

REGULARIZATION OF Γ_1 -STRUCTURES IN DIMENSION 3

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ABSTRACT. For Γ_1 -structures on 3-manifolds, we give a very simple proof of Thurston's regularization theorem, first proved in [13], without using Mather's homology equivalence. Moreover, in the co-orientable case, the resulting foliation can be chosen of a precise kind, namely an "open book foliation modified by suspension". There is also a model in the non co-orientable case.

1. INTRODUCTION

A Γ_1 -structure ξ , in the sense of A. Haefliger, on a manifold M is given by a line bundle $\nu = (E \rightarrow M)$, called the *normal bundle* to ξ , and a germ of codimension-one foliation \mathcal{F} along the zero section, which is required to be transverse to the fibers (see [8]). To fix ideas, consider the co-orientable case, that is, the normal bundle is trivial: $E \cong M \times \mathbb{R}$; for the general case see section 7. The Γ_1 -structure ξ is said to be *regular* when the foliation \mathcal{F} is transverse to the zero section, in which case the pullback of \mathcal{F} to M is a genuine foliation on M . A homotopy of ξ is defined as a Γ_1 -structure on $M \times [0, 1]$ inducing ξ on $M \times \{0\}$. A *regularization theorem* should claim that any Γ_1 -structure is homotopic to a regular one. It is not true in general. An obvious necessary condition is that ν must embed into the tangent bundle τM . When ν is trivial and $\dim M = 3$ this condition is fulfilled.

The C^∞ category is understood in the sequel, unless otherwise specified. In particular M is C^∞ . One calls ξ a Γ_1^r -structure ($r \geq 1$) if it is tangentially C^∞ and transversely C^r , that is, the foliation charts are C^r in the direction transverse to the leaves. We will prove the following theorem.

Theorem 1.1. *If M is a closed 3-manifold and ξ a Γ_1^r -structure, $r \geq 1$, whose normal bundle is trivial, then ξ is homotopic to a regular Γ_1^r -structure.*

Moreover, the resulting foliation of M may have its tangent plane field in a prescribed homotopy class (see proposition 6.1).

This theorem is a particular case of a general regularization theorem due to W. Thurston (see [13]). Thurston's proof was based on the deep result due to J. Mather [9], [10]: the homology equivalence between the classifying space of the group $\text{Diff}_c(\mathbb{R})$ endowed with the discrete topology and the loop space $\Omega B(\Gamma_1)_+$. We present a proof of this regularization theorem which does not need this result. A regularization theorem in all dimensions, still avoiding any difficult result, is provided in [12]. But there are reasons for considering the dimension 3 separately.

2000 *Mathematics Subject Classification.* 57R30.

Key words and phrases. Foliations, Haefliger's Γ -structures, open book.

FL supported by ANR Floer Power.

Our proof provides models realizing each homotopy class of Γ_1 -structure. The models are based on the notion of *open book decomposition*. Recall that such a structure on M consists of a link B in M , called the *binding*, and a fibration $p : M \setminus B \rightarrow S^1$ such that, for every $\theta \in S^1$, $p^{-1}(\theta)$ is the interior of an embedded surface, called the *page* P_θ , whose boundary is the binding. The existence of open book decomposition could be proved by J. Alexander when M is orientable, as a consequence of [1] (every orientable closed 3-manifold is a branched cover of the 3-sphere) and [2] (every link can be braided); but he was ignoring this concept which was introduced by H. Winkelnkemper in 1973 [16]. Henceforth, we refer to the more flexible construction by E. Giroux, which includes the non-orientable case (see section 3). An open book gives rise to a foliation \mathcal{O} constructed as follows. The pages endow B with a normal framing. So a tubular neighborhood T of B is trivialized: $T \cong B \times D^2$. Out of T the leaves are the pages modified by spiraling around T ; the boundary of T is a union of compact leaves; and the interior of T is foliated by a Reeb component, or a generalized Reeb component in the sense of Wood [17]. For technical reasons in the homotopy argument of section 4, the Reeb components of \mathcal{O} , instead of being usual Reeb components, will be *thick Reeb components* in which a neighborhood of the boundary is foliated by toric compact leaves. We call such a foliation an *open book foliation*.

The latter can be modified by inserting a so called *suspension foliation*. Precisely, let Σ be a compact sub-surface of some leaf of \mathcal{O} out of T and $\Sigma \times [-1, +1]$ be a foliated neighborhood of it (each $\Sigma \times \{t\}$ being contained in a leaf of \mathcal{O}). Let $\varphi : \pi_1(\Sigma) \rightarrow \text{Diff}_c([-1, +1])$ be some representation into the group of compactly supported diffeomorphisms; φ is assumed to be trivial on the peripheral elements. It allows us to construct a *suspension foliation* \mathcal{F}_φ on $\Sigma \times [-1, +1]$, whose leaves are transverse to the vertical segments $\{x\} \times [-1, +1]$ and whose holonomy is φ . The modification consists of removing \mathcal{O} from the interior of $\Sigma \times [-1, +1]$ and replacing it by \mathcal{F}_φ . The new foliation, denoted \mathcal{O}_φ , is an *open book foliation modified by suspension*. Theorem 1.1 can now be made more precise:

Theorem 1.2. *Every co-orientable Γ_1^r -structure, $r \geq 1$, is homotopic to an open book foliation modified by suspension.*

The proof of this theorem is given in sections 2 - 4 when $r \geq 2$. In section 5, we explain how to get the less regular case $1 \leq r < 2$. We have chosen to treat the case $r = 1 + bv$ (the holonomy local diffeomorphisms are C^1 and their first derivatives have a bounded variation). Indeed, Mather observed in [11] that $\text{Diff}_c^{1+bv}(\mathbb{R})$ is not a perfect group and it is often believed that the perfectness of $\text{Diff}_c^r(\mathbb{R})$ plays a role in the regularization theorem.

In section 6, the homotopy class of the tangent plane field will be discussed. Finally the case of non co-orientable Γ_1^r -structure will be sketched in section 7 where the corresponding models, based on *twisted open book*, will be presented.

We are very grateful to Vincent Colin, Étienne Ghys and Emmanuel Giroux for their comments, suggestions and explanations.

2. TSUBOI'S CONSTRUCTION

A Γ_1 -structure ξ on M is said to be *trivial* on a codimension 0 submanifold W when, for every $|t|$ small enough, $W \times \{t\}$ lies in a leaf of the associated foliation.

Every closed 3-manifold M has a so-called *Heegaard decomposition* $M = H_- \cup_{\Sigma} H_+$, where H_{\pm} is a possibly non-orientable handlebody (a ball with handles of index 1 attached) and Σ is their common boundary. A *thick Heegaard decomposition* is a similar decomposition where the surface is thickened:

$$M = H'_- \cup_{\Sigma \times \{-1\}} \Sigma \times [-1, +1] \cup_{\Sigma \times \{+1\}} H'_+.$$

The following statement is due to T. Tsuboi in [14] where it is left to the reader as an exercise.

Proposition 2.1. *Given a Γ_1 -structure ξ of class C^r , $r \geq 2$, on a closed 3-manifold M , there exists a thick Heegaard decomposition and a homotopy $(\xi_t)_{t \in [0,1]}$ from ξ such that:*

- 1) ξ_1 is trivial on H'_{\pm} ;
- 2) ξ_1 is regular on $\Sigma \times [-1, +1]$ and the induced foliation is a suspension.

Proof. With ξ and its foliation \mathcal{F} defined on an open neighborhood of the zero section $M \times 0$ in $M \times \mathbb{R}$, there comes a covering of the zero section by boxes, open in $M \times \mathbb{R}$, bi-foliated with respect to \mathcal{F} and the fibers. We choose a C^1 -triangulation Tr of M so fine that each simplex lies entirely in a box. With Tr comes a vector field X defined as follows.

First, on the standard k -simplex there is a smooth vector field X_{Δ^k} , tangent to each face, which is the (descending) gradient of a Morse function having one critical point of index k at the barycenter and one critical point of index i at the barycenter of each i -face. When $\Delta^i \subset \Delta^k$ is an i -face, X_{Δ^i} is the restriction of X_{Δ^k} to Δ^i . Now, if σ is a k -simplex of Tr , thought of as a C^1 -embedding $\sigma : \Delta^k \rightarrow M$, we define $X_{\sigma} := \sigma_*(X_{\Delta^k})$. The union of the X_{σ} 's is a C^0 vector field X which is uniquely integrable. After a reparametrization of each simplex we may assume that the stable manifold $W^s(b(\sigma))$ of the barycenter $b(\sigma)$ is C^1 .

The Γ_1 -structure ξ (co-oriented by the \mathbb{R} factor of $M \times \mathbb{R}$) is said to be in *Morse position* with respect to Tr if:

- (i) it has a smooth Morse type singularity of index k at the barycenter of each k -simplex and it is regular elsewhere;
- (ii) X is (negatively) transverse to ξ out of the singularities.

Lemma 2.2. *Let \mathcal{F} be the foliation associated to ξ . There exists a smooth section s such that $s^*\mathcal{F}$ is in Morse position with respect to Tr .*

Note that, as s is homotopic to the zero section, the Γ_1 -structure $s^*\mathcal{F}$ on M is homotopic to ξ .

Proof. Assume that s is already built near the $(k-1)$ -skeleton. Let σ be a k -simplex. We explain how to extend s on a neighborhood of σ . After a fibered isotopy of $M \times \mathbb{R}$ over the identity of M , we may assume that \mathcal{F} is trivial near $\{b(\sigma)\} \times \mathbb{R}$. Now, near $b(\sigma)$, we ask s to coincide with the graph of some local positive Morse function f_{σ} whose Hessian is negative definite on $T_{b(\sigma)}\sigma$ and positive definite on $T_{b(\sigma)}W^s(b(\sigma))$. This function is now fixed up to a positive constant factor. We will extend s as the graph of some function h in the \mathcal{F} -foliated chart over a neighborhood of σ . This function is already given on a neighborhood $N(\partial\sigma)$ of $\partial\sigma$ where it is C^r , the regularity of ξ , and satisfies $X.h < 0$ except at the barycenter of each face.

On the one hand, choose an arbitrary extension h_0 of h to a neighborhood of σ vanishing near $b(\sigma)$. On the other hand, choose a nonnegative function g_{σ} such that:

- $g_\sigma = 0$ near $\partial\sigma$;
- $g_\sigma = f_\sigma$ near $b(\sigma)$;
- $X.g_\sigma < 0$ when $g_\sigma > 0$ except at $b(\sigma)$.

Then, if $c > 0$ is a large enough constant, $h := h_0 + cg_\sigma$ has the required properties, except smoothness. Returning to $M \times \mathbb{R}$, the section s we have built is C^r , smooth near the singularities, and X is transverse to $s^*\xi$ except at the singularities. Therefore, there exists a smooth C^r -approximation of s , relative to a neighborhood of the barycenters which meets all the required properties. \square

Thus, by a deformation of the zero section which induces a homotopy of ξ , we have put ξ in *Morse position* with respect to Tr . In the same way, applying lemma 2.2 to the trivial Γ_1 -structure ξ_0 , we also have a Morse function f such that $X.f < 0$ except at the barycenters.

Let G_- (resp. G_+) denote the closure of the union of the unstable (resp. stable) manifolds of the singularities of X of index 1 (resp. 2). The following properties are clear:

- (a) In M , the subset G_- (resp. G_+) is a C^1 -complex of dimension 1.
- (b) It admits arbitrarily small handlebody neighborhoods H'_- (resp. H'_+) whose boundary is transverse to X .
- (c) Every orbit of X outside H'_\pm has one end point on $\partial H'_-$ and the other on $\partial H'_+$. This also holds true for any smooth C^0 -approximation \tilde{X} of X (in particular \tilde{X} is still negatively transverse to ξ).

Given a (co-orientable) Γ_1 -structure ξ on a space G , by an *upper (resp. lower) completion* of ξ one means a foliation \mathcal{F} of $G \times (-\epsilon, 1]$ (resp. $G \times [-1, \epsilon)$), for some positive ϵ , which is transverse to every fiber $\{x\} \times (-\epsilon, 1]$ (resp. $\{x\} \times [-1, \epsilon)$), whose germ along $G \times \{0\}$ is ξ , and such that $G \times \{t\}$ is a leaf of \mathcal{F} for every t close enough to $+1$ (resp. -1).

Lemma 2.3. *Every co-orientable Γ_1^r -structure on a simplicial complex G of dimension 1, $r \geq 2$, admits an upper (resp. lower) completion of class C^r .*

PROOF. One reduces immediately to the case where G is a single edge. In that case, using a partition of unity, one builds a line field which fulfills the claim. This line field is integrable. \square

By (a), the Γ_1 -structure ξ admits an upper (resp. lower) completion over G_+ (resp. G_-), and thus also over an open neighborhood N_+ (resp. N_-) of G_+ (resp. G_-). By (b), there is a handlebody neighborhood H'_\pm of G_\pm contained in N_\pm and whose boundary is transverse to X . So we have a foliation \mathcal{F} defined on a neighborhood of

$$(M \times \{0\}) \cup (H'_- \times [-1, 0]) \cup (H'_+ \times [0, 1])$$

which is transverse to X on $M \setminus (H'_- \cup H'_+)$ and tangent to $H'_\pm \times \{t\}$ for every t close to ± 1 .

By (c), there is a diffeomorphism $F : M \setminus \text{Int}(H'_- \cup H'_+) \rightarrow \Sigma \times [-1, +1]$ for some closed surface Σ , which maps orbit segments of \tilde{X} onto fibers.

For a small $\epsilon > 0$, choose a function $\psi : \mathbb{R} \rightarrow [-1, +1]$ which is smooth, odd, and such that:

- $\psi(t) = 0$ for $0 \leq t \leq 1 - 3\epsilon$ and $\psi(1 - 2\epsilon) = \epsilon$;
- ψ is affine on the interval $[1 - 2\epsilon, 1 - \epsilon]$;

- $\psi(1 - \epsilon) = 1 - \epsilon$ and $\psi(t) = 1$ for $t \geq 1$;
- $\psi' > 0$ on the interval $]1 - 3\epsilon, 1[$.

Let $s : M \rightarrow M \times \mathbb{R}$ be the graph of the function whose value is ± 1 on H'_\pm and $\psi(t)$ at the point $F^{-1}(x, t)$ for $(x, t) \in \Sigma \times [-1, +1]$. When ϵ is small enough, it is easily checked that, for every $x \in \Sigma$, the path $t \mapsto s \circ F^{-1}(x, t)$ is transverse to \mathcal{F} except at its end points. Then, $\xi_1 := s^*\mathcal{F}$ is homotopic to ξ and obviously fulfills the conditions required in proposition 2.1. \square

3. GIROUX'S CONSTRUCTION

We use here theorem III.2.7 from Giroux's article [5], which states the following:

Let M be a closed 3-manifold (orientable or not). There exist a Morse function $f : M \rightarrow \mathbb{R}$ and a co-orientable surface S which is f -essential in M .

Giroux says that S is f -essential when the restriction $f|_S$ has exactly the same critical points as f and the same local *extrema*. In the sequel, we call such a surface a *Giroux surface*.

Giroux explained to us [6] how this notion is related to open book decompositions. In the above statement, the function f can be easily chosen *self-indexing* (the value of a critical point is its Morse index in M). Thus, let N be the level set $f^{-1}(3/2)$. The smooth curve $B := N \cap S$ will be the binding of the open book decomposition we are looking for. It can be proved that the following holds for every regular value a , $0 < a \leq 3/2$:

- the level set $f^{-1}(a)$ is the union along their common boundaries of two surfaces, N_1^a and N_2^a , each one being diffeomorphic to the sub-level surface $S^a := S \cap f^{-1}([0, a])$;
- the sub-level $M^a := f^{-1}([0, a])$ is divided by S^a into two parts P_1^a and P_2^a which are isomorphic handlebodies (with corners);
- S^a is isotopic to N_i^a through P_i^a , for $i = 1, 2$, by an isotopy fixing its boundary curve $S^a \cap f^{-1}(a)$.

This claim is obvious when a is small and the property is preserved when crossing the critical level 1. In this way the handlebody $H_- := f^{-1}([0, 3/2])$ is divided by $S^{3/2}$ into two diffeomorphic parts $P_i^{3/2}$, $i = 1, 2$, and we have $N = N_1^{3/2} \cup N_2^{3/2}$. We take $S^{3/2}$, which is isotopic to $N_i^{3/2}$ in $P_i^{3/2}$, as a page. The figure is the same in $H_+ := f^{-1}([3/2, 3])$. The open book decomposition is now clear.

Proposition 3.1. *Let $K \subset M$ be a compact connected co-orientable surface whose boundary is not empty. Then there exists an open book decomposition whose some page contains K in its interior.*

Proof. (Giroux) According to the above discussion it is sufficient to find a Morse function f and a Giroux surface S (with respect to f) containing K . Let H_0 be the quotient of $K \times [-1, 1]$ by shrinking to a point each interval $\{x\} \times [-1, 1]$ when $x \in \partial K \times [-1, 1]$. After smoothing, it is a handlebody whose boundary is the double of K . On H_0 there exists a standard Morse function f_0 which is constant on ∂H_0 , having one minimum, the other critical points being of index 1. The surface $K \times \{0\}$ can be made f_0 -essential. This function is then extended to a

global Morse function \tilde{f}_0 on M . At this point we have to follow the proof of Theorem III.2.7 in [5]. The function \tilde{f}_0 is changed on the complement of H_0 , step by step when crossing its critical level, so that $K \times \{0\}$ extends as a Giroux surface in M . \square

Let now ξ be a Γ_1 -structure meeting the conclusion of proposition 2.1, up to rescaling the interval to $[-\varepsilon, +\varepsilon]$. Let \mathcal{F}_φ be the suspension foliation induced by ξ on $\Sigma \times [-\varepsilon, +\varepsilon]$. Choose $x_0 \in \Sigma \times \{0\}$; the segment $x_0 \times [-\varepsilon, +\varepsilon]$ is transverse to \mathcal{F}_φ . Let K be the surface obtained from $\Sigma \times 0$ by removing a small open disk centered at x_0 . The foliation \mathcal{F}_φ foliates $K \times [-\varepsilon, +\varepsilon]$ so that $K \times \{t\}$ lies in a leaf, when t is close to $\pm\varepsilon$, and $\partial K \times [-\varepsilon, +\varepsilon]$ is foliated by parallel circles. We apply proposition 3.1 to this K .

Corollary 3.2. *There exists an open book foliation \mathcal{O} of M inducing the trivial foliation on $K \times [-\varepsilon, +\varepsilon]$ (the leaves are $K \times \{t\}$, $t \in [-\varepsilon, +\varepsilon]$).*

Therefore, we have an open book foliation modified by suspension by replacing the above trivial foliation of $K \times [-\varepsilon, +\varepsilon]$ by $\tilde{\mathcal{F}}_\varphi$, the trace of \mathcal{F}_φ on $K \times [-\varepsilon, +\varepsilon]$. Let \mathcal{O}_φ be the resulting foliation of M and ξ_φ be its regular Γ_1 -structure. For proving theorem 1.2 (when $r \geq 2$) it is sufficient to prove that ξ and ξ_φ are homotopic. This is done in the next section.

4. HOMOTOPY OF Γ_1 -STRUCTURES

We are going to describe a homotopy from ξ_φ to ξ . Recall the tube T around the binding. For simplicity, we assume that each component of T is foliated by a standard Reeb foliation; the same holds true if T is foliated by Wood components (in the sense of [17]). Let T' be a slightly larger tube.

Lemma 4.1. *There exists a homotopy, relative to $M \setminus \text{int}(T')$, from ξ_φ to a new Γ_1 -structure ξ_1 on M such that:*

- 1) ξ_1 is trivial on T ;
- 2) ξ_1 is regular on $T' \setminus \text{int}(T)$ with compact toric leaves near ∂T and spiraling half-cylinder leaves with boundary in $\partial T'$ (as in an open book foliation).

Proof. Recall from the introduction that we only use *thick* Reeb components. So there is a third concentric tube T'' , $T \subset \text{int}(T'') \subset \text{int}(T')$, so that $T'' \setminus \text{int}(T)$ is foliated by toric leaves; and $\text{int}(T)$ is foliated by planes.

On each component of ∂T we have coordinates (x, y) coming from the framing of the binding, the x -axis being a parallel and the y -axis being a meridian. Let γ be a parallel in ∂T . The Γ_1 -structure which is induced by ξ_φ on γ is singular but not trivial and the germ of foliation \mathcal{G} along the zero section in the normal line bundle $A \cong \gamma \times \mathbb{R}$ is shown on figure 1.

The annulus A is endowed with coordinates $(x, z) \in \gamma \times \mathbb{R}$. The orientation of the z -axis, which is also the orientation of the normal bundle to the foliation \mathcal{O}_φ along γ , points to the interior of T . So the leaves of \mathcal{G} are parallel circles in $\{z \leq 0\}$ and spiraling leaves in $\{z > 0\}$. Take coordinates (x, y, r) on T where r is the distance to the binding; say that $r = 1$ on $\partial T''$ and $r = 1/2$ on ∂T . Let $\lambda(r^2)$ be an even smooth function with $r = 0$ as unique critical point, vanishing at $r = 1/2$ and $\lambda(1) < 0 < \lambda(0)$. Consider $g : T'' \rightarrow A$, $g(x, y, r) = (x, \lambda(r^2))$.

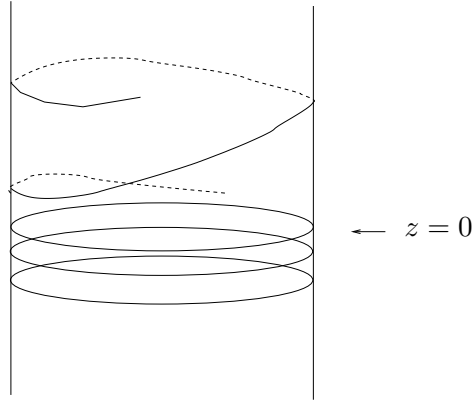


Figure 1

It is easily seen that $\xi_\varphi|_{\text{int}(T'')} \cong g^*\mathcal{G}$. Let now $\bar{\lambda}(r^2)$ be a new even function coinciding with $\lambda(r^2)$ near $r = 1$, having negative values everywhere and whose critical set is $\{r \in [0, 1/2]\}$. Let $\bar{g} : T'' \rightarrow A$, $\bar{g}(x, y, r) = (x, \bar{\lambda}(r^2))$. A barycentric combination of λ and $\bar{\lambda}$ yields a homotopy from g to \bar{g} which is relative to a neighborhood of $\partial T''$. The Γ_1 -structure ξ_1 we are looking for is defined by $\xi_1|_{\text{int}(T'')} = \bar{g}^*(\mathcal{G})$ and $\xi_1 = \xi_\varphi$ on a neighborhood of $M \setminus \text{int}(T'')$. \square

Recall the domain $K \times [-\varepsilon, +\varepsilon]$ from the previous section. After the following lemma we are done with the homotopy problem.

Lemma 4.2. *There exists a homotopy from ξ_1 to ξ relative to $K \times [-\varepsilon, +\varepsilon]$.*

Proof. Let us denote $M' := M \setminus \text{int}(K \times [-\varepsilon, +\varepsilon])$ which is a manifold with boundary and corners. It is equivalent to prove that the restrictions of ξ_1 and ξ to M' are homotopic relatively to $\partial M'$. Consider the standard closed 2-disk $D = D^2$ endowed with the Γ_1 -structure ξ_D which is shown on figure 2.

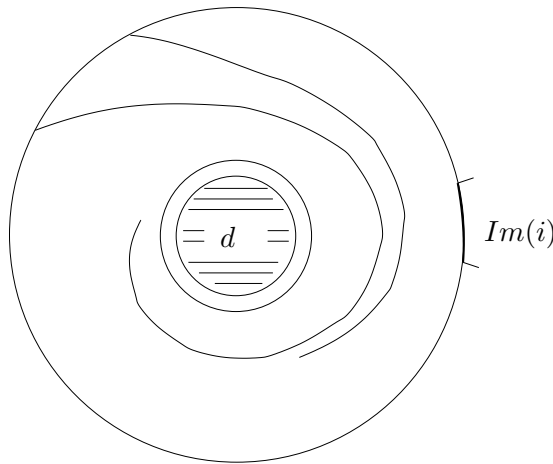


Figure 2

It is trivial on the small disk d and regular on the annulus $D \setminus \text{int}(d)$. In the regular part, the leaves are circles near ∂d and the other leaves are spiraling, crossing ∂D transversely. One checks that the restriction of ξ_1 to M' has the form $f^*\xi_D$ from some map $f : M' \rightarrow D$. We take $f|_T : T \rightarrow d$ to be the open book trivialization of T (recall the binding has a canonical framing); $f|_{\partial K \times [-\varepsilon, +\varepsilon]}$ to be the projection pr_2 onto $[-\varepsilon, +\varepsilon]$ composed with an embedding $i : [-\varepsilon, +\varepsilon] \rightarrow \partial D$ and f maps each leaf of the regular part of ξ_1 to a leaf of the regular part of ξ_D . As K does not approach T , we can take $f(\partial M') = i([-\varepsilon, +\varepsilon])$; actually, except near T , f is given by the fibration over $S^1 = \partial D^2$ of the open book decomposition.

Once ξ has Tsuboi's form (according to proposition 2.1), the restriction of ξ to M' has a similar form: $\xi = k^*\xi_D$ for some map $k : M' \rightarrow D^2$. Recall that M' is the union of two handlebodies and a solid cylinder $D^2 \times [-\varepsilon, +\varepsilon]$. Take k to be $i \circ pr_2$ on the cylinder and k to be constant on each handlebody. Observe that f and k coincide on $\partial M'$. As D retracts by deformation onto the image of i , one deduces that f and k are homotopic relatively to $\partial M'$. \square

This finishes the proof of theorem 1.2 when $r \geq 2$.

5. THE CASE C^{1+bv}

A co-oriented Γ_1^r -structure ξ on M can be realized by a foliation \mathcal{F} defined on a neighborhood of the 0-section in $M \times \mathbb{R}$; it is made of bi-foliated charts which are C^∞ in the direction of the leaves and C^r in the direction of the fibers. Consider such a box \mathcal{U} over an open disk D centered at $x_0 \in M$; its trace on the x_0 -fiber is an interval I . Each leaf of \mathcal{U} reads $z = f(x, t)$, $x \in D$, for some $t \in I$. Here f is a function which is smooth in x and C^r in t , with $f(x_0, t) = t$; the foliating property is equivalent to $\frac{\partial f}{\partial t} > 0$. When $r = 1 + bv$, there is a positive measure $\mu(x, t)$ on I , without atoms and depending smoothly on x , such that:

$$(*) \quad \frac{\partial}{\partial t} f(x, t) - \frac{\partial}{\partial t} f(x, t_0) = \int_{t_0}^t \mu(x, t).$$

Proposition 5.1. *Theorem 1.2 holds true for any class of regularity $r \geq 1$ including the class $r = 1 + bv$.*

Proof. The only part of the proof which requires some care of regularity is section 2, especially the proof of lemma 2.3. Indeed, we have to avoid integrating C^0 vector fields. For proving lemma 2.3 with weak regularity we use the lemmas below which we shall prove in the case $r = 1 + bv$ only.

Lemma 5.2. *Let $f : D \times I \rightarrow \mathbb{R}$ be a C^r -function as above. Assume $I =] - 2\varepsilon, +\varepsilon[$ and $f(x, -2\varepsilon) > -1$ for every $x \in D$. Then there exists a function $F : D \times [-1, 0] \rightarrow \mathbb{R}$ of class C^r such that:*

- 1) $F(x, t) = f(x, t)$ when $t \in [-\varepsilon, 0]$,
- 2) $F(x, t) = t$ when t is close to -1 ,
- 3) $\frac{\partial F}{\partial t} > 0$.

Proof. Let $\mu(x, t)$ be the positive measure whose support is $[-\varepsilon, 0]$ such that formula (*) holds for every $(x, t) \in D \times [-\varepsilon, 0]$ and $t_0 = 0$. There exists another positive measure $\nu(x, t)$, smooth in x and whose support is contained in $] -1, -\varepsilon]$, such that

$$(**) \quad f(x, -\varepsilon) = -1 + \int_{-1}^{-\varepsilon} \left(\int_{-1}^t \nu(x, \tau) \right) dt.$$

Then a solution is

$$F(x, t) = -1 + \int_{-1}^t \left(\int_{-1}^s (\mu(x, \tau) + \nu(x, \tau)) \right) ds.$$

□

Lemma 5.3. *Let A_1 and A_2 two disjoint closed sub-disks of D . Let F_1 and F_2 be two solutions of lemma 5.2. Then there exists a third solution which equals F_1 when $x \in A_1$ and F_2 when $x \in A_2$.*

Proof. Both solutions F_1 and F_2 differ by the choice of the measure $\nu(x, t)$ in formula (**), which is ν_i for F_i . Choose a partition of unity $1 = \lambda_1(x) + \lambda_2(x)$ with $\lambda_i = 1$ on A_i . Then $\nu(x, t) = \lambda_1(x)\nu_1(x, t) + \lambda_2(x)\nu_2(x, t)$ yields the desired solution. □

The proof of proposition 5.1 is now easy. As already said, it is sufficient to prove lemma 2.3 in class C^r , $r \geq 1$. It is an extension problem of a foliation given near the 1-skeleton $Tr^{[1]} \times \{0\}$ to $Tr^{[1]} \times [-1, 0]$. One covers $Tr^{[1]}$ by finitely many n -disks D_j . The problem is solved in each $D_j \times [-1, 0]$ by applying lemma 5.2. By applying lemma 5.3 one makes the different extensions match together. □

6. HOMOTOPY CLASS OF PLANE FIELDS

It is possible to enhance theorem 1.1 by prescribing the homotopy class of the underlying co-oriented plane field (see proposition 6.1 below). The question of doing the same with respect to theorem 1.2 is more subtil (see proposition 6.3).

Proposition 6.1. *Given a co-oriented Γ_1 -structure ξ on the closed 3-manifold M and a homotopy class $[\nu]$ of co-oriented plane field in the tangent space τM , there exists a regular Γ_1 -structure ξ_{reg} homotopic to ξ whose underlying foliation $\mathcal{F}_{reg} \cap M$ has a tangent co-oriented plane field in the class $[\nu]$.*

Before proving it we first recall some well-known facts on co-oriented plane fields (see [4]). Given a base plane field ν_0 , a suitable Thom-Pontryagin construction yields a natural bijection between the set of homotopy classes of plane fields on M and $\Omega_1^{\nu_0}(M)$, the group of (co)bordism classes of ν_0 -framed and oriented closed (maybe non-connected) curves in M . A ν_0 -framing of the curve γ is an isomorphism of fiber bundles $\varepsilon : \nu(\gamma, M) \rightarrow \nu_0|_\gamma$, whose source is the normal bundle to γ in M . We denote γ^ε the curve endowed with this framing. Moreover, given γ^ε , if γ' is homologous to γ in M there exists a ν_0 -framing ε' such that $(\gamma')^{\varepsilon'}$ is cobordant to γ^ε .

Proof of 6.1. We can start with an open book foliation \mathcal{O}_φ yielded by theorem 1.2. Let ν_0 be its tangent plane field. Near the binding, the meridian loops (out of T) are transverse to \mathcal{O}_φ and homotopic to zero in M . As a consequence, each 1-homology class may be represented as well by a (multi)-curve in a page or by a connected curve out of T positively transverse to all pages. We do the second choice for γ^ε , the ν_0 -framed curve whose cobordism class encodes $[\nu]$ with respect to ν_0 .

Hence we are allowed to *turbulize* \mathcal{O}_φ along γ . In a small tube $T(\gamma)$ about γ , we put a Wood component. Outside, the leaves are spiraling around $\partial T(\gamma)$. Let $\mathcal{O}_\varphi^{turb}$ be the resulting foliation. Whatever the chosen type of Wood component is, the Γ_1 -structures of $\mathcal{O}_\varphi^{turb}$ and \mathcal{O}_φ are homotopic by arguing as in section 4. But the framing ε tells us which sort of Wood component will be convenient for getting the desired class $[\nu]$ (see lemma 6.1 in [17]). \square

In the previous statement, we have lost the nice model we found in theorem 1.2. Actually, thanks to a lemma of Vincent Colin [3], it is possible to recover our model, at least when M is orientable (see below proposition 6.3).

Lemma 6.2. (Colin) *Let (B, p) be an open book decomposition of M and γ be a simple connected curve in some page P . Assume γ is orientation preserving. Then there exist a positive stabilization (B', p') of (B, p) and a curve γ' in B' which is isotopic to γ in M . When γ is a multi-curve, the same holds true after a sequence of stabilizations.*

The *positive Hopf open book decomposition* of the 3-sphere is the one whose binding is made of two unknots with linking number $+1$; a page is an annulus foliated by fibers of the Hopf fibration $S^3 \rightarrow S^2$. A positive stabilization is a “connected sum” with this open book. The new page P' is obtained from P by *plumbing* an annulus A whose core bounds a disk in M (see [7] for more details and other references).

Proof. If γ is connected, only one stabilization is needed. We are going to explain this case only. A tubular neighborhood of γ in P is an annulus.

Choose a simple arc α in P joining γ to some component β of B without crossing γ again. Let $\tilde{\gamma}$ be a simple arc from β to itself which follows $\alpha^{-1} * \gamma * \alpha$. The orientation assumption implies that the surgery of β by $\tilde{\gamma}$ in P provides a curve with two connected components, one of them being isotopic to γ in P . Let P_π be the page opposite to P and $R : P \rightarrow P_\pi$ the time π of a flow transverse to the pages (and stationary on B). The core curve C of the annulus A that we use for the plumbing is the union $\tilde{\gamma} \cup R(\tilde{\gamma})$. And A is $(+1)$ -twisted around C (with respect to its unknot framing) as in the Hopf open book. Let H be the 1-handle which is the closure of $A \setminus P$. Surgering B by H provides the new binding. By construction, one of its components is isotopic to γ . \square

Proposition 6.3. *Let \mathcal{O}_φ be an open book foliation modified by suspension, whose underlying open book is denoted (B, p) . Let ν_0 be its tangent co-oriented plane field. Let γ^ε be a ν_0 -framed curve in M and $[\nu]$ be its associated class of plane field. Assume γ is orientation preserving. Then there exists an open book foliation \mathcal{O}'_φ with the following properties:*

- 1) *its tangent plane field is in the class $[\nu]$;*
- 2) *the suspension modification is the same for \mathcal{O}'_φ as for \mathcal{O}_φ and is supported in $K \times [-\varepsilon, +\varepsilon]$;*

3) as Γ_1 -structures, \mathcal{O}_φ and \mathcal{O}'_φ are homotopic.

Proof. As said in the proof of 6.1, up to framed cobordism, γ^ε may be chosen as a simple (multi)-curve in one page P of (B, p) . Applying Colin's lemma provides a stabilization (B', p') such that, up to isotopy, γ lies in the new binding. Observe that, if K is in P , K is still in the new page P' ; hence 2) holds for any open book foliation carried by (B', p') . Once γ^ε is in the binding, for a suitable Wood component foliating a tube about γ^ε , item 1) is fulfilled. Finally item 3) follows from item 2) and the proofs in section 4. \square

7. CASE OF A Γ_1 -STRUCTURE WITH A TWISTED NORMAL BUNDLE

What happens when the bundle ν normal to ξ is twisted? It is known that a necessary condition to regularization is the existence of a fibered embedding $i : \nu \rightarrow \tau M$ into the tangent fiber bundle to M . Conversely, assuming that this condition is fulfilled, we are going to state a normal form theorem analogous to theorem 1.2. Since no step of the previous proof can be immediately adapted to this situation, we believe that it deserves a sketch of proof.

7.1. In the first step (Tsuboi's construction), we do not have "Morse position" with respect to a triangulation, since index and co-index of a singularity cannot be distinguished. Instead of lemma 2.2, we have the following statement.

After some homotopy, ξ has Morse singularities and admits a pseudo-gradient whose dynamics has no recurrence (that is, every orbit has a finite length).

Here, by a pseudo-gradient, it is meant a smooth section X of $\text{Hom}(\nu, \tau M)$, a *twisted vector field* indeed, such that $X \cdot \xi < 0$ except at the singularities (this sign is well-defined whatever a local orientation of ν , or co-orientation of ξ , is chosen); such a pseudo-gradient always exists by using an auxiliary Riemannian metric.

Sketch of proof. Generically ξ has Morse singularities. Let X_0 be a first pseudo-gradient, which is required to be the usual negative gradient in Morse coordinates near each singularity. Finitely many mutually disjoint 2-disks of M are chosen in regular leaves of ξ such that every orbit of X_0 crosses one of them. Following Wilson's idea [15], ξ and X_0 are changed in a neighborhood $D^2 \times [-1, +1]$ of each disk into a *plug* such that every orbit of the modified pseudo-gradient X is trapped by one of the plugs. The plug has the mirror symmetry with respect to $D^2 \times \{0\}$. In $D^2 \times [0, 1]$ we just modify ξ by introducing a cancelling pair of singularities, center-saddle. \square

Let G be the closure of the one-dimensional invariant manifold of all saddles. It is a graph. We claim: $\nu|_G$ is orientable. Indeed, we orient each edge from its saddle end point to its center end point. This is an orientation of $\nu|_G$ over the complement of the vertices. It is easily checked that this orientation extends over the vertices. Thus X becomes a usual vector field near G and we have an arbitrarily small tubular neighborhood H of G whose boundary is transverse to X , and X enters H . Now, the negative completion of $\xi|_H$ can be performed as in lemma 2.3.

The complement \hat{M} of $\text{int } H$ in M is fibered over a surface Σ , the fibers being intervals ($\cong [-1, 1]$) tangent to X . By taking a section we think of Σ as a surface in $M \setminus H$. Since

ξ is not co-orientable, Σ is one-sided and G is connected. Arguing as in section 2, after some homotopy, ξ becomes trivial on H and transverse to X on \hat{M} , hence a suspension foliation corresponding to a representation $\varphi : \pi_1(\Sigma) \rightarrow \text{Diff}_c([-1, 1])$.

7.2. In the second step (Giroux's construction), we have to leave the open books and we need a *twisted open book*. It is made of the following:

- a binding B which is a 1-dimensional closed co-orientable submanifold in M ;
- a *Seifert fibration* $p : M \setminus B \rightarrow [-1, +1]$ which has two one-sided exceptional surface fibers $p^{-1}(\pm 1)$ and which is a proper smooth submersion over the open interval;
- when t goes to ± 1 , $p^{-1}(t)$ tends to a 2-fold covering of $p^{-1}(\pm 1)$;
- near B the foliation looks like an open book.

The exceptional fibers are compactified by B as smooth surfaces with boundary. But, for $t \in]-1, +1[$, $p^{-1}(t)$ is compactified by B as a closed surface showing (in general) an angle along B . Notice that, since B is co-orientable, a twisted open book gives rise to a smooth foliation where each component of the binding is replaced by a Reeb component, the pages spiraling around it.

Such an open book is generated by a *one-sided Giroux surface*, which is the union of the compactified exceptional fibers. Abstractly, a one-sided Giroux surface in M with respect to a Morse function $f : M \rightarrow \mathbb{R}$ is a one-sided surface S such that $f|_S$ has the same critical points and the same extrema as f and fulfills the extra condition: for every regular value $t \in \mathbb{R}$, $f^{-1}(t) \cap S$ is a two-sided curve in the level set $f^{-1}(t)$. Starting with (S, f) where f is a self-indexing Morse function, a twisted open book is easily constructed. Its binding is the co-orientable curve $f^{-1}(3/2) \cap S$. In general such a one-sided Giroux surface (or twisted open book) does not exist on M ; the obstruction lies in the existence of a twisted line subbundle of τM . Nevertheless, with a suitable assumption, we have an analogue of proposition 3.1:

Let $i : \nu \rightarrow \tau M$ be an embedding of a twisted line bundle. Let $K \subset M$ be a compact connected one-sided surface whose boundary is not empty. Assume the following: $\nu|_K$ is twisted, $\nu|_{\partial K}$ is trivial and the normal bundle $\nu(K, M)$ is homotopic to $i(\nu)|_K$. Then there exists a twisted open book with one exceptional page containing K in its interior.

Sketch of proof. We give the proof only in the setting of 7.1 by taking K to be the closure of $\Sigma \setminus d_0$ where d_0 is a small closed 2-disk in Σ . Recall the fibration $\rho : \hat{M} \rightarrow \Sigma$. Let H' be the handlebody made of H to which is glued the 1-handle $\rho^{-1}(d_0)$. Let M' be the complement of $\text{int } H'$. Consider a minimal system of mutually disjoint compression disks d_1, \dots, d_g of H , so that cutting H along them yields a ball; g is the genus of H . Thus d_0, d_1, \dots, d_g is a minimal system of mutually disjoint compression disks of H' .

We claim: $g + 1$ is even. Indeed, by assumption there exists a non vanishing section of the bundle $\text{Hom}(\nu, \tau M)$. Thus, the number of zeroes of the pseudo-gradient X is even and the Euler characteristic $\chi(G)$ of the graph G is even. As G is connected, the genus g is odd, which proves the claim.

Now, one follows Giroux's algorithm for completing K to a closed Giroux's surface. On the surface $\partial M'$ the union of the attaching curves $\partial d_0, \partial d_1, \dots, \partial d_g$ is not separating. Then, after some isotopy, for each $i = 0, \dots, g$, ∂d_i crosses ∂K in exactly two points a_i, b_i linked by an arc α_i

(resp. α'_i) in ∂d_i (resp. ∂K), so that $\alpha_i \cup \alpha'_i$ bounds a disk in $\partial M'$. Moreover, one can arrange that all the arcs $\alpha_0, \dots, \alpha_g$ are parallel. Also we link a_i, b_i by a simple arc in d_i . Now, each compression disk defines, simultaneously, a 1-handle which is glued to K and a 2-handle which is glued to M' , yielding a proper surface K_1 in some 3-submanifold M'' of M , whose complement is a ball. The boundary of K_1 is made of $g + 2$ parallel curves in the sphere $\partial M''$. As this number is odd, Giroux described a process of adding cancelling pairs of 1- and 2- handles whose effect is to change K_1 into $K_2 \subset M''$ such that ∂K_2 is made of one curve only ([5], p. 676-677). Hence, K_2 can be closed into a Giroux's surface. \square

Theorem 7.3. *Let ξ be a non co-orientable Γ_1 -structure on M^3 whose normal bundle ν embeds into τM . Then ξ is homotopic to a twisted open book foliation modified by suspension.*

Sketch of proof. Let $K = \Sigma \setminus \text{int } D^2$ be the surface with a hole, where Σ was built in the first step 7.1; it meets the required assumptions for building a twisted open book.

The twisted open book built in the second step gives rise to a foliation \mathcal{O} . Indeed, as the binding B is co-orientable, it is allowed to spiral the pages around a tubular neighborhood of B . The tube itself is foliated by thick Reeb components. As in the co-orientable case, we can modify the open book foliation in a neighborhood of K using the representation φ , yielding the foliation \mathcal{O}_φ and its associated regular Γ_1 -structure ξ_φ . We have to prove that ξ and ξ_φ are homotopic. We may suppose that ξ is in Tsuboi form (7.1).

We observe that the total space of ν has a foliation \mathcal{F}_0 (unique up to isomorphism) transverse to the fibers, having the zero section as a leaf and whose all non trivial holonomy elements have order 2. It defines the trivial Γ_1 -structure ξ_0 in the twisted sense. Using notation of 7.2, one can prove that $\xi|_{H'}$ and $\xi_\varphi|_{H'}$ are both homotopic to $\xi_0|_{H'}$. Moreover, both homotopies coincide on the boundary $\partial H'$ (on H' , it is sufficient to think of the case when φ is the trivial representation. Thus $\xi|_{H'}$ and $\xi_\varphi|_{H'}$ are homotopic relative to the boundary. \square

7.4. Plane field homotopy class. *By turbulizing \mathcal{O}_φ , it is possible to have the normal field in any homotopy class of embeddings $\nu \rightarrow \tau M$.*

Indeed, a curve in M is homotopic to a curve transverse to \mathcal{O}_φ if and only if it does not twist ν . But these homology classes are exactly those which appear as a first difference homology class when comparing two embeddings $j_1, j_2 : \nu \rightarrow \tau M$, since a closed curve which twists ν is not a cycle in $H_1(M, \mathbb{Z}_{or(\nu)}) \cong H^2(M, \mathbb{Z}_{or(\nu^* \otimes \tau M)})$.

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