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# REGULARIZATION OF $\Gamma_1$ -STRUCTURES IN DIMENSION 3

FRANÇOIS LAUDENBACH AND GAËL MEIGNIEZ

ABSTRACT. For  $\Gamma_1$ -structures on 3-manifolds, we give a very simple proof of Thurston's regularization theorem, first proved in [13], without using Mather's homology equivalence. Moreover, in the co-orientable case, the resulting foliation can be chosen of a precise kind, namely an "open book foliation modified by suspension". There is also a model in the non co-orientable case.

## 1. INTRODUCTION

A  $\Gamma_1$ -structure  $\xi$ , in the sense of A. Haefliger, on a manifold  $M$  is given by a line bundle  $\nu = (E \rightarrow M)$ , called the *normal bundle* to  $\xi$ , and a germ of codimension-one foliation  $\mathcal{F}$  along the zero section, which is required to be transverse to the fibers (see [8]). To fix ideas, consider the co-orientable case, that is, the normal bundle is trivial:  $E \cong M \times \mathbb{R}$ ; for the general case see section 7. The  $\Gamma_1$ -structure  $\xi$  is said to be *regular* when the foliation  $\mathcal{F}$  is transverse to the zero section, in which case the pullback of  $\mathcal{F}$  to  $M$  is a genuine foliation on  $M$ . A homotopy of  $\xi$  is defined as a  $\Gamma_1$ -structure on  $M \times [0, 1]$  inducing  $\xi$  on  $M \times \{0\}$ . A *regularization theorem* should claim that any  $\Gamma_1$ -structure is homotopic to a regular one. It is not true in general. An obvious necessary condition is that  $\nu$  must embed into the tangent bundle  $\tau M$ . When  $\nu$  is trivial and  $\dim M = 3$  this condition is fulfilled.

The  $C^\infty$  category is understood in the sequel, unless otherwise specified. In particular  $M$  is  $C^\infty$ . One calls  $\xi$  a  $\Gamma_1^r$ -structure ( $r \geq 1$ ) if it is tangentially  $C^\infty$  and transversely  $C^r$ , that is, the foliation charts are  $C^r$  in the direction transverse to the leaves. We will prove the following theorem.

**Theorem 1.1.** *If  $M$  is a closed 3-manifold and  $\xi$  a  $\Gamma_1^r$ -structure,  $r \geq 1$ , whose normal bundle is trivial, then  $\xi$  is homotopic to a regular  $\Gamma_1^r$ -structure.*

Moreover, the resulting foliation of  $M$  may have its tangent plane field in a prescribed homotopy class (see proposition 6.1).

This theorem is a particular case of a general regularization theorem due to W. Thurston (see [13]). Thurston's proof was based on the deep result due to J. Mather [9], [10]: the homology equivalence between the classifying space of the group  $\text{Diff}_c(\mathbb{R})$  endowed with the discrete topology and the loop space  $\Omega B(\Gamma_1)_+$ . We present a proof of this regularization theorem which does not need this result. A regularization theorem in all dimensions, still avoiding any difficult result, is provided in [12]. But there are reasons for considering the dimension 3 separately.

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Our proof provides models realizing each homotopy class of  $\Gamma_1$ -structure. The models are based on the notion of *open book decomposition*. Recall that such a structure on  $M$  consists of a link  $B$  in  $M$ , called the *binding*, and a fibration  $p : M \setminus B \rightarrow S^1$  such that, for every  $\theta \in S^1$ ,  $p^{-1}(\theta)$  is the interior of an embedded surface, called the *page*  $P_\theta$ , whose boundary is the binding. The existence of open book decomposition could be proved by J. Alexander when  $M$  is orientable, as a consequence of [1] (every orientable closed 3-manifold is a branched cover of the 3-sphere) and [2] (every link can be braided); but he was ignoring this concept which was introduced by H. Winkelnkemper in 1973 [16]. Henceforth, we refer to the more flexible construction by E. Giroux, which includes the non-orientable case (see section 3). An open book gives rise to a foliation  $\mathcal{O}$  constructed as follows. The pages endow  $B$  with a normal framing. So a tubular neighborhood  $T$  of  $B$  is trivialized:  $T \cong B \times D^2$ . Out of  $T$  the leaves are the pages modified by spiraling around  $T$ ; the boundary of  $T$  is a union of compact leaves; and the interior of  $T$  is foliated by a Reeb component, or a generalized Reeb component in the sense of Wood [17]. For technical reasons in the homotopy argument of section 4, the Reeb components of  $\mathcal{O}$ , instead of being usual Reeb components, will be *thick Reeb components* in which a neighborhood of the boundary is foliated by toric compact leaves. We call such a foliation an *open book foliation*.

The latter can be modified by inserting a so called *suspension foliation*. Precisely, let  $\Sigma$  be a compact sub-surface of some leaf of  $\mathcal{O}$  out of  $T$  and  $\Sigma \times [-1, +1]$  be a foliated neighborhood of it (each  $\Sigma \times \{t\}$  being contained in a leaf of  $\mathcal{O}$ ). Let  $\varphi : \pi_1(\Sigma) \rightarrow \text{Diff}_c([-1, +1])$  be some representation into the group of compactly supported diffeomorphisms;  $\varphi$  is assumed to be trivial on the peripheral elements. It allows us to construct a *suspension foliation*  $\mathcal{F}_\varphi$  on  $\Sigma \times [-1, +1]$ , whose leaves are transverse to the vertical segments  $\{x\} \times [-1, +1]$  and whose holonomy is  $\varphi$ . The modification consists of removing  $\mathcal{O}$  from the interior of  $\Sigma \times [-1, +1]$  and replacing it by  $\mathcal{F}_\varphi$ . The new foliation, denoted  $\mathcal{O}_\varphi$ , is an *open book foliation modified by suspension*. Theorem 1.1 can now be made more precise:

**Theorem 1.2.** *Every co-orientable  $\Gamma_1^r$ -structure,  $r \geq 1$ , is homotopic to an open book foliation modified by suspension.*

The proof of this theorem is given in sections 2 - 4 when  $r \geq 2$ . In section 5, we explain how to get the less regular case  $1 \leq r < 2$ . We have chosen to treat the case  $r = 1 + bv$  (the holonomy local diffeomorphisms are  $C^1$  and their first derivatives have a bounded variation). Indeed, Mather observed in [11] that  $\text{Diff}_c^{1+bv}(\mathbb{R})$  is not a perfect group and it is often believed that the perfectness of  $\text{Diff}_c^r(\mathbb{R})$  plays a role in the regularization theorem.

In section 6, the homotopy class of the tangent plane field will be discussed. Finally the case of non co-orientable  $\Gamma_1^r$ -structure will be sketched in section 7 where the corresponding models, based on *twisted open book*, will be presented.

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## 2. TSUBOI'S CONSTRUCTION

A  $\Gamma_1$ -structure  $\xi$  on  $M$  is said to be *trivial* on a codimension 0 submanifold  $W$  when, for every  $|t|$  small enough,  $W \times \{t\}$  lies in a leaf of the associated foliation.

Every closed 3-manifold  $M$  has a so-called *Heegaard decomposition*  $M = H_- \cup_{\Sigma} H_+$ , where  $H_{\pm}$  is a possibly non-orientable handlebody (a ball with handles of index 1 attached) and  $\Sigma$  is their common boundary. A *thick Heegaard decomposition* is a similar decomposition where the surface is thickened:

$$M = H'_- \cup_{\Sigma \times \{-1\}} \Sigma \times [-1, +1] \cup_{\Sigma \times \{+1\}} H'_+.$$

The following statement is due to T. Tsuboi in [14] where it is left to the reader as an exercise.

**Proposition 2.1.** *Given a  $\Gamma_1$ -structure  $\xi$  of class  $C^r$ ,  $r \geq 2$ , on a closed 3-manifold  $M$ , there exists a thick Heegaard decomposition and a homotopy  $(\xi_t)_{t \in [0,1]}$  from  $\xi$  such that:*

- 1)  $\xi_1$  is trivial on  $H'_{\pm}$ ;
- 2)  $\xi_1$  is regular on  $\Sigma \times [-1, +1]$  and the induced foliation is a suspension.

**Proof.** With  $\xi$  and its foliation  $\mathcal{F}$  defined on an open neighborhood of the zero section  $M \times 0$  in  $M \times \mathbb{R}$ , there comes a covering of the zero section by boxes, open in  $M \times \mathbb{R}$ , bi-foliated with respect to  $\mathcal{F}$  and the fibers. We choose a  $C^1$ -triangulation  $Tr$  of  $M$  so fine that each simplex lies entirely in a box. With  $Tr$  comes a vector field  $X$  defined as follows.

First, on the standard  $k$ -simplex there is a smooth vector field  $X_{\Delta^k}$ , tangent to each face, which is the (descending) gradient of a Morse function having one critical point of index  $k$  at the barycenter and one critical point of index  $i$  at the barycenter of each  $i$ -face. When  $\Delta^i \subset \Delta^k$  is an  $i$ -face,  $X_{\Delta^i}$  is the restriction of  $X_{\Delta^k}$  to  $\Delta^i$ . Now, if  $\sigma$  is a  $k$ -simplex of  $Tr$ , thought of as a  $C^1$ -embedding  $\sigma : \Delta^k \rightarrow M$ , we define  $X_{\sigma} := \sigma_*(X_{\Delta^k})$ . The union of the  $X_{\sigma}$ 's is a  $C^0$  vector field  $X$  which is uniquely integrable. After a reparametrization of each simplex we may assume that the stable manifold  $W^s(b(\sigma))$  of the barycenter  $b(\sigma)$  is  $C^1$ .

The  $\Gamma_1$ -structure  $\xi$  (co-oriented by the  $\mathbb{R}$  factor of  $M \times \mathbb{R}$ ) is said to be in *Morse position* with respect to  $Tr$  if:

- (i) it has a smooth Morse type singularity of index  $k$  at the barycenter of each  $k$ -simplex and it is regular elsewhere;
- (ii)  $X$  is (negatively) transverse to  $\xi$  out of the singularities.

**Lemma 2.2.** *Let  $\mathcal{F}$  be the foliation associated to  $\xi$ . There exists a smooth section  $s$  such that  $s^*\mathcal{F}$  is in Morse position with respect to  $Tr$ .*

Note that, as  $s$  is homotopic to the zero section, the  $\Gamma_1$ -structure  $s^*\mathcal{F}$  on  $M$  is homotopic to  $\xi$ .

**Proof.** Assume that  $s$  is already built near the  $(k-1)$ -skeleton. Let  $\sigma$  be a  $k$ -simplex. We explain how to extend  $s$  on a neighborhood of  $\sigma$ . After a fibered isotopy of  $M \times \mathbb{R}$  over the identity of  $M$ , we may assume that  $\mathcal{F}$  is trivial near  $\{b(\sigma)\} \times \mathbb{R}$ . Now, near  $b(\sigma)$ , we ask  $s$  to coincide with the graph of some local positive Morse function  $f_{\sigma}$  whose Hessian is negative definite on  $T_{b(\sigma)}\sigma$  and positive definite on  $T_{b(\sigma)}W^s(b(\sigma))$ . This function is now fixed up to a positive constant factor. We will extend  $s$  as the graph of some function  $h$  in the  $\mathcal{F}$ -foliated chart over a neighborhood of  $\sigma$ . This function is already given on a neighborhood  $N(\partial\sigma)$  of  $\partial\sigma$  where it is  $C^r$ , the regularity of  $\xi$ , and satisfies  $X.h < 0$  except at the barycenter of each face.

On the one hand, choose an arbitrary extension  $h_0$  of  $h$  to a neighborhood of  $\sigma$  vanishing near  $b(\sigma)$ . On the other hand, choose a nonnegative function  $g_{\sigma}$  such that:

- $g_\sigma = 0$  near  $\partial\sigma$ ;
- $g_\sigma = f_\sigma$  near  $b(\sigma)$ ;
- $X.g_\sigma < 0$  when  $g_\sigma > 0$  except at  $b(\sigma)$ .

Then, if  $c > 0$  is a large enough constant,  $h := h_0 + cg_\sigma$  has the required properties, except smoothness. Returning to  $M \times \mathbb{R}$ , the section  $s$  we have built is  $C^r$ , smooth near the singularities, and  $X$  is transverse to  $s^*\xi$  except at the singularities. Therefore, there exists a smooth  $C^r$ -approximation of  $s$ , relative to a neighborhood of the barycenters which meets all the required properties.  $\square$

Thus, by a deformation of the zero section which induces a homotopy of  $\xi$ , we have put  $\xi$  in *Morse position* with respect to  $Tr$ . In the same way, applying lemma 2.2 to the trivial  $\Gamma_1$ -structure  $\xi_0$ , we also have a Morse function  $f$  such that  $X.f < 0$  except at the barycenters.

Let  $G_-$  (resp.  $G_+$ ) denote the closure of the union of the unstable (resp. stable) manifolds of the singularities of  $X$  of index 1 (resp. 2). The following properties are clear:

- (a) In  $M$ , the subset  $G_-$  (resp.  $G_+$ ) is a  $C^1$ -complex of dimension 1.
- (b) It admits arbitrarily small handlebody neighborhoods  $H'_-$  (resp.  $H'_+$ ) whose boundary is transverse to  $X$ .
- (c) Every orbit of  $X$  outside  $H'_\pm$  has one end point on  $\partial H'_-$  and the other on  $\partial H'_+$ . This also holds true for any smooth  $C^0$ -approximation  $\tilde{X}$  of  $X$  (in particular  $\tilde{X}$  is still negatively transverse to  $\xi$ ).

Given a (co-orientable)  $\Gamma_1$ -structure  $\xi$  on a space  $G$ , by an *upper (resp. lower) completion* of  $\xi$  one means a foliation  $\mathcal{F}$  of  $G \times (-\epsilon, 1]$  (resp.  $G \times [-1, \epsilon)$ ), for some positive  $\epsilon$ , which is transverse to every fiber  $\{x\} \times (-\epsilon, 1]$  (resp.  $\{x\} \times [-1, \epsilon)$ ), whose germ along  $G \times \{0\}$  is  $\xi$ , and such that  $G \times \{t\}$  is a leaf of  $\mathcal{F}$  for every  $t$  close enough to  $+1$  (resp.  $-1$ ).

**Lemma 2.3.** *Every co-orientable  $\Gamma_1^r$ -structure on a simplicial complex  $G$  of dimension 1,  $r \geq 2$ , admits an upper (resp. lower) completion of class  $C^r$ .*

PROOF. One reduces immediately to the case where  $G$  is a single edge. In that case, using a partition of unity, one builds a line field which fulfills the claim. This line field is integrable.  $\square$

By (a), the  $\Gamma_1$ -structure  $\xi$  admits an upper (resp. lower) completion over  $G_+$  (resp.  $G_-$ ), and thus also over an open neighborhood  $N_+$  (resp.  $N_-$ ) of  $G_+$  (resp.  $G_-$ ). By (b), there is a handlebody neighborhood  $H'_\pm$  of  $G_\pm$  contained in  $N_\pm$  and whose boundary is transverse to  $X$ . So we have a foliation  $\mathcal{F}$  defined on a neighborhood of

$$(M \times \{0\}) \cup (H'_- \times [-1, 0]) \cup (H'_+ \times [0, 1])$$

which is transverse to  $X$  on  $M \setminus (H'_- \cup H'_+)$  and tangent to  $H'_\pm \times \{t\}$  for every  $t$  close to  $\pm 1$ .

By (c), there is a diffeomorphism  $F : M \setminus \text{Int}(H'_- \cup H'_+) \rightarrow \Sigma \times [-1, +1]$  for some closed surface  $\Sigma$ , which maps orbit segments of  $\tilde{X}$  onto fibers.

For a small  $\epsilon > 0$ , choose a function  $\psi : \mathbb{R} \rightarrow [-1, +1]$  which is smooth, odd, and such that:

- $\psi(t) = 0$  for  $0 \leq t \leq 1 - 3\epsilon$  and  $\psi(1 - 2\epsilon) = \epsilon$ ;
- $\psi$  is affine on the interval  $[1 - 2\epsilon, 1 - \epsilon]$ ;

- $\psi(1 - \epsilon) = 1 - \epsilon$  and  $\psi(t) = 1$  for  $t \geq 1$ ;
- $\psi' > 0$  on the interval  $]1 - 3\epsilon, 1[$ .

Let  $s : M \rightarrow M \times \mathbb{R}$  be the graph of the function whose value is  $\pm 1$  on  $H'_\pm$  and  $\psi(t)$  at the point  $F^{-1}(x, t)$  for  $(x, t) \in \Sigma \times [-1, +1]$ . When  $\epsilon$  is small enough, it is easily checked that, for every  $x \in \Sigma$ , the path  $t \mapsto s \circ F^{-1}(x, t)$  is transverse to  $\mathcal{F}$  except at its end points. Then,  $\xi_1 := s^*\mathcal{F}$  is homotopic to  $\xi$  and obviously fulfills the conditions required in proposition 2.1.  $\square$

### 3. GIROUX'S CONSTRUCTION

We use here theorem III.2.7 from Giroux's article [5], which states the following:

*Let  $M$  be a closed 3-manifold (orientable or not). There exist a Morse function  $f : M \rightarrow \mathbb{R}$  and a co-orientable surface  $S$  which is  $f$ -essential in  $M$ .*

Giroux says that  $S$  is  $f$ -essential when the restriction  $f|_S$  has exactly the same critical points as  $f$  and the same local *extrema*. In the sequel, we call such a surface a *Giroux surface*.

Giroux explained to us [6] how this notion is related to open book decompositions. In the above statement, the function  $f$  can be easily chosen *self-indexing* (the value of a critical point is its Morse index in  $M$ ). Thus, let  $N$  be the level set  $f^{-1}(3/2)$ . The smooth curve  $B := N \cap S$  will be the binding of the open book decomposition we are looking for. It can be proved that the following holds for every regular value  $a$ ,  $0 < a \leq 3/2$  :

- the level set  $f^{-1}(a)$  is the union along their common boundaries of two surfaces,  $N_1^a$  and  $N_2^a$ , each one being diffeomorphic to the sub-level surface  $S^a := S \cap f^{-1}([0, a])$ ;
- the sub-level  $M^a := f^{-1}([0, a])$  is divided by  $S^a$  into two parts  $P_1^a$  and  $P_2^a$  which are isomorphic handlebodies (with corners);
- $S^a$  is isotopic to  $N_i^a$  through  $P_i^a$ , for  $i = 1, 2$ , by an isotopy fixing its boundary curve  $S^a \cap f^{-1}(a)$ .

This claim is obvious when  $a$  is small and the property is preserved when crossing the critical level 1. In this way the handlebody  $H_- := f^{-1}([0, 3/2])$  is divided by  $S^{3/2}$  into two diffeomorphic parts  $P_i^{3/2}$ ,  $i = 1, 2$ , and we have  $N = N_1^{3/2} \cup N_2^{3/2}$ . We take  $S^{3/2}$ , which is isotopic to  $N_i^{3/2}$  in  $P_i^{3/2}$ , as a page. The figure is the same in  $H_+ := f^{-1}([3/2, 3])$ . The open book decomposition is now clear.

**Proposition 3.1.** *Let  $K \subset M$  be a compact connected co-orientable surface whose boundary is not empty. Then there exists an open book decomposition whose some page contains  $K$  in its interior.*

**Proof.** (Giroux) According to the above discussion it is sufficient to find a Morse function  $f$  and a Giroux surface  $S$  (with respect to  $f$ ) containing  $K$ . Let  $H_0$  be the quotient of  $K \times [-1, 1]$  by shrinking to a point each interval  $\{x\} \times [-1, 1]$  when  $x \in \partial K \times [-1, 1]$ . After smoothing, it is a handlebody whose boundary is the double of  $K$ . On  $H_0$  there exists a standard Morse function  $f_0$  which is constant on  $\partial H_0$ , having one minimum, the other critical points being of index 1. The surface  $K \times \{0\}$  can be made  $f_0$ -essential. This function is then extended to a



global Morse function  $\tilde{f}_0$  on  $M$ . At this point we have to follow the proof of Theorem III.2.7 in [5]. The function  $\tilde{f}_0$  is changed on the complement of  $H_0$ , step by step when crossing its critical level, so that  $K \times \{0\}$  extends as a Giroux surface in  $M$ .  $\square$

Let now  $\xi$  be a  $\Gamma_1$ -structure meeting the conclusion of proposition 2.1, up to rescaling the interval to  $[-\varepsilon, +\varepsilon]$ . Let  $\mathcal{F}_\varphi$  be the suspension foliation induced by  $\xi$  on  $\Sigma \times [-\varepsilon, +\varepsilon]$ . Choose  $x_0 \in \Sigma \times \{0\}$ ; the segment  $x_0 \times [-\varepsilon, +\varepsilon]$  is transverse to  $\mathcal{F}_\varphi$ . Let  $K$  be the surface obtained from  $\Sigma \times 0$  by removing a small open disk centered at  $x_0$ . The foliation  $\mathcal{F}_\varphi$  foliates  $K \times [-\varepsilon, +\varepsilon]$  so that  $K \times \{t\}$  lies in a leaf, when  $t$  is close to  $\pm\varepsilon$ , and  $\partial K \times [-\varepsilon, +\varepsilon]$  is foliated by parallel circles. We apply proposition 3.1 to this  $K$ .

**Corollary 3.2.** *There exists an open book foliation  $\mathcal{O}$  of  $M$  inducing the trivial foliation on  $K \times [-\varepsilon, +\varepsilon]$  (the leaves are  $K \times \{t\}$ ,  $t \in [-\varepsilon, +\varepsilon]$ ).*

Therefore, we have an open book foliation modified by suspension by replacing the above trivial foliation of  $K \times [-\varepsilon, +\varepsilon]$  by  $\tilde{\mathcal{F}}_\varphi$ , the trace of  $\mathcal{F}_\varphi$  on  $K \times [-\varepsilon, +\varepsilon]$ . Let  $\mathcal{O}_\varphi$  be the resulting foliation of  $M$  and  $\xi_\varphi$  be its regular  $\Gamma_1$ -structure. For proving theorem 1.2 (when  $r \geq 2$ ) it is sufficient to prove that  $\xi$  and  $\xi_\varphi$  are homotopic. This is done in the next section.

#### 4. HOMOTOPY OF $\Gamma_1$ -STRUCTURES

We are going to describe a homotopy from  $\xi_\varphi$  to  $\xi$ . Recall the tube  $T$  around the binding. For simplicity, we assume that each component of  $T$  is foliated by a standard Reeb foliation; the same holds true if  $T$  is foliated by Wood components (in the sense of [17]). Let  $T'$  be a slightly larger tube.

**Lemma 4.1.** *There exists a homotopy, relative to  $M \setminus \text{int}(T')$ , from  $\xi_\varphi$  to a new  $\Gamma_1$ -structure  $\xi_1$  on  $M$  such that:*

- 1)  $\xi_1$  is trivial on  $T$ ;
- 2)  $\xi_1$  is regular on  $T' \setminus \text{int}(T)$  with compact toric leaves near  $\partial T$  and spiraling half-cylinder leaves with boundary in  $\partial T'$  (as in an open book foliation).

**Proof.** Recall from the introduction that we only use *thick* Reeb components. So there is a third concentric tube  $T''$ ,  $T \subset \text{int}(T'') \subset \text{int}(T')$ , so that  $T'' \setminus \text{int}(T)$  is foliated by toric leaves; and  $\text{int}(T)$  is foliated by planes.

On each component of  $\partial T$  we have coordinates  $(x, y)$  coming from the framing of the binding, the  $x$ -axis being a parallel and the  $y$ -axis being a meridian. Let  $\gamma$  be a parallel in  $\partial T$ . The  $\Gamma_1$ -structure which is induced by  $\xi_\varphi$  on  $\gamma$  is singular but not trivial and the germ of foliation  $\mathcal{G}$  along the zero section in the normal line bundle  $A \cong \gamma \times \mathbb{R}$  is shown on figure 1.

The annulus  $A$  is endowed with coordinates  $(x, z) \in \gamma \times \mathbb{R}$ . The orientation of the  $z$ -axis, which is also the orientation of the normal bundle to the foliation  $\mathcal{O}_\varphi$  along  $\gamma$ , points to the interior of  $T$ . So the leaves of  $\mathcal{G}$  are parallel circles in  $\{z \leq 0\}$  and spiraling leaves in  $\{z > 0\}$ . Take coordinates  $(x, y, r)$  on  $T$  where  $r$  is the distance to the binding; say that  $r = 1$  on  $\partial T''$  and  $r = 1/2$  on  $\partial T$ . Let  $\lambda(r^2)$  be an even smooth function with  $r = 0$  as unique critical point, vanishing at  $r = 1/2$  and  $\lambda(1) < 0 < \lambda(0)$ . Consider  $g : T'' \rightarrow A$ ,  $g(x, y, r) = (x, \lambda(r^2))$ .

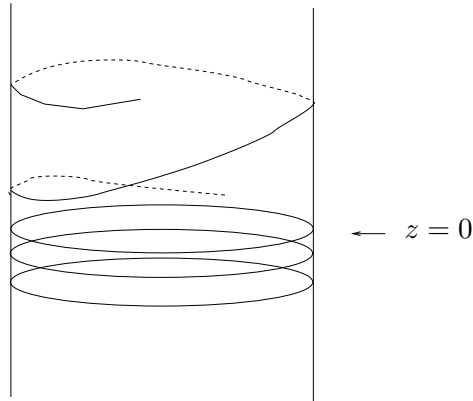


Figure 1

It is easily seen that  $\xi_\varphi|_{\text{int}(T'')} \cong g^*\mathcal{G}$ . Let now  $\bar{\lambda}(r^2)$  be a new even function coinciding with  $\lambda(r^2)$  near  $r = 1$ , having negative values everywhere and whose critical set is  $\{r \in [0, 1/2]\}$ . Let  $\bar{g} : T'' \rightarrow A$ ,  $\bar{g}(x, y, r) = (x, \bar{\lambda}(r^2))$ . A barycentric combination of  $\lambda$  and  $\bar{\lambda}$  yields a homotopy from  $g$  to  $\bar{g}$  which is relative to a neighborhood of  $\partial T''$ . The  $\Gamma_1$ -structure  $\xi_1$  we are looking for is defined by  $\xi_1|_{\text{int}(T'')} = \bar{g}^*(\mathcal{G})$  and  $\xi_1 = \xi_\varphi$  on a neighborhood of  $M \setminus \text{int}(T'')$ .  $\square$

Recall the domain  $K \times [-\varepsilon, +\varepsilon]$  from the previous section. After the following lemma we are done with the homotopy problem.

**Lemma 4.2.** *There exists a homotopy from  $\xi_1$  to  $\xi$  relative to  $K \times [-\varepsilon, +\varepsilon]$ .*

**Proof.** Let us denote  $M' := M \setminus \text{int}(K \times [-\varepsilon, +\varepsilon])$  which is a manifold with boundary and corners. It is equivalent to prove that the restrictions of  $\xi_1$  and  $\xi$  to  $M'$  are homotopic relatively to  $\partial M'$ . Consider the standard closed 2-disk  $D = D^2$  endowed with the  $\Gamma_1$ -structure  $\xi_D$  which is shown on figure 2.

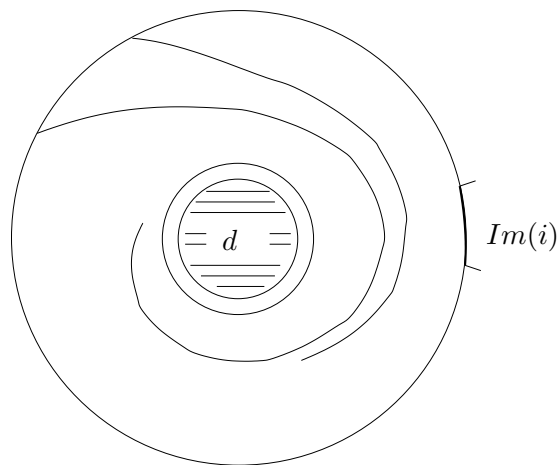


Figure 2



It is trivial on the small disk  $d$  and regular on the annulus  $D \setminus \text{int}(d)$ . In the regular part, the leaves are circles near  $\partial d$  and the other leaves are spiraling, crossing  $\partial D$  transversely. One checks that the restriction of  $\xi_1$  to  $M'$  has the form  $f^*\xi_D$  from some map  $f : M' \rightarrow D$ . We take  $f|_T : T \rightarrow d$  to be the open book trivialization of  $T$  (recall the binding has a canonical framing);  $f|\partial K \times [-\varepsilon, +\varepsilon]$  to be the projection  $pr_2$  onto  $[-\varepsilon, +\varepsilon]$  composed with an embedding  $i : [-\varepsilon, +\varepsilon] \rightarrow \partial D$  and  $f$  maps each leaf of the regular part of  $\xi_1$  to a leaf of the regular part of  $\xi_D$ . As  $K$  does not approach  $T$ , we can take  $f(\partial M') = i([-\varepsilon, +\varepsilon])$ ; actually, except near  $T$ ,  $f$  is given by the fibration over  $S^1 = \partial D^2$  of the open book decomposition.

Once  $\xi$  has Tsuboi's form (according to proposition 2.1), the restriction of  $\xi$  to  $M'$  has a similar form:  $\xi = k^*\xi_D$  for some map  $k : M' \rightarrow D^2$ . Recall that  $M'$  is the union of two handlebodies and a solid cylinder  $D^2 \times [-\varepsilon, +\varepsilon]$ . Take  $k$  to be  $i \circ pr_2$  on the cylinder and  $k$  to be constant on each handlebody. Observe that  $f$  and  $k$  coincide on  $\partial M'$ . As  $D$  retracts by deformation onto the image of  $i$ , one deduces that  $f$  and  $k$  are homotopic relatively to  $\partial M'$ .  $\square$

This finishes the proof of theorem 1.2 when  $r \geq 2$ .

## 5. THE CASE $C^{1+bv}$

A co-oriented  $\Gamma_1^r$ -structure  $\xi$  on  $M$  can be realized by a foliation  $\mathcal{F}$  defined on a neighborhood of the 0-section in  $M \times \mathbb{R}$ ; it is made of bi-foliated charts which are  $C^\infty$  in the direction of the leaves and  $C^r$  in the direction of the fibers. Consider such a box  $\mathcal{U}$  over an open disk  $D$  centered at  $x_0 \in M$ ; its trace on the  $x_0$ -fiber is an interval  $I$ . Each leaf of  $\mathcal{U}$  reads  $z = f(x, t)$ ,  $x \in D$ , for some  $t \in I$ . Here  $f$  is a function which is smooth in  $x$  and  $C^r$  in  $t$ , with  $f(x_0, t) = t$ ; the foliating property is equivalent to  $\frac{\partial f}{\partial t} > 0$ . When  $r = 1 + bv$ , there is a positive measure  $\mu(x, t)$  on  $I$ , without atoms and depending smoothly on  $x$ , such that:

$$(*) \quad \frac{\partial}{\partial t} f(x, t) - \frac{\partial}{\partial t} f(x, t_0) = \int_{t_0}^t \mu(x, t).$$

**Proposition 5.1.** *Theorem 1.2 holds true for any class of regularity  $r \geq 1$  including the class  $r = 1 + bv$ .*

**Proof.** The only part of the proof which requires some care of regularity is section 2, especially the proof of lemma 2.3. Indeed, we have to avoid integrating  $C^0$  vector fields. For proving lemma 2.3 with weak regularity we use the lemmas below which we shall prove in the case  $r = 1 + bv$  only.

**Lemma 5.2.** *Let  $f : D \times I \rightarrow \mathbb{R}$  be a  $C^r$ -function as above. Assume  $I = ] - 2\varepsilon, +\varepsilon[$  and  $f(x, -2\varepsilon) > -1$  for every  $x \in D$ . Then there exists a function  $F : D \times [-1, 0] \rightarrow \mathbb{R}$  of class  $C^r$  such that:*

- 1)  $F(x, t) = f(x, t)$  when  $t \in [-\varepsilon, 0]$ ,
- 2)  $F(x, t) = t$  when  $t$  is close to  $-1$ ,
- 3)  $\frac{\partial F}{\partial t} > 0$ .

**Proof.** Let  $\mu(x, t)$  be the positive measure whose support is  $[-\varepsilon, 0]$  such that formula (\*) holds for every  $(x, t) \in D \times [-\varepsilon, 0]$  and  $t_0 = 0$ . There exists another positive measure  $\nu(x, t)$ , smooth in  $x$  and whose support is contained in  $] - 1, -\varepsilon]$ , such that

$$(**) \quad f(x, -\varepsilon) = -1 + \int_{-1}^{-\varepsilon} \left( \int_{-1}^t \nu(x, \tau) \right) dt.$$

Then a solution is

$$F(x, t) = -1 + \int_{-1}^t \left( \int_{-1}^s (\mu(x, \tau) + \nu(x, \tau)) \right) ds.$$

□

**Lemma 5.3.** *Let  $A_1$  and  $A_2$  two disjoint closed sub-disks of  $D$ . Let  $F_1$  and  $F_2$  be two solutions of lemma 5.2. Then there exists a third solution which equals  $F_1$  when  $x \in A_1$  and  $F_2$  when  $x \in A_2$ .*

**Proof.** Both solutions  $F_1$  and  $F_2$  differ by the choice of the measure  $\nu(x, t)$  in formula (\*\*), which is  $\nu_i$  for  $F_i$ . Choose a partition of unity  $1 = \lambda_1(x) + \lambda_2(x)$  with  $\lambda_i = 1$  on  $A_i$ . Then  $\nu(x, t) = \lambda_1(x)\nu_1(x, t) + \lambda_2(x)\nu_2(x, t)$  yields the desired solution. □

The proof of proposition 5.1 is now easy. As already said, it is sufficient to prove lemma 2.3 in class  $C^r$ ,  $r \geq 1$ . It is an extension problem of a foliation given near the 1-skeleton  $Tr^{[1]} \times \{0\}$  to  $Tr^{[1]} \times [-1, 0]$ . One covers  $Tr^{[1]}$  by finitely many  $n$ -disks  $D_j$ . The problem is solved in each  $D_j \times [-1, 0]$  by applying lemma 5.2. By applying lemma 5.3 one makes the different extensions match together. □

## 6. HOMOTOPY CLASS OF PLANE FIELDS

It is possible to enhance theorem 1.1 by prescribing the homotopy class of the underlying co-oriented plane field (see proposition 6.1 below). The question of doing the same with respect to theorem 1.2 is more subtil (see proposition 6.3).

**Proposition 6.1.** *Given a co-oriented  $\Gamma_1$ -structure  $\xi$  on the closed 3-manifold  $M$  and a homotopy class  $[\nu]$  of co-oriented plane field in the tangent space  $\tau M$ , there exists a regular  $\Gamma_1$ -structure  $\xi_{reg}$  homotopic to  $\xi$  whose underlying foliation  $\mathcal{F}_{reg} \cap M$  has a tangent co-oriented plane field in the class  $[\nu]$ .*

Before proving it we first recall some well-known facts on co-oriented plane fields (see [4]). Given a base plane field  $\nu_0$ , a suitable Thom-Pontryagin construction yields a natural bijection between the set of homotopy classes of plane fields on  $M$  and  $\Omega_1^{\nu_0}(M)$ , the group of (co)bordism classes of  $\nu_0$ -framed and oriented closed (maybe non-connected) curves in  $M$ . A  $\nu_0$ -framing of the curve  $\gamma$  is an isomorphism of fiber bundles  $\varepsilon : \nu(\gamma, M) \rightarrow \nu_0|_{\gamma}$ , whose source is the normal bundle to  $\gamma$  in  $M$ . We denote  $\gamma^\varepsilon$  the curve endowed with this framing. Moreover, given  $\gamma^\varepsilon$ , if  $\gamma'$  is homologous to  $\gamma$  in  $M$  there exists a  $\nu_0$ -framing  $\varepsilon'$  such that  $(\gamma')^{\varepsilon'}$  is cobordant to  $\gamma^\varepsilon$ .

**Proof of 6.1.** We can start with an open book foliation  $\mathcal{O}_\varphi$  yielded by theorem 1.2. Let  $\nu_0$  be its tangent plane field. Near the binding, the meridian loops (out of  $T$ ) are transverse to  $\mathcal{O}_\varphi$  and homotopic to zero in  $M$ . As a consequence, each 1-homology class may be represented as well by a (multi)-curve in a page or by a connected curve out of  $T$  positively transverse to all pages. We do the second choice for  $\gamma^\varepsilon$ , the  $\nu_0$ -framed curve whose cobordism class encodes  $[\nu]$  with respect to  $\nu_0$ .

Hence we are allowed to *turbulize*  $\mathcal{O}_\varphi$  along  $\gamma$ . In a small tube  $T(\gamma)$  about  $\gamma$ , we put a Wood component. Outside, the leaves are spiraling around  $\partial T(\gamma)$ . Let  $\mathcal{O}_\varphi^{turb}$  be the resulting foliation. Whatever the chosen type of Wood component is, the  $\Gamma_1$ -structures of  $\mathcal{O}_\varphi^{turb}$  and  $\mathcal{O}_\varphi$  are homotopic by arguing as in section 4. But the framing  $\varepsilon$  tells us which sort of Wood component will be convenient for getting the desired class  $[\nu]$  (see lemma 6.1 in [17]).  $\square$

In the previous statement, we have lost the nice model we found in theorem 1.2. Actually, thanks to a lemma of Vincent Colin [3], it is possible to recover our model, at least when  $M$  is orientable (see below proposition 6.3).

**Lemma 6.2.** (Colin) *Let  $(B, p)$  be an open book decomposition of  $M$  and  $\gamma$  be a simple connected curve in some page  $P$ . Assume  $\gamma$  is orientation preserving. Then there exist a positive stabilization  $(B', p')$  of  $(B, p)$  and a curve  $\gamma'$  in  $B'$  which is isotopic to  $\gamma$  in  $M$ . When  $\gamma$  is a multi-curve, the same holds true after a sequence of stabilizations.*

The *positive Hopf open book decomposition* of the 3-sphere is the one whose binding is made of two unknots with linking number  $+1$ ; a page is an annulus foliated by fibers of the Hopf fibration  $S^3 \rightarrow S^2$ . A positive stabilization is a “connected sum” with this open book. The new page  $P'$  is obtained from  $P$  by *plumbing* an annulus  $A$  whose core bounds a disk in  $M$  (see [7] for more details and other references).

**Proof.** If  $\gamma$  is connected, only one stabilization is needed. We are going to explain this case only. A tubular neighborhood of  $\gamma$  in  $P$  is an annulus.

Choose a simple arc  $\alpha$  in  $P$  joining  $\gamma$  to some component  $\beta$  of  $B$  without crossing  $\gamma$  again. Let  $\tilde{\gamma}$  be a simple arc from  $\beta$  to itself which follows  $\alpha^{-1} * \gamma * \alpha$ . The orientation assumption implies that the surgery of  $\beta$  by  $\tilde{\gamma}$  in  $P$  provides a curve with two connected components, one of them being isotopic to  $\gamma$  in  $P$ . Let  $P_\pi$  be the page opposite to  $P$  and  $R: P \rightarrow P_\pi$  the time  $\pi$  of a flow transverse to the pages (and stationary on  $B$ ). The core curve  $C$  of the annulus  $A$  that we use for the plumbing is the union  $\tilde{\gamma} \cup R(\tilde{\gamma})$ . And  $A$  is  $(+1)$ -twisted around  $C$  (with respect to its unknot framing) as in the Hopf open book. Let  $H$  be the 1-handle which is the closure of  $A \setminus P$ . Surgering  $B$  by  $H$  provides the new binding. By construction, one of its components is isotopic to  $\gamma$ .  $\square$

**Proposition 6.3.** *Let  $\mathcal{O}_\varphi$  be an open book foliation modified by suspension, whose its underlying open book is denoted  $(B, p)$ . Let  $\nu_0$  be its tangent co-oriented plane field. Let  $\gamma^\varepsilon$  be a  $\nu_0$ -framed curve in  $M$  and  $[\nu]$  be its associated class of plane field. Assume  $\gamma$  is orientation preserving. Then there exists an open book foliation  $\mathcal{O}'_\varphi$  with the following properties:*

- 1) *its tangent plane field is in the class  $[\nu]$ ;*
- 2) *the suspension modification is the same for  $\mathcal{O}'_\varphi$  as for  $\mathcal{O}_\varphi$  and is supported in  $K \times [-\varepsilon, +\varepsilon]$ ;*

3) as  $\Gamma_1$ -structures,  $\mathcal{O}_\varphi$  and  $\mathcal{O}'_\varphi$  are homotopic.

**Proof.** As said in the proof of 6.1, up to framed cobordism,  $\gamma^\varepsilon$  may be chosen as a simple (multi)-curve in one page  $P$  of  $(B, p)$ . Applying Colin's lemma provides a stabilization  $(B', p')$  such that, up to isotopy,  $\gamma$  lies in the new binding. Observe that, if  $K$  is in  $P$ ,  $K$  is still in the new page  $P'$ ; hence 2) holds for any open book foliation carried by  $(B', p')$ . Once  $\gamma^\varepsilon$  is in the binding, for a suitable Wood component foliating a tube about  $\gamma^\varepsilon$ , item 1) is fulfilled. Finally item 3) follows from item 2) and the proofs in section 4.  $\square$

## 7. CASE OF A $\Gamma_1$ -STRUCTURE WITH A TWISTED NORMAL BUNDLE

What happens when the bundle  $\nu$  normal to  $\xi$  is twisted? It is known that a necessary condition to regularization is the existence of a fibered embedding  $i : \nu \rightarrow \tau M$  into the tangent fiber bundle to  $M$ . Conversely, assuming that this condition is fulfilled, we are going to state a normal form theorem analogous to theorem 1.2. Since no step of the previous proof can be immediately adapted to this situation, we believe that it deserves a sketch of proof.

**7.1.** In the first step (Tsuboi's construction), we do not have "Morse position" with respect to a triangulation, since index and co-index of a singularity cannot be distinguished. Instead of lemma 2.2, we have the following statement.

*After some homotopy,  $\xi$  has Morse singularities and admits a pseudo-gradient whose dynamics has no recurrence (that is, every orbit has a finite length).*

Here, by a pseudo-gradient, it is meant a smooth section  $X$  of  $Hom(\nu, \tau M)$ , a *twisted vector field* indeed, such that  $X \cdot \xi < 0$  except at the singularities (this sign is well-defined whatever a local orientation of  $\nu$ , or co-orientation of  $\xi$ , is chosen); such a pseudo-gradient always exists by using an auxiliary Riemannian metric.

**Sketch of proof.** Generically  $\xi$  has Morse singularities. Let  $X_0$  be a first pseudo-gradient, which is required to be the usual negative gradient in Morse coordinates near each singularity. Finitely many mutually disjoint 2-disks of  $M$  are chosen in regular leaves of  $\xi$  such that every orbit of  $X_0$  crosses one of them. Following Wilson's idea [15],  $\xi$  and  $X_0$  are changed in a neighborhood  $D^2 \times [-1, +1]$  of each disk into a *plug* such that every orbit of the modified pseudo-gradient  $X$  is trapped by one of the plugs. The plug has the mirror symmetry with respect to  $D^2 \times \{0\}$ . In  $D^2 \times [0, 1]$  we just modify  $\xi$  by introducing a cancelling pair of singularities, center-saddle.  $\square$

Let  $G$  be the closure of the one-dimensional invariant manifold of all saddles. It is a graph. We claim:  $\nu|_G$  is orientable. Indeed, we orient each edge from its saddle end point to its center end point. This is an orientation of  $\nu|_G$  over the complement of the vertices. It is easily checked that this orientation extends over the vertices. Thus  $X$  becomes a usual vector field near  $G$  and we have an arbitrarily small tubular neighborhood  $H$  of  $G$  whose boundary is transverse to  $X$ , and  $X$  enters  $H$ . Now, the negative completion of  $\xi|_H$  can be performed as in lemma 2.3.

The complement  $\hat{M}$  of  $\text{int } H$  in  $M$  is fibered over a surface  $\Sigma$ , the fibers being intervals ( $\cong [-1, 1]$ ) tangent to  $X$ . By taking a section we think of  $\Sigma$  as a surface in  $M \setminus H$ . Since

$\xi$  is not co-orientable,  $\Sigma$  is one-sided and  $G$  is connected. Arguing as in section 2, after some homotopy,  $\xi$  becomes trivial on  $H$  and transverse to  $X$  on  $\hat{M}$ , hence a suspension foliation corresponding to a representation  $\varphi : \pi_1(\Sigma) \rightarrow \text{Diff}_c([-1, 1])$ .

**7.2.** In the second step (Giroux's construction), we have to leave the open books and we need a *twisted open book*. It is made of the following:

- a binding  $B$  which is a 1-dimensional closed co-orientable submanifold in  $M$ ;
- a *Seifert fibration*  $p : M \setminus B \rightarrow [-1, +1]$  which has two one-sided exceptional surface fibers  $p^{-1}(\pm 1)$  and which is a proper smooth submersion over the open interval;
- when  $t$  goes to  $\pm 1$ ,  $p^{-1}(t)$  tends to a 2-fold covering of  $p^{-1}(\pm 1)$ ;
- near  $B$  the foliation looks like an open book.

The exceptional fibers are compactified by  $B$  as smooth surfaces with boundary. But, for  $t \in ]-1, +1[$ ,  $p^{-1}(t)$  is compactified by  $B$  as a closed surface showing (in general) an angle along  $B$ . Notice that, since  $B$  is co-orientable, a twisted open book gives rise to a smooth foliation where each component of the binding is replaced by a Reeb component, the pages spiraling around it.

Such an open book is generated by a *one-sided Giroux surface*, which is the union of the compactified exceptional fibers. Abstractly, a one-sided Giroux surface in  $M$  with respect to a Morse function  $f : M \rightarrow \mathbb{R}$  is a one-sided surface  $S$  such that  $f|_S$  has the same critical points and the same extrema as  $f$  and fulfills the extra condition: for every regular value  $t \in \mathbb{R}$ ,  $f^{-1}(t) \cap S$  is a two-sided curve in the level set  $f^{-1}(t)$ . Starting with  $(S, f)$  where  $f$  is a self-indexing Morse function, a twisted open book is easily constructed. Its binding is the co-orientable curve  $f^{-1}(3/2) \cap S$ . In general such a one-sided Giroux surface (or twisted open book) does not exist on  $M$ ; the obstruction lies in the existence of a twisted line subbundle of  $\tau M$ . Nevertheless, with a suitable assumption, we have an analogue of proposition 3.1:

*Let  $i : \nu \rightarrow \tau M$  be an embedding of a twisted line bundle. Let  $K \subset M$  be a compact connected one-sided surface whose boundary is not empty. Assume the following:  $\nu|_K$  is twisted,  $\nu|\partial K$  is trivial and the normal bundle  $\nu(K, M)$  is homotopic to  $i(\nu)|_K$ . Then there exists a twisted open book with one exceptional page containing  $K$  in its interior.*

**Sketch of proof.** We give the proof only in the setting of 7.1 by taking  $K$  to be the closure of  $\Sigma \setminus d_0$  where  $d_0$  is a small closed 2-disk in  $\Sigma$ . Recall the fibration  $\rho : \hat{M} \rightarrow \Sigma$ . Let  $H'$  be the handlebody made of  $H$  to which is glued the 1-handle  $\rho^{-1}(d_0)$ . Let  $M'$  be the complement of  $\text{int } H'$ . Consider a minimal system of mutually disjoint compression disks  $d_1, \dots, d_g$  of  $H$ , so that cutting  $H$  along them yields a ball;  $g$  is the genus of  $H$ . Thus  $d_0, d_1, \dots, d_g$  is a minimal system of mutually disjoint compression disks of  $H'$ .

We claim:  $g + 1$  is even. Indeed, by assumption there exists a non vanishing section of the bundle  $\text{Hom}(\nu, \tau M)$ . Thus, the number of zeroes of the pseudo-gradient  $X$  is even and the Euler characteristic  $\chi(G)$  of the graph  $G$  is even. As  $G$  is connected, the genus  $g$  is odd, which proves the claim.

Now, one follows Giroux's algorithm for completing  $K$  to a closed Giroux's surface. On the surface  $\partial M'$  the union of the attaching curves  $\partial d_0, \partial d_1, \dots, \partial d_g$  is not separating. Then, after some isotopy, for each  $i = 0, \dots, g$ ,  $\partial d_i$  crosses  $\partial K$  in exactly two points  $a_i, b_i$  linked by an arc  $\alpha_i$

(resp.  $\alpha'_i$ ) in  $\partial d_i$  (resp.  $\partial K$ ), so that  $\alpha_i \cup \alpha'_i$  bounds a disk in  $\partial M'$ . Moreover, one can arrange that all the arcs  $\alpha_0, \dots, \alpha_g$  are parallel. Also we link  $a_i, b_i$  by a simple arc in  $d_i$ . Now, each compression disk defines, simultaneously, a 1-handle which is glued to  $K$  and a 2-handle which is glued to  $M'$ , yielding a proper surface  $K_1$  in some 3-submanifold  $M''$  of  $M$ , whose complement is a ball. The boundary of  $K_1$  is made of  $g + 2$  parallel curves in the sphere  $\partial M''$ . As this number is odd, Giroux described a process of adding cancelling pairs of 1- and 2- handles whose effect is to change  $K_1$  into  $K_2 \subset M''$  such that  $\partial K_2$  is made of one curve only ([5], p. 676-677). Hence,  $K_2$  can be closed into a Giroux's surface.  $\square$

**Theorem 7.3.** *Let  $\xi$  be a non co-orientable  $\Gamma_1$ -structure on  $M^3$  whose normal bundle  $\nu$  embeds into  $\tau M$ . Then  $\xi$  is homotopic to a twisted open book foliation modified by suspension.*

**Sketch of proof.** Let  $K = \Sigma \setminus \text{int } D^2$  be the surface with a hole, where  $\Sigma$  was built in the first step 7.1; it meets the required assumptions for building a twisted open book.

The twisted open book built in the second step gives rise to a foliation  $\mathcal{O}$ . Indeed, as the binding  $B$  is co-orientable, it is allowed to spiral the pages around a tubular neighborhood of  $B$ . The tube itself is foliated by thick Reeb components. As in the co-orientable case, we can modify the open book foliation in a neighborhood of  $K$  using the representation  $\varphi$ , yielding the foliation  $\mathcal{O}_\varphi$  and its associated regular  $\Gamma_1$ -structure  $\xi_\varphi$ . We have to prove that  $\xi$  and  $\xi_\varphi$  are homotopic. We may suppose that  $\xi$  is in Tsuboi form (7.1).

We observe that the total space of  $\nu$  has a foliation  $\mathcal{F}_0$  (unique up to isomorphism) transverse to the fibers, having the zero section as a leaf and whose all non trivial holonomy elements have order 2. It defines the trivial  $\Gamma_1$ -structure  $\xi_0$  in the twisted sense. Using notation of 7.2, one can prove that  $\xi|_{H'}$  and  $\xi_\varphi|_{H'}$  are both homotopic to  $\xi_0|_{H'}$ . Moreover, both homotopies coincide on the boundary  $\partial H'$  (on  $H'$ , it is sufficient to think of the case when  $\varphi$  is the trivial representation). Thus  $\xi|_{H'}$  and  $\xi_\varphi|_{H'}$  are homotopic relative to the boundary.  $\square$

**7.4. Plane field homotopy class.** *By turbulizing  $\mathcal{O}_\varphi$ , it is possible to have the normal field in any homotopy class of embeddings  $\nu \rightarrow \tau M$ .*

Indeed, a curve in  $M$  is homotopic to a curve transverse to  $\mathcal{O}_\varphi$  if and only if it does not twist  $\nu$ . But these homology classes are exactly those which appear as a first difference homology class when comparing two embeddings  $j_1, j_2 : \nu \rightarrow \tau M$ , since a closed curve which twists  $\nu$  is not a cycle in  $H_1(M, \mathbb{Z}_{\text{or}(\nu)}) \cong H^2(M, \mathbb{Z}_{\text{or}(\nu^* \otimes \tau M)})$ .

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