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# REGULARIZATION OF $\Gamma_1$ -STRUCTURES IN DIMENSION 3

FRANÇOIS LAUDENBACH AND GAËL MEIGNIEZ

ABSTRACT. For co-orientable  $\Gamma_1$ -structures on 3-manifolds, we give a very simple proof of Thurston's regularization theorem, first proved in [16], without using the perfectness of the group of diffeomorphisms of the circle, nor Mather's homology equivalence. In particular it applies in classes of regularity where perfectness does not hold (or is not known to hold). Moreover the resulting foliation can be chosen of a precise kind, namely an "open book foliation modified by suspension".

## 1. INTRODUCTION

A  $\Gamma_1$ -structure  $\xi$ , in the sense of A. Haefliger, on a manifold  $M$  is given by a line bundle  $\nu = (E \rightarrow M)$ , called the *normal bundle* to  $\xi$ , and a germ of codimension-one foliation  $\mathcal{F}$  along the zero section, which is required to be transverse to the fibers (see [8]). Consider first the co-orientable case, that is, the normal bundle is trivial:  $E \cong M \times \mathbb{R}$ . The  $\Gamma_1$ -structure  $\xi$  is said to be *regular* when the foliation  $\mathcal{F}$  is transverse to the zero section, in which case the pullback of  $\mathcal{F}$  to  $M$  is a genuine foliation on  $M$ . A homotopy of  $\xi$  is defined as a  $\Gamma_1$ -structure on  $M \times [0, 1]$  inducing  $\xi$  on  $M \times \{0\}$ . A *regularization theorem* should claim that any  $\Gamma_1$ -structure is homotopic to a regular one. It is not true in general. An obvious necessary condition is that  $\nu$  must embed into the tangent bundle  $\tau M$ . When  $\nu$  is trivial and  $\dim M = 3$  this condition is fulfilled.

The  $C^\infty$  category is understood in the sequel, unless otherwise specified. In particular  $M$  is  $C^\infty$ . One calls  $\xi$  a  $\Gamma_1^r$ -structure ( $r \geq 1$ ) if it is tangentially  $C^\infty$  and transversely  $C^r$ , that is, the foliation charts are  $C^r$  in the direction transverse to the leaves. We will prove the following theorem.

**Theorem 1.1.** *If  $M$  is a closed 3-manifold and  $\xi$  a  $\Gamma_1^r$ -structure,  $r \geq 2$ , whose normal bundle is trivial, then  $\xi$  is homotopic to a regular  $\Gamma_1$ -structure.*

Moreover, the resulting foliation of  $M$  may have its tangent plane field in a prescribed homotopy class (see proposition 6.1).

When  $r > 2$  this theorem is a particular case of a general regularization theorem due to W. Thurston (see [16]). His proof was based on two deep results: the homology equivalence between  $\text{BDiff}_c(\mathbb{R})$  and  $\Omega\text{B}(\Gamma_1)_+$ , due to J. Mather [11]; and also, in dimension 3, the famous perfectness of the *discrete* group  $\text{Diff}^r(S^1)$ , due to M. Herman when  $r = \infty$  and to J. Mather when  $r > 2$  (see [9, 10] [12, 13]), and which is unknown at the present time when  $r = 2$ . We present a proof of this regularization theorem which is not based on any form of those results.

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As Étienne Ghys pointed out to us, the proof also works in the class of regularity  $r = 1 + bv$  (see proposition 5.1), the class he calls the Denjoy class (as it is the class of regularity where Denjoy's conjugation theorem holds). Here  $\text{Diff}^{1+bv}(S^1)$  means the set of  $C^1$ -diffeomorphisms  $f$  whose derivative has a bounded variation, that is,  $df$  (or  $\log(df)$ ) is the primitive of a Radon measure. J. Mather proved in [14] that the group  $\text{Diff}^{1+bv}(S^1)$  is not perfect: there is a surjective homomorphism  $\text{Diff}^{1+bv}(S^1) \rightarrow \mathbb{R}$ .

A regularization theorem in all dimensions, still avoiding any difficult results, is provided in [15]. But there are several reasons for considering the dimension 3 separately.

Our proof not only solves a regularity problem but it provides a model for each homotopy class of  $\Gamma_1$ -structure. This model relies on the notion of *open book decomposition*. Recall that such a structure on  $M$  consists of a link  $B$  in  $M$ , called the *binding*, and a fibration  $p : M \setminus B \rightarrow S^1$  such that, for every  $\theta \in S^1$ ,  $p^{-1}(\theta)$  is the interior of an embedded surface, called the *page*  $P_\theta$ , whose boundary is the binding. The existence of open book decomposition could be proved by J. Alexander when  $M$  is orientable, as a consequence of [1] (every orientable closed 3-manifold is a branched cover of the 3-sphere) and [2] (every link can be braided); but he was ignoring this concept which was introduced by H. Winkelnkemper in 1973 [19]. Henceforth, we refer to the more flexible construction by E. Giroux, which includes the non-orientable case (see section 3). An open book gives rise to a foliation  $\mathcal{O}$  constructed as follows. The pages endow  $B$  with a normal framing. So a tubular neighborhood  $T$  of  $B$  is trivialized:  $T \cong B \times D^2$ . Out of  $T$  the leaves are the pages modified by spiraling around  $T$ ; the boundary of  $T$  is a union of compact leaves; and the interior of  $T$  is foliated by a Reeb component, or a generalized Reeb component in the sense of Wood [20]. For technical reasons in the homotopy argument of section 4, the Reeb components of  $\mathcal{O}$ , instead of being usual Reeb components, will be *thick Reeb components* in which a neighborhood of the boundary is foliated by toric compact leaves. We call such a foliation an *open book foliation*.

The latter can be modified by inserting a so called *suspension foliation*. Precisely, let  $\Sigma$  be a compact sub-surface of some leaf of  $\mathcal{O}$  out of  $T$  and  $\Sigma \times [-1, +1]$  be a foliated neighborhood of it (each  $\Sigma \times \{t\}$  being contained in a leaf of  $\mathcal{O}$ ). Let  $\varphi : \pi_1(\Sigma) \rightarrow \text{Diff}^+([-1, +1])$  be some representation. It allows us to construct a *suspension foliation*  $\mathcal{F}_\varphi$  on  $\Sigma \times [-1, +1]$ , whose leaves are transverse to the vertical segments  $\{x\} \times [-1, +1]$  and whose holonomy is  $\varphi$ . The modification consists of removing  $\mathcal{O}$  from the interior of  $\Sigma \times [-1, +1]$  and replacing it by  $\mathcal{F}_\varphi$ . The new foliation, denoted  $\mathcal{O}_\varphi$ , is an *open book foliation modified by suspension*. Theorem 1.1 can now be made more precise:

**Theorem 1.2.** *Every co-orientable  $\Gamma_1^r$ -structure,  $r \geq 2$ , is homotopic to an open book foliation modified by suspension.*

In section 5, this theorem will be extended to  $r \geq 1$  including the case  $r = 1 + bv$ . In section 6, the homotopy class of the tangent plane field will be discussed. In this version we limit ourselves to the co-orientable case; the non co-orientable case will be treated in a forthcoming version.

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## 2. TSUBOI'S CONSTRUCTION

A  $\Gamma_1$ -structure  $\xi$  on  $M$  is said to be *trivial* on a codimension 0 submanifold  $W$  when, for every  $|t|$  small enough,  $W \times \{t\}$  lies in a leaf of the associated foliation.

Every closed 3-manifold  $M$  has a so-called *Heegaard decomposition*  $M = H_- \cup_{\Sigma} H_+$ , where  $H_{\pm}$  is a possibly non-orientable handlebody (a ball with handles of index 1 attached) and  $\Sigma$  is their common boundary. A *thick Heegaard decomposition* is a similar decomposition where the surface is thickened:

$$M = H'_- \cup_{\Sigma \times \{-1\}} \Sigma \times [-1, +1] \cup_{\Sigma \times \{+1\}} H'_+.$$

The following statement is due to T. Tsuboi in [17] where it is left to the reader as an exercise.

**Proposition 2.1.** *Given a  $\Gamma_1$ -structure  $\xi$  of class  $C^r$ ,  $r \geq 2$ , on a closed 3-manifold  $M$ , there exists a thick Heegaard decomposition and a homotopy  $(\xi_t)_{t \in [0,1]}$  from  $\xi$  such that:*

- 1)  $\xi_1$  is trivial on  $H'_{\pm}$ ;
- 2)  $\xi_1$  is regular on  $\Sigma \times [-1, +1]$  and the induced foliation is a suspension.

**Proof.** With  $\xi$  and its foliation  $\mathcal{F}$  defined on an open neighborhood of the zero section  $M \times 0$  in  $M \times \mathbb{R}$ , there comes a covering of the zero section by boxes, open in  $M \times \mathbb{R}$ , bi-foliated with respect to  $\mathcal{F}$  and the fibers. We choose a  $C^1$ -triangulation  $Tr$  of  $M$  so fine that each simplex lies entirely in a box. With  $Tr$  comes a vector field  $X$  defined as follows.

First, on the standard  $k$ -simplex there is a smooth vector field  $X_{\Delta^k}$ , tangent to each face, which is the (descending) gradient of a Morse function having one critical point of index  $k$  at the barycenter and one critical point of index  $i$  at the barycenter of each  $i$ -face. When  $\Delta^i \subset \Delta^k$  is an  $i$ -face,  $X_{\Delta^i}$  is the restriction of  $X_{\Delta^k}$  to  $\Delta^i$ . Now, if  $\sigma$  is a  $k$ -simplex of  $Tr$ , thought of as a  $C^1$ -embedding  $\sigma : \Delta^k \rightarrow M$ , we define  $X_{\sigma} := \sigma_*(X_{\Delta^k})$ . The union of the  $X_{\sigma}$ 's is a  $C^0$  vector field  $X$  which is uniquely integrable. After a reparametrization of each simplex we may assume that the stable manifold  $W^s(b(\sigma))$  of the barycenter  $b(\sigma)$  is  $C^1$ .

The  $\Gamma_1$ -structure  $\xi$  (co-oriented by the  $\mathbb{R}$  factor of  $M \times \mathbb{R}$ ) is said to be in *Morse position* with respect to  $Tr$  if:

- (i) it has a smooth Morse type singularity of index  $k$  at the barycenter of each  $k$ -simplex and it is regular elsewhere;
- (ii)  $X$  is (negatively) transverse to  $\xi$  out of the singularities.

**Lemma 2.2.** *Let  $\mathcal{F}$  be the foliation associated to  $\xi$ . There exists a smooth section  $s$  such that  $s^*\mathcal{F}$  is in Morse position with respect to  $Tr$ .*

Note that, as  $s$  is homotopic to the zero section, the  $\Gamma_1$ -structure  $s^*\mathcal{F}$  on  $M$  is homotopic to  $\xi$ .

**Proof.** Assume that  $s$  is already built near the  $(k-1)$ -skeleton. Let  $\sigma$  be a  $k$ -simplex. We explain how to extend  $s$  on a neighborhood of  $\sigma$ . After a fibered isotopy of  $M \times \mathbb{R}$  over the identity of  $M$ , we may assume that  $\mathcal{F}$  is trivial near  $\{b(\sigma)\} \times \mathbb{R}$ . Now, near  $b(\sigma)$ , we ask  $s$  to coincide with the graph of some local positive Morse function  $f_{\sigma}$  whose Hessian is negative definite on  $T_{b(\sigma)}\sigma$  and positive definite on  $T_{b(\sigma)}W^s(b(\sigma))$ . This function is now fixed up to a positive constant factor. We will extend  $s$  as the graph of some function  $h$  in the  $\mathcal{F}$ -foliated

chart over a neighborhood of  $\sigma$ . This function is already given on a neighborhood  $N(\partial\sigma)$  of  $\partial\sigma$  where it is  $C^r$ , the regularity of  $\xi$ , and satisfies  $X.h < 0$  except at the barycenter of each face.

On the one hand, choose an arbitrary extension  $h_0$  of  $h$  to a neighborhood of  $\sigma$ . On the other hand, choose a nonnegative function  $g_\sigma$  such that:

- $g_\sigma = 0$  near  $\partial\sigma$ ;
- $g_\sigma = f_\sigma$  near  $b(\sigma)$ ;
- $X.g_\sigma < 0$  when  $g_\sigma > 0$  except at  $b(\sigma)$ .

Then, if  $c > 0$  is a large enough constant,  $h := h_0 + cg_\sigma$  has the required properties, except smoothness. Returning to  $M \times \mathbb{R}$ , the section  $s$  we have built is  $C^r$ , smooth near the singularities, and  $X$  is transverse to  $s^*\xi$  except at the singularities. Therefore, there exists a smooth  $C^r$ -approximation of  $s$ , relative to a neighborhood of the barycenters which meets all the required properties.  $\square$

Thus, by a deformation of the zero section which induces a homotopy of  $\xi$ , we have put  $\xi$  in *Morse position* with respect to  $Tr$ . In the same way, applying lemma 2.2 to the trivial  $\Gamma_1$ -structure  $\xi_0$ , we also have a Morse function  $f$  such that  $X.f < 0$  except at the barycenters.

Let  $G_-$  (resp.  $G_+$ ) denote the closure of the union of the unstable (resp. stable) manifolds of the singularities of  $X$  of index 1 (resp. 2). The following properties are clear:

- (a) In  $M$ , the subset  $G_-$  (resp.  $G_+$ ) is a  $C^1$ -complex of dimension 1.
- (b) It admits arbitrarily small handlebody neighborhoods  $H'_-$  (resp.  $H'_+$ ) whose boundary is transverse to  $X$ .
- (c) Every orbit of  $X$  outside  $H'_\pm$  has one end point on  $\partial H'_-$  and the other on  $\partial H'_+$ . This also holds true for any smooth  $C^0$ -approximation  $\tilde{X}$  of  $X$  (in particular  $\tilde{X}$  is still negatively transverse to  $\xi$ ).

Given a (co-orientable)  $\Gamma_1$ -structure  $\xi$  on a space  $G$ , by an *upper* (resp. *lower*) *completion* of  $\xi$  one means a foliation  $\mathcal{F}$  of  $G \times (-\epsilon, 1]$  (resp.  $G \times [-1, \epsilon)$ ), for some positive  $\epsilon$ , which is transverse to every fiber  $\{x\} \times (-\epsilon, 1]$  (resp.  $\{x\} \times [-1, \epsilon)$ ), whose germ along  $G \times \{0\}$  is  $\xi$ , and such that  $G \times \{t\}$  is a leaf of  $\mathcal{F}$  for every  $t$  close enough to  $+1$  (resp.  $-1$ ).

**Lemma 2.3.** *Every co-orientable  $\Gamma_1^r$ -structure on a simplicial complex  $G$  of dimension 1,  $r \geq 2$ , admits an upper (resp. lower) completion of class  $C^r$ .*

PROOF. One reduces immediately to the case where  $G$  is a single edge. In that case, using a partition of unity, one builds a line field which fulfills the claim. This line field is integrable.  $\square$

By (a), the  $\Gamma_1$ -structure  $\xi$  admits an upper (resp. lower) completion over  $G_+$  (resp.  $G_-$ ), and thus also over an open neighborhood  $N_+$  (resp.  $N_-$ ) of  $G_+$  (resp.  $G_-$ ). By (b), there is a handlebody neighborhood  $H'_\pm$  of  $G_\pm$  contained in  $N_\pm$  and whose boundary is transverse to  $X$ . So we have a foliation  $\mathcal{F}$  defined on a neighborhood of

$$(M \times \{0\}) \cup (H'_- \times [-1, 0]) \cup (H'_+ \times [0, 1])$$

which is transverse to  $X$  on  $M \setminus (H'_- \cup H'_+)$  and tangent to  $H'_\pm \times \{t\}$  for every  $t$  close to  $\pm 1$ .

By (c), there is a diffeomorphism  $F : M \setminus \text{Int}(H'_- \cup H'_+) \rightarrow \Sigma \times [-1, +1]$  for some closed surface  $\Sigma$ , which maps orbit segments of  $\tilde{X}$  onto fibers.

For a small  $\epsilon > 0$ , choose a function  $\psi : \mathbb{R} \rightarrow [-1, +1]$  which is smooth, odd, and such that:

- $\psi(t) = 0$  for  $0 \leq t \leq 1 - 3\epsilon$  and  $\psi(1 - 2\epsilon) = \epsilon$ ;
- $\psi$  is affine on the interval  $[1 - 2\epsilon, 1 - \epsilon]$ ;
- $\psi(1 - \epsilon) = 1 - \epsilon$  and  $\psi(t) = 1$  for  $t \geq 1$ ;
- $\psi' > 0$  on the interval  $]1 - 3\epsilon, 1[$ .

Let  $s : M \rightarrow M \times \mathbb{R}$  be the graph of the function whose value is  $\pm 1$  on  $H'_\pm$  and  $\psi(t)$  at the point  $F^{-1}(x, t)$  for  $(x, t) \in \Sigma \times [-1, +1]$ . When  $\epsilon$  is small enough, it is easily checked that, for every  $x \in \Sigma$ , the path  $t \mapsto s \circ F^{-1}(x, t)$  is transverse to  $\mathcal{F}$  except at its end points. Then,  $\xi_1 := s^*\mathcal{F}$  is homotopic to  $\xi$  and obviously fulfills the conditions required in proposition 2.1.  $\square$

### 3. GIROUX'S CONSTRUCTION

We use here theorem III.2.7 from Giroux's article [5], which states the following:

*Let  $M$  be a closed 3-manifold (orientable or not). There exist a Morse function  $f : M \rightarrow \mathbb{R}$  and a co-orientable surface  $S$  which is  $f$ -essential in  $M$ .*

Giroux says that  $S$  is  $f$ -essential when the restriction  $f|_S$  has exactly the same critical points as  $f$  and the same local *extrema*. In the sequel, we call such a surface a *Giroux surface*.

Giroux explained to us [6] how this notion is related to open book decompositions. In the above statement, the function  $f$  can be easily chosen *self-indexing* (the value of a critical point is its Morse index in  $M$ ). Thus, let  $N$  be the level set  $f^{-1}(3/2)$ . The smooth curve  $B := N \cap S$  will be the binding of the open book decomposition we are looking for. It can be proved that the following holds for every regular value  $a$ ,  $0 < a \leq 3/2$ :

- the level set  $f^{-1}(a)$  is the union along their common boundaries of two surfaces,  $N_1^a$  and  $N_2^a$ , each one being diffeomorphic to the sub-level surface  $S^a := S \cap f^{-1}([0, a])$ ;
- the sub-level  $M^a := f^{-1}([0, a])$  is divided by  $S^a$  into two parts  $P_1^a$  and  $P_2^a$  which are isomorphic handlebodies (with corners);
- $S^a$  is isotopic to  $N_i^a$  through  $P_i^a$ , for  $i = 1, 2$ , by an isotopy fixing its boundary curve  $S^a \cap f^{-1}(a)$ .

This claim is obvious when  $a$  is small and the property is preserved when crossing the critical level 1. In this way the handlebody  $H_- := f^{-1}([0, 3/2])$  is divided by  $S^{3/2}$  into two diffeomorphic parts  $P_i^{3/2}$ ,  $i = 1, 2$ , and we have  $N = N_1^{3/2} \cup N_2^{3/2}$ . We take  $S^{3/2}$ , which is isotopic to  $N_i^{3/2}$  in  $P_i^{3/2}$ , as a page. The figure is the same in  $H_+ := f^{-1}([3/2, 3])$ . The open book decomposition is now clear.

**Proposition 3.1.** *Let  $K \subset M$  be a compact connected co-orientable surface whose boundary is not empty. Then there exists an open book decomposition whose some page contains  $K$  in its interior.*

**Proof.** (Giroux) According to the above discussion it is sufficient to find a Morse function  $f$  and a Giroux surface  $S$  (with respect to  $f$ ) containing  $K$ . Let  $H_0$  be the quotient of  $K \times [-1, 1]$  by shrinking to a point each interval  $\{x\} \times [-1, 1]$  when  $x \in \partial K \times [-1, 1]$ . After smoothing, it is a handlebody whose boundary is the double of  $K$ . On  $H_0$  there exists a standard Morse function  $f_0$  which is constant on  $\partial H_0$ , having one minimum, the other critical points being of index 1. The surface  $K \times \{0\}$  can be made  $f_0$ -essential. This function is then extended to a global Morse function  $\tilde{f}_0$  on  $M$ . At this point we have to follow the proof of Theorem III.2.7 in [5]. The function  $\tilde{f}_0$  is changed on the complement of  $H_0$ , step by step when crossing its critical level, so that  $K \times \{0\}$  extends as a Giroux surface in  $M$ .  $\square$

Let now  $\xi$  be a  $\Gamma_1$ -structure meeting the conclusion of proposition 2.1, up to rescaling the interval to  $[-\varepsilon, +\varepsilon]$ . Let  $\mathcal{F}_\varphi$  be the suspension foliation induced by  $\xi$  on  $\Sigma \times [-\varepsilon, +\varepsilon]$ . Choose  $x_0 \in \Sigma \times \{0\}$ ; the segment  $x_0 \times [-\varepsilon, +\varepsilon]$  is transverse to  $\mathcal{F}_\varphi$ . Let  $K$  be the surface obtained from  $\Sigma \times 0$  by removing a small open disk centered at  $x_0$ . The foliation  $\mathcal{F}_\varphi$  foliates  $K \times [-\varepsilon, +\varepsilon]$  so that  $K \times \{\pm\varepsilon\}$  lies in a leaf and  $\partial K \times [-\varepsilon, +\varepsilon]$  is foliated by parallel circles. We apply proposition 3.1 to this  $K$ .

**Corollary 3.2.** *There exists an open book foliation  $\mathcal{O}$  of  $M$  inducing the trivial foliation on  $K \times [-\varepsilon, +\varepsilon]$  (the leaves are  $K \times \{t\}$ ,  $t \in [-\varepsilon, +\varepsilon]$ ).*

Therefore, we have an open book foliation modified by suspension by replacing the above trivial foliation of  $K \times [-\varepsilon, +\varepsilon]$  by  $\tilde{\mathcal{F}}_\varphi$ , the trace of  $\mathcal{F}_\varphi$  on  $K \times [-\varepsilon, +\varepsilon]$ . Let  $\mathcal{O}_\varphi$  be the resulting foliation of  $M$  and  $\xi_\varphi$  be its regular  $\Gamma_1$ -structure. For proving theorem 1.2 it is sufficient to prove that  $\xi$  and  $\xi_\varphi$  are homotopic. This is done in the next section.

#### 4. HOMOTOPY OF $\Gamma_1$ -STRUCTURES

We are going to describe a homotopy from  $\xi_\varphi$  to  $\xi$ . Recall the tube  $T$  around the binding. For simplicity, we assume that each component of  $T$  is foliated by a standard Reeb foliation; the same holds true if  $T$  is foliated by Wood components (in the sense of [20]). Let  $T'$  be a slightly larger tube.

**Lemma 4.1.** *There exists a homotopy, relative to  $M \setminus \text{int}(T')$ , from  $\xi_\varphi$  to a new  $\Gamma_1$ -structure  $\xi_1$  on  $M$  such that:*

- 1)  $\xi_1$  is trivial on  $T$ ;
- 2)  $\xi_1$  is regular on  $T' \setminus \text{int}(T)$  with compact toric leaves near  $\partial T$  and spiraling half-cylinder leaves with boundary in  $\partial T'$  (as in an open book foliation).

**Proof.** Recall from the introduction that we only use *thick* Reeb components. So there is a third concentric tube  $T''$ ,  $T \subset \text{int}(T'') \subset \text{int}(T')$ , so that  $T'' \setminus \text{int}(T)$  is foliated by toric leaves; and  $\text{int}(T)$  is foliated by planes.

On each component of  $\partial T$  we have coordinates  $(x, y)$  coming from the framing of the binding, the  $x$ -axis being a parallel and the  $y$ -axis being a meridian. Let  $\gamma$  be a parallel in  $\partial T$ . The  $\Gamma_1$ -structure which is induced by  $\xi_\varphi$  on  $\gamma$  is singular but not trivial and the germ of foliation  $\mathcal{G}$  along the zero section in the normal line bundle  $A \cong \gamma \times \mathbb{R}$  is shown on figure 1.



The annulus  $A$  is endowed with coordinates  $(x, z) \in \gamma \times \mathbb{R}$ . The orientation of the  $z$ -axis, which is also the orientation of the normal bundle to the foliation  $\mathcal{O}_\varphi$  along  $\gamma$ , points to the interior of  $T$ . So the leaves of  $\mathcal{G}$  are parallel circles in  $\{z \leq 0\}$  and spiraling leaves in  $\{z > 0\}$ . Take coordinates  $(x, y, r)$  on  $T$  where  $r$  is the distance to the binding; say that  $r = 1$  on  $\partial T''$  and  $r = 1/2$  on  $\partial T$ . Let  $\lambda(r^2)$  be an even smooth function with  $r = 0$  as unique critical point, vanishing at  $r = 1/2$  and  $\lambda(1) < 0 < \lambda(0)$ . Consider  $g : T'' \rightarrow A$ ,  $g(x, y, r) = (x, \lambda(r^2))$ .

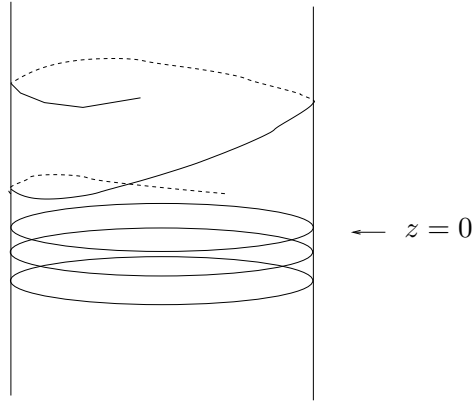


Figure 1

It is easily seen that  $\xi_\varphi|_{\text{int}(T'')} \cong g^*\mathcal{G}$ . Let now  $\bar{\lambda}(r^2)$  be a new even function coinciding with  $\lambda(r^2)$  near  $r = 1$ , having negative values everywhere and whose critical set is  $\{r \in [0, 1/2]\}$ . Let  $\bar{g} : T'' \rightarrow A$ ,  $\bar{g}(x, y, r) = (x, \bar{\lambda}(r^2))$ . A barycentric combination of  $\lambda$  and  $\bar{\lambda}$  yields a homotopy from  $g$  to  $\bar{g}$  which is relative to a neighborhood of  $\partial T''$ . The  $\Gamma_1$ -structure  $\xi_1$  we are looking for is defined by  $\xi_1|_{\text{int}(T'')} = \bar{g}^*(\mathcal{G})$  and  $\xi_1 = \xi_\varphi$  on a neighborhood of  $M \setminus \text{int}(T'')$ .  $\square$

Recall the domain  $K \times [-\varepsilon, +\varepsilon]$  from the previous section. After the following lemma we are done with the homotopy problem.

**Lemma 4.2.** *There exists a homotopy from  $\xi_1$  to  $\xi$  relative to  $K \times [-\varepsilon, +\varepsilon]$ .*

**Proof.** Let us denote  $M' := M \setminus \text{int}(K \times [-\varepsilon, +\varepsilon])$  which is a manifold with boundary and corners. It is equivalent to prove that the restrictions of  $\xi_1$  and  $\xi$  to  $M'$  are homotopic relatively to  $\partial M'$ . Consider the standard closed 2-disk  $D = D^2$  endowed with the  $\Gamma_1$ -structure  $\xi_D$  which is shown on figure 2.



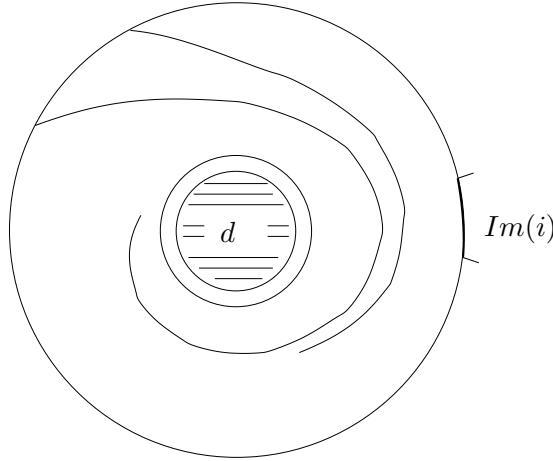


Figure 2

It is trivial on the small disk  $d$  and regular on the annulus  $D \setminus \text{int}(d)$ . In the regular part, the leaves are circles near  $\partial d$  and the other leaves are spiraling, crossing  $\partial D$  transversely. One checks that the restriction of  $\xi_1$  to  $M'$  has the form  $f^*\xi_D$  from some map  $f : M' \rightarrow D$ . We take  $f|_T : T \rightarrow d$  to be the open book trivialization of  $T$  (recall the binding has a canonical framing);  $f|\partial K \times [-\varepsilon, +\varepsilon]$  to be the projection  $pr_2$  onto  $[-\varepsilon, +\varepsilon]$  composed with an embedding  $i : [-\varepsilon, +\varepsilon] \rightarrow \partial D$  and  $f$  maps each leaf of the regular part of  $\xi_1$  to a leaf of the regular part of  $\xi_D$ . As  $K$  does not approach  $T$ , we can take  $f(\partial M') = i([-\varepsilon, +\varepsilon])$ ; actually, except near  $T$ ,  $f$  is given by the fibration over  $S^1 = \partial D^2$  of the open book decomposition.

Once  $\xi$  has Tsuboi's form (according to proposition 2.1), the restriction of  $\xi$  to  $M'$  has a similar form:  $\xi = k^*\xi_D$  for some map  $k : M' \rightarrow D^2$ . Recall that  $M'$  is the union of two handlebodies and a solid cylinder  $D^2 \times [-\varepsilon, +\varepsilon]$ . Take  $k$  to be  $i \circ pr_2$  on the cylinder and  $k$  to be constant on each handlebody. Observe that  $f$  and  $k$  coincide on  $\partial M'$ . As  $D$  retracts by deformation onto the image of  $i$ , one deduces that  $f$  and  $k$  are homotopic relatively to  $\partial M'$ .  $\square$

This finishes the proof of theorem 1.2.

## 5. THE CASE $C^{1+bv}$

A co-oriented  $\Gamma_1^r$ -structure  $\xi$  on  $M$  can be realized by a foliation  $\mathcal{F}$  defined on a neighborhood of the 0-section in  $M \times \mathbb{R}$ ; it is made of bi-foliated charts which are  $C^\infty$  in the direction of the leaves and  $C^r$  in the direction of the fibers. Consider such a box  $\mathcal{U}$  over an open disk  $D$  centered at  $x_0 \in M$ ; its trace on the  $x_0$ -fiber is an interval  $I$ . Each leaf of  $\mathcal{U}$  reads  $z = f(x, t)$ ,  $x \in D$ , for some  $t \in I$ . Here  $f$  is a function which is smooth in  $x$  and  $C^r$  in  $t$ , with  $f(x_0, t) = t$ ; the foliating property is equivalent to  $\frac{\partial f}{\partial t} > 0$ . When  $r = 1 + bv$ , there is a positive measure  $\mu(x, t)$  on  $I$ , without atoms and depending smoothly on  $x$ , such that:

$$(*) \quad \frac{\partial}{\partial t} f(x, t) - \frac{\partial}{\partial t} f(x, t_0) = \int_{t_0}^t \mu(x, t).$$

**Proposition 5.1.** *Theorem 1.2 holds true for any class of regularity  $r \geq 1$  including the class  $r = 1 + bv$ .*

**Proof.** The only part of the proof which requires some care of regularity is section 2, especially the proof of lemma 2.3. Indeed, we have to avoid integrating  $C^0$  vector fields. For proving lemma 2.3 with weak regularity we use the lemmas below which we shall prove in the case  $r = 1 + bv$  only.

**Lemma 5.2.** *Let  $f : D \times I \rightarrow \mathbb{R}$  be a  $C^r$ -function as above. Assume  $I = ] - 2\varepsilon, +\varepsilon[$  and  $f(x, -2\varepsilon) > -1$  for every  $x \in D$ . Then there exists a function  $F : D \times [-1, 0] \rightarrow \mathbb{R}$  of class  $C^r$  such that:*

- 1)  $F(x, t) = f(x, t)$  when  $t \in [-\varepsilon, 0]$ ,
- 2)  $F(x, t) = t$  when  $t$  is close to  $-1$ ,
- 3)  $\frac{\partial F}{\partial t} > 0$ .

**Proof.** Let  $\mu(x, t)$  be the positive measure whose support is  $[-\varepsilon, 0]$  such that formula (\*) holds for every  $(x, t) \in D \times [-\varepsilon, 0]$  and  $t_0 = 0$ . There exists another positive measure  $\nu(x, t)$ , smooth in  $x$  and whose support is contained in  $] - 1, -\varepsilon]$ , such that

$$(**) \quad f(x, -\varepsilon) = -1 + \int_{-1}^{-\varepsilon} \left( \int_{-1}^t \nu(x, \tau) \right) dt.$$

Then a solution is

$$F(x, t) = -1 + \int_{-1}^t \left( \int_{-1}^s (\mu(x, \tau) + \nu(x, \tau)) \right) ds.$$

□

**Lemma 5.3.** *Let  $A_1$  and  $A_2$  two disjoint closed sub-disks of  $D$ . Let  $F_1$  and  $F_2$  be two solutions of lemma 5.2. Then there exists a third solution which equals  $F_1$  when  $x \in A_1$  and  $F_2$  when  $x \in A_2$ .*

**Proof.** Both solutions  $F_1$  and  $F_2$  differ by the choice of the measure  $\nu(x, t)$  in formula (\*\*), which is  $\nu_i$  for  $F_i$ . Choose a partition of unity  $1 = \lambda_1(x) + \lambda_2(x)$  with  $\lambda_i = 1$  on  $A_i$ . Then  $\nu(x, t) = \lambda_1(x)\nu_1(x, t) + \lambda_2(x)\nu_2(x, t)$  yields the desired solution. □

The proof of proposition 5.1 is now easy. As already said, it is sufficient to prove lemma 2.3 in class  $C^r$ ,  $r \geq 1$ . It is an extension problem of a foliation given near the 1-skeleton  $Tr^{[1]} \times \{0\}$  to  $Tr^{[1]} \times [-1, 0]$ . One covers  $Tr^{[1]}$  by finitely many  $n$ -disks  $D_j$ . The problem is solved in each  $D_j \times [-1, 0]$  by applying lemma 5.2. By applying lemma 5.3 one makes the different extensions match together. □

## 6. HOMOTOPY CLASS OF PLANE FIELDS

It is possible to enhance theorem 1.1 by prescribing the homotopy class of the underlying co-oriented plane field (see proposition 6.1 below). The question of doing the same with respect

to theorem 1.2 is more subtil (see proposition 6.3).

**Proposition 6.1.** *Given a co-oriented  $\Gamma_1$ -structure  $\xi$  on the closed 3-manifold  $M$  and a homotopy class  $[\nu]$  of co-oriented plane field in the tangent space  $\tau M$ , there exists a regular  $\Gamma_1$ -structure  $\xi_{reg}$  homotopic to  $\xi$  whose underlying foliation  $\mathcal{F}_{reg} \cap M$  has a tangent co-oriented plane field in the class  $[\nu]$ .*

Before proving it we first recall some well-known facts on co-oriented plane fields (see [4]). Given a base plane field  $\nu_0$ , a suitable Thom-Pontryagin construction yields a natural bijection between the set of homotopy classes of plane fields on  $M$  and  $\Omega_1^{\nu_0}(M)$ , the group of (co)bordism classes of  $\nu_0$ -framed and oriented closed (maybe non-connected) curves in  $M$ . A  $\nu_0$ -framing of the curve  $\gamma$  is an isomorphism of fiber bundles  $\varepsilon : \nu(\gamma, M) \rightarrow \nu_0|_\gamma$ , whose source is the normal bundle to  $\gamma$  in  $M$ . We denote  $\gamma^\varepsilon$  the curve endowed with this framing. Moreover, given  $\gamma^\varepsilon$ , if  $\gamma'$  is homologous to  $\gamma$  in  $M$  there exists a  $\nu_0$ -framing  $\varepsilon'$  such that  $(\gamma')^{\varepsilon'}$  is cobordant to  $\gamma^\varepsilon$ .

**Proof of 6.1.** We can start with an open book foliation  $\mathcal{O}_\varphi$  yielded by theorem 1.2. Let  $\nu_0$  be its tangent plane field. Near the binding, the meridian loops (out of  $T$ ) are transverse to  $\mathcal{O}_\varphi$  and homotopic to zero in  $M$ . As a consequence, each 1-homology class may be represented as well by a (multi)-curve in a page or by a connected curve out of  $T$  positively transverse to all pages. We do the second choice for  $\gamma^\varepsilon$ , the  $\nu_0$ -framed curve whose cobordism class encodes  $[\nu]$  with respect to  $\nu_0$ .

Hence we are allowed to *turbulize*  $\mathcal{O}_\varphi$  along  $\gamma$ . In a small tube  $T(\gamma)$  about  $\gamma$ , we put a Wood component. Outside, the leaves are spiraling around  $\partial T(\gamma)$ . Let  $\mathcal{O}_\varphi^{turb}$  be the resulting foliation. Whatever the chosen type of Wood component is, the  $\Gamma_1$ -structures of  $\mathcal{O}_\varphi^{turb}$  and  $\mathcal{O}_\varphi$  are homotopic by arguing as in section 4. But the framing  $\varepsilon$  tells us which sort of Wood component will be convenient for getting the desired class  $[\nu]$  (see lemma 6.1 in [20]).  $\square$

In the previous statement, we have lost the nice model we found in theorem 1.2. Actually, thanks to a lemma of Vincent Colin [3], it is possible to recover our model, at least when  $M$  is orientable (see below proposition 6.3).

**Lemma 6.2.** (Colin) *Let  $(B, p)$  be an open book decomposition of  $M$  and  $\gamma$  be a simple connected curve in some page  $P$ . Assume  $\gamma$  is orientation preserving. Then there exist a positive stabilization  $(B', p')$  of  $(B, p)$  and a curve  $\gamma'$  in  $B'$  which is isotopic to  $\gamma$  in  $M$ . When  $\gamma$  is a multi-curve, the same holds true after a sequence of stabilizations.*

The *positive Hopf open book decomposition* of the 3-sphere is the one whose binding is made of two unknots with linking number  $+1$ ; a page is an annulus foliated by fibers of the Hopf fibration  $S^3 \rightarrow S^2$ . A positive stabilization is a “connected sum” with this open book. The new page  $P'$  is obtained from  $P$  by *plumbing* an annulus  $A$  whose core bounds a disk in  $M$  (see [7] for more details and other references).

**Proof.** If  $\gamma$  is connected, only one stabilization is needed. We are going to explain this case only. A tubular neighborhood of  $\gamma$  in  $P$  is an annulus.

Choose a simple arc  $\alpha$  in  $P$  joining  $\gamma$  to some component  $\beta$  of  $B$  without crossing  $\gamma$  again. Let  $\tilde{\gamma}$  be a simple arc from  $\beta$  to itself which follows  $\alpha^{-1} * \gamma * \alpha$ . The orientation assumption

implies that the surgery of  $\beta$  by  $\tilde{\gamma}$  in  $P$  provides a curve with two connected components, one of them being isotopic to  $\gamma$  in  $P$ . Let  $P_\pi$  be the page opposite to  $P$  and  $R : P \rightarrow P_\pi$  the time  $\pi$  of a flow transverse to the pages (and stationary on  $B$ ). The core curve  $C$  of the annulus  $A$  that we use for the plumbing is the union  $\tilde{\gamma} \cup R(\tilde{\gamma})$ . And  $A$  is  $(+1)$ -twisted around  $C$  (with respect to its unknot framing) as in the Hopf open book. Let  $H$  be the 1-handle which is the closure of  $A \setminus P$ . Surgering  $B$  by  $H$  provides the new binding. By construction, one of its components is isotopic to  $\gamma$ .  $\square$

**Proposition 6.3.** *Let  $\mathcal{O}_\varphi$  be an open book foliation modified by suspension, whose underlying open book is denoted  $(B, p)$ . Let  $\nu_0$  be its tangent co-oriented plane field. Let  $\gamma^\varepsilon$  be a  $\nu_0$ -framed curve in  $M$  and  $[\nu]$  be its associated class of plane field. Assume  $\gamma$  is orientation preserving. Then there exists an open book foliation  $\mathcal{O}'_\varphi$  with the following properties:*

- 1) *its tangent plane field is in the class  $[\nu]$ ;*
- 2) *the suspension modification is the same for  $\mathcal{O}'_\varphi$  as for  $\mathcal{O}_\varphi$  and is supported in  $K \times [-\varepsilon, +\varepsilon]$ ;*
- 3) *as  $\Gamma_1$ -structures,  $\mathcal{O}_\varphi$  and  $\mathcal{O}'_\varphi$  are homotopic.*

**Proof.** As said in the proof of 6.1, up to framed cobordism,  $\gamma^\varepsilon$  may be chosen as a simple (multi)-curve in one page  $P$  of  $(B, p)$ . Applying Colin's lemma provides a stabilization  $(B', p')$  such that, up to isotopy,  $\gamma$  lies in the new binding. Observe that, if  $K$  is in  $P$ ,  $K$  is still in the new page  $P'$ ; hence 2) holds for any open book foliation carried by  $(B', p')$ . Once  $\gamma^\varepsilon$  is in the binding, for a suitable Wood component foliating a tube about  $\gamma^\varepsilon$ , item 1) is fulfilled. Finally item 3) follows from item 2) and the proofs in section 4.  $\square$

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