, the two first terms of the decompositions reflect the mean of the deformation over the thickness and the rotations of the fibers of the shell in the direction \mathbf{n} . For the above decomposition, it worth noting that the fields \mathcal{V} , \mathbf{R} and \bar{v} are estimated in terms of the "strain energy" $\|dist(\nabla_x v, SO(3))\|_{L^2(S \times]-\delta, \delta])}$ and the thickness of the shell.

In the first decomposition, the field \mathbf{R} satisfies

$$\|dist(\mathbf{R}, SO(3))\|_{L^2(S)} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(S \times]-\delta, \delta[)}$$

which shows that the field \mathbf{R} is close to a rotation field for small energies.

In the second decomposition, for which we assume from the beginning that $\|dist(\nabla_x v, SO(3))\|_{L^2} \leq C(S)\delta^{3/2}$ where $C(S)$ is a geometrical constant, the field \mathbf{R} is valued in $SO(3)$.

For thin structures, the usual technique in order to rescale the applied forces to obtain a certain level of energy is to established nonlinear Korn's type inequalities. In order to simplify the analysis, we consider here that the deformation v is equal to the identity on a part of the lateral boundary of the shell (clamped condition). Using Poincaré's inequality as done in [4] (see also [8] and Subsection 4.1 of the present paper) leads in the case of a shell to the following inequality

$$\|v - I_d\|_{(L^2(S \times]-\delta, \delta[))^3} + \|\nabla_x v - \mathbf{I}_3\|_{(L^2(S \times]-\delta, \delta[))^9} \leq C(\delta^{1/2} + \|dist(\nabla_x v, SO(3))\|_{L^2(S \times]-\delta, \delta[)}).$$

The first important consequence of the decomposition technique together with its estimates is the following nonlinear Korn's inequality for shells

$$\|v - I_d\|_{(L^2(S \times]-\delta, \delta[))^3} + \|\nabla_x v - \mathbf{I}_3\|_{(L^2(S \times]-\delta, \delta[))^9} \leq \frac{C}{\delta} \|dist(\nabla_x v, SO(3))\|_{L^2(S \times]-\delta, \delta[)}.$$

Another important technical argument involved in the proof of the above inequalities is the possible extension of a deformation in a neighborhood of the lateral boundary without increasing the order (with respect to δ) of the strain energy. Indeed the two inequalities identify for energies of order $\delta^{3/2}$ which is the first interesting critical case. For smaller levels of energy, the second estimate is more relevant. We also establish the following estimate for the linear part of the strain tensor

$$\|\nabla_x v + (\nabla_x v)^T - 2\mathbf{I}_3\|_{(L^2(S \times]-\delta, \delta[))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(S \times]-\delta, \delta[)} \left\{ 1 + \frac{1}{\delta^{5/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(S \times]-\delta, \delta[)} \right\}$$

which shows that $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(S \times]-\delta, \delta[)} \sim \delta^{5/2}$ is another critical case. For such level of energy, our Korn's inequality for shells turns out to appear as an important tool. We have established and used the analogue of these inequalities for rods in [4].

In the second part of the paper, we strongly use the results of the first part in order to derive the asymptotic behavior of the Green-St Venant's strain tensor. We focus on the case where the "strain energy" $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(S \times]-\delta, \delta[)}$ is of order $\delta^{\kappa-1/2}$ ($\kappa \geq 2$). The order $\delta^{3/2}$ is the highest level of energy which can be analyzed through our technique.

For $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(S \times]-\delta, \delta[)} \sim \delta^{3/2}$, we deduce the expression of the limit of the Green-St Venant's strain tensor from the decompositions, the associated estimates and a standard rescaling and the result is the same using the two decompositions. In this case the limit deformation is pure bending but the limit Green-St Venant's strain tensor contains a field which measures the defect between the mean deformation and a pure bending deformation.

For $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(S \times]-\delta, \delta[)} \sim \delta^{\kappa-1/2}$ with $\kappa > 2$, the displacements of the fibers of the shell are rigid displacements. To describe the limit behavior, we introduce the inextentional and extentional displacements which correspond respectively to the bending and to a generalization of membrane displacements for a plate. The value $\kappa = 3$ is a critical case. For $2 < \kappa < 3$, the inextentional and extentional displacements are coupled. If $\kappa \geq 3$, the defect field mentioned above can be expressed in terms of the extentional displacement ($\kappa > 3$) and also of the inextentional displacement ($\kappa = 3$).

A byproduct of the decomposition technique and the derivation of the limit of the Green-St Venant's strain tensor introduced in this paper is a simplification of the obtention of limit elastic shell models through Γ -convergence (that we will present in a forthcoming paper).

As general references on the theory of shells, we refer to [5], [6], [7], [9], [10], [12], [16], [18], [19], [20]. The rigidity theorem and its applications to thin structures using Γ -convergence arguments are developed in [11], [12], [17], [18]. The decomposition of the deformations in thin structures is introduced in [14], [15] and a few applications to the junctions of multi-structures and homogenization are given in [1], [2], [3].

The paper is organized as follows. Section 2 is devoted to describe the geometry of the shell and to give a few notations. In Section 3 we introduce the two decompositions of the deformations of a thin shell and we derive the estimates on the terms of these decompositions. We precise the boundary conditions on the deformation and we establish a nonlinear Korn's inequality for shells in Section 4. Section 5 is concerned with a standard rescaling. We derive the limit of the Green-St Venant's strain tensor of a sequence of deformations such that $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(S \times]-\delta, \delta])} \sim \delta^{\kappa-1/2}$ in Section 6 for $\kappa = 2$ and in Section 7 for $\kappa > 2$. At last the appendix contains a few technical results on the interpolation of rotations.

2. The geometry and notations.

Let us introduce a few notations and definitions concerning the geometry of the shell (see [14] for a detailed presentation).

Let ω be a bounded domain in \mathbb{R}^2 with lipschitzian boundary and let ϕ be an injective mapping from $\bar{\omega}$ into \mathbb{R}^3 of class \mathcal{C}^2 . We denote S the surface $\phi(\bar{\omega})$. We assume that the two vectors $\frac{\partial \phi}{\partial s_1}(s_1, s_2)$ and $\frac{\partial \phi}{\partial s_2}(s_1, s_2)$ are linearly independent at each point $(s_1, s_2) \in \bar{\omega}$.

We set

$$(2.1) \quad \mathbf{t}_1 = \frac{\partial \phi}{\partial s_1}, \quad \mathbf{t}_2 = \frac{\partial \phi}{\partial s_2}, \quad \mathbf{n} = \frac{\mathbf{t}_1 \wedge \mathbf{t}_2}{\|\mathbf{t}_1 \wedge \mathbf{t}_2\|_2}.$$

The vectors \mathbf{t}_1 and \mathbf{t}_2 are tangential vectors to the surface S and the vector \mathbf{n} is a unit normal vector to this surface. The reference fiber of the shell is the segment $] -\delta, \delta[$. We set

$$\Omega_\delta = \omega \times] -\delta, \delta[.$$

Now we consider the mapping $\Phi : \bar{\omega} \times \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$(2.2) \quad \Phi : (s_1, s_2, s_3) \longmapsto x = \phi(s_1, s_2) + s_3 \mathbf{n}(s_1, s_2).$$

There exists $\delta_0 \in (0, 1]$ depending only on S , such that the restriction of Φ to the compact set $\bar{\Omega}_{\delta_0} = \bar{\omega} \times]-\delta_0, \delta_0]$ is a \mathcal{C}^1 -diffeomorphism of that set onto its range (see e.g. [6]). Hence, there exist two constants $c_0 > 0$ and $c_1 \geq c_0$, which depend only on ϕ , such that

$$\forall \delta \in (0, \delta_0], \quad \forall s \in \Omega_{\delta_0}, \quad c_0 \leq \|\nabla_s \Phi(s)\| \leq c_1, \quad \text{and for } x = \Phi(s) \quad c_0 \leq \|\nabla_x \Phi^{-1}(x)\| \leq c_1.$$

Definition 2.1. For $\delta \in (0, \delta_0]$, the shell \mathcal{Q}_δ is defined as follows:

$$\mathcal{Q}_\delta = \Phi(\Omega_\delta).$$

The mid-surface of the shell is S . The lateral boundary of the shell is $\Gamma_\delta = \Phi(\partial\omega \times]-\delta, \delta[)$. The fibers of the shell are the segments $\Phi(\{(s_1, s_2)\} \times]-\delta, \delta[)$, $(s_1, s_2) \in \omega$. We respectively denote by x and s the running

points of \mathcal{Q}_δ and of Ω_δ . A function v defined on \mathcal{Q}_δ can be also considered as a function defined on Ω_δ which we will also denote by v . As far as the gradients of v are concerned we have $\nabla_x v$ and $\nabla_s v = \nabla_x v \cdot \nabla \Phi$ and for a.e. $x = \Phi(s)$

$$c \|\nabla_x v(x)\| \leq \|\nabla_s v(s)\| \leq C \|\nabla_x v(x)\|,$$

where the constants are strictly positive and do not depend on δ .

Since we will need to extend a deformation defined over the shell \mathcal{Q}_δ , we also assume the following.

For any $\eta > 0$, let us denote the open set

$$\omega_\eta = \{(s_1, s_2) \in \mathbb{R}^2 \mid \text{dist}((s_1, s_2), \omega) < \eta\}.$$

We assume that there exist $\eta_0 > 0$ and an extension of the mapping ϕ (still denoted ϕ) belonging to $(\mathcal{C}^2(\bar{\omega}_{\eta_0}))^3$ which remains injective and such that the vectors $\frac{\partial \phi}{\partial s_1}(s_1, s_2)$ and $\frac{\partial \phi}{\partial s_2}(s_1, s_2)$ are linearly independent at each point $(s_1, s_2) \in \bar{\omega}_{\eta_0}$. The function Φ (introduced above) is now defined on $\bar{\omega}_{\eta_0} \times [-\delta_0, \delta_0]$ and we still assume that it is a \mathcal{C}^1 -diffeomorphism of that set onto its range. Then there exist four constants c'_0, c'_1, c' and C' such that

$$(2.3) \quad \begin{cases} \forall s \in \bar{\omega}_{\eta_0} \times [-\delta_0, \delta_0], & c'_0 \leq \|\nabla_s \Phi(s)\| \leq c'_1, \text{ and for } x = \Phi(s) & c'_0 \leq \|\nabla_x \Phi^{-1}(x)\| \leq c'_1 \\ c' \|\nabla_x v(x)\| \leq \|\nabla_s v(s)\| \leq C' \|\nabla_x v(x)\|, & \text{for a.e. } x = \Phi(s). \end{cases}$$

At the end we denote by I_d the identity map of \mathbb{R}^3 .

3. Decompositions of a deformation.

In this Section, we recall the theorem of rigidity established in [11] (Theorem 3.1 of Section 3.1). In Subsection 3.2 we recall that any deformation can be extended in a neighborhood of the lateral boundary of the shell with the same level of "energy". Then we apply Theorem 3.1 to a covering of the shell. In Subsections 3.4 and 3.5, we introduce the two decompositions of a deformation and we established estimates on these decompositions in term of $\|\text{dist}(\nabla_x v, SO(3))\|_{L^2}$.

3.1. Theorem of rigidity.

We equip the vector space $\mathbb{R}^{n \times n}$ of $n \times n$ matrices with the Frobenius norm defined by

$$\mathbf{A} = (a_{ij})_{1 \leq i, j \leq n}, \quad \|\mathbf{A}\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}.$$

We just recall the following theorem established in [11] in the version given in [4].

Theorem 3.1. *Let Ω be an open set of \mathbb{R}^n contained in the ball $B(O; R)$ and star-shaped with respect to the ball $B(O; R_1)$ ($0 < R_1 \leq R$). For any $v \in (H^1(\Omega))^n$, there exist $\mathbf{R} \in SO(n)$ and $\mathbf{a} \in \mathbb{R}^n$ such that*

$$(3.1) \quad \begin{cases} \|\nabla_x v - \mathbf{R}\|_{(L^2(\Omega))^{n \times n}} \leq C \|\text{dist}(\nabla_x v; SO(n))\|_{L^2(\Omega)}, \\ \|v - \mathbf{a} - \mathbf{R}x\|_{(L^2(\Omega))^n} \leq CR \|\text{dist}(\nabla_x v; SO(n))\|_{L^2(\Omega)}, \end{cases}$$

where the constant C depends only on n and $\frac{R}{R_1}$.

3.2. Extension of a deformation and splitting of the shell.

In order to make easier the decomposition of a deformation as the sum of an elementary deformation given via an approximate field of rotations (see Subsection 3.4) or a field of rotations (see Subsection 3.5)

and a residual one, we must extend any deformation belonging to $(H^1(\mathcal{Q}_\delta))^3$ in a neighborhood of the lateral boundary Γ_δ of the shell. To this end we will use Lemma 3.2 below. The proof of this lemma is identical to the one of Lemma 3.2 of [14] upon replacing the strain semi-norm of a displacement field by the norm of the distance between the gradient of a deformation v and $SO(3)$.

Lemma 3.2. *Let δ be fixed in $(0, \delta_0]$ such that $3\delta \leq \eta_0$ and set*

$$\mathcal{Q}'_\delta = \Phi(\omega_{3\delta} \times] - \delta, \delta[).$$

There exists an extension operator P_δ from $(H^1(\mathcal{Q}_\delta))^3$ into $(H^1(\mathcal{Q}'_\delta))^3$ such that

$$\begin{aligned} \forall v \in (H^1(\mathcal{Q}_\delta))^3, \quad P_\delta(v) \in (H^1(\mathcal{Q}'_\delta))^3, \quad P_\delta(v)|_{\mathcal{Q}_\delta} = v, \\ \|\text{dist}(\nabla_x P_\delta(v), SO(3))\|_{L^2(\mathcal{Q}'_\delta)} \leq c \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \end{aligned}$$

with a constant c which only depends on $\partial\omega$ and on the constants appearing in inequalities (2.3).

Let us now precise the extension operator P_δ near a part of the boundary where $v = I_d$. Let γ_0 be an open subset of $\partial\omega$ which is made of a finite number of connected components (whose closure are disjoint). Let us denote the lateral part of the boundary by

$$\Gamma_{0,\delta} = \Phi(\gamma_0 \times] - \delta, \delta[).$$

Consider now a deformation v such that $v = I_d$ on $\Gamma_{0,\delta}$. Let $\gamma'_{0,\delta}$ be the domain

$$\gamma'_{0,\delta} = \{(s_1, s_2) \in \gamma_0 \mid \text{dist}((s_1, s_2), E_0) > 3\delta\}$$

where E_0 denotes the extremities of γ_0 . We set

$$\begin{aligned} \mathcal{Q}_\delta^1 &= \Phi(\{(s_1, s_2) \in (\omega_{3\delta} \setminus \bar{\omega}) \mid \text{dist}((s_1, s_2), \gamma'_{0,\delta}) < 3\delta\} \times] - \delta, \delta[), \\ \mathcal{Q}_\delta^2 &= \Phi(\{(s_1, s_2) \in \omega_{3\delta} \mid \text{dist}((s_1, s_2), \gamma_0) < 6\delta\} \times] - \delta, \delta[). \end{aligned}$$

Indeed, up to choosing δ_0 small enough, we can assume that \mathcal{Q}_δ^2 has the same number of connected components as γ_0 . The open set \mathcal{Q}_δ^1 is included into $\mathcal{Q}'_\delta \setminus \mathcal{Q}_\delta$. According to the construction of P_δ given in [14], we can extend the deformation v by choosing $P_\delta(v) = I_d$ in \mathcal{Q}_δ^1 together with the following estimates

$$(3.2) \quad \begin{cases} \|\nabla_x P_\delta(v) - \mathbf{I}_3\|_{(L^2(\mathcal{Q}_\delta^2))^9} \leq C \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \\ \|P_\delta(v) - I_d\|_{(L^2(\mathcal{Q}_\delta^2))^3} \leq C\delta \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}. \end{cases}$$

From now on we assume that $3\delta \leq \eta_0$ and then any deformation v belonging to $(H^1(\mathcal{Q}_\delta))^3$ is extended to a deformation belonging to $(H^1(\mathcal{Q}'_\delta))^3$ which we still denote by v .

Now we are in a position to reproduce the technique developed in [14] in order to obtain a covering of the shell (the reader is referred to Section 3.3 of this paper for further details). Let \mathcal{N}_δ be the set of every $(k, l) \in \mathbb{Z}^2$ such that the open set

$$\omega_{\delta, (k, l)} =]k\delta, (k+1)\delta[\times]l\delta, (l+1)\delta[$$

is included in $\omega_{3\delta}$ and let \mathcal{N}'_δ be the set of every $(k, l) \in \mathcal{N}_\delta$ such that

$$((k+1)\delta, l\delta), (k\delta, (l+1)\delta), (k+1)\delta, (l+1)\delta \text{ are in } \mathcal{N}_\delta.$$

We set $\Omega_{\delta, (k, l)} = \omega_{\delta, (k, l)} \times]-\delta, \delta[$.

By construction of the above covering, we have

$$\omega \subset \bigcup_{(k, l) \in \mathcal{N}'_\delta} \bar{\omega}_{\delta, (k, l)}.$$

According to [14], there exist two constants R and R_1 , which depend on ω and on the constants c'_0, c'_1, c' and C' (see(2.3)), such that for any $\delta \leq (0, \eta_0/3]$ the open set $\mathcal{Q}_{\delta, (k, l)} = \Phi(\Omega_{\delta, (k, l)})$ has a diameter less than $R\delta$ and is star-shaped with respect to a ball of radius $R_1\delta$.

As a convention and from now on, we will say that a constant C which depends only upon $\partial\omega$ and on the constants c'_0, c'_1, c' and C' depends on the mid-surface S and we write $C(S)$.

Since the ratio $\frac{R\delta}{R_1\delta}$ of each part $\mathcal{Q}_{\delta, (k, l)}$ does not depend on δ , Theorem 3.1 gives a constant $C(S)$. Let v be a deformation in $(H^1(\mathcal{Q}_\delta))^3$ extended to a deformation belonging to $(H^1(\mathcal{Q}'_\delta))^3$. Applying Theorem 3.1 upon each part $\mathcal{Q}_{\delta, (k, l)}$ for $(k, l) \in \mathcal{N}_\delta$, there exist $\mathbf{R}_{\delta, (k, l)} \in SO(3)$ and $\mathbf{a}_{\delta, (k, l)} \in \mathbb{R}^3$ such that

$$(3.3) \quad \begin{cases} \|\nabla_x v - \mathbf{R}_{\delta, (k, l)}\|_{(L^2(\mathcal{Q}_{\delta, (k, l)}))^{3 \times 3}} \leq C(S) \|\text{dist}(\nabla_x v; SO(3))\|_{L^2(\mathcal{Q}_{\delta, (k, l)})} \\ \|v - \mathbf{a}_{\delta, (k, l)} - \mathbf{R}_{\delta, (k, l)}(x - \phi(k\delta, l\delta))\|_{(L^2(\mathcal{Q}_{\delta, (k, l)}))^3} \leq C(S)R(S)\delta \|\text{dist}(\nabla_x v; SO(3))\|_{L^2(\mathcal{Q}_{\delta, (k, l)})}. \end{cases}$$

For any $(k, l) \in \mathcal{N}_\delta$ such that $(k+1, l) \in \mathcal{N}_\delta$, the open set $\mathcal{Q}'_{\delta, (k, l)} = \Phi(](k+1/2)\delta, (k+3/2)\delta[\times]l\delta, (l+1)\delta[\times]-\delta, \delta[)$ also have a diameter less than $R(S)\delta$ and it is also star-shaped with respect to a ball of radius $R_1(S)\delta$ (see Section 3.3 in [14]). We apply again Theorem 2.1 in the domain $\mathcal{Q}'_{\delta, (k, l)}$. This gives a rotation $\mathbf{R}'_{\delta, (k, l)}$. In the domain $\mathcal{Q}'_{\delta, (k, l)} \cap \mathcal{Q}_{\delta, (k, l)}$ we eliminate $\nabla_x v$ in order to evaluate $\|\|\mathbf{R}_{\delta, (k, l)} - \mathbf{R}'_{\delta, (k, l)}\|\|$. Then we evaluate $\|\|\mathbf{R}_{\delta, (k+1, l)} - \mathbf{R}'_{\delta, (k, l)}\|\|$. Finally it leads to

$$(3.4) \quad \|\|\mathbf{R}_{\delta, (k+1, l)} - \mathbf{R}_{\delta, (k, l)}\|\| \leq \frac{C(S)}{\delta^{3/2}} \left\{ \|\|\text{dist}(\nabla_x v; SO(3))\|\|_{L^2(\mathcal{Q}_{\delta, (k, l)})} + \|\|\text{dist}(\nabla_x v; SO(3))\|\|_{L^2(\mathcal{Q}_{\delta, (k+1, l)})} \right\}.$$

In the same way, we prove that for any $(k, l) \in \mathcal{N}_\delta$ such that $(k, l+1) \in \mathcal{N}_\delta$ we have

$$(3.5) \quad \|\|\mathbf{R}_{\delta, (k, l+1)} - \mathbf{R}_{\delta, (k, l)}\|\| \leq \frac{C(S)}{\delta^{3/2}} \left\{ \|\|\text{dist}(\nabla_x v; SO(3))\|\|_{L^2(\mathcal{Q}_{\delta, (k, l)})} + \|\|\text{dist}(\nabla_x v; SO(3))\|\|_{L^2(\mathcal{Q}_{\delta, (k, l+1)})} \right\}$$

3.3. First decomposition of a deformation

In this section any deformation $v \in (H^1(\mathcal{Q}_\delta))^3$ of the shell \mathcal{Q}_δ is decomposed as

$$(3.6) \quad v(s) = \mathcal{V}(s_1, s_2) + s_3 \mathbf{R}_a(s_1, s_2) \mathbf{n}(s_1, s_2) + \bar{v}_a(s), \quad s \in \Omega_\delta,$$

where \mathcal{V} belongs to $(H^1(\omega))^3$, \mathbf{R}_a belongs to $(H^1(\omega))^{3 \times 3}$ and \bar{v}_a belongs to $(H^1(\mathcal{Q}_\delta))^3$. The map \mathcal{V} is the mean value of v over the fibers while the second term $s_3 \mathbf{R}_a(s_1, s_2) \mathbf{n}(s_1, s_2)$ is an approximation of the rotation of the fiber (of the shell) which contains the point $\phi(s_1, s_2)$. The sum of the two first terms $\mathcal{V}(s_1, s_2) + s_3 \mathbf{R}_a(s_1, s_2) \mathbf{n}(s_1, s_2)$ is called the elementary deformation of first type of the shell.

The matrix \mathbf{R}_a is defined as the Q_1 interpolate at the vertices of the cell $\omega_{\delta,(k,l)} =]k\delta, (k+1)\delta[\times]l\delta, (l+1)\delta[$ of the four elements $\mathbf{R}_{\delta,(k,l)}$, $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k,l+1)}$ and $\mathbf{R}_{\delta,(k+1,l+1)}$ belonging to $SO(3)$ (see the previous subsection). We can always define paths in $SO(3)$ from $\mathbf{R}_{\delta,(k,l)}$ to $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k,l)}$ to $\mathbf{R}_{\delta,(k,l+1)}$, $\mathbf{R}_{\delta,(k+1,l)}$ to $\mathbf{R}_{\delta,(k+1,l+1)}$ and $\mathbf{R}_{\delta,(k,l+1)}$ to $\mathbf{R}_{\delta,(k+1,l+1)}$. That gives continuous maps from the edges of the domain $\omega_{\delta,(k,l)}$ into $SO(3)$. If it is possible to extend these maps in order to obtain a continuous function from $\omega_{\delta,(k,l)}$ into $SO(3)$, then it means that the loop passing through $\mathbf{R}_{\delta,(k,l)}$, $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k+1,l+1)}$ and $\mathbf{R}_{\delta,(k,l+1)}$ is homotopic to the constant loop equal to $\mathbf{R}_{\delta,(k,l)}$. But the fundamental group $\pi_1(SO(3), \mathbf{R}_{\delta,(k,l)})$ is isomorphic to \mathbb{Z}_2 (the group of odd and even integers), hence the extension does not always exist. That is the reason why we use here a Q_1 interpolate in order to define an approximate field \mathbf{R}_a of rotations. In the next subsection we show that if the matrices $\mathbf{R}_{\delta,(k+1,l)}$, $\mathbf{R}_{\delta,(k+1,l+1)}$ and $\mathbf{R}_{\delta,(k,l+1)}$ are in a neighborhood of $\mathbf{R}_{\delta,(k,l)}$ then this extension exists and we give in Theorem 3.4 a simple condition to do so.

Theorem 3.3. *Let $v \in (H^1(\mathcal{Q}_\delta))^3$, there exist an elementary deformation (of first type) $\mathcal{V} + s_3 \mathbf{R}_a \mathbf{n}$ and a deformation \bar{v}_a satisfying (3.6) and such that*

$$(3.7) \quad \left\{ \begin{array}{l} \|\bar{v}_a\|_{(L^2(\Omega_\delta))^3} \leq C\delta \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \|\nabla_s \bar{v}_a\|_{(L^2(\Omega_\delta))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \mathbf{R}_a}{\partial s_\alpha} \right\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{R}_a \mathbf{t}_\alpha \right\|_{(L^2(\omega))^3} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \|\nabla_x v - \mathbf{R}_a\|_{(L^2(\Omega_\delta))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \|dist(\mathbf{R}_a, SO(3))\|_{L^2(\omega)} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \end{array} \right.$$

where the constant C does not depend on δ .

Proof. The field \mathcal{V} is defined by

$$(3.8) \quad \mathcal{V}(s_1, s_2) = \frac{1}{2\delta} \int_{-\delta}^{\delta} v(s_1, s_2, s_3) ds_3, \quad \text{a.e. in } \omega.$$

Then we define the field \mathbf{R}_a as following

$$\forall (k, l) \in \mathcal{N}_\delta, \quad \mathbf{R}_a(k\delta, l\delta) = \mathbf{R}_{\delta,(k,l)}$$

and for any $(s_1, s_2) \in \omega_{\delta,(k,l)}$, $\mathbf{R}_a(s_1, s_2)$ is the Q_1 interpolate of the values of \mathbf{R}_a at the vertices of the cell $\omega_{\delta,(k,l)}$.

Finally we define the field \bar{v}_a by

$$\bar{v}_a(s) = v(s) - \mathcal{V}(s_1, s_2) - s_3 \mathbf{R}_a(s_1, s_2) \mathbf{n}(s_1, s_2) \quad \text{a.e. in } \Omega_\delta.$$

From (3.4) and (3.5) we get the third estimate in (3.7). By definition of \mathbf{R}_a we obtain

$$(3.9) \quad \sum_{(k,l) \in \mathcal{N}'_\delta} \|\mathbf{R}_a - \mathbf{R}_{\delta,(k,l)}\|_{(L^2(\omega_{\delta,(k,l)}))^9}^2 \leq \frac{C}{\delta} \|dist(\nabla_x v; SO(3))\|_{L^2(\mathcal{Q}_\delta)}^2.$$

Taking the mean value of v on the fibers and using definition (3.8) of \mathcal{V} lead to

$$(3.10) \quad \sum_{(k,l) \in \mathcal{N}'_\delta} \|\mathcal{V} - \mathbf{a}_{\delta,(k,l)} - \mathbf{R}_{\delta,(k,l)}(\phi - \phi(k\delta, l\delta))\|_{(L^2(\omega_{\delta,(k,l)}))^3}^2 \leq C\delta \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}^2.$$

From (3.3), (3.9), (3.10) and the definition of \bar{v}_a we get the first estimate in (3.7).

We compute the derivatives of the deformation v to get

$$(3.11) \quad \frac{\partial v}{\partial s_1} = \nabla_x v \left(\mathbf{t}_1 + s_3 \frac{\partial \mathbf{n}}{\partial s_1} \right), \quad \frac{\partial v}{\partial s_2} = \nabla_x v \left(\mathbf{t}_2 + s_3 \frac{\partial \mathbf{n}}{\partial s_2} \right), \quad \frac{\partial v}{\partial s_3} = \nabla_x v \mathbf{n}.$$

We consider the restrictions of these derivatives to $\Omega_{\delta,(k,l)}$. Then, from (3.3) and (3.9) we have

$$(3.12) \quad \left\| \frac{\partial v}{\partial s_\alpha} - \mathbf{R}_a \left(\mathbf{t}_\alpha + s_3 \frac{\partial \mathbf{n}}{\partial s_\alpha} \right) \right\|_{(L^2(\Omega_\delta))^3}^2 + \left\| \frac{\partial v}{\partial s_3} - \mathbf{R}_a \mathbf{n} \right\|_{(L^2(\Omega_\delta))^3}^2 \leq C \| \text{dist}(\nabla_x v, SO(3)) \|_{L^2(\mathcal{Q}_\delta)}^2.$$

By taking the mean value of $\frac{\partial v}{\partial s_\alpha} - \mathbf{R}_a \left(\mathbf{t}_\alpha + s_3 \frac{\partial \mathbf{n}}{\partial s_\alpha} \right)$ on the fibers we obtain the fourth inequality in (3.7). Observe now that

$$(3.13) \quad \frac{\partial \bar{v}_a}{\partial s_\alpha} = \frac{\partial v}{\partial s_\alpha} - \frac{\partial \mathcal{V}}{\partial s_\alpha} - s_3 \mathbf{R}_a \frac{\partial \mathbf{n}}{\partial s_\alpha} - s_3 \frac{\partial \mathbf{R}_a}{\partial s_\alpha} \mathbf{n}, \quad \frac{\partial \bar{v}_a}{\partial s_3} = \frac{\partial v}{\partial s_3} - \mathbf{R}_a \mathbf{n}.$$

Then, from (3.12) and the third and fourth inequalities in (3.7) we obtain the second estimate in (3.7). The fifth inequality in (3.7) is an immediate consequence of (3.3) and (3.9). The last estimate of (3.7) is due to (3.4), (3.5) and to the very definition of the field \mathbf{R}_a . \square

Since the matrices $\mathbf{R}_{\delta,(k,l)}$ belong to $SO(3)$, the function \mathbf{R}_a is uniformly bounded and satisfies

$$\| \mathbf{R}_a \|_{(L^\infty(\omega))^9} \leq \sqrt{3}.$$

Let (k, l) be in \mathcal{N}_δ . By a straightforward computation, for any $(s_1, s_2) \in \omega_{\delta,(k,l)}$ we obtain

$$\begin{aligned} \| | \mathbf{R}_a(s_1, s_2) \mathbf{R}_a^T(s_1, s_2) - \mathbf{I}_3 | \| &\leq C \{ \| | \mathbf{R}_{\delta,(k,l)} - \mathbf{R}_{\delta,(k+1,l)} | \| + \| | \mathbf{R}_{\delta,(k,l)} - \mathbf{R}_{\delta,(k,l+1)} | \| \\ &\quad + \| | \mathbf{R}_{\delta,(k,l+1)} - \mathbf{R}_{\delta,(k+1,l+1)} | \| + \| | \mathbf{R}_{\delta,(k+1,l)} - \mathbf{R}_{\delta,(k+1,l+1)} | \| \} \\ | \det(\mathbf{R}_a(s_1, s_2)) - 1 | &\leq C \{ \| | \mathbf{R}_{\delta,(k,l)} - \mathbf{R}_{\delta,(k+1,l)} | \| + \| | \mathbf{R}_{\delta,(k,l)} - \mathbf{R}_{\delta,(k,l+1)} | \| \\ &\quad + \| | \mathbf{R}_{\delta,(k,l+1)} - \mathbf{R}_{\delta,(k+1,l+1)} | \| + \| | \mathbf{R}_{\delta,(k+1,l)} - \mathbf{R}_{\delta,(k+1,l+1)} | \| \}, \end{aligned}$$

where C is an absolute constant. Hence, from (3.4) and (3.5) we deduce that

$$(3.14) \quad \begin{cases} \| | \mathbf{R}_a \mathbf{R}_a^T - \mathbf{I}_3 | \|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{1/2}} \| \text{dist}(\nabla_x v, SO(3)) \|_{L^2(\mathcal{Q}_\delta)}, \\ \| \det(\mathbf{R}_a) - 1 \|_{L^2(\omega)} \leq \frac{C}{\delta^{1/2}} \| \text{dist}(\nabla_x v, SO(3)) \|_{L^2(\mathcal{Q}_\delta)}. \end{cases}$$

Notice that the function $\mathbf{R}_a \mathbf{R}_a^T$ belongs to $(H^1(\omega))^{3 \times 3}$ and satisfies

$$(3.15) \quad \left\| \frac{\partial \mathbf{R}_a \mathbf{R}_a^T}{\partial s_\alpha} \right\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{3/2}} \| \text{dist}(\nabla_x v, SO(3)) \|_{L^2(\mathcal{Q}_\delta)}.$$

3.4. Second decomposition of a deformation.

In this section any deformation $v \in (H^1(\mathcal{Q}_\delta))^3$ of the shell \mathcal{Q}_δ is decomposed as

$$(3.16) \quad v(s) = \mathcal{V}(s_1, s_2) + s_3 \mathbf{R}(s_1, s_2) \mathbf{n}(s_1, s_2) + \bar{v}(s), \quad s \in \Omega_\delta,$$

where \mathcal{V} belongs to $(H^1(\omega))^3$, \mathbf{R} belongs to $(H^1(\omega))^{3 \times 3}$ and satisfies for a.e. $(s_1, s_2) \in \omega$: $\mathbf{R}(s_1, s_2) \in SO(3)$ and \bar{v} belongs to $(H^1(\mathcal{Q}_\delta))^3$. The first term \mathcal{V} is still the mean value of v over the fibers. Now, the second one $s_3 \mathbf{R}(s_1, s_2) \mathbf{n}(s_1, s_2)$ describes the rotation of the fiber (of the shell) which contains the point $\phi(s_1, s_2)$. The sum of the two first terms $\mathcal{V}(s_1, s_2) + s_3 \mathbf{R}(s_1, s_2) \mathbf{n}(s_1, s_2)$ is called the elementary deformation of second type of the shell.

Theorem 3.4. *There exists a constant $C(S)$ (which depends only on the mid-surface S) such that for any $v \in (H^1(\mathcal{Q}_\delta))^3$ verifying*

$$(3.17) \quad \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leq C(S) \delta^{3/2}$$

then there exist an elementary deformation of second type $\mathcal{V} + s_3 \mathbf{R} \mathbf{n}$ and a deformation \bar{v} satisfying (3.16) and such that

$$(3.18) \quad \begin{cases} \|\bar{v}\|_{(L^2(\Omega_\delta))^3} \leq C \delta \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \|\nabla_s \bar{v}\|_{(L^2(\Omega_\delta))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \right\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{R} \mathbf{t}_\alpha \right\|_{(L^2(\omega))^3} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \|\nabla_x v - \mathbf{R}\|_{(L^2(\Omega_\delta))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \end{cases}$$

where the constant C does not depend on δ .

Proof. In this proof let us denote by $C_1(S)$ the constant appearing in estimates (3.4) and (3.5). If we assume that

$$(3.19) \quad \frac{\sqrt{2} C_1(S)}{\delta^{3/2}} \|dist(\nabla_x v; SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leq \frac{1}{2}.$$

then, for each $(k, l) \in \mathcal{N}'_\delta$ we have using (3.4) and (3.5)

$$\| \mathbf{R}_{\delta, (k+1, l)} - \mathbf{R}_{\delta, (k, l)} \| \leq \frac{1}{2}, \quad \| \mathbf{R}_{\delta, (k, l+1)} - \mathbf{R}_{\delta, (k, l)} \| \leq \frac{1}{2}.$$

Thanks to Lemma A.2 in Appendix A there exists a function $\mathbf{R} \in (W^{1, \infty}(\omega))^{3 \times 3}$ such that for any $(s_1, s_2) \in \omega$ the matrix $\mathbf{R}(s_1, s_2)$ belongs to $SO(3)$ and such that

$$\forall (k, l) \in \mathcal{N}_\delta, \quad \mathbf{R}(k\delta, l\delta) = \mathbf{R}_{\delta, (k, l)}.$$

From (3.4), (3.5) and Lemma A.2 we obtain the estimates (3.18) of the derivatives of \mathbf{R} . Due to the corollary of Lemma A.2 we have

$$(3.20) \quad \|\mathbf{R} - \mathbf{R}_a\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v; SO(3))\|_{L^2(\mathcal{Q}_\delta)}.$$

All remainder estimates in (3.18) are consequences of (3.7) and (3.20). \square

4. Two nonlinear Korn's inequalities for shells

In this section, we first precise the boundary conditions on the deformations. We discuss essentially the usual case of the clamped condition on $\Gamma_{0,\delta}$ (see Subsection 3.2). In Subsection 4.1, we deduce the first estimates on v and ∇v . Then we show that the elementary deformations of the decompositions can be imposed on the same boundary than v . The main result of Subsection 4.2 is the Korn's inequality for shells given.

Let v be in $(H^1(\mathcal{Q}_\delta))^3$ such that

$$v(x) = x \quad \text{on } \Gamma_{0,\delta}.$$

Due to the definition (3.3) of \mathcal{V} , we have

$$(4.1) \quad \mathcal{V} = \phi \quad \text{on } \gamma_0.$$

4.1. First H^1 - Estimates

Using the boundary condition (4.1), estimates (3.7) or (3.18) and the fact that $\|\mathbf{R}_a\|_{(L^\infty(\omega))^{3 \times 3}} \leq \sqrt{3}$ and $\|\mathbf{R}\|_{(L^\infty(\omega))^{3 \times 3}} \leq \sqrt{3}$, it leads to

$$(4.2) \quad \|\mathcal{V}\|_{(H^1(\omega))^3} \leq C \left(1 + \frac{1}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \right).$$

With the help of the decompositions (3.6) or (3.16), estimates (3.7) or (3.18) and (4.2) we deduce that

$$\|v\|_{(L^2(\mathcal{Q}_\delta))^3} + \frac{1}{\delta} \|v - \mathcal{V}\|_{(L^2(\mathcal{Q}_\delta))^3} + \|\nabla_x v\|_{(L^2(\mathcal{Q}_\delta))^9} \leq C \left(\delta^{1/2} + \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \right).$$

The above inequality leads to the following first "nonlinear Korn's inequality for shells":

$$(4.3) \quad \|v - I_d\|_{(L^2(\mathcal{Q}_\delta))^3} + \|\nabla_x v - \mathbf{I}_3\|_{(L^2(\mathcal{Q}_\delta))^9} \leq C \left(\delta^{1/2} + \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \right)$$

together with

$$\|(v - I_d) - (\mathcal{V} - \phi)\|_{(L^2(\mathcal{Q}_\delta))^3} \leq C \delta \left(\delta^{1/2} + \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \right).$$

Let us notice that inequality (4.3) can be obtained without using the decomposition of the deformation. Indeed, we first have

$$\| |\nabla v(x)| \| \leq dist(\nabla v(x), SO(3)) + \sqrt{3}, \quad \text{for a.e. } x$$

so that by integration

$$\|\nabla_x v\|_{(L^2(\mathcal{Q}_\delta))^9} \leq C \left(\delta^{1/2} + \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \right).$$

Poincaré's inequality then leads to (4.3). This is the technique used to derive estimates in [13].

4.2. Further H^1 - Estimates

In this subsection, we derive a boundary condition on \mathbf{R}_a and \mathbf{R} on γ_0 using the extension given in Subsection 3.2. We prove the following lemma:

Lemma 4.1. *In Theorem 3.3 (respectively in Theorem 3.4), we can choose \mathbf{R}_a (resp. \mathbf{R}) such that*

$$\mathbf{R}_a = \mathbf{I}_3 \quad \text{on } \gamma_0, \quad (\text{resp. } \mathbf{R} = \mathbf{I}_3 \quad \text{on } \gamma_0),$$

without modifications in the estimates of these theorems.

Proof. Recall that $\gamma'_{0,\delta}$, \mathcal{Q}_δ^1 and \mathcal{Q}_δ^2 are defined in subsection 3.2. We also set

$$\mathcal{Q}_\delta^3 = \Phi(\{(s_1, s_2) \in \omega_{3\delta} \mid \text{dist}((s_1, s_2), \gamma_0) < 3\delta\} \times] - \delta, \delta[)$$

Let us consider the following function

$$\rho_\delta(s_1, s_2) = \inf \left\{ 1, \sup \left(0, \frac{1}{3\delta} \text{dist}((s_1, s_2), \gamma_0) - 1 \right) \right\}, \quad (s_1, s_2) \in \mathbb{R}^2.$$

This function belongs to $W^{1,\infty}(\mathbb{R}^2)$ and it is equal to 1 if $\text{dist}((s_1, s_2), \gamma_0) > 6\delta$ and to 0 if $\text{dist}((s_1, s_2), \gamma_0) < 3\delta$. Let v_δ be the deformation defined by

$$v_\delta(s) = \phi(s_1, s_2) + s_3 \mathbf{n}(s_1, s_2) + \rho_\delta(s_1, s_2)(v(s) - \phi(s_1, s_2) - s_3 \mathbf{n}(s_1, s_2)) \quad \text{for a.e. } s \in \omega_{3\delta} \times] - \delta, \delta[.$$

By definition of v_δ , we have

$$v_\delta = v \quad \text{in } \mathcal{Q}'_\delta \setminus \mathcal{Q}_\delta^2, \quad v_\delta = I_d \quad \text{in } \mathcal{Q}_\delta^3.$$

Recall that $v = I_d$ on \mathcal{Q}_δ^1 . Since the L^∞ -norm of ρ_δ is of order $1/\delta$ and the two estimates in (3.2) lead to

$$(4.4) \quad \begin{cases} \|\nabla_x v - \nabla_x v_\delta\|_{(L^2(\mathcal{Q}'_\delta))^9} \leq C \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \\ \|v - v_\delta\|_{(L^2(\mathcal{Q}'_\delta))^3} \leq C\delta \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}. \end{cases}$$

Hence

$$(4.5) \quad \begin{cases} \|\text{dist}(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{Q}'_\delta)} \leq \|\nabla_x v - \nabla_x v_\delta\|_{(L^2(\mathcal{Q}'_\delta))^9} + \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}'_\delta)} \\ \leq C \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \end{cases}$$

where the constant does not depend on δ .

Since $v_\delta = I_d$ in \mathcal{Q}_δ^2 , the \mathbf{R}_a 's and the \mathbf{R} 's given by application of Theorem 3.3 or 3.4 to the deformation v_δ are both equal to \mathbf{I}_3 over γ_0 . Estimate(3.7) and (3.18) of these theorem together with (4.4)-(4.5) show that Theorems 3.3 and 3.4 hold true for v with $\mathbf{R}_a = \mathbf{I}_3$ and $\mathbf{R} = \mathbf{I}_3$ on γ_0 . \square

The next theorem gives a second nonlinear Korn's inequalities, which is an improvement of (4.3) for energies of order smaller than $\delta^{3/2}$ and an estimate on $v - \mathcal{V}$ which permit to precise the scaling of the applied forces in Section 7.

Theorem 4.2. (A second nonlinear Korn's inequality for shells) *There exists a constant C which does not depend upon δ such that for all $v \in (H^1(\mathcal{Q}_\delta))^3$ such that $v = I_d$ on $\Gamma_{0,\delta}$*

$$(4.6) \quad \|v - I_d\|_{(L^2(\mathcal{Q}_\delta))^3} + \|\nabla_x v - \mathbf{I}_3\|_{(L^2(\mathcal{Q}_\delta))^9} \leq \frac{C}{\delta} \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)},$$

and

$$(4.7) \quad \|(v - I_d) - (\mathcal{V} - \phi)\|_{(L^2(\mathcal{Q}_\delta))^3} \leq C \|\text{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)},$$

where \mathcal{V} is given by (3.8).

Proof. From the decomposition (3.6), Theorem 3.3 and the boundary condition on \mathbf{R}_a given by Lemma 4.1, the use of Poincaré's inequality gives

$$(4.8) \quad \begin{cases} \|\mathbf{R}_a - \mathbf{I}_3\|_{(H^1(\omega))^9} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{t}_\alpha \right\|_{(L^2(\omega))^3} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \end{cases}$$

Using the fact that $\mathbf{t}_\alpha = \frac{\partial \phi}{\partial s_\alpha}$ and the boundary condition (4.1) on \mathcal{V} , it leads to

$$\|\mathcal{V} - \phi\|_{(L^2(\omega))^3} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}.$$

Using again the decomposition (3.6) and Theorem 3.3, the above estimate implies that $v - I_d$ satisfies the nonlinear Korn's inequality (4.6). At last the decomposition (3.6), which implies that $(v - I_d) - (\mathcal{V} - \phi) = (\mathbf{R}_a - \mathbf{I}_3)s_3\mathbf{n} + \bar{v}_a$, the first estimate in (3.7) and (4.8) permit to obtain (4.7). \square

Let us compare the two inequalities (4.3) and (4.6). Indeed they are equivalent for energies of order $\delta^{3/2}$. For energies order smaller than $\delta^{3/2}$, (4.6) is better (4.3) which is then more relevant in general for thin structures.

The decomposition technique given in Section 3 also allows to estimate the linearized strain tensor of an admissible deformation. This is the object of the lemma below.

Lemma 4.3 *There exists a constant C which does not depend upon δ such that for all $v \in (H^1(\mathcal{Q}_\delta))^3$ such that $v = I_d$ on $\Gamma_{0,\delta}$*

$$(4.9) \quad \|\nabla_x v + (\nabla_x v)^T - 2\mathbf{I}_3\|_{(L^2(\mathcal{Q}_\delta))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \left\{ 1 + \frac{1}{\delta^{5/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \right\}.$$

Proof. In view of the decomposition (3.6) and Theorem 3.3 we have

$$(4.10) \quad \|\nabla_x v + (\nabla_x v)^T - 2\mathbf{I}_3\|_{(L^2(\Omega_\delta))^9} \leq C \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} + C\delta^{1/2} \|\mathbf{R}_a + \mathbf{R}_a^T - 2\mathbf{I}_3\|_{(L^2(\omega))^9}.$$

Due to the equalities

$$\begin{aligned} \mathbf{R}_a + \mathbf{R}_a^T - 2\mathbf{I}_3 &= \mathbf{R}_a \mathbf{R}_a \mathbf{R}_a^T + \mathbf{R}_a^T - 2\mathbf{R}_a \mathbf{R}_a^T + \mathbf{R}_a (\mathbf{I}_3 - \mathbf{R}_a \mathbf{R}_a^T) + 2(\mathbf{R}_a \mathbf{R}_a^T - \mathbf{I}_3) \\ &= (\mathbf{R}_a - \mathbf{I}_3)^2 \mathbf{R}_a^T + \mathbf{R}_a (\mathbf{I}_3 - \mathbf{R}_a \mathbf{R}_a^T) + 2(\mathbf{R}_a \mathbf{R}_a^T - \mathbf{I}_3) \end{aligned}$$

and to the first estimate in (3.14), it follows that

$$(4.11) \quad \|\mathbf{R}_a + \mathbf{R}_a^T - 2\mathbf{I}_3\|_{(L^2(\omega))^9} \leq C \|(\mathbf{R}_a - \mathbf{I}_3)^2\|_{(L^2(\omega))^9} + \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}.$$

Since $\|(\mathbf{R}_a - \mathbf{I}_3)^2\|_{(L^2(\omega))^9} \leq C \|\mathbf{R}_a - \mathbf{I}_3\|_{(L^4(\omega))^9}^2$ and the fact that the space $(H^1(\omega))^{3 \times 3}$ is continuously imbedded in $(L^4(\omega))^{3 \times 3}$, we deduce that

$$(4.12) \quad \|(\mathbf{R}_a - \mathbf{I}_3)^2\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^3} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}^2.$$

From (4.10), (4.11) and (4.12) we finally get (4.9). \square

Remark 4.4. In view of (3.7) and since the field \mathbf{R}_a belongs to $(L^\infty(\omega))^{3 \times 3}$, the function $(\mathbf{R}_a - \mathbf{I}_3)^2$ belongs to $(H^1(\omega))^{3 \times 3}$ with

$$\left\| \frac{\partial(\mathbf{R}_a - \mathbf{I}_3)^2}{\partial s_\alpha} \right\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}.$$

Hence, with Lemma 4.1, $\|(\mathbf{R}_a - \mathbf{I}_3)^2\|_{(L^2(\omega))^9} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}$ which gives together with (4.10)-(4.11)

$$\|\nabla_x v + (\nabla_x v)^T - 2\mathbf{I}_3\|_{(L^2(\mathcal{Q}_\delta))^9} \leq \frac{C}{\delta} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}.$$

Notice that the above estimate is worse than (4.9) at least as soon as the energy is smaller than $\delta^{1/2}$. \square

Let us emphasize that in view of estimates (3.7)-(3.18), (4.3) and (4.9) one can distinguish two critical cases for the behavior of the quantity $\|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}$

$$\|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} = \begin{cases} O(\delta^{3/2}), \\ O(\delta^{5/2}). \end{cases}$$

Estimates (4.2)-(4.3) show that the behavior $\|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \sim O(\delta^{1/2})$ also corresponds to an interesting case, but the estimates (3.7) and (4.8) show that the decompositions given in Theorems 3.3 and 3.4 are not relevant in this case which, as a consequence, must be analyzed by a different approach.

In the following we will describe the asymptotic behavior of a sequence of deformations v_δ which satisfies $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \sim O(\delta^{\kappa-1/2})$, $\kappa \geq 2$.

5. Rescaling Ω_δ

As usual when dealing with a thin shell, we rescale Ω_δ using the operator

$$(\Pi_\delta w)(s_1, s_2, S_3) = w(s_1, s_2, s_3) \text{ for any } s \in \Omega_\delta$$

defined for e.g. $w \in L^2(\Omega_\delta)$ for which $(\Pi_\delta w) \in L^2(\Omega)$. The estimates (3.7) on \bar{v}_a transposed over Ω lead to

$$(5.1) \quad \begin{cases} \|\Pi_\delta \bar{v}_a\|_{(L^2(\Omega))^3} \leq C\delta^{1/2} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \Pi_\delta \bar{v}_a}{\partial s_1} \right\|_{(L^2(\Omega))^3} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \Pi_\delta \bar{v}_a}{\partial s_2} \right\|_{(L^2(\Omega))^3} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \Pi_\delta \bar{v}_a}{\partial S_3} \right\|_{(L^2(\Omega))^3} \leq C\delta^{1/2} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \end{cases}$$

and estimates (4.6) on $v - I_d$ give

$$(5.2) \quad \begin{cases} \|\Pi_\delta(v - I_d)\|_{(L^2(\Omega))^3} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \Pi_\delta(v - I_d)}{\partial s_1} \right\|_{(L^2(\Omega))^3} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \Pi_\delta(v - I_d)}{\partial s_2} \right\|_{(L^2(\Omega))^3} \leq \frac{C}{\delta^{3/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \\ \left\| \frac{\partial \Pi_\delta(v - I_d)}{\partial S_3} \right\|_{(L^2(\Omega))^3} \leq \frac{C}{\delta^{1/2}} \|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}. \end{cases}$$

6. Asymptotic behavior of the Green-St Venant's strain tensor in the case $\kappa = 2$.

Let us consider a sequence of deformations v_δ of $(H^1(\mathcal{Q}_\delta))^3$ such that

$$(6.1) \quad \|\text{dist}(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{3/2}.$$

For fixed $\delta > 0$, the deformation v_δ is decomposed as in Theorem 3.3 and the terms of this decomposition are denoted by \mathcal{V}_δ , $\mathbf{R}_{a,\delta}$ and $\bar{v}_{a,\delta}$. If moreover the hypothesis (3.17) holds true for the sequence v_δ , then v_δ can be alternatively decomposed through (3.16) in terms of \mathcal{V}_δ , \mathbf{R}_δ and \bar{v}_δ so that the estimates (3.18) of Theorem 3.5 are satisfied uniformly in δ .

In what follows we investigate the behavior of the sequences \mathcal{V}_δ , $\mathbf{R}_{a,\delta}$ and $\bar{v}_{a,\delta}$. Indeed due to (3.20) all the result of this section can be easily transposed in terms of the sequence \mathbf{R}_δ and the details are left to the reader.

The estimates (3.7), (5.1) and (5.2) lead to the following lemma.

Lemma 6.1. *There exists a subsequence still indexed by δ such that*

$$(6.2) \quad \left\{ \begin{array}{l} \mathcal{V}_\delta \longrightarrow \mathcal{V} \text{ strongly in } (H^1(\omega))^3, \\ \mathbf{R}_{a,\delta} \rightharpoonup \mathbf{R} \text{ weakly in } (H^1(\omega))^{3 \times 3} \text{ and strongly in } (L^2(\omega))^{3 \times 3}, \\ \frac{1}{\delta^2} \Pi_\delta \bar{v}_{a,\delta} \rightharpoonup \bar{v} \text{ weakly in } (L^2(\omega; H^1(-1, 1)))^3, \\ \frac{1}{\delta} \left(\frac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_{a,\delta} \mathbf{t}_\alpha \right) \rightharpoonup \mathcal{Z}_\alpha \text{ weakly in } (L^2(\omega))^3, \\ \frac{1}{\delta} \left(\mathbf{R}_{a,\delta}^T \mathbf{R}_{a,\delta} - \mathbf{I}_3 \right) \rightharpoonup 0 \text{ weakly in } (L^2(\omega))^{3 \times 3}, \end{array} \right.$$

where \mathbf{R} belongs $SO(3)$ for a.e. $(s_1, s_2) \in \omega$. We also have $\mathcal{V} \in (H^2(\omega))^3$ and

$$(6.3) \quad \frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R} \mathbf{t}_\alpha.$$

The boundaries conditions

$$(6.4) \quad \mathcal{V} = \phi, \quad \mathbf{R} = \mathbf{I}_3 \quad \text{on } \gamma_0,$$

hold true. Moreover, we have

$$(6.5) \quad \left\{ \begin{array}{l} \Pi_\delta v_\delta \longrightarrow \mathcal{V} \text{ strongly in } (H^1(\Omega))^3, \\ \Pi_\delta(\nabla_x v_\delta) \longrightarrow \mathbf{R} \text{ strongly in } (L^2(\Omega))^9. \end{array} \right.$$

Proof. The convergences (6.2) are direct consequences of Theorem 3.3 and estimate (4.8) excepted for what concerns the last convergence which will be established below. The compact imbedding of $(H^1(\omega))^{3 \times 3}$ in $(L^4(\omega))^{3 \times 3}$ and the first convergence in (6.2) permit to obtain

$$(6.6) \quad \left\{ \begin{array}{l} \mathbf{R}_{a,\delta} \longrightarrow \mathbf{R} \text{ strongly in } (L^4(\omega))^{3 \times 3}, \\ \det(\mathbf{R}_{a,\delta}) \longrightarrow \det(\mathbf{R}) \text{ strongly in } L^{4/3}(\omega). \end{array} \right.$$

These convergences and estimates (3.14) prove that for a.e. $(s_1, s_2) \in \omega$: $\mathbf{R}(s_1, s_2) \in SO(3)$. The relation (6.3) and (6.4) and the convergences (6.5) are immediate consequences of Theorem 3.3 and of the above results. We now turn to the proof of the last convergence in (6.2). We first set

$$\tilde{\mathbf{R}}_{a,\delta}(s_1, s_2) = \mathbf{R}_{a,\delta}\left(\delta\left[\frac{s_1}{\delta}\right], \delta\left[\frac{s_2}{\delta}\right]\right) \quad \text{a.e. in } \omega$$

where $[t]$ denote the integer part of the real t . From (3.4), (3.5) and (6.1) we have

$$(6.7) \quad \|\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta}\|_{(L^2(\omega))^{3 \times 3}} \leq C\delta.$$

From (6.6) and the above estimate, we deduce that

$$(6.8) \quad \tilde{\mathbf{R}}_{a,\delta} \longrightarrow \mathbf{R} \quad \text{strongly in } (L^2(\omega))^{3 \times 3}.$$

Now we derive the weak limit of the sequence $\frac{1}{\delta}(\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta})$. Let Φ be in $C_0^\infty(\Omega)^{3 \times 3}$ and set $M_\delta(\Phi)(s_1, s_2) = \int_{]0,1]^2} \Phi\left(\delta\left[\frac{s_1}{\delta}\right] + z_1\delta, \delta\left[\frac{s_2}{\delta}\right] + z_2\delta\right) dz_1 dz_2$ for a.e. (s_1, s_2) in ω . We recall that (see [2])

$$\begin{aligned} \frac{1}{\delta}(\Phi - M_\delta(\Phi)) &\rightharpoonup 0 \quad \text{weakly in } (L^2(\omega))^{3 \times 3} \\ M_\delta(\Phi) &\longrightarrow \Phi \quad \text{strongly in } (L^2(\omega))^{3 \times 3} \end{aligned}$$

We write

$$\begin{aligned} \int_\omega \frac{1}{\delta}(\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta})\Phi &= \int_\omega \mathbf{R}_{a,\delta} \frac{1}{\delta}(\Phi - M_\delta(\Phi)) + \int_\omega \frac{1}{\delta}(\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta})M_\delta(\Phi) \\ &= \int_\omega \mathbf{R}_{a,\delta} \frac{1}{\delta}(\Phi - M_\delta(\Phi)) + \frac{1}{2} \int_\omega \left(\frac{\partial \mathbf{R}_{a,\delta}}{\partial s_1} + \frac{\partial \mathbf{R}_{a,\delta}}{\partial s_2} \right) M_\delta(\Phi) + K_\delta \end{aligned}$$

where $|K_\delta| \leq C\delta \|\nabla \mathbf{R}_{a,\delta}\|_{(L^2(\omega))^{3 \times 3}} \|\nabla \Phi\|_{(L^2(\omega))^{3 \times 3}}$. In view of the properties of $M_\delta(\Phi)$ recalled above, of (6.2) and (6.4), we deduce from the above equality that

$$\frac{1}{\delta}(\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta}) \rightharpoonup \frac{1}{2} \left(\frac{\partial \mathbf{R}}{\partial s_1} + \frac{\partial \mathbf{R}}{\partial s_2} \right) \quad \text{weakly in } (L^2(\omega))^{3 \times 3}.$$

In order to prove the last convergence of (6.2), we write

$$\frac{1}{\delta}(\mathbf{R}_{a,\delta}^T \mathbf{R}_{a,\delta} - \mathbf{I}_3) = \frac{1}{\delta} \left(\tilde{\mathbf{R}}_{a,\delta}^T (\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta}) + (\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta})^T \tilde{\mathbf{R}}_{a,\delta} + (\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta})^T (\mathbf{R}_{a,\delta} - \tilde{\mathbf{R}}_{a,\delta}) \right)$$

and we use estimates (3.14) and (6.7), the strong convergence (6.8) and the above weak convergence. \square

The following Corollary gives the limit of the Green-St Venant's strain tensor of the sequence v_δ .

Theorem 6.2. *For the same subsequence as in Lemma 6.1 we have*

$$(6.9) \quad \frac{1}{2\delta} \Pi_\delta((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3) \rightharpoonup (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-T} \mathbf{E}(\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-1} \quad \text{weakly in } (L^1(\Omega))^9,$$

where the symmetric matrix \mathbf{E} is equal to

$$(6.10) \quad \begin{pmatrix} S_3 \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} \cdot \mathbf{Rt}_1 + \mathcal{Z}_1 \cdot \mathbf{Rt}_1 & S_3 \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} \cdot \mathbf{Rt}_2 + \frac{1}{2} \{ \mathcal{Z}_2 \cdot \mathbf{Rt}_1 + \mathcal{Z}_1 \cdot \mathbf{Rt}_2 \} & \frac{1}{2} \frac{\partial \bar{v}}{\partial S_3} \cdot \mathbf{Rt}_1 + \frac{1}{2} \mathcal{Z}_1 \cdot \mathbf{Rn} \\ * & S_3 \frac{\partial \mathbf{R}}{\partial s_2} \mathbf{n} \cdot \mathbf{Rt}_2 + \mathcal{Z}_2 \cdot \mathbf{Rt}_2 & \frac{1}{2} \frac{\partial \bar{v}}{\partial S_3} \cdot \mathbf{Rt}_2 + \frac{1}{2} \mathcal{Z}_2 \cdot \mathbf{Rn} \\ * & * & \frac{\partial \bar{v}}{\partial S_3} \cdot \mathbf{Rn} \end{pmatrix}$$

and where $(\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})$ denotes the 3×3 matrix with first column \mathbf{t}_1 , second column \mathbf{t}_2 and third column \mathbf{n} and where $(\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-T} = ((\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-1})^T$.

Proof. First from estimate (3.7), equalities (3.13) and the convergences in Lemma 6.1, we obtain

$$\begin{aligned} \frac{1}{\delta} (\Pi_\delta(\nabla_x v_\delta) - \mathbf{R}_{a,\delta}) \mathbf{t}_\alpha &\rightharpoonup S_3 \frac{\partial \mathbf{R}}{\partial s_\alpha} \mathbf{n} + \mathcal{Z}_\alpha \quad \text{weakly in } (L^2(\Omega))^3, \\ \frac{1}{\delta} (\Pi_\delta(\nabla_x v_\delta) - \mathbf{R}_{a,\delta}) \mathbf{n} &\rightharpoonup \frac{\partial \bar{v}}{\partial S_3} \quad \text{weakly in } (L^2(\Omega))^3. \end{aligned}$$

Then thanks to the identity

$$\begin{aligned} \frac{1}{2\delta} \Pi_\delta((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3) &= \frac{1}{2\delta} \Pi_\delta((\nabla_x v_\delta - \mathbf{R}_{a,\delta})^T (\nabla_x v_\delta - \mathbf{R}_{a,\delta})) + \frac{1}{2\delta} \mathbf{R}_{a,\delta}^T \Pi_\delta(\nabla_x v_\delta - \mathbf{R}_{a,\delta}) \\ &\quad + \frac{1}{2\delta} \Pi_\delta(\nabla_x v_\delta - \mathbf{R}_{a,\delta})^T \mathbf{R}_{a,\delta} + \frac{1}{2\delta} (\mathbf{R}_{a,\delta}^T \mathbf{R}_{a,\delta} - \mathbf{I}_3), \end{aligned}$$

and again to estimate (3.7) and Lemma 6.1 we deduce that

$$(6.11) \quad \left\{ \begin{aligned} \frac{1}{\delta} \Pi_\delta((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3) &\rightharpoonup (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-T} \left(S_3 \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} + \mathcal{Z}_1 \mid S_3 \frac{\partial \mathbf{R}}{\partial s_2} \mathbf{n} + \mathcal{Z}_2 \mid \frac{\partial \bar{v}}{\partial S_3} \right)^T \mathbf{R} \\ &\quad + \mathbf{R}^T \left(S_3 \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} + \mathcal{Z}_1 \mid S_3 \frac{\partial \mathbf{R}}{\partial s_2} \mathbf{n} + \mathcal{Z}_2 \mid \frac{\partial \bar{v}}{\partial S_3} \right) (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-1} \\ &\quad \text{weakly in } (L^1(\Omega))^9. \end{aligned} \right.$$

Now remark that

$$(6.12) \quad \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} \cdot \mathbf{R} \mathbf{t}_2 = \frac{\partial \mathbf{R}}{\partial s_2} \mathbf{n} \cdot \mathbf{R} \mathbf{t}_1.$$

Indeed, deriving the relation $\mathbf{R}^T \mathbf{R} = \mathbf{I}_3$ with respect to s_α shows that $\mathbf{R}^T \frac{\partial \mathbf{R}}{\partial s_\alpha} + \frac{\partial \mathbf{R}^T}{\partial s_\alpha} \mathbf{R} = 0$. Hence, there exists an antisymmetric matrix field $\mathbf{A}_\alpha \in L^2(\omega; \mathbb{R}^{3 \times 3})$ such that $\frac{\partial \mathbf{R}}{\partial s_\alpha} = \mathbf{R} \mathbf{A}_\alpha$. Moreover there exists a field \mathbf{a}_α belonging to $(L^2(\omega))^3$ such that

$$\forall x \in \mathbb{R}^3, \quad \mathbf{A}_\alpha x = \mathbf{a}_\alpha \wedge x.$$

Now we derive the equality $\frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R} \mathbf{t}_\alpha$ with respect to s_β and we obtain

$$\frac{\partial^2 \mathcal{V}}{\partial s_\alpha \partial s_\beta} = \frac{\partial \mathbf{R}}{\partial s_\beta} \mathbf{t}_\alpha + \mathbf{R} \frac{\partial \mathbf{t}_\alpha}{\partial s_\beta} = \mathbf{R} \mathbf{A}_\beta \mathbf{t}_\alpha + \mathbf{R} \frac{\partial^2 \phi}{\partial s_\alpha \partial s_\beta}.$$

It implies that $\mathbf{A}_1 \mathbf{t}_2 = \mathbf{A}_2 \mathbf{t}_1$ from which (6.12) follows. Taking into account the definition of the matrix \mathbf{E} , convergence (6.11) and the equality (6.12) show that (6.9) holds true. \square

Remark 6.3. *There exists a constant C such that*

$$\left\| \frac{\partial \mathbf{R}}{\partial s_1} \right\|_{(L^2(\omega))^9} + \left\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \right\|_{(L^2(\omega))^9} \leq C \left(\left\| \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} \cdot \mathbf{R} \mathbf{t}_1 \right\|_{L^2(\omega)} + \left\| \frac{\partial \mathbf{R}}{\partial s_2} \mathbf{n} \cdot \mathbf{R} \mathbf{t}_2 \right\|_{L^2(\omega)} + \left\| \frac{\partial \mathbf{R}}{\partial s_1} \mathbf{n} \cdot \mathbf{R} \mathbf{t}_2 \right\|_{L^2(\omega)} \right).$$

With the same notation as in the proof of Theorem 6.2, we have

$$\left\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \right\|_{(L^2(\omega))^9}^2 = \|\mathbf{A}_\alpha\|_{(L^2(\omega))^9}^2 = 2 \|\mathbf{a}_\alpha\|_{(L^2(\omega))^3}^2.$$

Recalling that $\mathbf{a}_1 \wedge \mathbf{t}_2 = \mathbf{a}_2 \wedge \mathbf{t}_1$, we obtain $\mathbf{a}_\alpha \cdot \mathbf{n} = 0$ and then

$$\left\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \mathbf{n} \right\|_{(L^2(\omega))^3}^2 = \|\mathbf{a}_\alpha \wedge \mathbf{n}\|_{(L^2(\omega))^3}^2 = \|\mathbf{a}_\alpha\|_{(L^2(\omega))^3}^2 = \frac{1}{2} \left\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \right\|_{(L^2(\omega))^9}^2.$$

Remark. It is well known that the constraint $\frac{\partial \mathcal{V}}{\partial s_1} = \mathbf{Rt}_1$ and $\frac{\partial \mathcal{V}}{\partial s_2} = \mathbf{Rt}_2$ together the boundary conditions are strong limitations on the possible deformation for the limit 2d shell. Actually for a plate or as soon as S is a developable surface, the configuration after deformation must also be a developable surface. In the general case, it is an open problem to know if the set \mathbf{V}_{nin} contains other deformations than identity mapping or very special isometries (as for example symetries).

7. Asymptotic behavior of the Green-St Venant's strain tensor in the case $\kappa > 2$.

Let us consider a sequence of deformations v_δ of $(H^1(\mathcal{Q}_\delta))^3$ such that

$$(7.1) \quad \|\text{dist}(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{\kappa-1/2}, \quad \text{with } \kappa > 2.$$

We use the decomposition (3.16) of a deformation and the estimates (3.18) of Theorem 3.4. These estimates and the boundary condition (Lemma 4.1) lead to the following convergences:

$$(7.2) \quad \begin{cases} \mathbf{R}_\delta \longrightarrow \mathbf{I}_3 & \text{strongly in } (H^1(\omega))^9, \\ \Pi_\delta v_\delta \longrightarrow \phi & \text{strongly in } (H^1(\Omega))^3, \\ \Pi_\delta(\nabla_x v_\delta) \longrightarrow \mathbf{I}_3 & \text{strongly in } (L^2(\Omega))^9. \end{cases}$$

In view of these convergences, we now study the asymptotic behavior of the sequence of displacements belonging to $(H^1(\mathcal{Q}_\delta))^3$

$$u_\delta(x) = v_\delta(x) - x.$$

Due to the decomposition (3.16) we write

$$(7.3) \quad u_\delta(s) = \mathcal{U}_\delta(s_1, s_2) + s_3(\mathbf{R}_\delta - \mathbf{I}_3)(s_1, s_2)\mathbf{n}(s_1, s_2) + \bar{v}_\delta(s), \quad s \in \Omega_\delta,$$

where $\mathcal{U}_\delta(s_1, s_2) = \mathcal{V}_\delta(s_1, s_2) - \phi(s_1, s_2)$.

Thanks to the estimates (3.18) we obtain the following lemma:

Lemma 7.1. *There exists a subsequence still indexed by δ such that*

$$(7.4) \quad \begin{cases} \frac{1}{\delta^{\kappa-2}}(\mathbf{R}_\delta - \mathbf{I}_3) \rightharpoonup \mathbf{A} & \text{weakly in } (H^1(\omega))^9 \text{ and strongly in } (L^2(\omega))^9, \\ \frac{1}{\delta^{\kappa-2}}\mathcal{U}_\delta \longrightarrow \mathcal{U} & \text{strongly in } (H^1(\omega))^3, \\ \frac{1}{\delta^\kappa}\Pi_\delta\bar{v}_\delta \rightharpoonup \bar{v} & \text{weakly in } (L^2(\omega; H^1(-1, 1)))^3, \\ \frac{1}{\delta^{\kappa-1}}\left(\frac{\partial \mathcal{U}_\delta}{\partial s_\alpha} - (\mathbf{R}_\delta - \mathbf{I}_3)\mathbf{t}_\alpha\right) \rightharpoonup \mathcal{Z}_\alpha & \text{weakly in } (L^2(\omega))^3, \end{cases}$$

and

$$(7.5) \quad \begin{cases} \frac{1}{\delta^{\kappa-2}}\Pi_\delta u_\delta \longrightarrow \mathcal{U} & \text{strongly in } (H^1(\Omega))^3, \\ \frac{1}{\delta^{\kappa-2}}\Pi_\delta(\nabla_x u_\delta) \longrightarrow \mathbf{A} & \text{strongly in } (L^2(\Omega))^9 \end{cases}$$

where $\mathbf{A} \in (H^1(\omega))^9$, $\mathcal{U} \in (H^1(\omega))^3$, $\bar{v} \in (L^2(\omega; H^1(-1, 1)))^3$ and $\mathcal{Z}_\alpha \in (L^2(\omega))^3$. Moreover, we have

$$(7.6) \quad \mathcal{U} = 0, \quad \mathbf{A} = 0 \quad \text{on } \gamma_0, \quad \text{and} \quad \frac{\partial \mathcal{U}}{\partial s_\alpha} = \mathbf{A} \mathbf{t}_\alpha,$$

and $\mathcal{U} \in (H^2(\omega))^3$.

We now show that the matrix \mathbf{A} is actually an antisymmetric matrix. Using the first convergence in (7.2) and the first convergence in (7.4) we get

$$\frac{1}{\delta^{\kappa-2}} \mathbf{R}_\delta^T (\mathbf{R}_\delta - \mathbf{I}_3) \rightharpoonup \mathbf{A} \quad \text{weakly in } (H^1(\omega))^{3 \times 3}.$$

The matrix \mathbf{R}_δ belongs to $SO(3)$, hence $\mathbf{R}_\delta^T (\mathbf{R}_\delta - \mathbf{I}_3) = \mathbf{I}_3 - \mathbf{R}_\delta^T$. It follows that the matrix \mathbf{A} is an antisymmetric matrix. There exists a field $\mathcal{R} \in (H^1(\omega))^3$ (with $\mathcal{R} = 0$ on γ_0 due to (7.6)) such that for all $x \in \mathbb{R}^3$ we have

$$(7.7) \quad \mathbf{A}x = \mathcal{R} \wedge x.$$

From (7.6) and the above equality we obtain

$$(7.8) \quad \frac{\partial \mathcal{U}}{\partial s_\alpha} = \mathcal{R} \wedge \mathbf{t}_\alpha.$$

7.1. Inextensional and extensional displacements of the shell.

Now we define the inextensional displacements and extensional displacements sets of the mid-surface of the shell. We set

$$H_{\gamma_0}^1 = \left\{ v \in H^1(\omega) \mid v = 0 \quad \text{on } \gamma_0 \right\}.$$

We equip $(H_{\gamma_0}^1(\omega))^3$ with the following inner product:

$$\forall (U, V) \in (H_{\gamma_0}^1(\omega))^3 \times (H_{\gamma_0}^1(\omega))^3, \quad \langle U, V \rangle = \int_\omega \left[\frac{\partial U}{\partial s_1} \cdot \frac{\partial V}{\partial s_1} + \frac{\partial U}{\partial s_2} \cdot \frac{\partial V}{\partial s_2} \right].$$

For any $U \in (H^1(\omega))^3$ we set

$$e_{11}(U) = \frac{\partial U}{\partial s_1} \cdot \mathbf{t}_1, \quad e_{12}(U) = \frac{1}{2} \left\{ \frac{\partial U}{\partial s_1} \cdot \mathbf{t}_2 + \frac{\partial U}{\partial s_2} \cdot \mathbf{t}_1 \right\}, \quad e_{22}(U) = \frac{\partial U}{\partial s_2} \cdot \mathbf{t}_2.$$

The spaces of inextensional and extensional displacements are respectively defined by

$$D_{In} = \left\{ U \in (H_{\gamma_0}^1(\omega))^3 \mid e_{11}(U) = e_{12}(U) = e_{22}(U) = 0 \right\} \quad D_{Ex} = (D_{In})^\perp,$$

where $(D_{In})^\perp$ is the orthogonal of D_{In} in the space $(H_{\gamma_0}^1(\omega))^3$. For all $U \in D_{In}$ there exists a unique field $\mathcal{R} \in (L^2(\omega))^3$ such that

$$\frac{\partial U}{\partial s_1} = \mathcal{R} \wedge \mathbf{t}_1, \quad \frac{\partial U}{\partial s_2} = \mathcal{R} \wedge \mathbf{t}_2,$$

and $\{\mathcal{R} \mid U \in D_{In}\}$ is a closed subspace of $(L^2(\omega))^3$.

We equip D_{Ex} with the norm

$$(7.9) \quad \forall U \in D_{Ex}, \quad \|U\|_{Ex} = \sqrt{\|e_{11}(U)\|_{L^2(\omega)}^2 + \|e_{12}(U)\|_{L^2(\omega)}^2 + \|e_{22}(U)\|_{L^2(\omega)}^2}.$$

Generally, D_{Ex} is not a Hilbert space. We denote by \mathcal{D}_{Ex} a Hilbert space in which D_{Ex} is a dense subspace. In the general case an element belonging to \mathcal{D}_{Ex} is neither a function nor a distribution. If a sequence $\{U_n\}_{n \in \mathbb{N}}$ converges to $U \in \mathcal{D}_{Ex}$, the sequences $\{e_{11}(U_n)\}_{n \in \mathbb{N}}$, $\{e_{12}(U_n)\}_{n \in \mathbb{N}}$ and $\{e_{22}(U_n)\}_{n \in \mathbb{N}}$ strongly converge in $L^2(\omega)$ and their limits depend only on U . That is the reason why we will denote these limits $e_{11}(U)$, $e_{12}(U)$ and $e_{22}(U)$. But notice that we use here improper notations because the element U has not always derivatives in the distribution sense.

If the shell is a plate, we have $\phi(s_1, s_2) = (s_1, s_2)$ hence $\mathbf{t}_1 = \mathbf{e}_1$, $\mathbf{t}_2 = \mathbf{e}_2$ and $\mathbf{n} = \mathbf{e}_3$. In this case the extensional displacements are the membrane displacements and the inextensional displacements have the form $U_3 \mathbf{e}_3$ where U_3 is the bending. We have $D_{Ex} = \mathcal{D}_{Ex} = (H_{\gamma_0}^1(\omega))^2$ and due to Korn's inequality in $(H_{\gamma_0}^1(\omega))^2$, the norm $\|\cdot\|_{Ex}$ is equivalent to the H^1 norm on D_{Ex} .

7.2. Limit of the Green-St Venant's strain tensor for $\kappa > 2$.

We consider the sequence v_δ introduced in Section 7 which satisfies $\|dist(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{\kappa-1/2}$ and the associated displacement $u_\delta = v_\delta - I_d$.

We write the displacement \mathcal{U}_δ of the mid-surface as the sum of an inextensional displacement and an extensional one as in Section 6

$$(7.10) \quad \mathcal{U}_\delta = U_{I,\delta} + U_{E,\delta} \quad U_{I,\delta} \in D_{In}, \quad U_{E,\delta} \in D_{Ex}.$$

We first give the estimates on $U_{I,\delta}$ and $U_{E,\delta}$ in the following lemma.

Lemma 7.2. *We have*

$$(7.11) \quad \|U_{I,\delta}\|_{(H^1(\omega))^3} \leq C\delta^{\kappa-2}, \quad \|U_{E,\delta}\|_{(H^1(\omega))^3} \leq C\delta^{\kappa-2}, \quad \|U_{E,\delta}\|_{Ex} \leq C\delta^{\kappa-1}(1 + \delta^{\kappa-3}).$$

The constants do not depend on δ . Then, we can choose the subsequence in Lemma 7.1 such that

$$(7.12) \quad \begin{cases} \frac{1}{\delta^{\kappa-2}} U_{I,\delta} \rightharpoonup \mathcal{U} & \text{weakly in } (H^1(\omega))^3 \\ \frac{1}{\delta^{\kappa-2}} U_{E,\delta} \rightharpoonup 0 & \text{weakly in } (H^1(\omega))^3 \end{cases}$$

and moreover

$$(7.13) \quad \begin{cases} \text{if } 2 < \kappa < 3, & \frac{1}{\delta^{2\kappa-4}} U_{E,\delta} \rightharpoonup U_E & \text{weakly in } \mathcal{D}_{Ex} \\ \text{if } \kappa \geq 3, & \frac{1}{\delta^{\kappa-1}} U_{E,\delta} \rightharpoonup U_E & \text{weakly in } \mathcal{D}_{Ex} \end{cases}$$

The convergences in (7.13) are equivalent to the weak convergences in $L^2(\omega)$ of $e_{11}(U_{E,\delta})$, $e_{12}(U_{E,\delta})$ and $e_{22}(U_{E,\delta})$.

Proof. The two first estimates of (7.11) follow from (7.4) and from the orthogonality of $U_{I,\delta}$ and $U_{E,\delta}$. Now notice that

$$(7.14) \quad e_{\alpha\alpha}(\mathcal{U}_\delta) = e_{\alpha\alpha}(U_{E,\delta}) \quad \text{and} \quad e_{\alpha\beta}(\mathcal{U}_\delta) = e_{12}(U_{E,\delta}).$$

We denote by \mathbf{A}_δ the antisymmetric part of \mathbf{R}_δ . Notice that $\|\mathbf{R}_\delta + \mathbf{R}_\delta^T - 2\mathbf{I}_3\|_{(L^2(\omega))^{3 \times 3}} \leq \|\mathbf{R}_\delta - \mathbf{I}_3\|_{(L^4(\omega))^{3 \times 3}}^2 \leq C\|\nabla \mathbf{R}_\delta\|_{(L^2(\omega))^{18}}^2$. Then, from estimates (3.18) and (7.1) we deduce that

$$\left\| \frac{\partial \mathcal{U}_\delta}{\partial s_\alpha} - \mathbf{A}_\delta \mathbf{t}_\alpha \right\|_{(L^2(\omega))^3} \leq C\delta^{\kappa-1} + C\delta^{2\kappa-4}.$$

Then by definition of the norm $\|\cdot\|_{E_x}$ we get the third estimate in (7.11) and then the convergences in (7.13). \square

The following theorem gives the expression of the limit of the Green-St Venant's tensor.

Theorem 7.3. *Let us set*

$$(7.15) \quad \mathcal{Z}_{\alpha\beta} = \frac{1}{2} \{ \mathcal{Z}_\alpha \cdot \mathbf{t}_\beta + \mathcal{Z}_\beta \cdot \mathbf{t}_\alpha \}, \quad \bar{u} = \bar{v} + \frac{S_3}{2} (\mathcal{Z}_1 \cdot \mathbf{n}) \mathbf{t}'_1 + \frac{S_3}{2} (\mathcal{Z}_2 \cdot \mathbf{n}) \mathbf{t}'_2$$

where $(\mathbf{t}'_1, \mathbf{t}'_2)$ is the contravariant basis of $(\mathbf{t}_1, \mathbf{t}_2)$.

For a subsequence we have

$$(7.16) \quad \frac{1}{2\delta^{\kappa-1}} \Pi_\delta ((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3) \rightharpoonup (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-T} \mathbf{E} (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-1} \quad \text{weakly in } (L^1(\Omega))^9,$$

where the symmetric matrix \mathbf{E} is defined by

$$(7.17) \quad \mathbf{E} = \begin{pmatrix} S_3 \left[\frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n} \right] \cdot \mathbf{t}_1 + \mathcal{Z}_{11} & S_3 \left[\frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n} \right] \cdot \mathbf{t}_2 + \mathcal{Z}_{12} & \frac{1}{2} \frac{\partial \bar{u}}{\partial S_3} \cdot \mathbf{t}_1 \\ * & S_3 \left[\frac{\partial \mathcal{R}}{\partial s_2} \wedge \mathbf{n} \right] \cdot \mathbf{t}_2 + \mathcal{Z}_{22} & \frac{1}{2} \frac{\partial \bar{u}}{\partial S_3} \cdot \mathbf{t}_2 \\ * & * & \frac{\partial \bar{u}}{\partial S_3} \cdot \mathbf{n} \end{pmatrix}$$

Moreover, if $2 < \kappa < 3$ then we have

$$(7.18) \quad e_{\alpha\beta}(U_E) + \frac{1}{2} \frac{\partial \mathcal{U}}{\partial s_\alpha} \cdot \frac{\partial \mathcal{U}}{\partial s_\beta} = 0$$

and if $\kappa \geq 3$ then we have

$$(7.19) \quad \mathcal{Z}_{\alpha\beta} = \begin{cases} e_{\alpha\beta}(U_E) + \frac{1}{2} \frac{\partial \mathcal{U}}{\partial s_\alpha} \cdot \frac{\partial \mathcal{U}}{\partial s_\beta} & \text{if } \kappa = 3 \\ e_{\alpha\beta}(U_E) & \text{if } \kappa > 3 \end{cases}$$

Proof. First we have

$$\frac{\partial v}{\partial s_1} = \nabla_x v (\mathbf{t}_1 + s_3 \frac{\partial \mathbf{n}}{\partial s_1}), \quad \frac{\partial v}{\partial s_2} = \nabla_x v (\mathbf{t}_2 + s_3 \frac{\partial \mathbf{n}}{\partial s_2}), \quad \frac{\partial v}{\partial s_3} = \nabla_x v \mathbf{n}.$$

As a consequence of the above formulae, of (3.3) and of the convergences in Lemma 5.1, we have

$$(7.20) \quad \begin{cases} \frac{1}{\delta^{\kappa-1}} (\Pi_\delta (\nabla_x v_\delta) - \mathbf{R}_\delta) \mathbf{t}_\alpha \rightharpoonup S_3 \frac{\partial \mathcal{R}}{\partial s_\alpha} \wedge \mathbf{n} + \mathcal{Z}_\alpha & \text{weakly in } (L^2(\Omega))^3, \\ \frac{1}{\delta^{\kappa-1}} (\Pi_\delta (\nabla_x v_\delta) - \mathbf{R}_\delta) \mathbf{n} \rightharpoonup \frac{\partial \bar{v}}{\partial S_3} & \text{weakly in } (L^2(\Omega))^3. \end{cases}$$

Then thanks to the identity

$$\frac{1}{2\delta^{\kappa-1}}\Pi_\delta((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3) = \frac{1}{2\delta^{\kappa-1}}\Pi_\delta((\nabla_x v_\delta - \mathbf{R}_\delta)^T \nabla_x v_\delta) + \frac{1}{2\delta^{\kappa-1}}\mathbf{R}_\delta^T \Pi_\delta(\nabla_x v_\delta - \mathbf{R}_\delta)$$

and to convergences (7.2) and (7.20), we deduce that

$$\frac{1}{2\delta^{\kappa-1}}\Pi_\delta((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3) \rightharpoonup (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-T} \mathbf{E} (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^{-1} \quad \text{weakly in } (L^1(\Omega))^9,$$

where the symmetric matrix \mathbf{E} is equal to

$$\begin{aligned} \mathbf{E} &= \frac{1}{2} \left(S_3 \frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n} + \mathcal{Z}_1 | S_3 \frac{\partial \mathcal{R}}{\partial s_2} \wedge \mathbf{n} + \mathcal{Z}_2 | \frac{\partial \bar{v}}{\partial S_3} \right)^T (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n}) \\ &\quad + \frac{1}{2} (\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{n})^T \left(S_3 \frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n} + \mathcal{Z}_1 | S_3 \frac{\partial \mathcal{R}}{\partial s_2} \wedge \mathbf{n} + \mathcal{Z}_2 | \frac{\partial \bar{v}}{\partial S_3} \right). \end{aligned}$$

Deriving the equality (7.8) with respect to s_1 and s_2 gives

$$\frac{\partial^2 \mathcal{U}}{\partial s_1 \partial s_2} = \frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{t}_2 + \mathcal{R} \wedge \frac{\partial^2 \phi}{\partial s_1 \partial s_2} = \frac{\partial \mathcal{R}}{\partial s_2} \wedge \mathbf{t}_1 + \mathcal{R} \wedge \frac{\partial^2 \phi}{\partial s_1 \partial s_2},$$

hence

$$\left[\frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n} \right] \mathbf{t}_2 = \left[\frac{\partial \mathcal{R}}{\partial s_2} \wedge \mathbf{n} \right] \mathbf{t}_1.$$

Introducing $\mathcal{Z}_{\alpha\beta}$ and \bar{u} we obtain the expression (7.17) for \mathbf{E} .

Below we show (7.18) and (7.19). Due to (7.4), we first have

$$(7.21) \quad \frac{1}{\delta^{\kappa-1}} \left[e_{\alpha\beta}(\mathcal{U}_\delta) - \frac{1}{2} (\mathbf{R}_\delta + \mathbf{R}_\delta^T - 2\mathbf{I}_3) \mathbf{t}_\alpha \cdot \mathbf{t}_\beta \right] \rightharpoonup \mathcal{Z}_{\alpha\beta} \quad \text{weakly in } L^2(\omega).$$

Recalling the identity $(\mathbf{R}_\delta + \mathbf{R}_\delta^T - 2\mathbf{I}_3) \mathbf{t}_\alpha \cdot \mathbf{t}_\beta = -(\mathbf{R}_\delta - \mathbf{I}_3) \mathbf{t}_\alpha \cdot (\mathbf{R}_\delta - \mathbf{I}_3) \mathbf{t}_\beta$ and using the first convergence in (7.4), we deduce that

$$(7.22) \quad \frac{1}{\delta^{2\kappa-4}} (\mathbf{R}_\delta + \mathbf{R}_\delta^T - 2\mathbf{I}_3) \mathbf{t}_\alpha \cdot \mathbf{t}_\beta \longrightarrow -\frac{1}{2} \mathbf{A} \mathbf{t}_\alpha \cdot \mathbf{A} \mathbf{t}_\beta \quad \text{strongly in } L^2(\omega).$$

In the case $2 < \kappa < 3$ we have

$$\begin{aligned} \frac{1}{\delta^{2\kappa-4}} \left[e_{\alpha\beta}(\mathcal{U}_\delta) - \frac{1}{2} (\mathbf{R}_\delta + \mathbf{R}_\delta^T - 2\mathbf{I}_3) \mathbf{t}_\alpha \cdot \mathbf{t}_\beta \right] &\longrightarrow 0 \quad \text{strongly in } L^2(\omega) \\ \frac{1}{\delta^{2\kappa-4}} e_{\alpha\beta}(\mathcal{U}_\delta) &\rightharpoonup e_{\alpha\beta}(U_E) \quad \text{weakly in } L^2(\omega). \end{aligned}$$

Thanks to (7.6) and (7.22) we obtain (7.18).

In the case $\kappa \geq 3$ then convergences (7.13), (7.21) and (7.22) permit to obtain the expression of $\mathcal{Z}_{\alpha\beta}$ in terms of $e_{\alpha\beta}(U_E)$ and \mathcal{U} . \square

Appendix A

In this section the vector space $\mathbb{R}^{n \times n}$ of all matrices with n rows and n is equipped with the Frobenius norm. We set

$$Y =]0, 1[^2, \quad B_3 = \left\{ \mathbf{x} \in \mathbb{R}^3 ; \|\mathbf{x}\|_2 \leq 1 \right\}, \quad S_3 = \left\{ \mathbf{x} \in \mathbb{R}^3 ; \|\mathbf{x}\|_2 = 1 \right\}.$$

We denote $\mathcal{R}_{\mathbf{a},\theta}$ the rotation with axis directed by the vector $\mathbf{a} \in S_3$ and with angle of rotation about this axis $\theta \in \mathbb{R}$,

$$(A.1) \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad \mathcal{R}_{\mathbf{a},\theta}(\mathbf{x}) = \cos(\theta)\mathbf{x} + (1 - \cos(\theta)) \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a} + \sin(\theta) \mathbf{a} \wedge \mathbf{x}.$$

Let \mathbf{R}_0 and \mathbf{R}_1 be two matrices in $SO(3)$. Matrix \mathbf{R}_0 represent the rotation $\mathcal{R}_{\mathbf{a}_0,\theta_0}$ and matrix \mathbf{R}_1 represent the rotation $\mathcal{R}_{\mathbf{a}_1,\theta_1}$. The linear transformation in \mathbb{R}^3

$$x \longmapsto 2(\sin(\theta_1)\mathbf{a}_1 - \sin(\theta_0)\mathbf{a}_0) \wedge x$$

has for matrix $\mathbf{R}_1 - \mathbf{R}_0 - (\mathbf{R}_1 - \mathbf{R}_0)^T$ and we have

$$\|\sin(\theta_1)\mathbf{a}_1 - \sin(\theta_0)\mathbf{a}_0\|_2 = \frac{1}{2\sqrt{2}}\|\mathbf{R}_1 - \mathbf{R}_0 - (\mathbf{R}_1 - \mathbf{R}_0)^T\| \leq \frac{1}{\sqrt{2}}\|\mathbf{R}_1 - \mathbf{R}_0\|.$$

To any matrix \mathbf{R} in $SO(3)$ we associate the vector $\mathbf{b} = \sin(\theta)\mathbf{a}$ where \mathbf{R} is the matrix of the rotation $\mathcal{R}_{\mathbf{a},\theta}$. This map is continuous from $SO(3)$ into B_3 and from the above inequality, we obtain

$$\|\mathbf{b}\|_2 \leq \frac{1}{\sqrt{2}}\|\mathbf{R} - \mathbf{I}_3\|.$$

If $\cos(\theta) \neq -1$, using the vector \mathbf{b} we can write the rotation $\mathcal{R}_{\mathbf{a},\theta}$ as

$$(A.2) \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad \mathcal{R}_{\mathbf{a},\theta}(\mathbf{x}) = \cos(\theta)\mathbf{x} + \frac{1}{1 + \cos(\theta)} \langle \mathbf{x}, \mathbf{b} \rangle \mathbf{b} + \mathbf{b} \wedge \mathbf{x}.$$

Let \mathbf{R}_0 and \mathbf{R}_1 be two matrices in $SO(3)$ such that

$$\|\mathbf{R}_0 - \mathbf{R}_1\| < 2\sqrt{2}.$$

Now we define a path \mathbf{f} from \mathbf{R}_0 to \mathbf{R}_1 :

- if $\mathbf{R}_1 = \mathbf{R}_0$ we choose the constant function $\mathbf{f}(t) = \mathbf{R}_0$, $t \in [0, 1]$,
- if $\mathbf{R}_1 \neq \mathbf{R}_0$, we set $\mathbf{R}_2 = \mathbf{R}_0^{-1}\mathbf{R}_1$, there exists a unique pair $(\mathbf{a}_2, \theta_2) \in S_3 \times]0, \pi[$ such that the matrix \mathbf{R}_2 represent the rotation with axis directed by the vector \mathbf{a}_2 and with the angle θ_2 . We consider the rotations field $\mathcal{R}_{\mathbf{a}(t),\theta(t)}$ given by formula (A.1) where

$$\mathbf{a}(t) = \mathbf{a}_2, \quad \theta(t) = t\theta_2, \quad t \in [0, 1]$$

and we define $\mathbf{f}(t)$ as the matrix of the rotation $\mathcal{R}_0 \circ \mathcal{R}_{\mathbf{a}(t),\theta(t)}$ where \mathcal{R}_0 is the rotation with matrix \mathbf{R}_0 .

Lemma A.1. *The path \mathbf{f} belongs to $W^{1,\infty}(0, 1; SO(3))$ and satisfies*

$$(A.3) \quad \begin{cases} \mathbf{f}(0) = \mathbf{R}_0, & \mathbf{f}(1) = \mathbf{R}_1, & \left\| \frac{d\mathbf{f}}{dt} \right\|_{(L^\infty(0,1))^9} \leq \frac{\pi}{2} \|\mathbf{R}_1 - \mathbf{R}_0\|, \\ \|\mathbf{R}_0 - \mathbf{f}(t)\| \leq \|\mathbf{R}_0 - \mathbf{R}_1\|. \end{cases}$$

Proof One has

$$\left\| \frac{d\mathbf{f}}{dt} \right\|_{(L^\infty(0,1))^9} = \sqrt{2}\theta_2 \leq \frac{\pi}{2} \|\mathbf{R}_2 - \mathbf{I}_3\| = \frac{\pi}{2} \|\mathbf{R}_1 - \mathbf{R}_0\|.$$

Moreover

$$\| \|\mathbf{R}_0 - \mathbf{f}(t)\| \| = \| \|I_3 - \mathbf{R}_0^{-1}\mathbf{f}(t)\| \| = 2\sqrt{2} \sin\left(\frac{\theta_2 t}{2}\right) \leq 2\sqrt{2} \sin\left(\frac{\theta_2}{2}\right) = \| \|I_3 - \mathbf{R}_2\| \| = \| \|\mathbf{R}_0 - \mathbf{R}_1\| \|.$$

□

Lemma A.2. *Let \mathbf{R}_{00} , \mathbf{R}_{01} , \mathbf{R}_{10} and \mathbf{R}_{11} be four matrices belonging to $SO(3)$ and satisfying*

$$(A.4) \quad \| \|\mathbf{R}_{10} - \mathbf{R}_{00}\| \| \leq \frac{1}{2}, \quad \| \|\mathbf{R}_{01} - \mathbf{R}_{00}\| \| \leq \frac{1}{2}, \quad \| \|\mathbf{R}_{11} - \mathbf{R}_{01}\| \| \leq \frac{1}{2}, \quad \| \|\mathbf{R}_{11} - \mathbf{R}_{10}\| \| \leq \frac{1}{2}.$$

There exists a function $\mathbf{R} \in W^{1,\infty}(Y; SO(3))$ such that

$$(A.5) \quad \begin{cases} \mathbf{R}(0,0) = \mathbf{R}_{00}, & \mathbf{R}(0,1) = \mathbf{R}_{01}, & \mathbf{R}(1,0) = \mathbf{R}_{10}, & \mathbf{R}(1,1) = \mathbf{R}_{11}, \\ \| \|\nabla \mathbf{R}\|_{(L^\infty(Y))^9} \| \leq C \{ \| \|\mathbf{R}_{10} - \mathbf{R}_{00}\| \| + \| \|\mathbf{R}_{01} - \mathbf{R}_{00}\| \| + \| \|\mathbf{R}_{11} - \mathbf{R}_{01}\| \| + \| \|\mathbf{R}_{11} - \mathbf{R}_{10}\| \| \}. \end{cases}$$

and where the functions $x_1 \rightarrow \mathbf{R}(x_1, 0)$, $x_1 \rightarrow \mathbf{R}(x_1, 1)$, $x_2 \rightarrow \mathbf{R}(0, x_2)$ and $x_2 \rightarrow \mathbf{R}(1, x_2)$ are paths given by Lemma A.1.

Proof. We denote

- $\mathbf{f}_{00,01}$ the path from \mathbf{R}_{00} to \mathbf{R}_{01} ,
- $\mathbf{f}_{01,11}$ the path from \mathbf{R}_{01} to \mathbf{R}_{11} ,
- $\mathbf{f}_{00,10}$ the path from \mathbf{R}_{00} to \mathbf{R}_{10} and
- $\mathbf{f}_{01,11}$ the path from \mathbf{R}_{01} to \mathbf{R}_{11} given by Lemma A.

From Lemma A.1, we have

$$\forall t \in [0, 1], \quad \begin{cases} \| \|\mathbf{f}_{00,01}(t) - \mathbf{R}_{00}\| \| \leq 1, & \| \|\mathbf{f}_{01,11}(t) - \mathbf{R}_{00}\| \| \leq 1, \\ \| \|\mathbf{f}_{00,10}(t) - \mathbf{R}_{00}\| \| \leq 1, & \| \|\mathbf{f}_{01,11}(t) - \mathbf{R}_{00}\| \| \leq 1. \end{cases}$$

For any $t \in [0, 1]$,

- to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{00,01}(t)$ we associate the vector $\mathbf{b}_{00,01}(t)$,
- to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{01,11}(t)$ we associate the vector $\mathbf{b}_{01,11}(t)$,
- to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{00,10}(t)$ we associate the vector $\mathbf{b}_{00,10}(t)$ and
- to matrix $\mathbf{R}_{00}^{-1}\mathbf{f}_{01,11}(t)$ we associate the vector $\mathbf{b}_{01,11}(t)$.

Let \mathbf{b} be the vectors field defined by

$$\mathbf{b}(x_1, x_2) = \begin{cases} \mathbf{b}_{00,10}(0) (= \mathbf{b}_{00,01}(0)) & \text{if } (x_1, x_2) = (0, 0), \\ \frac{x_1}{x_1 + x_2} \mathbf{b}_{00,10}(x_2) + \frac{x_2}{x_1 + x_2} \mathbf{b}_{00,01}(x_1) & \text{if } 0 < x_1 + x_2 \leq 1 \\ \frac{1 - x_1}{2 - x_1 - x_2} \mathbf{b}_{10,11}(x_2) + \frac{1 - x_2}{2 - x_1 - x_2} \mathbf{b}_{01,11}(x_1) & \text{if } 1 \leq x_1 + x_2 < 2 \\ \mathbf{b}_{01,11}(1) (= \mathbf{b}_{10,11}(1)) & \text{if } (x_1, x_2) = (1, 1). \end{cases}$$

This function belongs to $(W^{1,\infty}(Y))^3$ and satisfies

$$\forall (x_1, x_2) \in \bar{Y}, \quad \| \|\mathbf{b}(x_1, x_2)\|_2 \| \leq \frac{1}{\sqrt{2}}.$$

Now we introduce the rotations field $\mathcal{R}(x_1, x_2)$ given by formula (A.2) where $\mathbf{b}(x_1, x_2)$ is defined above and where

$$\theta(x_1, x_2) = \arccos \sqrt{1 - \langle \mathbf{b}(x_1, x_2), \mathbf{b}(x_1, x_2) \rangle}, \quad (x_1, x_2) \in \bar{Y}.$$

Let $\mathbf{R}(x_1, x_2)$ be the matrix of the rotation $\mathcal{R}_{00} \circ \mathcal{R}(x_1, x_2)$ where \mathcal{R}_{00} is the rotation with matrix \mathbf{R}_{00} . It is easy to check that \mathbf{R} satisfies the conditions (A.5). \square

Corollary of Lemma A.2. Let \mathbf{R}_a be the Q_1 interpolate of the matrices \mathbf{R}_{00} , \mathbf{R}_{01} , \mathbf{R}_{10} and \mathbf{R}_{11} . There exists a strictly positive constant C such that

$$\|\mathbf{R} - \mathbf{R}_a\|_{(L^2(\mathcal{Y}))^9} \leq C\{\|\mathbf{R}_{10} - \mathbf{R}_{00}\| + \|\mathbf{R}_{11} - \mathbf{R}_{01}\| + \|\mathbf{R}_{01} - \mathbf{R}_{00}\| + \|\mathbf{R}_{11} - \mathbf{R}_{10}\|\}.$$

\square

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